7.13 *G is m-colorable if and only if $\alpha(G \Box K_m) \geq n(G)$.*

Proof. *Necessity* $\chi(G\Box K_m) = max\{\chi(G), \chi(K_m)\} = m$ since G is m-colorable. And since $\chi(H) \geq \frac{n(H)}{\alpha(H)}$, $\alpha(G\Box K_m) \geq \frac{n(G)m}{m} = n(G)$.

Sufficiency If $\alpha(G\Box K_m) \geq n(G)$, $\alpha(G\Box K_m) = n(G)$, because $G\Box K_m$ contain $n(G)$ copies of K_m. $\alpha(G\Box K_m)$ contains at most one vertex of each copy of K_m.
Not complete. □

Question: Why not does this proposition be stated as: *G is m-colorable if and only if $\alpha(G\Box K_m) = n(G)$?*

7.16 *For the graph G_n with even n, every proper 4-coloring of G_n uses each color on exactly n vertices.*

Proof. G_n is constructed by adding a new 4-cycle surrounding G_{n-1}. G_2 is as Figure 1. When n is even, two consecutive 4-cycles will be isomorphic to G_2. If such G_2 is proved each color class covers exactly 2 vertices, adjustment could be implemented to color all consecutive G_2. Following is the proof of proper 4-coloring of G_2 using each color on exactly 2 vertices.
It is easy to give a proper 4-coloring of G_2. What's more, one color class could cover at most 2 vertices since we cannot find three vertices which are dis-adjacent to each other. Therefore, one color covers exactly 2 vertices. □

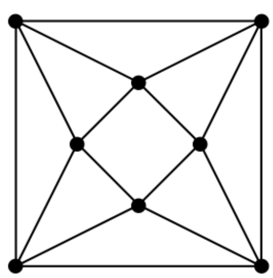

Figure 1: G_2

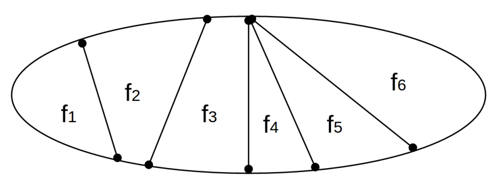

Figure 2: Outerplanar graph

7.17

(a) *Every outerplanar graph is 3-chromatic.*

Proof. Outerplanar graph is □

7.10 *$\chi(G_{n,k}) = k + 1$ if $k + 1$ divides n and $\chi(G_{n,k}) = k + 2$ if $k + 1$ does not divide n.*

Proof. According to the construction of the graph $G_{n,k}$, cliques of size $k+1$ are formed. Then color the graph $G_{n,k}$ with $k+1$ colors. If $k+1$ divides n, $1, 2, 3..., k+1, 1, 2, 3..., k+1$ will be a proper coloring of graph $G_{n,k}$.

If k+1 does not divides n, let $n = q(k+1)+r = (q-r)(k+1)+r(k+2), 1 \leq r < (k+1)$. Since $n \geq k(k+1)$, $q \geq k \geq r$. Therefore, a proper coloring exists that color $q-r$ times with $k+1$ colors and r times with $k+2$ colors. $\chi(G_{n,k}) = k+2$. □

7.12 *For unit-distance graph G, $4 \leq \chi(G) \leq 7$*

Proof. Lower bound. From the definition of unit-distance graph, we find the $\omega(G) = 3$. We claim 3-coloring is not a proper coloring for G. Consider two equilateral triangles of side-length one sharing an edge. Two corners of the common edge are color 1 and color 2. The other two disjoint corners are color 3, since their distance is $\sqrt{3}$. Then consider a circle with radius $\sqrt{3}$. The vertex on the circle must be the same with the center. However, two vertices with distance 1 exist on the circle. Therefore 3-coloring is not a proper coloring.

Upper bound. Give an explicit 7-coloring of graph G as Figure **??**. Color 7 hexagons(six hexagons surround one, each has maximal diameter 1) with 7 colors. Move such a 7-color panel to fill the plane. The shortest distance between two hexagons with the same color is greater than 1. It is a proper coloring. □