8.7 *Proof.* (1) $\Rightarrow$ (2): Because $d_k \geq n/2$, $deg(u) + deg(v) \geq n$. $\square$

Proof. (2) $\Rightarrow$ (3): Prove the contrapositive. If $\exists k < n/2, d_k \leq k$. We claim a contradiction that $\exists uv \notin E(G), deg(u) + deg(v) < n$. Assume v_k as the smallest k for $d_k \leq k$. Then $k - 1 < d_{k-1} \leq k$, i.e., $d_{k-1} = d_k = k$. We prove that such uv mentioned above exists under worst case. For $1 \leq i \leq k$, it must be a clique, if not, such an edge uv exists among them.

The remaining $n-k$ vertices form a clique with degree of $n-k+1$. From above, for $v_i, 1 \leq i \leq k$, there are at most k edges join them with $v_j, j > k$. Because edges joining $v_i, 1 \leq i \leq k$ with $v_j, k < j \leq n$ join in backward order(from v_n to v_{k+1}) and $k < n/2$, no edge will join $v_{k+1}$with $v_i, 1 \leq i \leq k$, i.e.,$deg(v_{k+1}) = n - k - 1$. Then $v_k v_{k+1}, deg(v_k) + deg(v_{k+1}) = n - 1 < n$, which forms the contradiction. $\square$

Proof. (3) $\Rightarrow$ (4): From condition, we have $i \geq \lfloor n/2 \rfloor, d_i \geq n/2$. For $d_j \leq j, d_k < k$, $j \geq n/2, k \geq n/2$. Then $d_j + d_k \geq n$. $\square$

Proof. (4) $\Rightarrow$ (5): We prove the contrapositive. If $d_k \leq k < n/2$ and $d_{n-k} < n - k$, since $n - k > k$, then $d_k + d_{n-k} < n$, which is a contradiction. $\square$

Proof. (5) $\Rightarrow$ (6): We prove the contrapositive. If i, j satisfy those conditions, we assume $d_i + d_j < n$. Then contradiction against (5) should be found. Without loss of generality, assume $i < j$.

If $i + j = n$, $i < n/2$ which satisfies condition of (5). Then $d_{n-i} = d_j \geq j$, a contradiction. If $i + j > n$, we assume $d_i = i$. Since $2d_i \leq d_i + d_j < n$, $i < n/2$. Then $n = i + (n - i) \leq d_i + d_{n-i} \leq d_i + d_j < n$, a contradiction. For $d_i < i$, since d_i is increasing sequence, the property also hold. $\square$

Proof. (6) $\Rightarrow$ (7): Assume i, j as $\max\{i+j\}$ which satisfies $1 \leq i, j \leq n, v_i v_j \notin E(G)$. Without lack of generality, we assume $i < j$. $v_j \ldots v_n$ form a clique, otherwise a bigger pair exists. And $d_j \geq n - i - 1, d_i \geq n - j$. Then, if $d_i > i$, $d_i + d_j > n - 1$, i.e., $d_i + d_j \geq n$. If $d_j \geq j$, $d_i + d_j \geq n$. If $i + j < n$, $d_i + d_j \geq 2n - (i + j) - 1 \geq n$. Hence, v_i, v_j will be linked according to definition of Hamiltonian closure.

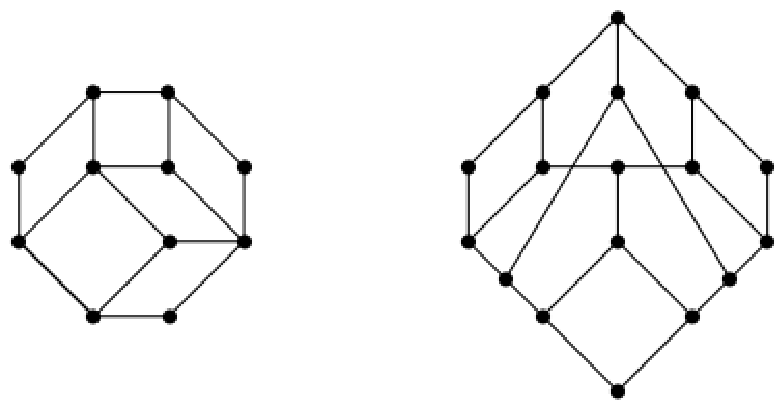
After adding the edge $v_i v_j$ forming G', G' still satisfy the **Las Vergnas Condition**. Iteratively adding such edges yields K_n. $\square$

8.9 *Proof. Every unique 3-edge-coloring 3-regular graph has Hamiltonian cycle.* Graph G is uniquely 3-edge-colored and 3-regular. One color class will be a perfect matching. Two classes named H is 2-factor, which is union of cycles. If H is just a single cycle, it will be a Hamiltonian cycle. If not, choose one of the cycle, switching the color yields another 3-edge-coloring, a contradiction against the uniquely 3-edge-coloring. $\square$

8.10 *Proof. A plane triangulation has a vertex partition into two sets inducing forests if and only if the dual is Hamiltonian.* Let the plane triangulation be G, $G^* = F$.

"$\Leftarrow$" F has a Hamiltonian cycle C. H consisting of C and some edges of F inside C is outerplanar. Then the H^* without the vertex of outer face of H is one set of G, inducing forests. This argument also applies to the graph consisting of cycle C and edges of F outside C.

"$\Rightarrow$" We can always find a vertex partition to get two trees $S, \bar{S}$. Then $[S, \bar{S}]$ is a bond. Dual of edges $[S, \bar{S}]$ will form a cycle. We claim that this cycle is a spanning cycle. Since triangulation G has $3n-6$ edges and two trees consist of $n-2$ edges, $|[S, \bar{S}]| = 2n-4$ and $n(F) = f(G) = 2n-4$. Such a cycle is Hamiltonian cycle. □



8.11

(a) *Proof.* Both the graphs are non-Hamiltonian graphs. The **Grinberg Theorem** requires that $\sum_i (i-2)(f_i' - f_i'') = 0$. The graph G_1 has six 4-edge faces and one 8-edge face. Then we have $2(f_4' - f_4'') + 6(f_8' - f_8'') = 0$. Rewrite the equation to get $(f_4' - f_4'') + 3(f_8' - f_8'') = 0$. And $f_4' + f_4'' = 6, f_8' + f_8'' = 1$. The first term will be even while the second term is odd, so the equation will not be satisfied.

For the second graph G_2, redraw it as a plan graph. G_2 has three 3-edge faces and six 6-edge faces. The equation $(f_3' - f_3'') + 4(f_6' - f_6'') = 0$ will not hold. Because $f_3' + f_3'' = 3, f_6' + f_6'' = 6$, $f_3' - f_3'' = \pm 1, \pm 3$ and $f_6' - f_6'' = 0, \pm 2, \pm 4$. The first term of the equation is odd while the second is even. □

8.13 *Proof.* Similar with problem **8.11**, Hamiltonian graph requires the equation $3(f_5' - f_5'') + 6(f_8' - f_8'') + 7(f_9' - f_9'') = 0$. $f_5' + f_5'' = 21, f_8' + f_8'' = 3, f_9' - f_9'' = 1$. Let $x = f_5' - f_5'', y = f_8' - f_8'', z = f_9' - f_9''$. x is odd, y is odd, $z = \pm 1$. Rewrite the equation, $3(x + 2y) + 7z = 0$. The first term is multiple of 3, while the second is ± 7. The equation doesn't hold. □