# ECE276A: Sensing & Estimation in Robotics
# Lecture 3: Rotations

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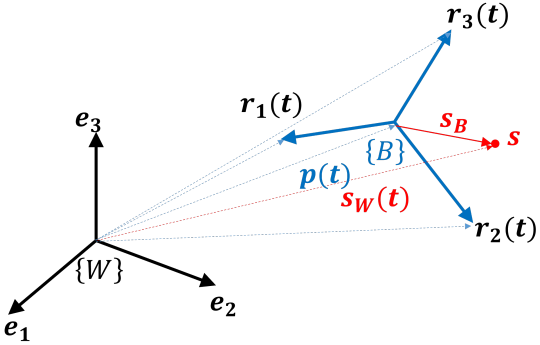
# Outline

# Rigid Body Motion

- Consider a rigid body moving in a fixed **world reference frame** $\{W\}$
- **Body reference frame** $\{B\}$: it is sufficient to specify the motion of one point $\mathbf{p}(t) \in \mathbb{R}^3$ and 3 coordinate axes $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{r}_3(t)$ attached to the point




- A point $\mathbf{s}$ on the rigid body has fixed coordinates $\mathbf{s}_B \in \mathbb{R}^3$ in the body frame $\{B\}$ but time-varying coordinates $\mathbf{s}_W(t) \in \mathbb{R}^3$ in the world frame $\{W\}$

# Rigid Body Motion

- A rigid body in 3D is free to translate (3 degrees of freedom) and rotate (3 degrees of freedom)
- The space of rotations in 3D is denoted $SO(3)$
- The space of translation and rotation transformations in 3D is denoted $SE(3)$
- The **pose** $T(t) \in SE(3)$ of a reference frame $\{B\}$ at time $t$ in a fixed world frame $\{W\}$ is determined by:
  1. the position $\mathbf{p}(t) \in \mathbb{R}^3$ of $\{B\}$ relative to $\{W\}$
  2. the orientation $R(t) \in SO(3)$ of $\{B\}$ relative to $\{W\}$, determined by the 3 coordinate axes $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{r}_3(t)$
- The space of positions $\mathbb{R}^3$ is familiar
- How do we describe the space of orientations $SO(3)$ and the space of poses $SE(3)$?

## Special Euclidean Group

- Rigid body motion is described by functions that specify how the coordinates of 3-D points on a rigid body change with time

- Rigid body motion preserves distances (preserves vector norms) and does not allow reflection of the coordinate system (preserves vector cross products)

- **Euclidean Group** $E(3)$: set of functions $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm of any two vectors

- **Special Euclidean Group** $SE(3)$: set of functions $g : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the norm and the cross product of any two vectors
  1. Norm: $\|g_*(\mathbf{u}) - g_*(\mathbf{v})\| = \|\mathbf{v} - \mathbf{u}\|$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
  2. Cross product: $g_*(\mathbf{u}) \times g_*(\mathbf{v}) = g_*(\mathbf{u} \times \mathbf{v})$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$

  where $g_*(\mathbf{x}) := g(\mathbf{x}) - g(\mathbf{0})$.

- **Corollary**: $SE(3)$ elements $g$ also preserve:
  1. Angle: $\mathbf{u}^\top \mathbf{v} = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right) \Rightarrow \mathbf{u}^\top \mathbf{v} = g_*(\mathbf{u})^\top g_*(\mathbf{v})$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
  2. Volume: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $g_*(\mathbf{u})^\top (g_*(\mathbf{v}) \times g_*(\mathbf{w})) = \mathbf{u}^\top (\mathbf{v} \times \mathbf{w})$

     (volume of parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$)

## Orientation and Rotation

- Pure rotational motion is a special case of rigid body motion
- The orientation of a body frame $\{B\}$ in the world frame $\{W\}$ is determined by three orthogonal unit vectors $\mathbf{r}_1 = g(\mathbf{e}_1)$, $\mathbf{r}_2 = g(\mathbf{e}_2)$, $\mathbf{r}_3 = g(\mathbf{e}_3)$ with coordinates transformed from $\{B\}$ to $\{W\}$
- The vectors organized in a 3×3 matrix specify the orientation of $\{B\}$ in $\{W\}$:

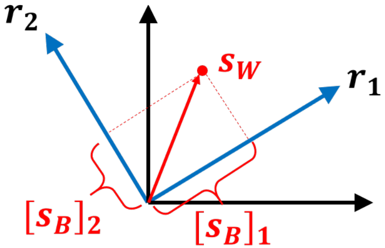
$$_{\{W\}}R_{\{B\}} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- Consider a point with coordinates $\mathbf{s}_B \in \mathbb{R}^3$ in $\{B\}$
- Its coordinates $\mathbf{s}_W$ in $\{W\}$ are:

$$\begin{aligned} \mathbf{s}_W &= [s_B]_1\mathbf{r}_1 + [s_B]_2\mathbf{r}_2 + [s_B]_3\mathbf{r}_3 \\ &= R\mathbf{s}_B \end{aligned}$$

- The rotation transformation $g$ from $\{B\}$ to $\{W\}$ is a linear function:

$$g(\mathbf{s}) = R\mathbf{s}$$



## Special Orthogonal Group $SO(3)$

- $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis: $\mathbf{r}_i^\top \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Since $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis, the inverse of $R$ is its transpose:

$$R^\top R = I \qquad\qquad R^{-1} = R^\top$$

- $R$ belongs to the **orthogonal group**:

$$O(3) := \{R \in \mathbb{R}^{3\times 3} \mid R^\top R = RR^\top = I\}$$

- Distances are preserved since $R^\top R = I$:

$$\|R(\mathbf{x}-\mathbf{y})\|_2^2 = (\mathbf{x}-\mathbf{y})^\top R^\top R(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y})^\top(\mathbf{x}-\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_2^2$$

- Reflections are not allowed since $\det(R) = \mathbf{r}_1^\top(\mathbf{r}_2 \times \mathbf{r}_3) = 1$:

$$R(\mathbf{x}\times\mathbf{y}) = R\left(\mathbf{x}\times(R^\top R\mathbf{y})\right) = (R\hat{\mathbf{x}}R^\top)R\mathbf{y} = \frac{1}{\det(R)}(R\mathbf{x})\times(R\mathbf{y})$$

- $R$ belongs to the **special orthogonal group**:

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\}$$

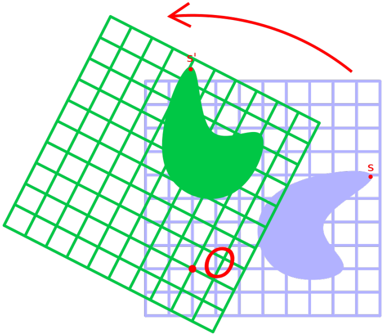
# Parametrizing 2-D Rotations

- There are 2 common ways to parametrize a rotation matrix $R \in SO(2)$

- **Rotation Angle**: a 2-D rotation of a point $\mathbf{s}_B \in \mathbb{R}^2$ can be parametrized by an angle $\theta$ around the $z$-axis:

$$\mathbf{s}_W = R(\theta)\mathbf{s}_B := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{s}_B$$

- $\theta > 0$: counterclockwise rotation




- **Unit-Norm Complex Number**: a 2-D rotation of $[s_B]_1 + i[s_B]_2 \in \mathbb{C}$ can be parametrized by a unit-norm complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta) \in \mathbb{C}$:

$$e^{i\theta}([s_B]_1 + i[s_B]_2) = ([s_B]_1\cos\theta - [s_B]_2\sin\theta) + i([s_B]_1\sin\theta + [s_B]_2\cos\theta)$$

# Parametrizing 2-D Rotations

- The two algebraic representations of $SO(2)$ are closely related
- There is a one-to-one correspondence between unit-norm complex numbers and 2-D rotation matrices:

$$e^{j\theta} = a + jb \quad \Leftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = aI + bJ \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad J^2 = -I$$

- Note also that $J$ is related to the rate of change of $R(\theta)$

$$J = \frac{d}{d\theta} R(\theta) \bigg|_{\theta=0} = \begin{bmatrix} -\sin(0) & -\cos(0) \\ \cos(0) & -\sin(0) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# Parametrizing 3-D Rotations

- There are 3 common ways to parametrize a rotation matrix $R \in SO(3)$
- **Euler Angles**: an extension of the rotation angle parametrization of 2-D rotations that specifies rotation angles around the three principal axes
- **Axis-Angle**: an extension of the rotation angle parametrization of 2-D rotations that allows the axis of rotation to be chosen freely instead of being a fixed principal axis
- **Unit-Norm Quaternion**: an extension of the unit-norm complex number parametrization of 2-D rotations

# Outline

# Euler Angle Parametrization

- Three angles specify rotations around the three principal axes
- There are 24 different ways to apply these rotations
  - **Extrinsic axes**: the rotation axes remain static
  - **Intrinsic axes**: the rotation axes move with the rotations
  - Each of the two groups (intrinsic and extrinsic) can be divided into:
    - **Euler Angles**: rotation about one axis, then a second, and then the first
    - **Tait-Bryan Angles**: rotation about all three axes
  - Euler and Tait-Bryan angles each have 6 possible choices for each of the extrinsic/intrinsic groups leading to $2 * 2 * 6 = 24$ possible conventions to specify a rotation sequence with three given angles
- For simplicity, we refer to all 24 conventions as **Euler Angles** and specify:
  - $r$ (**rotating = intrinsic**) or $s$ (**static = extrinsic**)
  - $xyz$ or $zyx$ or $zxz$, etc. (**order of rotation axes**)
- An extrinsic rotation is equivalent to an intrinsic rotation by the same angles but with inverted rotation order:

$$sxyz = rzyx$$

## Principal 3-D Rotations

- A rotation by an angle $\phi$ around the $x$-axis is represented by:

$$R_x(\phi) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

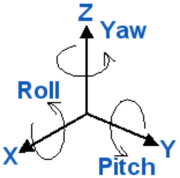
- A rotation by an angle $\theta$ around the $y$-axis is represented by:

$$R_y(\theta) := \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- A rotation by an angle $\psi$ around the $z$-axis is represented by:

$$R_z(\psi) := \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Roll Pitch Yaw Convention




- **Roll** ($\phi$), **pitch** ($\theta$), **yaw** ($\psi$) angles are used in aerospace engineering to specify rotation of an aircraft around the $x$, $y$, and $z$ axes, respectively

- Intrinsic yaw ($\psi$), pitch ($\theta$), roll ($\phi$) rotation (*rzyx*):
  - A rotation $\psi$ about the original $z$-axis
  - A rotation $\theta$ about the intermediate $y$-axis
  - A rotation $\phi$ about the transformed $x$-axis

- Extrinsic roll ($\phi$), pitch ($\theta$), yaw ($\psi$) rotation (*sxyz*):
  - A rotation $\phi$ about the global $x$-axis
  - A rotation $\theta$ about the global $y$-axis
  - A rotation $\psi$ about the global $z$-axis

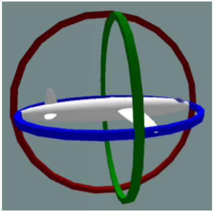
- Both conventions define the following body-to-world rotation:

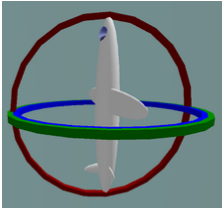
$$
\begin{aligned}
R &= R_z(\psi) R_y(\theta) R_x(\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\end{aligned}
$$

## Gimbal Lock

- Angle parametrizations are widely used due to their simplicity
- Unfortunately, in 3-D, angle parametrizations are not one-to-one and lead to **singularities** known as **gimbal lock**
- Example: if the pitch becomes $\theta = 90^\circ$, the roll and yaw become associated with the same degree of freedom and cannot be uniquely determined.
- The following leads to the same rotation matrix $R$ for any choice of $\delta$:

$$R = R_z(\psi)R_y(\pi/2)R_x(\phi + \delta)$$





https://www.youtube.com/watch?v=-7v0OeN7sdI
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# Outline

# Cross Product and Hat Map

- The **cross product** of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is also a vector in $\mathbb{R}^3$:

$$\mathbf{x} \times \mathbf{y} := \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \hat{\mathbf{x}}\mathbf{y}$$

- The cross product $\mathbf{x} \times \mathbf{y}$ can be represented by a *linear* map $\hat{\mathbf{x}}$ called the **hat map**

- The **hat map** $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector $\mathbf{x} \in \mathbb{R}^3$ to a skew-symmetric matrix:

$$\hat{\mathbf{x}} := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \qquad \hat{\mathbf{x}}^\top = -\hat{\mathbf{x}}$$

- The vector space $\mathbb{R}^3$ and the space of skew-symmetric $3 \times 3$ matrices $\mathfrak{so}(3)$ are isomorphic, i.e., there exists a one-to-one map (the hat map) that preserves their structure

# Hat Map Properties

- **Lemma**: A matrix $M \in \mathbb{R}^{3\times 3}$ is skew-symmetric iff $M = \hat{\mathbf{x}}$ for some $\mathbf{x} \in \mathbb{R}^3$
- The inverse of the hat map is the **vee map**, $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$, that extracts the components of the vector $\mathbf{x} = \hat{\mathbf{x}}^\vee$ from the matrix $\hat{\mathbf{x}}$
- **Hat map properties**: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$:
  - $\hat{\mathbf{x}}\mathbf{y} = \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x} = -\hat{\mathbf{y}}\mathbf{x}$ (cross product)
  - $\hat{\mathbf{x}}\hat{\mathbf{y}} = \mathbf{y}\mathbf{x}^\top - \mathbf{x}^\top\mathbf{y}\, I$
  - $\mathbf{x} \times \mathbf{y} \times \mathbf{z} = \hat{\mathbf{x}}\hat{\mathbf{y}}\mathbf{z} = \mathbf{y}\mathbf{x}^\top\mathbf{z} - \mathbf{x}^\top\mathbf{y}\mathbf{z}$ (vector triple product)
  - $\hat{\mathbf{x}}\hat{\mathbf{y}}\mathbf{z} + \hat{\mathbf{y}}\hat{\mathbf{z}}\mathbf{x} + \hat{\mathbf{z}}\hat{\mathbf{x}}\mathbf{y} = 0$ (Jacobi identity)
  - $\hat{\mathbf{x}}^{2k+1} = (-\mathbf{x}^\top\mathbf{x})^k\hat{\mathbf{x}}$
  - $(\hat{\mathbf{x}}\mathbf{y})^\wedge = \hat{\mathbf{x}}\hat{\mathbf{y}} - \hat{\mathbf{y}}\hat{\mathbf{x}} = \mathbf{y}\mathbf{x}^\top - \mathbf{x}\mathbf{y}^\top$
  - $\hat{\mathbf{x}}A + A^\top\hat{\mathbf{x}} = ((\mathrm{tr}(A)I - A)\mathbf{x})^\wedge$
  - $\mathrm{tr}(\hat{\mathbf{x}}A) = \frac{1}{2}\mathrm{tr}(\hat{\mathbf{x}}(A - A^\top)) = -\mathbf{x}^\top(A - A^\top)^\vee$
  - $(A\mathbf{x})^\wedge = \det(A)A^{-\top}\hat{\mathbf{x}}A^{-1}$

## Axis-Angle Parametrization

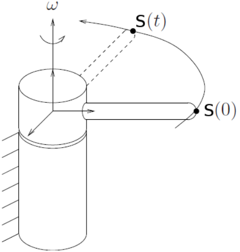
- If a point is rotating about a fixed axis with angular velocity $\boldsymbol{\omega}$ and position vector $\mathbf{r}$ (wrt rotation axis), its linear velocity (tangential to the circle) is:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

- Consider a point $\mathbf{s} \in \mathbb{R}^3$ rotating about an axis $\boldsymbol{\eta} \in \mathbb{R}^3$ at constant unit velocity:

$$\dot{\mathbf{s}}(t) = \boldsymbol{\eta} \times \mathbf{s}(t) = \hat{\boldsymbol{\eta}}\mathbf{s}(t)$$

- This is a linear time-invariant (LTI) system of ordinary differential equations determined by the skew-symmetric matrix $\hat{\boldsymbol{\eta}}$




- The solution to the LTI system $\dot{\mathbf{s}} = \hat{\boldsymbol{\eta}}\mathbf{s}$ is: $\mathbf{s}(t) = \exp(t\hat{\boldsymbol{\eta}})\mathbf{s}(0)$
- Since $\mathbf{s}$ undergoes pure rotation, we also know that: $\mathbf{s}(t) = R(t)\mathbf{s}(0)$
- Since the rotation has constant unit velocity, the elapsed time $t$ equals the angle of rotation $\theta$:

$$R(\theta) = \exp(\theta\hat{\boldsymbol{\eta}})$$

# Exponential Map from $\mathfrak{so}(3)$ to $SO(3)$

- Any rotation can be represented as a rotation about a unit-vector axis $\boldsymbol{\eta} \in \mathbb{R}^3$ through angle $\theta \in \mathbb{R}$
- The axis-angle parametrization is combined in a single rotation vector $\boldsymbol{\theta} := \theta\boldsymbol{\eta} \in \mathbb{R}^3$ with magnitude $\theta$ and direction $\boldsymbol{\eta}$
- **Exponential map** $\exp : \mathfrak{so}(3) \to SO(3)$ maps a skew-symmetric matrix $\hat{\boldsymbol{\theta}}$ obtained from an axis-angle vector $\boldsymbol{\theta}$ to a rotation matrix $R$:

$$R = \exp(\hat{\boldsymbol{\theta}}) := \sum_{n=0}^{\infty} \frac{1}{n!}\hat{\boldsymbol{\theta}}^n = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \ldots$$

- The matrix exponential defines a map from the space of skew-symmetric matrices $\mathfrak{so}(3)$ to the space of rotation matrices $SO(3)$
  - The exponential map is **surjective** but **not injective**: every element of $SO(3)$ can be generated from multiple elements of $\mathfrak{so}(3)$, e.g., any vector $(\|\boldsymbol{\theta}\| + 2\pi k)\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ for integer $k$ leads to the same $R$
  - The exponential map is **not commutative**: $e^{\hat{\boldsymbol{\theta}}_1}e^{\hat{\boldsymbol{\theta}}_2} \neq e^{\hat{\boldsymbol{\theta}}_2}e^{\hat{\boldsymbol{\theta}}_1} \neq e^{\hat{\boldsymbol{\theta}}_1+\hat{\boldsymbol{\theta}}_2}$, unless $\hat{\boldsymbol{\theta}}_1\hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_2\hat{\boldsymbol{\theta}}_1 = 0$

# Rodrigues Formula

- **Rodrigues Formula**: closed-from expression for the exponential map from $\mathfrak{so}(3)$ to $SO(3)$:

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2$$

- The formula is derived using $\hat{\boldsymbol{\theta}}^{2n+1} = (-\boldsymbol{\theta}^\top\boldsymbol{\theta})^n\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}
\exp(\hat{\boldsymbol{\theta}}) &= I + \sum_{n=1}^{\infty}\frac{1}{n!}\hat{\boldsymbol{\theta}}^n \\
&= I + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\hat{\boldsymbol{\theta}}^{2n+1} + \sum_{n=0}^{\infty}\frac{1}{(2n+2)!}\hat{\boldsymbol{\theta}}^{2n+2} \\
&= I + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+1)!}\right)\hat{\boldsymbol{\theta}} + \left(\sum_{n=0}^{\infty}\frac{(-1)^n\|\boldsymbol{\theta}\|^{2n}}{(2n+2)!}\right)\hat{\boldsymbol{\theta}}^2 \\
&= I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2
\end{aligned}$$

## Logarithm Map from $SO(3)$ to $\mathfrak{so}(3)$

- For any $R \in SO(3)$, there exists a (non-unique) $\boldsymbol{\theta} \in \mathbb{R}^3$ such that $R = \exp(\hat{\boldsymbol{\theta}})$
- **Logarithm map** $\log : SO(3) \to \mathfrak{so}(3)$ is the inverse of $\exp(\hat{\boldsymbol{\theta}})$:

$$\theta = \|\boldsymbol{\theta}\| = \arccos\left(\frac{\operatorname{tr}(R) - 1}{2}\right)$$

$$\boldsymbol{\eta} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} = \frac{1}{2\sin(\|\boldsymbol{\theta}\|)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

$$\hat{\boldsymbol{\theta}} = \log(R) = \frac{\|\boldsymbol{\theta}\|}{2\sin\|\boldsymbol{\theta}\|}(R - R^\top)$$

- If $R = I$, then $\theta = 0$ and $\boldsymbol{\eta}$ is undefined
- If $\operatorname{tr}(R) = -1$, then $\theta = \pi$ and for any $i \in \{1, 2, 3\}$:

$$\boldsymbol{\eta} = \frac{1}{\sqrt{2(1 + R_{ii})}}(I + R)e_i$$

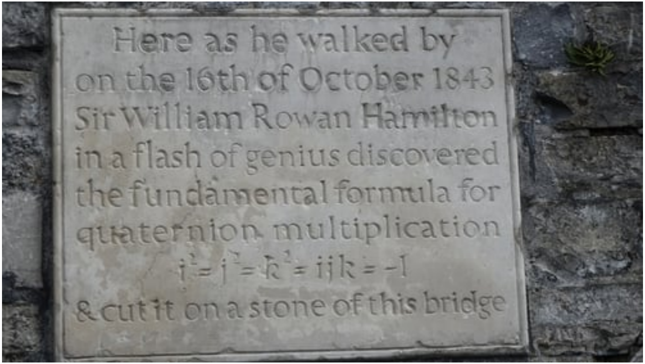
- The matrix exponential "integrates" $\hat{\boldsymbol{\theta}} \in \mathfrak{so}(3)$ for one second; the matrix logarithm "differentiates" $R \in SO(3)$ to obtain $\hat{\boldsymbol{\theta}} \in \mathfrak{so}(3)$

# Outline

# Quaternions

- William Hamilton attempted to extend complex numbers to three dimensions but struggled to define multiplication consistently. In 1843, he realized that this is possible in four dimensions. Overcome with excitement, he carved the fundamental formulas into the stone of the Broom Bridge:

$$i^2 = j^2 = k^2 = ijk = -1 \qquad ij = -ji = k$$



# Quaternions

- **Quaternions**: $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ generalize complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$:

$$\mathbf{q} = q_s + q_1 i + q_2 j + q_3 k = (q_s + q_1 i) + (q_2 + q_3 i)j = [q_s,\ \mathbf{q}_v]$$

- As in 2-D, 3-D rotations can be represented using "unit complex numbers", i.e., **unit-norm quaternions**:

$$\mathbb{H}_* := \{\mathbf{q} \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^T \mathbf{q}_v = 1\}$$

- To represent rotations without singularities, we embed a 3-D space $SO(3)$ into a 4-D space $\mathbb{H}$ and introduce a unit-norm constraint

- A rotation matrix $R \in SO(3)$ can be obtained from a unit quaternion $\mathbf{q}$:

$$R(\mathbf{q}) = E(\mathbf{q})G(\mathbf{q})^\top \qquad \begin{aligned} E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \\ G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \end{aligned}$$

- The space of quaternions $\mathbb{H}_*$ is a **double covering** of $SO(3)$ because two unit quaternions correspond to the same rotation: $R(\mathbf{q}) = R(-\mathbf{q})$

# Quaternion Axis-Angle Parametrization

- A rotation around a unit axis $\boldsymbol{\eta} := \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \in \mathbb{R}^3$ by angle $\theta := \|\boldsymbol{\theta}\|$ can be represented by a unit quaternion:

$$\mathbf{q} = \left[\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\boldsymbol{\eta}\right] \in \mathbb{H}_*$$

- A rotation around a unit axis $\boldsymbol{\eta} \in \mathbb{R}^3$ by angle $\theta$ can be recovered from a unit quaternion $\mathbf{q} = [q_s,\ \mathbf{q}_v] \in \mathbb{H}_*$:

$$\theta = 2\arccos(q_s) \qquad \boldsymbol{\eta} = \begin{cases} \frac{1}{\sin(\theta/2)}\mathbf{q}_v, & \text{if } \theta \neq 0 \\ 0, & \text{if } \theta = 0 \end{cases}$$

- The inverse transformation above has a singularity at $\theta = 0$ because the transformation from $\boldsymbol{\theta}$ to $\mathbf{q}$ is many-to-one and there are infinitely many rotation axes that can be used

# Quaternion Operations

| | |
|---|---|
| **Addition** | $\mathbf{q} + \mathbf{p} := [q_s + p_s,\ \mathbf{q}_v + \mathbf{p}_v]$ |
| **Multiplication** | $\mathbf{q} \circ \mathbf{p} := \left[q_s p_s - \mathbf{q}_v^T \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$ |
| **Conjugation** | $\bar{\mathbf{q}} := [q_s,\ -\mathbf{q}_v]$ |
| **Norm** | $\|\mathbf{q}\| := \sqrt{q_s^2 + \mathbf{q}_v^T \mathbf{q}_v} \qquad \|\mathbf{q} \circ \mathbf{p}\| = \|\mathbf{q}\|\|\mathbf{p}\|$ |
| **Inverse** | $\mathbf{q}^{-1} := \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|^2}$ |
| **Rotation** | $[0,\ \mathbf{x}'] = \mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [0,\ R(\mathbf{q})\mathbf{x}]$ |
| **Velocity** | $\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \circ [0,\ \boldsymbol{\omega}] = \frac{1}{2} G(\mathbf{q})^\top \boldsymbol{\omega}$ |
| **Exp** | $\exp(\mathbf{q}) := e^{q_s} \left[\cos \|\mathbf{q}_v\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \sin \|\mathbf{q}_v\|\right]$ |
| **Log** | $\log(\mathbf{q}) := \left[\log \|\mathbf{q}\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|} \arccos \frac{q_s}{\|\mathbf{q}\|}\right]$ |

- **Exp**: constructs $\mathbf{q} \in \mathbb{H}_*$ from rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$: $\mathbf{q} = \exp\left(\left[0,\ \frac{\boldsymbol{\theta}}{2}\right]\right)$
- **Log**: recovers a rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$ from $\mathbf{q} \in \mathbb{H}_*$: $[0,\ \boldsymbol{\theta}] = 2\log(\mathbf{q})$

## Quaternion Multiplication and Rotation

- Quaternion multiplication (combination of dot and cross products):

$$\mathbf{q} \circ \mathbf{p} := \left[q_s p_s - \mathbf{q}_v^\top \mathbf{p}_v,\ q_s \mathbf{p}_v + p_s \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v\right]$$

- Quaternion multiplication $\mathbf{q} \circ \mathbf{p}$ can be represented using linear operations:

$$\begin{aligned} \mathbf{q} \circ \mathbf{p} &= [\mathbf{q}]_L\, \mathbf{p} = [\mathbf{p}]_R\, \mathbf{q} \\ [\mathbf{q}]_L &:= \left[\mathbf{q} \quad G(\mathbf{q})^\top\right] & G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \\ [\mathbf{q}]_R &:= \left[\mathbf{q} \quad E(\mathbf{q})^\top\right] & E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \end{aligned}$$

- Rotating a vector $\mathbf{x} \in \mathbb{R}^3$ by quaternion $\mathbf{q} \in \mathbb{H}_*$ is performed as:

$$\mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [0,\ \mathbf{x}'] = [0,\ R(\mathbf{q})\mathbf{x}]$$

- This relates a quaternion $\mathbf{q}$ to a corresponding rotation matrix $R(\mathbf{q})$:

$$\begin{aligned} \begin{bmatrix} 0 \\ R(\mathbf{q})\mathbf{x} \end{bmatrix} &= \mathbf{q} \circ [0,\ \mathbf{x}] \circ \mathbf{q}^{-1} = [\bar{\mathbf{q}}]_R\, [\mathbf{q}]_L \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \\ &= \left[\bar{\mathbf{q}} \quad E(\bar{\mathbf{q}})^\top\right] \left[\mathbf{q} \quad G(\mathbf{q})^\top\right] \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^\top \\ E(\mathbf{q}) \end{bmatrix} \left[\mathbf{q} \quad G(\mathbf{q})^\top\right] \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{q}^\top \mathbf{q} & \mathbf{q}^\top G(\mathbf{q})^\top \\ E(\mathbf{q})\mathbf{q} & E(\mathbf{q}) G(\mathbf{q})^\top \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^\top G(\mathbf{q})^\top \mathbf{x} \\ E(\mathbf{q}) G(\mathbf{q})^\top \mathbf{x} \end{bmatrix} \end{aligned}$$

# Example: Rotation with a Quaternion

- Let $\mathbf{x} = \mathbf{e}_2 \in \mathbb{R}^3$ be a point in frame $\{A\}$
- What are the coordinates of $\mathbf{x}$ in frame $\{B\}$ which is rotated by $\theta = \pi/3$ with respect to $\{A\}$ around the $x$-axis?
- The quaternion corresponding to the rotation from $\{B\}$ to $\{A\}$ is:

$$ {}_A\mathbf{q}_B = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\boldsymbol{\eta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} $$

- The quaternion corresponding to the rotation from $\{A\}$ to $\{B\}$ is:

$$ {}_B\mathbf{q}_A = {}_A\mathbf{q}_B^{-1} = {}_A\bar{\mathbf{q}}_B = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -\mathbf{e}_1 \end{bmatrix} $$

- The coordinates of $\mathbf{x}$ in frame $\{B\}$ are:

$$ \begin{aligned} {}_B\mathbf{q}_A \circ [0,\ \mathbf{x}] \circ {}_B\mathbf{q}_A^{-1} &= \frac{1}{4}\begin{bmatrix} \sqrt{3} \\ -\mathbf{e}_1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ \mathbf{e}_2 \end{bmatrix} \circ \begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 0 \\ \sqrt{3}\mathbf{e}_2 - \mathbf{e}_1 \times \mathbf{e}_2 \end{bmatrix} \circ \begin{bmatrix} \sqrt{3} \\ \mathbf{e}_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ \mathbf{e}_2 - \sqrt{3}\mathbf{e}_3 \end{bmatrix} \end{aligned} $$

# Representations of Orientation (Summary)

- **Rotation Matrix**: an element of the **Special Orthogonal Group**:

$$R \in SO(3) := \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; \underbrace{R^\top R = I}_{\text{distances preserved}}, \underbrace{\det(R) = 1}_{\text{no reflection}} \right\}$$

- **Euler Angles**: roll $\phi$, pitch $\theta$, yaw $\psi$ specifying a **sxyz** or **rzyx** rotation:

$$R = R_z(\psi) R_y(\theta) R_x(\phi)$$

- **Axis-Angle**: $\boldsymbol{\theta} \in \mathbb{R}^3$ specifying rotation about axis $\boldsymbol{\eta} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$ by angle $\theta = \|\boldsymbol{\theta}\|$

$$R = \exp(\hat{\boldsymbol{\theta}}) = I + \hat{\boldsymbol{\theta}} + \frac{1}{2!}\hat{\boldsymbol{\theta}}^2 + \frac{1}{3!}\hat{\boldsymbol{\theta}}^3 + \ldots = I + \left(\frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1-\cos\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|^2}\right)\hat{\boldsymbol{\theta}}^2$$

- **Unit Quaternion**: $\mathbf{q} = [q_s,\ \mathbf{q}_v] \in \mathbb{H}_* := \left\{\mathbf{q} \in \mathbb{H} \mid q_s^2 + \mathbf{q}_v^\top \mathbf{q}_v = 1\right\}$:

$$R = E(\mathbf{q}) G(\mathbf{q})^\top \qquad \begin{aligned} E(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I + \hat{\mathbf{q}}_v] \\ G(\mathbf{q}) &= [-\mathbf{q}_v,\ q_s I - \hat{\mathbf{q}}_v] \end{aligned}$$

# Outline

## Rigid Body Pose

- Let $\{B\}$ be a body frame whose position and orientation with respect to the world frame $\{W\}$ are $\mathbf{p} \in \mathbb{R}^3$ and $R \in SO(3)$, respectively

- The coordinates of a point $\mathbf{s}_B \in \mathbb{R}^3$ can be converted to the world frame by first rotating the point and then translating it to the world frame:

$$\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p}$$

- The **homogeneous coordinates** of a point $\mathbf{s} \in \mathbb{R}^3$ are

$$\underline{\mathbf{s}} := \lambda \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix} \propto \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

  where the scale factor $\lambda$ allows representing points arbitrarily far away from the origin as $\lambda \to 0$, e.g., $\underline{\mathbf{s}} = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}^\top$

- Rigid-body transformations are linear in homogeneous coordinates:

$$\underline{\mathbf{s}}_W = \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix} = T\underline{\mathbf{s}}_B \qquad T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

# Special Euclidean Group $SE(3)$

- The pose of a rigid body can be described by a matrix $T$ in the **special Euclidean group**:

$$SE(3) := \left\{ T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \,\middle|\, R \in SO(3), \mathbf{p} \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}$$

- The pose of a rigid body $T$ specifies a transformation from the body frame $\{B\}$ to the world frame $\{W\}$:

$${}_{\{W\}}T_{\{B\}} := \begin{bmatrix} {}_{\{W\}}R_{\{B\}} & {}_{\{W\}}\mathbf{p}_{\{B\}} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- A point with body-frame coordinates $\mathbf{s}_B$, has world-frame coordinates:

$$\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p} \quad \text{equivalent to} \quad \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix}$$

- A point with world-frame coordinates $\mathbf{s}_W$, has body-frame coordinates:

$$\begin{bmatrix} \mathbf{s}_B \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix} = \begin{bmatrix} R^\top & -R^\top\mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_W \\ 1 \end{bmatrix}$$

## Composing Transformations

- Given a rigid body with pose ${}_{\{W\}}T_{\{1\}}$ at time $t_1$ and ${}_{\{W\}}T_{\{2\}}$ at time $t_2$, the relative transformation from inertial frame $\{2\}$ at time $t_2$ to inertial frame $\{1\}$ at time $t_1$ is:

$$
\begin{aligned}
{}_{\{1\}}T_{\{2\}} &= {}_{\{1\}}T_{\{W\}}\;{}_{\{W\}}T_{\{2\}} = \left({}_{\{W\}}T_{\{1\}}\right)^{-1}\;{}_{\{W\}}T_{\{2\}} \\
&= \begin{bmatrix} {}_{\{W\}}R_{\{1\}}^{\top} & -{}_{\{W\}}R_{\{1\}}^{\top} \times {}_{\{W\}}\mathbf{p}_{\{1\}} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} \begin{bmatrix} {}_{\{W\}}R_{\{2\}} & {}_{\{W\}}\mathbf{p}_{\{2\}} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}
\end{aligned}
$$

- The pose $T_k$ of a rigid body at time $t_k$ always specifies a transformation from the body frame at time $t_k$ to the world frame so we will not explicitly write the world frame subscript

- The relative transformation from inertial frame $\{2\}$ with world-frame pose $T_2$ to an inertial frame $\{1\}$ with world-frame pose $T_1$ is:

$$
{}_1T_2 = T_1^{-1}T_2
$$

# Summary

| | **Rotation** $SO(3)$ | **Pose** $SE(3)$ |
|---|---|---|
| **Representation** | $R : \begin{cases} R^T R = I \\ \det(R) = 1 \end{cases}$ | $T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$ |
| **Transformation** | $\mathbf{s}_W = R\mathbf{s}_B$ | $\mathbf{s}_W = R\mathbf{s}_B + \mathbf{p}$ |
| **Inverse** | $R^{-1} = R^\top$ | $T^{-1} = \begin{bmatrix} R^\top & -R^\top\mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix}$ |
| **Composition** | ${}_W R_B = {}_W R_A \, {}_A R_B$ | ${}_W T_B = {}_W T_A \, {}_A T_B$ |