HW1, Graph Theory, 2015 Fall

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1.1 Draw the squares on the chessboard with black and white so that adjacent pair of squares receive different colors. It is easy to see the two squares removed have the same color. Hence without lost of generality, we can say there are 30 black squares and 32 white ones. Each 1×2 domino must cover a black square and a white square, so by no means there is an arrangement of dominoes that covers the whole 62 squares.

In terms of graph theory,

$$\begin{aligned}
\text{chessboard graph} &\leftrightarrow \text{graph} \\
\text{chessboard can be colored with black and white} &\leftrightarrow \text{bipartite graph} \\
\text{domino} &\leftrightarrow \text{edge} \\
\text{nonoverlapping dominoes} &\leftrightarrow \text{matching} \\
\text{nonoverlapping dominoes covering all the squares} &\leftrightarrow \text{perfect matching}
\end{aligned}$$

Thus, the argument above actually demonstrates a result in graph theory: A bipartite graph has no perfect matching when the two parties have different numbers of vertices.

(The terms "matching" and "perfect matching" will be taught in chapter 4, so I don't require you to state it in the homework.)

1.5 For any vertex $x \in V(G)$, consider its neighbors $N(x) = \{v_1, v_2, \ldots, v_k\}$. Since every vertex is at distance at least 1 to another one on the plane, the angle formed by $v_i x v_j$ is at least $\pi/3$. By pigeonhole principle, $k \leq 6$. Thus

$$|E| = \frac{1}{2} \sum_{x \in V(G)} \deg(x) \leq \frac{1}{2} \sum_{x \in V(G)} 6 = 3|V(G)|.$$

1.8 For a vertex $x \in V(G)$, let the neighbors $N(x) = \{v_1, v_2, \ldots, v_k\}$. And $N(v_i) = \{v, u_{i,1}, u_{i,2}, \ldots, u_{i,k-1}\}$. The size of each set is k since G is k-regular.

Suppose $v_i = u_{j,\ell}$ for some i, j, ℓ. Then there will be a 3-cycle vv_iv_j, a contradiction. Suppose $u_{i,j} = u_{s,t}$ for some i, j, s, t. Then there will be a 4-cycle $vv_iu_{i,j}v_s$, a contradiction. Hence all the v_i, $u_{j,\ell}$ are distinct vertices.

$$|V| \geq |\{x\}| + |N(x)| + \sum_{i=1}^{k} |N(v_i) - v| = k^2 + 1$$

.

An example for $k = 2$ is C_5, while the one for $k = 3$ is the Petersen graph.

1.11 There are two components. The one C_0 contains $(0, 0, \ldots, 0)$ and the other one C_1 contains $(0, 0, \ldots, 0, 1)$. Actually, C_i is precisely those vertices $(a_1, a_2, \ldots, a_n)$ with $\sum_{i=1}^{n} a_i \equiv i \pmod 2$.

Notice that if $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) \in E(G)$, then

$$\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{n} b_i \pmod 2.$$

This amounts to say that C_1 and C_2 are both disjoint unions of some connected components. Now it suffices to show that both C_1 and C_2 are indeed connected.

For $x_0 = (a_1, a_2, \ldots, a_n) \in C_0$, let a_i and a_j be the first two occurrence of 1, i.e., those with the minimal indices. For example, when $x_0 = (1, 0, 1, 1, 1)$, we take $i = 1$ and $j = 3$. Then x is adjacent to

$$x_1 = (a_1, a_2, \ldots, a_{i-1}, 0, a_i, a_{i+1}, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_n) = (0, 0, \ldots, 0, a_{j+1}, a_{j+2}, \ldots, a_n).$$

Continue in this fashion (note that in each step, the sum $\sum_{i=1}^{n} a_i$ is decreased by 2), we finally find a walk from x to $(0, 0, \ldots, 0)$. Hence C_0 is connected.

As for C_1, we similar find a walk for $x_0 = (a_1, a_2, \ldots, a_n) \in C_1$, except that we terminate at some vertex $x_k = e_r = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, where the single 1 is in the r-th position. It is from the definition that x_k is adjacent to e_n when $r \neq n$, or is itself e_n when $r = n$, and hence C_1 is also connected.

1.16 **(a)** Suppose G is disconnected, with components $G_1, G_2, \ldots, G_k$. For each pair of vertices $x, y \in V(G_i)$, $i \in [k]$, xzy is a walk for any $z \notin V(G_i)$. And for each pair of vertices $x \in V(G_i)$, $y \in V(G_j)$ for $i \neq j$, xy itself is a walk. Hence $\overline{G}$ is connected.

(b) Suppose $G \neq K_1$ is the minimal graph such that G is P_4-free, and G and $\overline{G}$ are both connected. When $|V(G)| = 2$, it is to see that there is no such graph. Hence we can assume $|V(G)| > 2$. Pick any vertex $x \in V(G)$. $G - x$ is P_4-free and hence either $G - x$ or $\overline{G - x}$ is disconnected, due to the minimality of G. Since a graph G is P_4 free if and only if $\overline{G}$ is P_4-free, due to the fact $\overline{P_4} = P_4$. We can assume $G - x$ is disconnected without lost of generality.

Suppose the components of $G - x$ are $G_1, G_2 \ldots, G_k$. Since G is connected, for each $i \in [k]$, there is a vertex $v_i \in V(G_i)$ such that $xv_i \in E(G)$. Since $\overline{G}$ is connected, x cannot be adjacent to every vertex in $G - x$, say $u_s \in V(G_s)$ is one vertex such that $xu_s \notin E(G)$ for some s. We can suitably choose v_s and u_s so that they are adjacent since G_s is connected. So $G[\{u_s, v_s, x, v_t\}] \cong P_4$ for arbitrary $t \neq s$, which contradicts to the assumption that G is P_4-free.