5.2

(1) $|S \cap X| = a, |S \cap Y| = b, S, \overline{S} \neq \emptyset, a, b \in \mathbb{N}$. We have $0 \leq a \leq m, 0 \leq b \leq n, 0 < a + b < m + n$
$|[S, \overline{S}]| = a(n - b) + b(m - a) = an + bm - 2ab$

(2)

$$\begin{aligned} \kappa(K_{m,n}) &= \min |[S, \overline{S}]| \\ &= \min_{0<a+b<m+n} (an + bm - 2ab) \\ &= \min\{m, n\} \end{aligned} \tag{1}$$

Let $f(a, b) = an + bm - 2ab$. We have $\frac{\partial f}{\partial a} = n - 2b$, $\frac{\partial f^2}{\partial a^2} = 0$; $\frac{\partial f}{\partial b} = m - 2a$, $\frac{\partial f^2}{\partial b^2} = 0$
The domain of f is a plain of a, b except $(0, 0), (m, n)$, as in Figure 1. From above, fixing a or b, f is monotone on the other variable. The minimum of f will only get from the point in Figure 1.
Calculate value of these six points, $(1, 0), (0, 1), (m - 1, n), (m, n - 1)$ will let f get the m or n.

(3) *Arbitrary seven edges will be a disconnecting set.* $K3, 3$ has 9 edges in total. After removing arbitrary 7 edges, only 2 edges remain. At most 3 vertices are connected in a component. Therefore, set of arbitrary 7 edges will be a disconnecting set.
No such an edge cut contains seven edges. Because each vertex has degree of 3, the size of any edge cut will be times of 3.

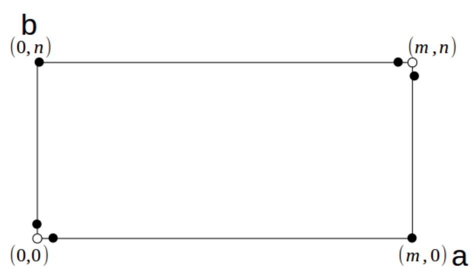

Figure 1: Domain of f

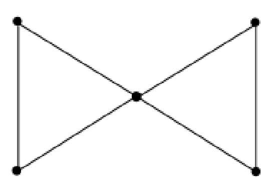

Figure 2: n=5

5.4 $\delta(G) \leq n(G) - 1$. When $\delta(G) = n(G) - 1$, G is K_n. It is true $\kappa(K_n) = \delta(K_n)$.
According to **Thm 5.4**, we have $\kappa(G) \leq \delta(G)$. When $\delta(G) = n(G) - 2$, two disconnected vertices x, y always exists. Let $S = V(G) \backslash x, y$. S will be a vertex cut. So $\kappa(G) \geq n(G) - 2 = \delta(G)$. Then $\kappa(G) = \delta(G)$.
With the result above, $\kappa(G) = \delta(G)$ given $\delta(G) \geq n(G) - 2$.
When n=5, graph of Figure 2 satisfies $\delta(G) = n - 3, \kappa(G) < n - 3$.

5.6 *Proof.* $\Delta(G) \leq 3$ means $deg(v) \leq 3, v \in G$. Similar with proof of **Thm 5.6**, let S be the smallest vertex cut. We just need to find a disconnecting set F with the same size of S. Then we will finish proof because $\kappa(G) \leq \kappa'(G)$.

First, if $\exists v : deg(v) = 1$, $\kappa(G) = \kappa'(G) = 1$. Otherwise, let H_1, H_2 be two components of $G - S$. For $v \in S$, because S is the smallest vertex cut. v has neighbors in both H_1, H_2. In the proof of **Thm 5.6**, for $v : deg(v) = 3$, we denote the set as S_1. We have selected $|S_1|$ edges as edges of F. Then We discuss about $v : deg(v) = 2$ denoted by set S_2. For $v : deg(v) = 2$, v has one neighbor in H_1, H_2 respectively. Then we select one edge of the two as edge in F. Therefore, F of size $|S_1 + S_2| = |S|$ will cut all paths between H_1 and H_2. □

5.8 We know that number of edges of a cycle with n vertices is n. Block of edge contains two vertices and one edge. Combination of blocks of edge will form a tree, which has $n-1$ edges. Combinations of blocks of cycle will have $n+k$ edges, k is the number of blocks. The cactus with the most edges must contains most blocks of cycle. So cycle of 3 nodes will be the choice.
As in Figure 3, all components share a common vertex. The number of edges will be $\lfloor 3(n-1)/2 \rfloor$.

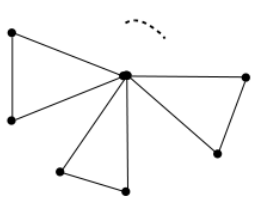

Figure 3: Cactus with most edges

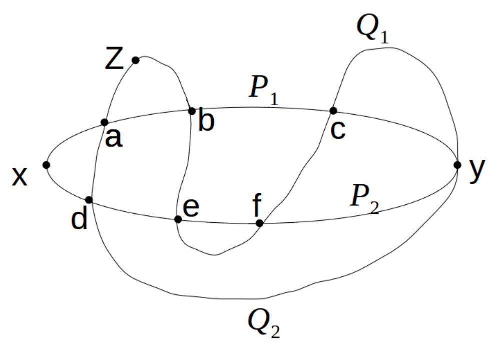

Figure 4: Case of $P_i \cap Q_j \neq \emptyset$

5.10 *Sufficiency.* Sufficiency is obvious. Remove arbitrary vertex x, we can always find a path between an arbitrary pair of vertex, i.e. connected. With **Thm 5.10(1)**, G is 2-connected.
Necessity. With **Thm 5.10(2)**, for any x, y, we can find two disjoint paths $x - y$, denoted by P_1, P_2. Similarly, two disjoint paths $y - z$ are denoted by Q_1, Q_2. If $P_i \cap Q_j = \emptyset, i, j = 1, 2$, such a path $x - z$ that go through y exists as $P_i + Q_j$.
If $P_i \cap Q_j \neq \emptyset, \forall i, j = 1, 2$, Figure 4 as an example, we can also find a path $x - z$ going through y. For example, path x go along P_2 to y, then along P_1 to b, finally along Q_1 to z is such a path.