7.5 *G is m-colorable if and only if $\alpha(G\Box K_m) \geq n(G)$.*

Proof. Necessity $\chi(G\Box K_m) = max\{\chi(G), \chi(K_m)\} = m$ since G is m-colorable. And since $\chi(H) \geq \frac{n(H)}{\alpha(H)}$, $\alpha(G\Box K_m) \geq \frac{n(G)m}{m} = n(G)$.

Sufficiency If $\alpha(G\Box K_m) \geq n(G)$, $\alpha(G\Box K_m) = n(G)$, because $G\Box K_m$ contain $n(G)$ copies of K_m. $\alpha(G\Box K_m)$ contains at most one vertex of each copy of K_m.
Not complete. □

Question: Why not is this proposition be stated as: *G is m-colorable if and only if* $\alpha(G\Box K_m) = n(G)$?

7.7

(a) *Proof.* Construct a graph G, area as vertex, adjacent if areas share a common edge. Because no three intersecting at a point, $\omega(G) = 2$ and no odd cycle exists; i.e. G is a bipartite. We can find an orientation D (from partite X to Y) such that $l(D) = 1$. According to **Gallai's Thm**, then $\omega(G) \leq \chi(G) \leq 1 + l(D) = 2$. □

(b) We can always map the vertices to a line disjointly, forming an ordering of vertex. Since no three intersects at a point, each vertex has at most 2 neighbors earlier in the ordering. Thus, $\chi(G) \leq 3$.

7.8 *For P_4-free graph, greedy algorithm produces optimal coloring whatever the ordering of vertices.*

Proof. Suppose that the algorithm uses k colors for the ordering $v_1, ..., v_n$ and let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Prove that $i = 1$ which means it is a optimal coloring. Let $Q = \{u_i, ..., u_k\}$ be such a clique. If $i > 1$, by greedy coloring algorithm, some elements of Q has earlier neighbours with color $i-1$. We assume x with color $i-1$ as the vertex having most neighbours of Q. Then find $z \in Q, z \not\leftrightarrow x$, if not, x would be included into Q. Next find w which is a neighbour of z with color $i-1$. And find $y \in (N(x) \cap Q) - N(w)$. Since x, w are not adjacent for their colors, $y \not\leftrightarrow x, y \not\leftrightarrow w$, w, z, y, x is a P_4. It contradicts the condition. Thus $i = 1$.

□

7.10 $\chi(G_{n,k}) = k+1$ *if* $k+1$ *divides* n *and* $\chi(G_{n,k})$ *does not divides* n.

Proof. According to the construction of the graph $G_{n,k}$, cliques of size $k+1$ are formed. Then color the graph $G_{n,k}$ with $k+1$ colors. If $k+1$ divides n, $1, 2, 3..., k+1, 1, 2, 3..., k+1$ will be a proper coloring of graph $G_{n,k}$.

If $k+1$ does not divides n, let $n = q(k+1)+r = (q-r)(k+1)+r(k+2), 1 \leq r < (k+1)$. Since $n \geq k(k+1)$, $q \geq k \geq r$. Therefore, a proper coloring exists that color $q-r$ times with $k+1$ colors and r times with $k+2$ colors. $\chi(G_{n,k}) = k+2$. □

7.12 *For unit-distance graph G, $4 \leq \chi(G) \leq 7$*

Proof. Lower bound. From the definition of unit-distance graph, we find the $\omega(G) = 3$. We claim 3-coloring is not a proper coloring for G. Consider two equilateral triangles of side-length one sharing an edge. Two corners of the common edge are color 1 and color 2. The other two disjoint corners are color 3, since their distance is $\sqrt{3}$. Then consider a circle with radius $\sqrt{3}$. The vertex on the circle must be the same with the center. However, two vertices with distance 1 exist on the circle. Therefore 3-coloring is not a proper coloring.

Upper bound. Give an explicit 7-coloring of graph G as 1. Color 7 hexagons(six hexagons surround one, each has maximal diameter 1) with 7 colors. Move such a 7-color panel to fill the plane. The shortest distance between two hexagons with the same color is greater than 1. It is a proper coloring. □

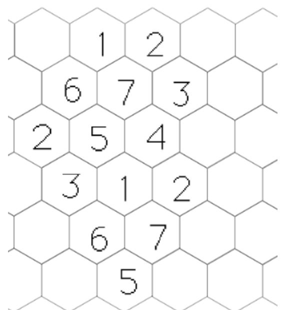

Figure 1: Proper 7-coloring of graph