7.13 *Proof. All planar graph with at most 12 vertices can be 4-colored.* We prove by induction on $n(G)$. For planar graph, we have $e \leq 3n - 6$ and $\delta(G) \leq 5$. $\sum_{v \in V} deg(v) = 2e \leq 60$. When $\delta(G) = 5$, it can only be icosahedron, which is 4-colorable.

For $\delta(G) \leq 4$, we can always find a vertex $\{v : deg(v) \leq 4\}$. By induction hypothesis, the remaining graph without v is 4-colorable. For $\delta(G) = 4$, we can find a Kempe chain. If the chain connects two non-sequential neighbours of v, we can always find another Kempe chain which will not connect non-sequential neighbours of v. Then exchanging the colors on the chain yields a color for v. That's all.

Planar graph with at most 32 edges is 4-colorable. For planar graph, $\delta(G) \leq 5$. When $\delta(G) = 5$, $5 \cdot 13 > 2e = 64$ which implies that $n(G) \leq 12$. Apply above conclusion, it is 4-colorable. When $\delta(G) \leq 4$, it is the same way with above to prove 4-colorable. □

7.16 *Proof. For the graph G_n with even n, every proper 4-coloring of G_n uses each color on exactly n vertices.* G_n is constructed by adding a new 4-cycle surrounding G_{n-1}. G_2 is as Figure 1. When n is even, two consecutive 4-cycles will be isomorphic to G_2. If such G_2 is proved each color class covers exactly 2 vertices, adjustment could be implemented to color all consecutive G_2. Following is the proof of proper 4-coloring of G_2 using each color on exactly 2 vertices.

It is easy to give a proper 4-coloring of G_2. What's more, one color class could cover at most 2 vertices since we cannot find three vertices which are dis-adjacent to each other. Therefore, one color covers exactly 2 vertices. □

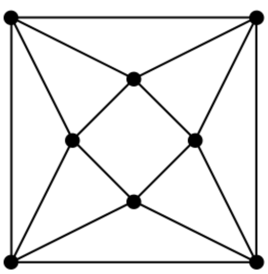

Figure 1: G_2

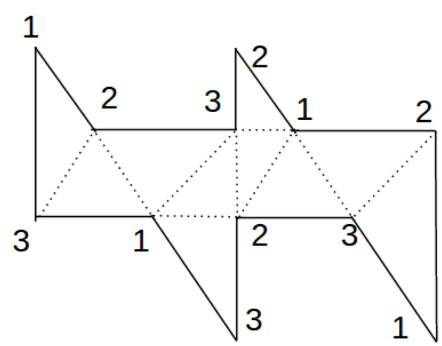

Figure 2: An example of Art Gallery

7.17

(a) *Proof.* (With **Four Color Thm**) *Every outerplanar graph is 3-chromatic.* Outerplanar graph is planar. So it can be 4-colorable according to **Four Color Thm**. Given a sequence of a outerplanar and using greedy coloring algorithm, only when four vertices v_a, v_b, v_c, v_d are adjacent to each other, i.e., subdivision of K_4 as a subgraph, 4 colors are needed(**Dirac's Thm**). However, outerplanar graph is K_4-free. Therefore, outerplanar graph is 3-chromatic. □

(b) *Proof.* (Without **Four Color Thm**) *Every outerplanar graph is 3-chromatic.* We prove by giving a 3-coloring of outerplanar graph. Outerplanar graph has all vertices as the boundary of exterior plane.

If an outerplanar graph G has cycles C_i, every two cycles share only one edge(if not, some vertex will not be the boundary of exterior plane). And see C_i as vertex sequence of graph G', no cycle exists in G', i.e., a tree.

We have known that C_i is 3-chromatic(when $n(C_i)$ is odd, 3-coloring is needed). When one cycle is colored, we can always color the adjacent cycles with 3 colors until all cycles are colored. The remaining parts, which are trees, can be 2-colored.

If no cycle exists, G will be a tree, which is 2-chromatic. Then, a 3-coloring is given. □

(c) *Art Gallery Theorem.* Given a polygon as Figure 2, we can add some straight lines to triangulate the polygons. Consider a 3-coloring of such a polygon so that each triangular is colored with 3 colors(see Figure 2). Then each color class has at least $\lfloor n/3 \rfloor$ elements. In each triangular, the corner can see all points of the triangle. Hence, put a guard at the place of one color class will see the whole art gallery, i.e., $\lfloor n/3 \rfloor$ guards are needed.

7.19 *Whether exist such an edge xy that* $\min_{xy \in E}(deg(x) + deg(y)) > 11$. Add a vertex in each plane of regular icosahedron and join the vertex with all the vertices of the plane $\min_{xy \in E}(deg(x) + deg(y)) = 13$.

7.20 *Proof. Every torus-embedded graph is 7 chromatic.* For a torus-embedded graph G, if we can find a region which has at most 6 adjacent regions, then induct on the $n(G)$. Given G, when we shrink a region to a point, then with the induction hypothesis, the remaining graph will be 7 chromatic. Add the removing region will always be colored with one of the 7 colors because it has only six adjacent regions.

Now, we prove such a region exists. Because each region has at most 3 edges, each region will have at least 3 adjacent regions, i.e., $3n \leq 2e$. With Eular's formula on torus $n - e + f = 0$, we have $e \leq 3f$. If every region has more than 7 adjacent regions, $7f \leq 2e$. Then $2e \geq 7f \geq 6f \geq 2e$, which is a contradiction. □