

7.13 G is m -colorable if and only if $\alpha(G \square K_m) \geq n(G)$.

Proof. Necessity $\chi(G \square K_m) = \max\{\chi(G), \chi(K_m)\} = m$ since G is m -colorable. And since $\chi(H) \geq \frac{n(H)}{\alpha(H)}$, $\alpha(G \square K_m) \geq \frac{n(G)m}{m} = n(G)$.

Sufficiency If $\alpha(G \square K_m) \geq n(G)$, $\alpha(G \square K_m) = n(G)$, because $G \square K_m$ contain $n(G)$ copies of K_m . $\alpha(G \square K_m)$ contains at most one vertex of each copy of K_m .

Not complete. \square

Question: Why not does this proposition be stated as: G is m -colorable if and only if $\alpha(G \square K_m) = n(G)$?

7.16 For the graph G_n with even n , every proper 4-coloring of G_n uses each color on exactly n vertices.

Proof. G_n is constructed by adding a new 4-cycle surrounding G_{n-1} . G_2 is as Figure 1. When n is even, two consecutive 4-cycles will be isomorphic to G_2 . If such G_2 is proved each color class covers exactly 2 vertices, adjustment could be implemented to color all consecutive G_2 . Following is the proof of proper 4-coloring of G_2 using each color on exactly 2 vertices.

It is easy to give a proper 4-coloring of G_2 . What's more, one color class could cover at most 2 vertices since we cannot find three vertices which are dis-adjacent to each other. Therefore, one color covers exactly 2 vertices. \square

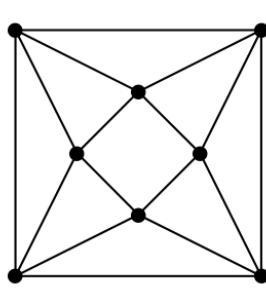


Figure 1: G_2

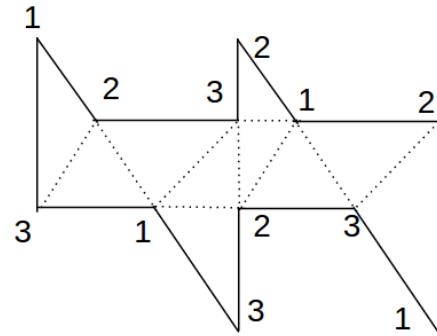


Figure 2: An example of Art Gallery

7.17

(a) Every outerplanar graph is 3-chromatic.

Proof. With **Four Color Thm**

Outerplanar graph is planar. So it can be 4-colorable according to **Four Color Thm**. Given a sequence of a outerplanar and using greedy coloring algorithm, only when four vertices v_a, v_b, v_c, v_d are adjacent to each other, i.e., subdivision of K_4 as a subgraph, 4 colors are needed(**Dirac's Thm**). However, outerplanar graph is K_4 -free. Therefore, outerplanar graph is 3-chromatic. \square

(b) Every outerplanar graph is 3-chromatic.

Proof. Without **Four Color Thm**

We prove by giving a 3-coloring of outerplanar graph. Outerplanar graph has all vertices as the boundary of exterior plane.

If an outerplanar graph G has cycles C_i , every two cycles share only one edge(if not, some vertex will not be the boundary of exterior plane). And see C_i as vertex sequence of graph G' , no cycle exists in G' , i.e., a tree. We have known that C_i is 3-chromatic(when $n(C_i)$ is odd, 3-coloring is needed). When one cycle is colored, we can always color the adjacent cycles with 3 colors until all cycles are colored. The remaining parts, which are trees, can be 2-colored.

If no cycle exists, G will be a tree, which is 2-chromatic. Then, a 3-coloring is given. \square

(c) *Art Gallery Theorem*

Given a polygon as Figure 2, we can add some straight lines to triangulate the polygons. Consider a 3-coloring of such a polygon so that each triangular is colored with 3 colors(see Figure 2). Then each color class has at least $\lfloor n/3 \rfloor$ elements. In each triangular, the corner can see all points of the triangle. Hence, put a guard at the place of one color class will see the whole art gallery, i.e., $\lfloor n/3 \rfloor$ guards are needed.

7.19 $\chi(G_{n,k}) = k + 1$ if $k + 1$ divides n and $\chi(G_{n,k}) = k + 2$ if $k + 1$ does not divide n .

Proof. According to the construction of the graph $G_{n,k}$, cliques of size $k + 1$ are formed. Then color the graph $G_{n,k}$ with $k + 1$ colors. If $k + 1$ divides n , $1, 2, 3, \dots, k + 1, 1, 2, 3, \dots, k + 1$ will be a proper coloring of graph $G_{n,k}$.

If $k+1$ does not divides n , let $n = q(k+1)+r = (q-r)(k+1)+r(k+2)$, $1 \leq r < (k+1)$. Since $n \geq k(k + 1)$, $q \geq k \geq r$. Therefore, a proper coloring exists that color $q - r$ times with $k + 1$ colors and r times with $k + 2$ colors. $\chi(G_{n,k}) = k + 2$. \square

7.20 Every torus-embedded graph is 7 chromatic.

Proof. For a torus-embedded graph G , if we can find a region which has at most 6 adjacent regions, then induct on the $n(G)$. Given G , when we shrink a region to a point, then with the induction hypothesis, the remaining graph will be 7 chromatic. Add the removing region will always be colored with one of the 7 colors because it has only six adjacent regions.

Now, we prove such a region exists. Because each region has at most 3 edges, each region will have at least 3 adjacent regions, i.e., $3n \leq 2e$. With Eular's formula on torus $n - e + f = 0$, we have $e \leq 3f$. If every region has more than 7 adjacent regions, $7f \leq 2e$. Then $2e \geq 7f \geq 6f \geq 2e$, which is a contradiction. \square