Notations: $n(G)$: number of vertices of graph G.

4.1 *Proof.* Let g_n denote number of perfect matching of Q_n.
Assume a series $h_n = 2^{2^{n-2}}$, we have $h_n = h_{n-1}^2$. First, $g_2 = h_2$. Then to prove $g_n \geq h_n = h_{n-1}^2$, it is sufficient to prove $g_n \geq g_{n-1}^2$.
Q_n is induced by Q_{n-1}. $n(Q_n) = 2n(Q_{n-1})$. Q_n is formed by Q_{n-1} with a more bit sequence and its counterpart Q'_{n-1} having the same topology with Q_{n-1}. Vertex in one part has 1-to-1 link with vertex of the other part.
In this way, Q_{n-1}, Q'_{n-1} both have g_{n-1} perfect matching. Therefore, $g_n \geq g_{n-1}^2$. □

4.6 We can get that G has at least r X-perfect matchings.

Proof. With some vertex x in X, it has at least r edges with Y. Select one edge xy_1, the remaining graph G' still satisfies that $N(S) \geq |S|$, i.e. a perfect matching exists in G'. We have r choices of xy_i. Therefore, G has at least r different X-perfect matching. □

4.8 *Proof.* Proof consists of two parts. The two people are denoted by A, B.
(1) If G has perfect matching M. The strategy of B is to select the vertex that the edge between this vertex and vertex selected by A last time belongs to M.
Whatever dose A selects, B can always find an vertex adjacent that the edge between them belongs to M, because G has a perfect matching.
(2) If G doesn't have perfect matching. The strategy of A is to select the vertex not joining with edge which belongs to maximum matching.
In this way, B can only select an vertex joining an edge which doesn't belong to the maximum matching. After B's turn, A can always find an vertex which joining an edge belongs to the maximum matching. □

4.10 Birkhoff Theorem Proof
Form associated graph of **doubly stochastic matrix** Q. Use following matrix as an example.

$$Q = \begin{pmatrix} 1/3 & 1/2 & 1/6 \\ 0 & 1/3 & 2/3 \\ 2/3 & 1/6 & 1/6 \end{pmatrix}$$

Associated graph is as following Figure 1.

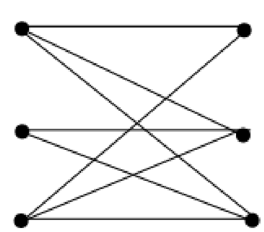

Figure 1: Associated graph of Q

Lemma 1. *Associated graph of doubly stochastic matrix has a perfect matching.*

Proof of lemma. There is a subset A of the vertices in one part such that the set $R(A)$ of all vertices connected to some vertex in A has strictly less than $|A|$ elements. Without loss of generality, A is a set of vertices representing rows while $R(A)$ represents columns.
We have $\sum_{i\in A, j\in R(A)} x_{i,j} = |A|$ and $\sum_{i\in A, j\in R(A)} x_{i,j} \leq |R(A)|$. Then Hall's condition is satisfied, a perfect matching exists. □

Proof of Birkhoff Theorem. We proceed by induction on non-zero entries in matrix. Let M_0 be doubly stochastic matrix. By the lemma, the associated graph has a perfect matching. Use above Q as an example, $(1,1),(2,2),(3,3)$ is a perfect matching. Let α_0 be the minimum of the entries indexed by the matching, $1/6$. Let P_0 be the permutation matrix with 1 exactly at position of the matching. Then $M_0 - \alpha_0 P_0$ has non-negative entries. Dividing each entry by $(1-\alpha_0)$ in $M_0 - \alpha_0 P_0$ gives a doubly stochastic matrix M_1. Then $M_0 = \alpha_0 P_0 + (1-\alpha_0)M_1$, where M_1 has less non-negative entries than M_0. By induction hypothesis, M_1 may be written as $M_1 = \alpha_1 P1 + \alpha_2 P_2 \cdots + \alpha_n P_n$. Therefore, we have

$$Q = M_0 = \alpha_0 P_1 + (1-\alpha_0)\alpha_1 P_1 + \cdots + (1-\alpha_0)\alpha_n P_n$$

Let $\alpha_0 = c_1, (1-\alpha_0)\alpha_1 = c_2, \ldots,$ where

$$\sum_i^m c_i = \alpha_0 + (1-\alpha_0)(\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1}) = \alpha_0 + (1-\alpha_0) = 1$$

□

4.13 *a Proof. Necessity.* S is an independent set. That means no edges joining vertices in S. Therefore, $\overline{S}$ is a vertex cover.
Sufficiency. $\overline{S}$ is an vertex cover. If S is not an independent set, i.e. edge $e : xy$ exists joining vertices x, y in S. It contradicts that $\overline{S}$ covers all edges of graph.
Hence, every maximum independent set is the complement of a minimum vertex cover, and $\alpha(G) + \beta(G) = |V(G)|$. □

b Proof. With a maximum matching M, we try to construct an edge cover of size $n(G) - |M|$. Since the minimum edge cover won't be larger than this cover, it implies that $\beta'(G) \leq n(G) - \alpha'(G)$. On the other hand, with a minimum edge cover L, we try to construct a maximum matching of size $n(G) - |L|$. Since maximum matching, $\alpha'(G) \geq n(G) - \beta'(G)$. Then we have $\alpha'(G) + \beta'(G) = n(G)$. Following are procedures to find such edge cover and mathcing.
Let M be the maximum matching of graph G. We construct an edge cover by adding to M one edge incident to each unsaturated vertex, one edge to one vertex. The number of unsaturated vertices is $n(G) - 2|M|$, including M we will get an edge cover of size $n(G) - 2|M| + |M| = n(G) - |M|$.

Let L be the minimum edge cover. If both endpoints of edge e belong to edge in L other than e, then $e \notin L$, since $L - e$ is also an edge cover. Hence each component of L has only one vertex with degree larger than 1, i.e. forming a star. Then we make a matching by selecting one edge of each star. Size of the matching will be $n(G) - |L|$. $\square$

c *Proof.* From above a, b, we have $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. From **Thm 4.9**, we have $\alpha'(G) = \beta(G)$. Hence, $\alpha(G) = \beta'(G)$. $\square$