HW2, Graph Theory, 2015 Fall

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1.20 Duplicate each edge e_i as e_i^1 and e_i^2. Each vertex in the resulting graph G' has even degree. Hence G' is Eulerian, and contains a closed trail

$$W' = e_{i_1}^{p_1} e_{i_2}^{p_2} \dots e_{i_m}^{p_m}$$

which goes through all the edges. Now forget all the superscript in W', i.e., consider

$$W = e_{i_1} e_{i_2} \dots e_{i_m}.$$

Then W is a closed walk of G that goes, and at most twice, through each edge.

1.22 The first proof in the literature: A. Lempel, m-ary closed sequences:
`http://dx.doi.org/10.1016/0097-3165(71)90029-X`
A more general result: M. Imori *et al.*, The line digraph of a regular and pancircular digraph is also regular and pancircular: `http://link.springer.com/article/10.1007/BF01864164` (Visible in NTU domain)

Notice that $G_{\sigma,n+1}$ can constructed from the line graph of $G_{\sigma,n}$. Hence

$(*)$ A closed trail of length ℓ in $G_{\sigma,n}$ corresponds to an ℓ-cycle in $G_{\sigma,n+1}$.

In order to prove that for each $1 \le \ell \le \sigma^{n-1}$, there is a cycle of length ℓ in $G_{\sigma,n}$, it suffices to prove that for each $1 \le \ell \le \sigma^{n-1}$, there is a closed trail of length ℓ in $G_{\sigma,n-1}$.

We prove by induction on n. The base case $G_{\sigma,1}$ or $G_{\sigma,2}$ is easy to check. By induction hypothesis, there is a cycle with ℓ' edges for any $1 \le \ell' \le \sigma^{n-1}$.

For a closed trail C, we denote $\ell(C)$ the length of C, i.e., the number of edges in it. Let $m = |E(G_{\sigma,n})|$.
Claim 1. If there are closed trails $C_1, C_2, \dots, C_r$ in $G_{\sigma,n}$, prove that there is a closed trail of length $\ell' = m - \sum_{i=1}^{r} \ell(C_i)$.

Proof. Let the set of "unions of closed trails with lengths summing to ℓ'" be S. Since every vertex in $E - \bigcup_{i=1}^{r} C_r$ has even degree, it is a union of Euler graphs, and hence S is nonempty. Now let $\cup_{i=1}^{t} C_i' \in S$ with t smallest. If $k > 1$, then one of the following two cases hold.

1. There are two trails C_i' and C_j' that go through a same vertex v. Then we can concatenate two trails into a longer one. Contradicts to the minimality of t.
2. There is a directed edge (u, v) (which means u to v) with $u \in V(C_i')$ and $v \in V(C_j')$. Pick edges (u, x) from C_i' and (y, v) from C_j'. By the definition of de Bruijn graph, there is an edge (y, x) in $G_{\sigma,n}$. Hence we have a closed trail by replacing (u, x) and (y, v) with (u, v) and (y, x) in $C_i' \cup C_j'$. Again contradicts to the minimality of t.

Claim 2. For any graph G, each degree of whose vertex is even and at least 2, (That is, each component of G is Eulerian.) we can find a vertex disjoint union of closed cycles $C_1, C_2, \ldots, C_r$ in G so that $\bigcup_{i=1}^r V(C_i) = V(G)$. (So $\forall x \in G, \deg_C(x) = 2$, where $C = \bigcup_{i=1}^r C_i$.) Briefly, each vertex is gone through exactly once in $C_1, C_2, \ldots, C_r$.

To show this, we need the 2-Factor Theorem... (See e.g. `https://en.wikipedia.org/wiki/2-factor_theorem`)

1.25 $(5,5,4,3,2,2,2,1)$, $(5,5,5,3,2,2,1,1)$ and $(5,5,4,4,2,2,1,1)$ are graphic sequences. The realizations are omitted here. You can construct by try-and-error, or use the algorithm provided in the textbook. However, $(5,5,5,4,2,1,1,1)$ is not a graphic sequence since

$$\begin{aligned} d_1 + d_2 + d_3 = 5+5+5 = 1514 &= 6+3+2+1+1+1 \\ &>= 3(3-1) + \min\{3, d_4\} + \min\{3, d_5\} + \min\{3, d_6\} + \min\{3, d_7\} + \min\{3, d_8\}. \end{aligned}$$

1.30 If there is a bipartite graph G with degree sequences of the two parts $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Let the two parts be A and B. Then

$$\begin{aligned} \sum_{i=1}^m a_i &= \sum_{x \in A} |N(x)| = \sum_{x \in A} \sum_{y \sim x} 1 \\ &= \sum_{y \in B} \sum_{x \sim y} 1 = \sum_{y \in B} |N(y)| = \sum_{i=1}^n b_i. \end{aligned}$$

In plain text, each edge has exactly one endpoint lie in each part. So $\sum_{i=1}^m a_i$ counts the edges from part A, while $\sum_{i=1}^n b_i$ counts the edges from part B.

Next, we consider the vertices $y_1, y_2, \ldots, y_k$ with degrees $b_1, b_2, \ldots, b_k$. Suppose we construct a matrix with m columns with initially all zeros, and fill in a_i 1's from the top of each column. Count the number of 1's that are in the first k rows. If we count by columns, we see the number is $\sum_{j=1}^m \min\{k, a_j\}$; while counting by rows gives the number $\sum_{j=1}^k a_j^*$. So these two numbers are the same.

Now

$$\begin{aligned} \sum_{i=1}^k b_i &= \sum_{i=1}^k |N(y_i)| = \sum_{i=1}^k \sum_{x \sim y_i} 1 \\ &= \sum_{j=1}^m \sum_{y_i \sim x_j} 1 \leq \sum_{j=1}^m \min\{k, \deg(x_j)\} \\ &= \sum_{j=1}^m \min\{k, a_j\} = \sum_{j=1}^k a_j^*. \end{aligned}$$

Conversely, consider a new degree sequence $\mathsf{d} = \{d_i\}_{i=1}^{n+m}$ so that $d_i = b_i + n - 1$ if $1 \leq i \leq n$, and $d_i = a_{i-n}$

if $i > n$. Now if $k \le n$, then

$$\begin{aligned}
\sum_{i=1}^{k} d_k &= \left(\sum_{i=1}^{k} b_k\right) + k(n-1) \le \left(\sum_{i=1}^{k} a_i^*\right) + k(n-1) \\
&= \left(\sum_{i=1}^{m} \min\{k, a_i\}\right) + k(n-1) = k(k-1) + k(n-k) + \left(\sum_{i=1}^{m} \min\{k, a_i\}\right) \\
&= k(k-1) + \left(\sum_{i=k+1}^{n} \min\{k, b_i + n - 1\}\right) + \left(\sum_{i=1}^{k} \min\{k, a_i\}\right) \\
&= k(k-1) + \left(\sum_{i=k+1}^{n+m} \min\{k, d_i\}\right)
\end{aligned}$$

So the degree sequence d satisfies the Erdős-Gallai Theorem, and there exists a graph G realizing d. Suppose in G, the vertices $x_1, x_2, \ldots, x_n$ have degrees $b_1, b_2, \ldots, b_n$, and $y_1, y_2, \ldots, y_n$ have degrees $a_1, a_2, \ldots, a_m$. Denote $A = \{y_1, y_2, \ldots, y_m\}$, $B = \{x_1, x_2, \ldots, x_n\}$, $E(X, Y)$ the set of edges with one end point in X and the other in Y, and $D(X)$ the set of edges with one end point in X. (Formally speaking, $E(X,Y) = \{e = uv \in E(G) : u \in X, v \in Y\}$ and $D(X) = \{e = uv \in E(G) : u \in X\}$.) Then

$$|D(A)| = \sum_{y \in A} \deg(y), \quad |D(B)| = \sum_{x \in B} \deg(x),$$

and

$$D(B) = E(A,B) \uplus E(B,B), \quad D(A) = E(A,B) \uplus E(A,A).$$

Incorporate these, we obtain that

$$\begin{aligned}
|E(B,B)| &= \left(\sum_{x \in B} \deg(x)\right) - |E(A,B)| = \left(\sum_{i=1}^{n} b_i + n - 1\right) - |E(A,B)| \\
&= \left(\sum_{i=1}^{m} a_i\right) + n(n-1) - |E(A,B)| = \left(\sum_{y \in A} \deg(y)\right) - |E(A,B)| + n(n-1) \\
&= |E(A,A)| + n(n-1) \ge |E(A,A)| + |E(B,B)| \ge |E(B,B)|.
\end{aligned}$$

So all equalities hold: $|E(B,B)| = n(n-1)$ and $|E(A,A)| = 0$. Thus if we delete $E(B,B)$ from G, we get a bipartite graph with degree sequence of part A being $a_1, a_2, \ldots, a_m$ and that of part B being $b_1, b_2, \ldots, b_n$.

Intuitively, we pretend to add edges in part B to make it a clique in the degree sequence. By Erdő-Gallai Theorem we verify that it is indeed a graphic sequence, and examine that part B must be a clique.

1.32 Let M be the set of points on the line of mountain, and $H(p)$ denote the height of the point p, i.e. $p = (x, H(p))$ for some x is on the line of mountain. The set S' collects all pairs $(p_1, p_2) \in M \times M$ such that $H(p_1) = H(p_2)$. Notice S has infinite elements so we have to simplify it. We call h a *critical height* if there is some $p \in M$ that is a peak or valley and $H(p) = h$. Let

$$S := \{(p_1, p_2) \in S' : H(p_1) = H(p_2) \text{ is a critical height.}\}$$

We call p a *critical point* if $H(p)$ is a critical height. For example,

Then the black nodes are critical points, and the critical heights are those of dashed lines. The points $(C, E), (D, N), (L, H), (B, A), (I, I) \in S$ while $(O, A), (H, I)$ are not. Construct a graph G with vertex set

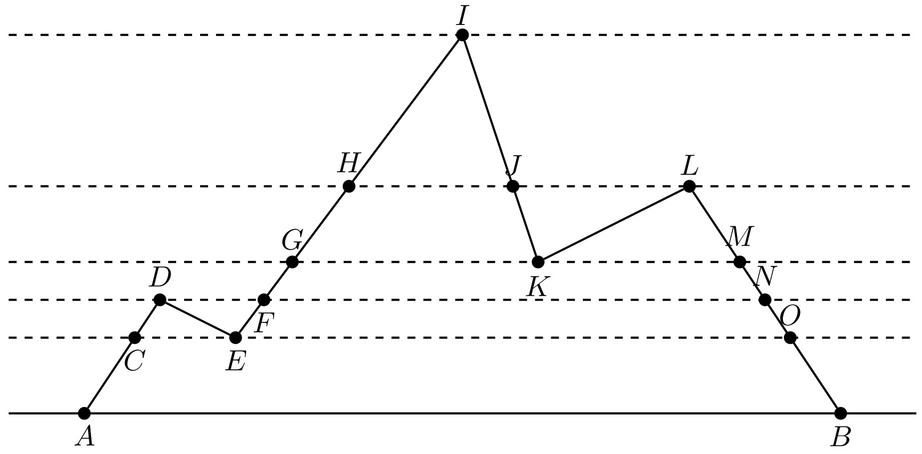

Figure 1: The line of a mountain.

S, and distinct vertices (p_1, p_2) and (p_3, p_4) form an edge if and only if when the first climber W_1 is at p_1, the second climber W_2 is at p_3, they can follow the rule and move to p_2 and p_4, respectively, without encountering any other critical point. Then it can be seen that $\deg((A, A)) = \deg((A, B))) = \deg((B, A)) = \deg((B, B)) = 1$, where $A = (a, 0)$ and $B = (b, 0)$, and any other vertex has degree an even number. Consider the connected component C containing (A, B), one of the other three vertices of odd degree must be in C by Handshaking Theorem: $2|E| = \sum_{x \in V(G)} \deg(x)$. If $(B, A) \in C$, then from the definition of "connected", there is a path in C from $(A, B) \to (B, A)$, and it implies there is a way of climbing so that W_1 goes from A to B while W_2 goes from B to A. It is clear that W_1 and W_2 will meet at some point during the climbing. If $(A, A) \in C$ or $(B, B) \in C$, it directly implies there is a way of climbing so that W_1 and W_2 meet each other at A or B.