

## 5.2

- (1)  $|S \cap X| = a, |S \cap Y| = b, S, \bar{S} \neq \emptyset, a, b \in \mathbb{N}$ . We have  $0 \leq a \leq m, 0 \leq b \leq n, 0 < a + b < m + n$

$$|[S, \bar{S}]| = a(n - b) + b(m - a) = an + bm - 2ab$$

(2)

$$\begin{aligned} \kappa(K_{m,n}) &= \min |[S, \bar{S}]| \\ &= \min_{0 < a+b < m+n} (an + bm - 2ab) \\ &= \min\{m, n\} \end{aligned} \quad (1)$$

Let  $f(a, b) = an + bm - 2ab$ . We have  $\frac{\partial f}{\partial a} = n - 2b, \frac{\partial f^2}{\partial a^2} = 0; \frac{\partial f}{\partial b} = m - 2a, \frac{\partial f^2}{\partial b^2} = 0$ . The domain of  $f$  is a plain of  $a, b$  except  $(0, 0), (m, n)$ , as in Figure ???. From above, fixing  $a$  or  $b$ ,  $f$  is monotone on the other variable. The minimum of  $f$  will only get from the point in Figure ???.

Calculate value of these six points,  $(1, 0), (0, 1), (m - 1, n), (m, n - 1)$  will let  $f$  get the  $m$  or  $n$ .

- (3) *Arbitrary seven edges will be a disconnecting set.*  $K_{3,3}$  has 9 edges in total. After removing arbitrary 7 edges, only 2 edges remain. At most 3 vertices are connected in a component. Therefore, set of arbitrary 7 edges will be a disconnecting set. *No such an edge cut contains seven edges.* Because each vertex has degree of 3, the size of any edge cut will be times of 3.

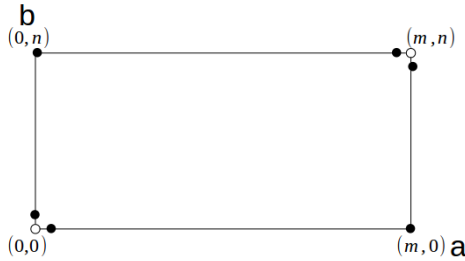


Figure 1: Domain of  $f$

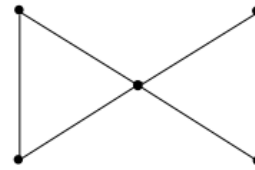


Figure 2:  $n=5$

- 5.4**  $\delta(G) \leq n(G) - 1$ . When  $\delta(G) = n(G) - 1$ ,  $G$  is  $K_n$ . It is true  $\kappa(K_n) = \delta(K_n)$ . According to **Thm 5.4**, we have  $\kappa(G) \leq \delta(G)$ . When  $\delta(G) = n(G) - 2$ , two disconnected vertices  $x, y$  always exists. Let  $S = V(G) \setminus \{x, y\}$ .  $S$  will be a vertex cut. So  $\kappa(G) \geq n(G) - 2 = \delta(G)$ . Then  $\kappa(G) = \delta(G)$ . With the result above,  $\kappa(G) = \delta(G)$  given  $\delta(G) \geq n(G) - 2$ . When  $n=5$ , graph of Figure ?? satisfies  $\delta(G) = n - 3, \kappa(G) < n - 3$ .

- 5.6** *Proof.*  $\Delta(G) \leq 3$  means  $\deg(v) \leq 3, v \in G$ . Similar with proof of **Thm 5.6**, let  $S$  be the smallest vertex cut. We just need to find a disconnecting set  $F$  with the same size of  $S$ . Then we will finish proof because  $\kappa(G) \leq \kappa'(G)$ .

First, if  $\exists v : \deg(v) = 1$ ,  $\kappa(G) = \kappa'(G) = 1$ . Otherwise, let  $H_1, H_2$  be two components of  $G - S$ . For  $v \in S$ , because  $S$  is the smallest vertex cut.  $v$  has neighbors in both  $H_1, H_2$ . In the proof of **Thm 5.6**, for  $v : \deg(v) = 3$ , we denote the set as  $S_1$ . We have selected  $|S_1|$  edges as edges of  $F$ . Then We discuss about  $v : \deg(v) = 2$  denoted by set  $S_2$ . For  $v : \deg(v) = 2$ ,  $v$  has one neighbor in  $H_1, H_2$  respectively. Then we select one edge of the two as edge in  $F$ . Therefore,  $F$  of size  $|S_1 + S_2| = |S|$  will cut all paths between  $H_1$  and  $H_2$ .  $\square$

**5.8** We know that number of edges of a cycle with  $n$  vertices is  $n$ . Block of edge contains two vertices and one edge. Combination of blocks of edge will form a tree, which has  $n - 1$  edges. Combinations of blocks of cycle will have  $n + k$  edges,  $k$  is the number of blocks. The cactus with the most edges must contains most blocks of cycle. So cycle of 3 nodes will be the choice.

As in Figure ??, all components share a common vertex. The number of edges will be  $\lfloor 3(n - 1)/2 \rfloor$ .

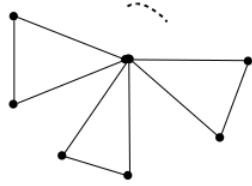


Figure 3: Cactus with most edges

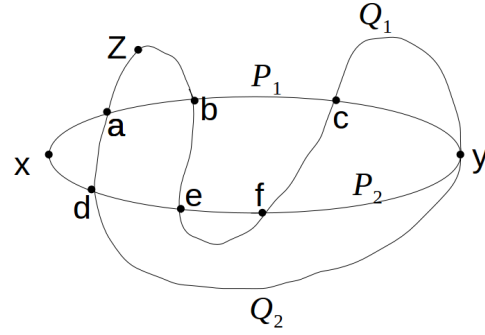


Figure 4: Case of  $P_i \cap Q_j \neq \emptyset$

**5.10 Sufficiency.** Sufficiency is obvious. Remove arbitrary vertex  $x$ , we can always find a path between an arbitrary pair of vertex, i.e. connected. With **Thm 5.10(1)**,  $G$  is 2-connected.

*Necessity.* With **Thm 5.10(2)**, for any  $x, y$ , we can find two disjoint paths  $x - y$ , denoted by  $P_1, P_2$ . Similarly, two disjoint paths  $y - z$  are denoted by  $Q_1, Q_2$ . If  $P_i \cap Q_j = \emptyset, i, j = 1, 2$ , such a path  $x - z$  that go through  $y$  exists as  $P_i + Q_j$ .

If  $P_i \cap Q_j \neq \emptyset, \forall i, j = 1, 2$ , Figure ?? as an example, we can also find a path  $x - z$  going through  $y$ . For example, path  $x$  go along  $P_2$  to  $y$ , then along  $P_1$  to  $b$ , finally along  $Q_1$  to  $z$  is such a path.