7.21 *Proof.* $\chi'(Q_k) = \triangle(Q_k)$ Prove by explicit coloring. The edges between vertices differing in coordinate j form a complete matching. Over the k choices of j, these partition the edges.

$\chi'(K_{m,n}) = \triangle(K_{m,n})$ Prove by explicit coloring. Assume $m \leq n$, then $\triangle(G) = n$. If the vertices are $X \cup Y$ with $X = x_1, \dots x_m, Y = y_1, \dots y_n$. We give the edge $x_i y_j$ with color labelled $i + j(modn)$. Since incident edges differ in the index of each vertex, they receive different colors. □

7.24 *Proof. Given a regular multigraph G with cut-vertex, $\chi'(G) > \delta(G)$.*

Since G is regular and has cut-vertex x, one color class will be 1-factor(a perfect matching). So $n(G)$ is even. $G - x$ has odd vertices, so we can find a component H which has odd vertices and a vertex $\{y : y \notin H, y \leftrightarrow x\}$. Then given a color class containing xy, it is also a 1-factor which contains a 1-factor of H. However, it is impossible for odd $n(H)$. □

7.25

(a) *Proof. If G doesn't have a component of odd cycle, G has a 2-edge coloring that uses both colors on each vertex $\{v : deg(v) \geq 2\}$.* If G is Eulerian, we can find a Eular Tour with endpoint s and alternate two colors along the tour. Then each vertex v except the endpoint has two colors on entering and leaving edges. When $e(G)$ is even, the endpoint also share two colors. When $e(G)$ is odd and $deg(s) > 2$, it works. When $e(G)$ is odd and no vertex with degree at least 4, G will be a odd cycle.

If G is not Eulerian, adding a vertex x and joining x with vertices with odd degree yield a Eulerian graph G'. Find a Euler Tour from x. Alternate two colors along the tour. For $\{v : deg_G(v) is even\}$, v have two colors. For $\{v : deg_G(v) is odd\}$, v also have two colors because $deg_{G'}(v) \geq 4$ among which only one edge is connected to x. □

(b) *Proof.* Subgraph H has components C_1 which contains u. If C_1 is not an odd cycle, according to part (a) we can recolor C_1 so that for $\{v : v \neq u\}$, $c(v)$ does not decrease. But $c(u)$ increases by one, which means f is not optimal k-edge-coloring. It is a contradiction. □

(c) *Proof. If G is bipartite, G could be $\triangle(G) - edge - coloring$.* We prove by contradiction. If a bipartite graph G can not be $\triangle(G) - edge - coloring$. The vertex $\{v : deg(v) = \triangle(G)\}$ will have at least one color missed and another color appears two times at v. Then with result of part (b), G has a subgraph of odd cycle which contradict with property of bipartite graph. □

8.2

(a) *Proof. Given Hamiltonian Bipartite graph G and two vertices $\{x, y : x, y \in V(G)\}$, $G - x - y$ has perfect matching if and only if x, y are in different partites.*

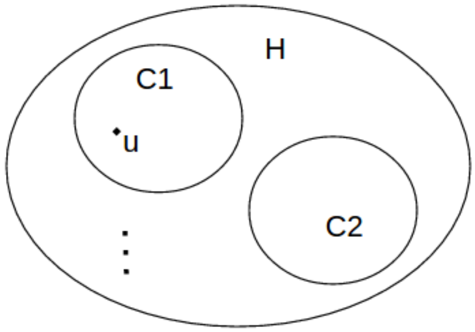

Figure 1: Subgraph H

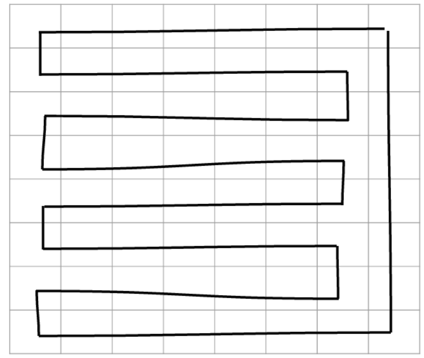

Figure 2: Hamiltonian cycle in 8x8

" $\Rightarrow$ " Because G is bipartite graph and has Hamilton cycle, the two partites G_X, G_Y have same number of vertices. Assume $G' = G - x - y$, since G' has perfect matching, $n(G'_X) = n(G'_Y)$. Hence, x, y are in different partites.

" $\Leftarrow$ " Assume C as the Hamilton cycle of G. Because x, y are in different partites, the two paths obtained by removing x, y both have even vertices. Then $G - x - y$ has an explicit perfect matching. $\square$

(b) *Proof.* This problem can be transferred to problem (a). Assume the chessboard as such a graph G. Grids on the chessboard are vertices. Join two grids if they share an edge. See Figure 2, a bold line which is a cycle is a Hamiltonian cycle. Vertices are divided into two partites according to their color(black and white). Hence, G is a Hamiltonian bipartite graph. Remove two grids so that remaining chessboard can be divided into multiple 1x2 rectangular is equal to $G-x-y$ has perfect matching. The two removed grids with different colors equals to two x, y are in different partites. Therefore, it is proved with problem (a). $\square$

8.5 *Proof. Cube of connected graph with at least 3 vertices has Hamiltonian cycle.* Because cube of G contains cube of spanning tree of G, it suffices to prove a stronger claim for trees T. Given an edge $xy \in E(T)$, remove xy. Two subtrees called R, S form containing x, y respectively. Find vertex $\{w : w \leftrightarrow x\}, \{z : z \leftrightarrow y\}$. Without loss of generality, $n(R) \leq n(S)$. With the induction hypothesis, R^3, S^3 both have Hamiltonian cycle. T^3 contains R^3 and S^3. If $n(R) \geq 3$, we can obtain Hamiltonian cycle of T^3 by replacing xw, yz with xy, wz. wz exists because $d(w, z) \leq 3$. If $n(R) = 2$, replace yz with xy, wz. If $n(R) = 1$, replace yz with xy, xz. $\square$