20220302

习题七

26(13)

因为对$k \in \mathbb{N}^*$有：

$$\int_k^{k+1} \ln{[k]}\mathrm{d}k = \ln k$$

原式等于：

$$\begin{aligned}\int_1^{1+n} \ln{[x]}\mathrm{d}x &= \sum_{k=1}^n \ln k \\ &= \ln \prod_{k=1}^n k \\ &= \boxed{\ln n!}\end{aligned}$$

(15)

$$\begin{aligned}\int_0^\pi x^2 \operatorname{sgn}(\cos x)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} x^2\mathrm{d}x - \int_{\frac{\pi}{2}}^\pi x^2\mathrm{d}x \\ &= \frac{1}{3}\left(\left(\frac{\pi}{2}\right)^3 - \left((\pi)^3 - \left(\frac{\pi}{2}\right)^3\right)\right) \\ &= -\frac{1}{3} \times \frac{3}{4}\pi^3 \\ &= \boxed{-\frac{1}{4}\pi^3}\end{aligned}$$

(16)

$$\begin{aligned}\int_0^1 x^m (\ln x)^n \mathrm{d}x &= \frac{1}{m+1}\int_0^1 (\ln x)^n \mathrm{d}(x^{m+1}) \\ &= \frac{1}{m+1}\left((\ln x)^n(x^{m+1})\Big|_0^1 - \int_0^1 x^{m+1}\mathrm{d}(\ln^n x)\right) \\ &= \frac{1}{m+1}\left((\ln x)^n(x^{m+1})\Big|_0^1 - n\int_0^1 x^m \ln^{n-1} x\mathrm{d}x\right)\end{aligned}$$

只需要求出 $(\ln x)^n(x^{m+1})\Big|_0^1$ 即得线性递推关系。因为易知 $(\ln x)^n(x^{m+1})|_{x=1} = 0$，只需要求 $(\ln x)^n(x^{m+1})|_{x=0}$，即 $\lim_{x\to 0+0}(\ln x)^n x^{m+1}$。

因为若 $\lim_{x\to 0+0}(\ln x)^n x$ 存在，即有

$$\lim_{x\to 0+0}(\ln x)^n x^{m+1} = \left(\lim_{x\to 0+0}(\ln x)^n x\right)\left(\lim_{x\to 0+0} x^m\right) = 0$$

故只需要证 $\lim_{x\to 0+0}(\ln x)^n x$ 存在。证明如下：

$$
\begin{aligned}
\lim_{x\to 0+0}(\ln x)^n x &= \lim_{x\to 0+0}(\ln x)^n x \\
&= \lim_{x\to +\infty}\frac{(\ln\frac{1}{x})^n}{x} \\
&= \frac{(-1)^n}{n^n}\left(\lim_{x\to+\infty}\frac{\ln\frac{x}{n}+\ln n}{\frac{x}{n}}\right)^n \\
&= 0
\end{aligned}
$$

存在性得证，因此 $\lim_{x\to 0+0}(\ln x)^n x^{m+1}=0$ ，所以

$$
(\ln x)^n(x^{m+1})\Big|_0^1 = 0
$$

可得

$$
\begin{aligned}
\int_0^1 x^m(\ln x)^n &= -\frac{n}{m+1}\int_0^1 x^m(\ln x)^{n-1}\mathrm{d}x \\
&= \cdots \\
&= (-1)^n\frac{n!}{(m+1)^n}\int_0^1 x^m\,\mathrm{d}x \\
&= \boxed{(-1)^n\frac{n!}{(m+1)^{n+1}}}
\end{aligned}
$$

28

设 $T=2\pi n+r\ (n\in\mathbb{N},0\le r<2\pi)$，有:

$$
\begin{aligned}
\frac{1}{T}\int_0^T f(x)\mathrm{d}x &= \frac{1}{2\pi n+r}\left(n\int_0^{2\pi}f(x)\mathrm{d}x+\int_0^r f(x)\mathrm{d}x\right) \\
&= \frac{2\pi n}{2\pi n+r}\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x+\frac{1}{T}\int_0^r f(x)\mathrm{d}x
\end{aligned}
$$

因为 $f(x)\in R[0,2\pi]$，$\int_0^r f(x)\mathrm{d}x$存在且$\le \int_0^{2\pi}|f(x)|\mathrm{d}x=M$。

因此

$$
\begin{aligned}
\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(x)\mathrm{d}x &= \lim_{T\to+\infty}\left(\frac{2\pi n}{2\pi n+r}\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x\right)+\lim_{T\to+\infty}\left(\frac{1}{T}\int_0^r f(x)\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x+0 \\
&= \frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x
\end{aligned}
$$

得证。■

31

对于特定的$x>0$，因为$e^{t^2}>0$，故 $\int_0^x e^{t^2}\mathrm{d}t>0$，有解

$$
\xi_x=\sqrt{\ln\left(\frac{\int_0^x e^{t^2}\mathrm{d}t}{x}\right)}
$$

故有存在性。因$xe^{\xi_x^2}$单调，有唯一性。

$$
\begin{aligned}
\lim_{x\to+\infty}\frac{\xi_x}{x} &= \lim_{x\to+\infty}\frac{\sqrt{\ln\left(\frac{\int_0^x e^{t^2}\mathrm{d}t}{x}\right)}}{x}\\
&= \sqrt{\lim_{x\to+\infty}\frac{\ln\left(\int_0^x e^{t^2}\mathrm{d}t\right)-\ln x}{x^2}}\\
&= \sqrt{\lim_{x\to+\infty}\frac{\frac{e^{x^2}}{\int_0^x e^{t^2}\mathrm{d}t}-\frac{1}{x}}{2x}} && \text{(洛必达)}\\
&= \sqrt{\lim_{x\to+\infty}\frac{\frac{e^{x^2}}{2x}}{\int_0^x e^{t^2}\mathrm{d}t}-0}\\
&= \sqrt{\lim_{x\to+\infty}\frac{\frac{4x^2e^{x^2}-2e^{x^2}}{4x^2}}{e^{x^2}}-0} && \text{(洛必达)}\\
&= \boxed{1}
\end{aligned}
$$

32

(1)

左边：

$$
\begin{aligned}
\int_0^1\frac{x^\alpha}{\sqrt{1+x}}\mathrm{d}x &= \frac{1}{\sqrt{1+\xi}}\int_0^1 x^\alpha\mathrm{d}x\\
&\geq \frac{1}{\sqrt{1+1}}\frac{1}{1+\alpha}\\
&= \frac{1}{\sqrt{2}(1+\alpha)}
\end{aligned}
$$

右边同理：

$$
\begin{aligned}
\int_0^1\frac{x^\alpha}{\sqrt{1+x}}\mathrm{d}x &= \frac{1}{\sqrt{1+\xi}}\int_0^1 x^\alpha\mathrm{d}x\\
&\leq \frac{1}{\sqrt{1+0}}\frac{1}{1+\alpha}\\
&= \frac{1}{(1+\alpha)}
\end{aligned}
$$

得证。■

(2)

由定积分第一中值定理：

$$
\int_0^{\frac{\pi}{4}}\frac{x\mathrm{d}x}{1+x^2\tan^2 x} = \frac{1}{1+\xi^2\tan^2\xi}\int_0^{\frac{\pi}{4}} x\mathrm{d}x
$$

其中$\frac{1}{1+x^2\tan^2 x}$在$[0,\frac{\pi}{4}]$严格单调递增，且$\frac{1}{2}<\frac{1}{1+\left(\frac{\pi}{4}\right)^2}\leq\frac{1}{1+x^2\tan^2 x}\leq\frac{1}{1+0}=1$。

因为$\frac{1}{1+x^2\tan^2 x}$严格单调，故由定积分第一中值定理的证明过程可知，$\xi\neq 0,\frac{1}{\pi}$

因此$\xi\in[0,\frac{4}{\pi}]$时，$\frac{1}{2}<\frac{1}{1+\xi^2\tan^2\xi}<1$。又因为$\int_0^{\frac{\pi}{4}}x\mathrm{d}x=\frac{\pi^2}{32}$，故

$$\frac{\pi^2}{64}<\int_0^{\frac{\pi}{4}}\frac{x\mathrm{d}x}{1+x^2\tan^2 x}<\frac{\pi^2}{32}$$

得证。■

20220307

习题七

33(2)

因为$\int x\sin x^2\mathrm{d}x=\frac{1}{2}\int\sin x^2\mathrm{d}(x^2)=-\frac{1}{2}\cos x^2$，有：

$$\begin{aligned}\left|\int_a^b\sin x^2\mathrm{d}x\right|&=\left|\int_a^b\frac{1}{x}(x\sin x^2)\mathrm{d}x\right|\\&=\left|\frac{1}{a}\int_a^\xi x\sin x^2\mathrm{d}x\right|\\&=\left|\frac{1}{2}\left(\frac{1}{a}(\cos a^2-\cos\xi^2)\right)\right|\\&\le\frac{1}{2a}|1-(-1)|\\&=\frac{1}{a}\end{aligned}$$

得证。■

34

(1)

引理：

$$\begin{aligned}\int_{\frac{k}{n}\pi}^{\frac{k+1}{n}\pi}f(x)\sin 2nx\mathrm{d}x&=\left(f(\frac{k+1}{n}\pi)\int_\xi^{\frac{k+1}{n}\pi}\sin 2nx\mathrm{d}x+f(\frac{k}{n}\pi)\int_{\frac{k}{n}\pi}^{\xi}\sin 2nx\mathrm{d}x\right)\\&=\frac{1}{2n}f(\frac{k+1}{n}\pi)(\cos 2n\xi-1)+\frac{1}{2n}f(\frac{k}{n}\pi)(1-\cos 2n\xi)\\&=\frac{1}{2n}(1-\cos 2n\xi)\left(f(\frac{k}{n}\pi)-f(\frac{k+1}{n}\pi)\right)\\&\ge 0\end{aligned}$$

证明：

$$\int_{-\pi}^{\pi}f(x)\sin 2nx\mathrm{d}x=\sum_{k=-n}^{n-1}\int_{\frac{k}{n}\pi}^{\frac{k+1}{n}\pi}f(x)\sin 2nx\mathrm{d}x\ge 0$$

(2)

同理。引理：

$$\begin{aligned}\int_{\frac{2k-1}{2n+1}\pi}^{\frac{2k+1}{2n+1}\pi} f(x)\sin{(2n+1)x}\mathrm{d}x &= \left(f(\tfrac{2k+1}{2n+1}\pi)\int_{\xi}^{\frac{2k+1}{2n+1}\pi}\sin{(2n+1)x}\mathrm{d}x + f(\tfrac{2k-1}{2n+1}\pi)\int_{\frac{2k-1}{2n+1}\pi}^{\xi}\sin{(2n+1)x}\mathrm{d}x\right)\\ &= \frac{1}{2n+1}\left(f(\tfrac{2k+1}{2n+1}\pi)(1+\cos{(2n+1)\xi}) + f(\tfrac{2k-1}{2n+1}\pi)(-1-\cos{(2n+1)\xi})\right)\\ &= -\frac{1}{2n}(1+\cos 2n\xi)\left(f(\frac{k}{n}\pi) - f(\frac{k+1}{n}\pi)\right)\\ &\leq 0\end{aligned}$$

证明：

$$\int_{-\pi}^{\pi} f(x)\sin{(2n+1)x}\mathrm{d}x = \sum_{k=-n}^{n}\int_{\frac{2k-1}{2n+1}\pi}^{\frac{2k+1}{2n+1}\pi} f(x)\sin{(2n+1)x}\mathrm{d}x \geq 0$$

36

> 设函数 $f(x)$ 在 $[0,1]$ 上连续，在 $(0,1)$ 内可导，并且满足 $f(1)=2\displaystyle\int_0^{\frac{1}{2}} e^{1-x}f(x)\mathrm{d}x$。证明：存在 $\xi\in(0,1)$ ，使得 $f(\xi)=f'(\xi)$。

由题目条件，

$$f(1)=2\int_0^{\frac{1}{2}} e^{1-x}f(x)\mathrm{d}x = e^{1-\xi}f(\xi)$$

令$g(x)=f(1)-e^{1-x}f(x)$，则有$g(1)=g(\xi)=0$，故$\exists y\in(\xi,1)$使得$g'(x)=0$，即

$$e^{1-x}(f(x)-f'(x))=0$$

因为$e^{1-x}>0$，有

$$f(x)=f'(x)$$

得证。■

20220309

习题七

41(2)

> 求由三叶线 $r=a\sin 3\theta$ 围成的平面图形的面积（其中参数 $a>0$）。

$$
\begin{aligned}
S &= \int \frac{1}{2} r^2 \mathrm{d}\theta \\
&= \frac{3}{2} \int_0^{\frac{\pi}{3}} a^2 \sin^2 3\theta \, \mathrm{d}\theta \\
&= \frac{1}{4} \int_0^{\frac{\pi}{3}} a^2 \frac{1-\cos 6\theta}{2} \, \mathrm{d}(6\theta) \\
&= \frac{a^2}{8} (6\theta - \sin 6\theta) \Big|_0^{\frac{\pi}{3}} \\
&= \boxed{\frac{a^2 \pi}{4}}
\end{aligned}
$$

42(2)

> 求由 $x^4 + y^4 = x^2 + y^2$ 围成的平面图形的面积。

将直角坐标系转换成极座标系：

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}
$$

将之代入，整理得到：

$$
r^2 = \frac{4}{3 + \cos 4\theta}
$$

用极座标下的面积公式得：

$$
\begin{aligned}
S &= \frac{1}{2} \int r^2 \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \int_0^{2\pi} \frac{4}{3+\cos 4\theta} \mathrm{d}\theta \\
&= 4 \int_0^{\frac{\pi}{4}} \frac{4}{3+\cos 4\theta} \mathrm{d}\theta
\end{aligned}
$$

令$\alpha = 4\theta, t = \tan\frac{\alpha}{2}$，有：

$$
\begin{aligned}
\int \frac{4}{3+\cos 4\theta} \mathrm{d}\theta &= \int \frac{1}{3+\cos\alpha} \mathrm{d}\alpha \\
&= \int \frac{1}{3 + \frac{\tan^2\frac{\alpha}{2} - 1}{\tan^2\frac{\alpha}{2} + 1}} \frac{2}{1+\tan^2\frac{\alpha}{2}} \mathrm{d}(\tan\frac{\alpha}{2}) \\
&= \int \frac{t^2+1}{4t^2+2} \frac{2}{1+t^2} \mathrm{d}t \\
&= \int \frac{1}{2t^2+1} \mathrm{d}t \\
&= \frac{\sqrt{2}}{2} \int \frac{1}{(\sqrt{2}t)^2 + 1} \mathrm{d}(\sqrt{2}t) \\
&= \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}t\right)
\end{aligned}
$$

因此，

$$
\begin{aligned}
S &= 4\int_0^{\frac{\pi}{4}} \frac{4}{3+\cos 4\theta}\mathrm{d}\theta \\
&= 4\frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan 2\theta\right)\Big|_0^{\frac{\pi}{4}} \\
&= 2\sqrt{2}(\frac{\pi}{2}) \\
&= \sqrt{2}\pi
\end{aligned}
$$

49(4)

> 求圆的渐伸线的弧长：$x=a(\cos t+t\sin t), y=a(\sin t-t\cos t)\ (0\le t\le 2\pi, a>0)$

$$
\begin{aligned}
L &= \int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\mathrm{d}t \\
&= \int_0^{2\pi}\sqrt{a^2(-\sin t+t\cos t+\sin t)^2+a^2(\cos t-\cos t+t\sin t)^2}\mathrm{d}t \\
&= \int_0^{2\pi}\sqrt{a^2(t\cos t)^2+a^2(t\sin t)^2}\mathrm{d}t \\
&= \int_0^{2\pi}|at|\mathrm{d}t \\
&= \left(\frac{at^2}{2}\right)\Big|_0^{2\pi} \\
&= 2a\pi^2
\end{aligned}
$$

51(6)

> 求曲线 $y=a\cosh\frac{x}{a}$ 绕 x 轴旋转一圈所得曲面的面积

$$
\begin{aligned}
S &= 2\pi\int_{-b}^{b} y(x)\sqrt{1+y'^2(x)}\mathrm{d}x \\
&= 2\pi\int_{-b}^{b} a\cosh\frac{x}{a}\sqrt{1+(\sinh\frac{x}{a})^2}\mathrm{d}x \\
&= 2\pi\int_{-b}^{b} a\cosh^2\frac{x}{a}\mathrm{d}x \\
&= 2\pi\int_{-b}^{b} a\frac{1}{2}\left(\cosh\left(\frac{2x}{a}\right)+1\right)\mathrm{d}x \\
&= a\pi\left(a\sinh\left(\frac{2x}{a}\right)+x\right)\Big|_{-b}^{b} \\
&= a\pi\left(2a\sinh\left(\frac{2b}{a}\right)+2b\right)
\end{aligned}
$$