6.10 Contracting edge $(0,1),(2,3),(4,5),(5,6),(6,7),(8,9),(10,11),(11,12),(12,13)$ and deleting edge $(4,13),(5,10)$ yield a K_5, i.e. $K_5 \leq G$. According to **Wagner Thm**, Headwood graph is not planar graph.

6.16

6.17 According to **Property 6.26**, because $K_{4,4}$ is triangular-free graph, $t(K_{4,4}) \geq \lceil \frac{16}{12} \rceil = 2$. $K_{4,4}$ can be formed by combine two $K_{4,2}$. Hence $t(K_{4,4}) = 2$.

Similarly, $t(K_{5,5}) \geq \lceil \frac{25}{16} \rceil = 2$. Construction of two planar graph to form $K_{5,5}$ as Figure 1.

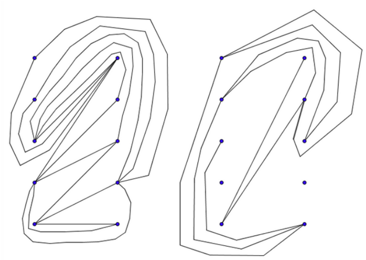

Figure 1: Construction of $K_{5,5}$

6.19

(a) $3c(K_{n,n}) \leq f(n) \leq 3\binom{n}{2}^2$. *Lower Bound* Consider 3 $K_{n,n}$ in $K_{n,n,n}$ and each crossing is calculated only once. Hence the lower bound is $3c(K_{n,n})$.
Upper Bound Consider $\binom{3}{2}$ selections of two partite sets and $\binom{n}{2}$ selections from each partite sets. Then yield the upper bound $3\binom{n}{2}^2$. $\square$

(b) *Proof.* $\square$

(c) $f(n) \geq n^3(n-1)/6$. $K_{n,n,n}$ contains n^3 $K_{n-1,n-1,n-1}$. Crossing formed by vertices two partite sets is counted $n(n-2)^2$ times. Crossing formed by vertices from three partite sets is counted $(n-1)^2(n-2)$ times. Then each crossing is counted at most $(n-1)^2(n-2)$ times. Then $f(n) \geq \frac{n^3}{(n-1)^2(n-2)} f(n-1)$. By recurrence, $f(n) \geq n^3(n-1)f(3)/54$. And from (2), substitute $f(3) \geq 9$ to last inequation yields the result. $\square$

(d) $f(n) \leq \frac{9}{16} + O(n^3)$. $\square$

6.20

(a) $c(K_{m,n}) \geq m\frac{m-1}{5}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. $K_{m,n}$ has $\binom{m}{6}$ $K_{6,n}$. Assume w,x,y,z exists a crossing; x,y are among the 6 vertices while w,z are among the n vertices. The crossing appears $\binom{m-2}{4}$ times.
Hence, $\binom{m-2}{4}c(K_{m,n}) \geq \binom{m}{6}c(K_{6,n})$, which substitutes $c(K_{6,n})$ will yield the target result. $\square$

(b) $c(K_n) \geq \frac{1}{80}n^4 + O(n^3)$. Consider copies $K_{6,n-6}$ in K_n. There will be $\binom{n}{6}$ copies. As for a crossing w, x, y, z, it appears $(\binom{4}{2} - 2)\binom{n-4}{4} = 4\binom{n-4}{4}$ times.
Hence $4\binom{n-4}{4}c(K_n) \geq \binom{n}{6}c(K_{6,n})$. Replacing the inequation will yield the result. $\square$

Bibliography

[1] Thickness of graph, retrieve from: mathworld.wolfram.com/GraphThickness.html