ECE276A: Sensing & Estimation in Robotics
Lecture 4: Robot Motion and Observation Models

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Outline

Rigid-Body Kinematics and Dynamics

Motion Models

Observation Models

Rotation Kinematics

- The trajectory $R(t)$ of continuous rotation motion satisfies:

$$R^\top(t)R(t) = I \quad \Rightarrow \quad \dot{R}^\top(t)R(t) + R^\top(t)\dot{R}(t) = \mathbf{0}$$

- Since $R^\top(t)\dot{R}(t)$ is **skew-symmetric**, there exists $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ such that:

$$R^\top(t)\dot{R}(t) = \hat{\boldsymbol{\omega}}(t)$$

- **Rotation kinematics**: the orientation of a rigid body $R(t) \in SO(3)$ rotating with angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ (in body-frame coordinates) satisfies:

$$\dot{R}(t) = R(t)\hat{\boldsymbol{\omega}}(t)$$

- **Discrete-time rotation kinematics**: if $\boldsymbol{\omega}(t) \equiv \boldsymbol{\omega}_k$ is constant for $t \in [t_k, t_{k+1})$ and $R_k := R(t_k)$, $\tau_k := t_{k+1} - t_k$:

$$R_{k+1} = R_k \exp(\tau_k \hat{\boldsymbol{\omega}}_k)$$

  where $\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$ is the matrix exponential function.

Quaternion Kinematics

- The trajectory $\mathbf{q}$ of continuous rotation motion satisfies:

$$\bar{\mathbf{q}} \circ \mathbf{q} = \mathbf{e}_1 \quad \Rightarrow \quad \dot{\bar{\mathbf{q}}} \circ \mathbf{q} + \bar{\mathbf{q}} \circ \dot{\mathbf{q}} = \mathbf{0}$$

- Thus, there exists some $\mathbf{r} \in \mathbb{H}$ such that:

$$\bar{\mathbf{q}} \circ \dot{\mathbf{q}} = \mathbf{r} \qquad \text{and} \qquad \bar{\mathbf{r}} = -\mathbf{r} \quad (\text{i.e, } r_s = 0)$$

- **Quaternion kinematics**: the orientation of a rigid body $\mathbf{q}(t) \in \mathbb{H}_*$ rotating with angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ (in body-frame coordinates) satisfies:

$$\dot{\mathbf{q}}(t) = \mathbf{q}(t) \circ [0, \boldsymbol{\omega}(t)/2]$$

- **Discrete-time quaternion kinematics**: if $\boldsymbol{\omega}(t) \equiv \boldsymbol{\omega}_k$ is constant for $t \in [t_k, t_{k+1})$ and $\mathbf{q}_k := \mathbf{q}(t_k)$, $\tau_k := t_{k+1} - t_k$:

$$\mathbf{q}_{k+1} = \mathbf{q}_k \circ \exp([0, \tau_k \boldsymbol{\omega}_k/2])$$

where $\exp([q_s,\ \mathbf{q}_v]) = e^{q_s}\left[\cos\|\mathbf{q}_v\|,\ \frac{\mathbf{q}_v}{\|\mathbf{q}_v\|}\sin\|\mathbf{q}_v\|\right]$ is the quaternion exponential function.

Pose Kinematics

- **Pose kinematics**: the pose of a rigid body $T(t) \in SE(3)$ moving with generalized velocity $\zeta(t) = \begin{bmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} \in \mathbb{R}^6$ (in body-frame coordinates) satisfies:

$$\dot{T}(t) = T(t)\hat{\zeta}(t) \qquad\qquad \hat{\zeta} = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}^{\wedge} := \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \quad (\textbf{twist})$$

- **Discrete-time pose kinematics**: if $\zeta(t) \equiv \zeta_k$ is constant for $t \in [t_k, t_{k+1})$ and $T_k := T(t_k)$, $\tau_k := t_{k+1} - t_k$:

$$T_{k+1} = T_k \exp(\tau_k \hat{\zeta}_k)$$

where $\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$ is the matrix exponential function.

Pose Dynamics

- **Pose dynamics**: the pose $T(t) \in SE(3)$ and twist $\boldsymbol{\zeta}(t) \in \mathbb{R}^6$ of a rigid body with mass $m \in \mathbb{R}_{>0}$ and moment of inertia $J \in \mathbb{R}^{3\times 3}$, moving with **wrench** (generalized force) $\mathbf{w}(t) = \begin{bmatrix} \mathbf{f}(t) \\ \boldsymbol{\tau}(t) \end{bmatrix} \in \mathbb{R}^6$ (in body-frame coordinates) satisfies:

$$\dot{T}(t) = T(t)\hat{\boldsymbol{\zeta}}(t) \qquad\qquad M := \begin{bmatrix} mI & 0 \\ 0 & J \end{bmatrix}$$

$$M\dot{\boldsymbol{\zeta}}(t) = \overset{\curlywedge}{\boldsymbol{\zeta}}(t)^\top M\boldsymbol{\zeta}(t) + \mathbf{w}(t) \qquad\qquad \overset{\curlywedge}{\boldsymbol{\zeta}} = \overset{\curlywedge}{\begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}} := \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\mathbf{v}} \\ \mathbf{0} & \hat{\boldsymbol{\omega}} \end{bmatrix}$$

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Rigid-Body Kinematics and Dynamics

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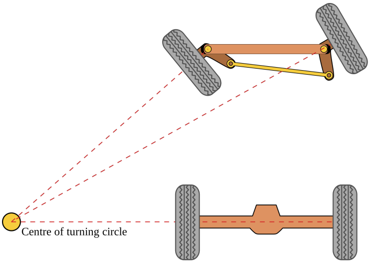
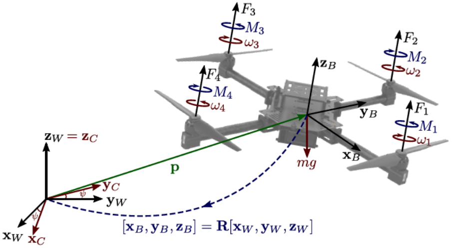
Observation Models

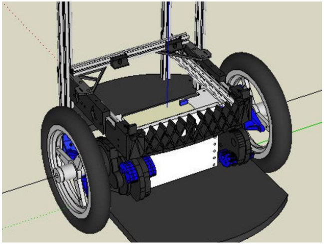
Motion Models



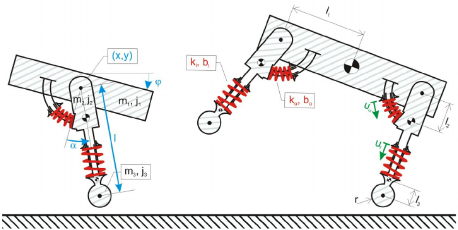
Ackermann Drive



Quadrotor



Differential Drive



Spring-Loaded Gait

Motion Model

- Variables describing a robot system:
  - time t (continuous or discrete)
  - state $\mathbf{x}$ (e.g., position, orientation)
  - control input $\mathbf{u}$ (e.g., velocity, force)
  - disturbance $\mathbf{w}$ (e.g., tire slip, wind)
- A **motion model** is a function f relating the current state $\mathbf{x}$ and input $\mathbf{u}$ of a robot with its state change
  - Continuous time: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$
  - Discrete time: $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$
- If the motion is affected by disturbance $\mathbf{w}$ modeled as a random variable, then the state $\mathbf{x}$ is also a random variable described either:
  - in function form: $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \mathbf{w}_t)$ or
  - with the probability density function $p_f(\cdot \mid \mathbf{x}_t, \mathbf{u}_t)$ of $\mathbf{x}_{t+1}$

Odometry Motion Model

- Consider a rigid-body robot with state $\mathbf{x}_t = T_t \in SE(3)$ representing the robot pose in the world frame at time t

- **Odometry**: onboard sensors (camera, lidar, encoders, imu, etc.) may be used to estimate the relative pose of the robot body frame at time $t+1$ with respect to the body frame at time t:

$$\mathbf{u}_t = {}_tT_{t+1} = \begin{bmatrix} {}_tR_{t+1} & {}_t\mathbf{p}_{t+1} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in SE(3)$$

- **Odometry motion model**: given the robot pose $\mathbf{x}_t$ and the odometry $\mathbf{u}_t$ at time t, the state at time $t+1$ satisfies:

$$T_{t+1} = \mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{x}_t\mathbf{u}_t = T_t\ {}_tT_{t+1}$$

- Given an initial pose $\mathbf{x}_0$ and odometry measurements $\mathbf{u}_0, \ldots, \mathbf{u}_t$, the robot pose at time $t+1$ can be estimated as:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t) = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})\mathbf{u}_t = \ldots = \mathbf{x}_0\mathbf{u}_0\mathbf{u}_1 \cdots \mathbf{u}_t$$

- An odometry estimate is "drifting" (gets worse over time) because small measurement errors in each $\mathbf{u}_t$ are accumulated

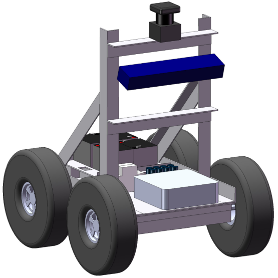
Differential-Drive Kinematic Model

- **State**: $\mathbf{x} = (\mathbf{p}, \theta)$, where $\mathbf{p} = (x, y) \in \mathbb{R}^2$ is the position and $\theta \in (-\pi, \pi]$ is the orientation (yaw angle) in the world frame

- **Control**: $\mathbf{u} = (v, \omega)$, where $v \in \mathbb{R}$ is the linear velocity and $\omega \in \mathbb{R}$ is the angular velocity (yaw rate) in the body frame

- **Continuous-time model**:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\mathbf{x}, \mathbf{u}) := \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$



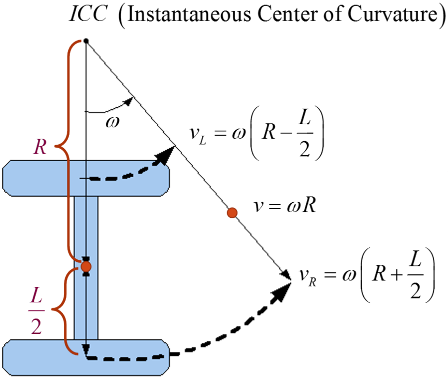
- The model is obtained using 2D pose kinematics with body-frame twist $\zeta = (v, 0, \omega)^\top$:

$$\begin{bmatrix} \dot{R}(\theta) & \dot{\mathbf{p}} \\ \mathbf{0}^\top & 0 \end{bmatrix} = \begin{bmatrix} R(\theta) & \mathbf{p} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} 0 & -\omega & v \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Differential-Drive Kinematic Model

- Let ℓ be the axle length (distance between wheels) and r be the radius of rotation, i.e., the distance from the ICC to the axle center
- The arc-length traveled is equal to the angle θ times the radius r

$$vt = r\theta \qquad \Rightarrow \qquad v = \frac{r\theta}{t} = r\omega$$



- Left wheel: $v_L = \omega(r - \ell/2)$
- Right wheel: $v_R = \omega(r + \ell/2)$
- Linear and angular velocity from wheel velocities:

$$\omega = \frac{v_R - v_L}{\ell}$$

$$r = \frac{\ell}{2}\left(\frac{v_L + v_R}{v_R - v_L}\right) = \frac{v}{\omega}$$

$$v = \frac{v_R + v_L}{2}$$

Discrete-Time Differential-Drive Kinematic Model

- **Euler discretization** over time interval of length τ_t:

$$\mathbf{x}_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = f_d(\mathbf{x}_t, \mathbf{u}_t) := \mathbf{x}_t + \tau_t \begin{bmatrix} v_t \cos(\theta_t) \\ v_t \sin(\theta_t) \\ \omega_t \end{bmatrix}$$

- **Exact integration** over time interval of length τ_t:

$$\mathbf{x}_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = f_d(\mathbf{x}_t, \mathbf{u}_t) := \mathbf{x}_t + \tau_t \begin{bmatrix} v_t \operatorname{sinc}\left(\frac{\omega_t \tau_t}{2}\right) \cos\left(\theta_t + \frac{\omega_t \tau_t}{2}\right) \\ v_t \operatorname{sinc}\left(\frac{\omega_t \tau_t}{2}\right) \sin\left(\theta_t + \frac{\omega_t \tau_t}{2}\right) \\ \omega_t \end{bmatrix}$$

- The exact integration is equivalent to the discrete-time pose kinematics:

$$\begin{bmatrix} R(\theta_{t+1}) & \mathbf{p}_{t+1} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} R(\theta_t) & \mathbf{p}_t \\ \mathbf{0}^\top & 1 \end{bmatrix} \exp\left(\tau_t \begin{bmatrix} 0 & -\omega_t & v_t \\ \omega_t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

Discrete-Time Differential-Drive Kinematic Model

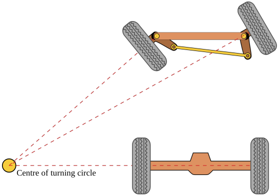
- What is the state after τ seconds if we apply constant linear velocity v and angular velocity ω at time t_0?
- To convert the continuous-time differential-drive model to discrete time, we solve the ordinary differential equations:

$$
\begin{aligned}
\dot{x}(t) &= v\cos\theta(t)\\
\dot{y}(t) &= v\sin\theta(t)\\
\dot{\theta}(t) &= \omega
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\theta(t_0+\tau) &= \theta(t_0) + \int_{t_0}^{t_0+\tau}\omega ds = \theta(t_0) + \omega\tau\\
x(t_0+\tau) &= x(t_0) + v\int_{t_0}^{t_0+\tau}\cos\theta(s)ds\\
&= x(t_0) + \frac{v}{\omega}\left(\sin\left(\omega\tau+\theta(t_0)\right) - \sin\theta(t_0)\right)\\
&= x(t_0) + v\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}\cos\left(\theta(t_0)+\frac{\omega\tau}{2}\right)\\
y(t_0+\tau) &= y(t_0) + v\int_{t_0}^{t_0+\tau}\sin\theta(s)ds\\
&= y(t_0) - \frac{v}{\omega}\left(\cos\theta(t_0) - \cos\left(\omega\tau+\theta(t_0)\right)\right)\\
&= y(t_0) + v\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}\sin\left(\theta(t_0)+\frac{\omega\tau}{2}\right)
\end{aligned}
$$

Ackermann-Drive Kinematic Model

- **State**: $\mathbf{x} = (\mathbf{p}, \theta)$, where $\mathbf{p} = (x, y) \in \mathbb{R}^2$ is the position and $\theta \in (-\pi, \pi]$ is the orientation (yaw angle) in the world frame
- **Control**: $\mathbf{u} = (v, \phi)$, where $v \in \mathbb{R}$ is the linear velocity and $\phi \in (-\pi, \pi]$ is the steering angle in the body frame
- **Continuous-time model**:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\mathbf{x}, \mathbf{u}) := \begin{bmatrix} v \cos\theta \\ v \sin\theta \\ \frac{v}{L} \tan\phi \end{bmatrix}$$



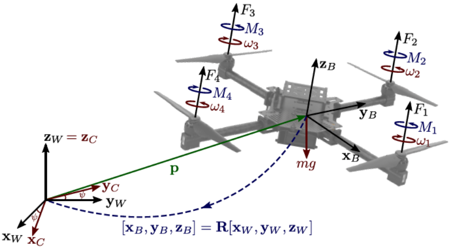
where L is the distance between the two wheel axles

- With the definition $\omega := \frac{v}{L} \tan\phi$, the Ackermann-drive model is equivalent to the differential-drive model and we can use the same discretized models

Quadrotor Dynamics Model

- **State**: $\mathbf{x} = (\mathbf{p}, R, \mathbf{v}, \omega)$ with position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, body-frame linear velocity $\mathbf{v} \in \mathbb{R}^3$, body-frame angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$
- **Control**: $\mathbf{u} = (\rho, \boldsymbol{\tau})$ with body-frame thrust force $\rho \in \mathbb{R}$ and torque $\boldsymbol{\tau} \in \mathbb{R}^3$
- **Continuous-time dynamics model** with mass m, gravity acceleration g, moment of inertia $J \in \mathbb{R}^{3\times3}$ and $\mathbf{e}_3 = (0, 0, 1)^\top$, obtained from rigid-body pose dynamics:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) = \begin{cases} \dot{\mathbf{p}} = R\mathbf{v} \\ \dot{R} = R\hat{\boldsymbol{\omega}} \\ m\dot{\mathbf{v}} = -\boldsymbol{\omega} \times m\mathbf{v} + \left(\rho\mathbf{e}_3 - mgR^\top\mathbf{e}_3\right) \\ J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times J\boldsymbol{\omega} + \boldsymbol{\tau} \end{cases}$$



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Rigid-Body Kinematics and Dynamics

Motion Models

Observation Models

Observation Models

Inertial Measurement Unit

RGB Camera

Global Positioning System

2-D Lidar

Observation Model

- Variables describing a sensor:
  - sensor state $\mathbf{x}$ (e.g., position, orientation)
  - environment state $\mathbf{m}$ (e.g., object position, orientation, shape)
  - measurement $\mathbf{z}$ (e.g., image)
  - noise $\mathbf{v}$ (e.g., blur)
- An **observation model** is a function h relating the sensor state $\mathbf{x}$ and the environment state $\mathbf{m}$ with the sensor measurement $\mathbf{z}$:

$$\mathbf{z} = h(\mathbf{x}, \mathbf{m})$$

- If the sensor is affected by noise $\mathbf{v}$ modeled as a random variable, then the measurement $\mathbf{z}$ is also a random variable described either:
  - in function form: $\mathbf{z} = h(\mathbf{x}, \mathbf{m}, \mathbf{v})$ or
  - with the probability density function $p_h(\cdot \mid \mathbf{x}, \mathbf{m})$ of $\mathbf{z}$

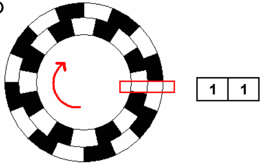
Common Sensor Models

- **Inertial or force sensor**: measures velocity, acceleration, or force, e.g., encoder, magnetometer, gyroscope, accelerometer
- **Position sensor**: measures position, e.g., GPS, RGBD camera, laser scanner
- **Bearing sensor**: measures angles, e.g., compass, RGB camera
- **Range sensor**: measures distance, e.g., radio received signal strength or time-of-flight

Encoder

- A magnetic encoder consists of a rotating gear, a permanent magnet, and a sensing element
- The sensor has two output channels with offset phase to determine the direction of rotation
- A microcontroller counts the number of transitions adding or subtracting 1 (depending on the direction of rotation) to the counter
- The distance traveled by the wheel, corresponding to one tick on the encoder is:

$$\text{meters per tick} = \frac{\pi \times (\text{wheel diameter})}{\text{ticks per revolution}}$$



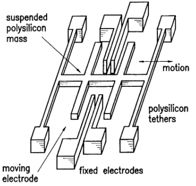
- The distance traveled during time τ for a given encoder count z, wheel diameter d, and n ticks on the sensor per revolution is:

$$\tau v \approx \frac{\pi d z}{n}$$

and can be used to measure the linear velocity for a differential-drive model

Inertial Measurement Unit

- **IMU**: inertial measurement unit:
  - triaxial accelerometer (measures linear acceleration)
  - triaxial gyroscope (measures angular velocity)

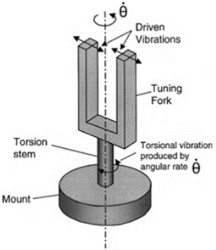


Surface Micromachined Accelerometer

- **Accelerometer**:
  - A mass m on a spring with constant k. The spring displacement is proportional to the system acceleration:

$$F = ma = kd \quad \Rightarrow \quad a = \frac{kd}{m}$$

  - VLSI fabrication: the displacement of a metal plate with mass m is measured with respect to another plate using capacitance
  - Used for car airbags (if the acceleration goes above $2g$, the car is hitting something!)



- **Gyroscope**: uses Coriolis force to detect rotational velocity from the changing mechanical resonsance of a tuning fork

IMU Observation Model

- **State**: $(\mathbf{p}, R, \mathbf{v}, \boldsymbol{\omega}, \mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}_g, \mathbf{b}_a)$ with position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, body-frame linear velocity $\mathbf{v} \in \mathbb{R}^3$, body-frame angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$, body-frame linear acceleration $\mathbf{a} \in \mathbb{R}^3$, body-frame angular acceleration $\boldsymbol{\alpha} \in \mathbb{R}^3$, gyroscope bias $\mathbf{b}_g \in \mathbb{R}^3$, accelerometer bias $\mathbf{b}_a \in \mathbb{R}^3$
- **Extrinsic Parameters**: IMU position ${}_B\mathbf{p}_I \in \mathbb{R}^3$ and orientation ${}_BR_I \in SO(3)$ in the body frame are assumed known or obtained from calibration
- **Strapdown IMU**: the IMU frame and the body frame are identical, i.e, ${}_B\mathbf{p}_I = \mathbf{0}$ and ${}_BR_I = I$
- **Measurement**: $(\mathbf{z}_\omega, \mathbf{z}_a)$ with angular velocity measurement $\mathbf{z}_\omega \in \mathbb{R}^3$ and linear acceleration measurement $\mathbf{z}_a \in \mathbb{R}^3$:

$$
\begin{aligned}
\mathbf{z}_\omega &= {}_BR_I^\top \boldsymbol{\omega} + \mathbf{b}_g + \mathbf{n}_g \\
\mathbf{z}_\mathbf{a} &= {}_BR_I^\top \left(\mathbf{a} - gR^\top \mathbf{e}_3 + \hat{\alpha}\, {}_B\mathbf{p}_I + \hat{\omega}^2\, {}_B\mathbf{p}_I \right) + \mathbf{b}_a + \mathbf{n}_a
\end{aligned}
$$

Laser Sensors

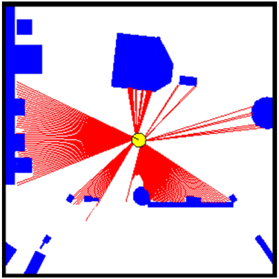
Single-Beam Garmin Lidar

2-D Hokuyo Lidar

3-D Velodyne Lidar

LIDAR Model

- **LIDAR**: LIght Detection And Ranging
- Illuminates the scene with pulsed laser light and measures the return times and wavelengths of the reflected pulses
- Mirrors are used to steer the laser beam in the *xy* plane (and *zy* plane for 3D lidars)
- LIDAR rays are emitted over a set of known horizontal (azimuth) and vertical (elevation) angles $\{\alpha_k, \epsilon_k\}$ and return range estimates $\{r_k\}$ to obstacles in the environment $\mathbf{m}$
- Example: Hokuyo URG-04LX; detectable range: 0.02 to 4m; 240° field of view with 0.36° angular resolution (666 beams); 100 ms/scan



Laser Range-Azimuth-Elevation Model

- Consider a Lidar with position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$ observing a point $\mathbf{m} \in \mathbb{R}^3$ in the world frame
- The point $\mathbf{m}$ has coordinates $\bar{\mathbf{m}} := R^\top(\mathbf{m} - \mathbf{p})$ in the lidar frame
- The lidar provides a spherical coordinate measurement of $\bar{\mathbf{m}}$:

$$\bar{\mathbf{m}} = R^\top(\mathbf{m} - \mathbf{p}) = \begin{bmatrix} r\cos\alpha\cos\epsilon \\ r\sin\alpha\cos\epsilon \\ r\sin\epsilon \end{bmatrix}$$

  where r is the range, α is the azimuth, and ϵ is the elevation

- **Inverse observation model**: expresses the lidar state $\mathbf{p}$, R and environment state $\mathbf{m}$, in terms of the measurement $\mathbf{z} = \begin{bmatrix} r & \alpha & \epsilon \end{bmatrix}^T$
- Inverting gives the **laser range-azimuth-elevation model**:

$$\mathbf{z} = \begin{bmatrix} r \\ \alpha \\ \epsilon \end{bmatrix} = \begin{bmatrix} \|\bar{\mathbf{m}}\|_2 \\ \arctan\left(\bar{\mathbf{m}}_y / \bar{\mathbf{m}}_x\right) \\ \arcsin\left(\bar{\mathbf{m}}_z / \|\bar{\mathbf{m}}\|_2\right) \end{bmatrix} \qquad \bar{\mathbf{m}} = R^\top(\mathbf{m} - \mathbf{p})$$

Common Observation Models

- **Position sensor**: state $\mathbf{x} = (\mathbf{p}, R)$, position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, observed point $\mathbf{m} \in \mathbb{R}^3$, measurement $\mathbf{z} \in \mathbb{R}^3$:

$$\mathbf{z} = h(\mathbf{x}, \mathbf{m}) = R^\top (\mathbf{m} - \mathbf{p})$$

- **Range sensor**: state $\mathbf{x} = (\mathbf{p}, R)$, position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, observed point $\mathbf{m} \in \mathbb{R}^3$, measurement $z \in \mathbb{R}$:

$$z = h(\mathbf{x}, \mathbf{m}) = \|R^\top (\mathbf{m} - \mathbf{p})\|_2 = \|\mathbf{m} - \mathbf{p}\|_2$$

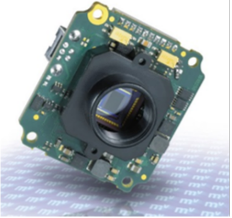
- **Bearing sensor**: state $\mathbf{x} = (\mathbf{p}, \theta)$, position $\mathbf{p} \in \mathbb{R}^2$, orientation $\theta \in (-\pi, \pi]$, observed point $\mathbf{m} \in \mathbb{R}^2$, bearing $z \in \mathbb{R}$:

$$z = h(\mathbf{x}, \mathbf{m}) = \arctan\left(\frac{m_2 - p_2}{m_1 - p_1}\right) - \theta$$

- **Camera sensor**: state $\mathbf{x} = (\mathbf{p}, R)$, position $\mathbf{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, intrinsic camera matrix $K \in \mathbb{R}^{3\times3}$, projection matrix $P := [I,\ \mathbf{0}] \in \mathbb{R}^{2\times3}$, observed point $\mathbf{m} \in \mathbb{R}^3$, pixel $\mathbf{z} \in \mathbb{R}^2$:

$$\mathbf{z} = h(\mathbf{x}, \mathbf{m}) = PK\pi(R^\top (\mathbf{m} - \mathbf{p})) \quad \text{projection:} \quad \pi(\mathbf{m}) := \frac{1}{\mathbf{e}_3^\top \mathbf{m}} \mathbf{m}$$

Camera Sensors



Global shutter

Stereo (+ IMU)

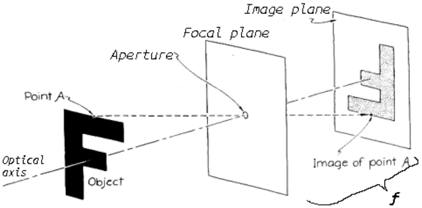
RGBD

Event-based

Image Formation

- **Image formation model**: must trade-off physical accuracy and mathematical simplicity
- The values of an image depend on the shape and reflectance of the scene as well as the distribution of light
- **Image intensity** $I(u, v)$ describes the energy falling onto a small patch of the imaging sensor (integrated both over the shutter interval and over a region of space) and is measured in power per unit area (W/m^2)
- A camera uses a set of lenses to control the direction of light propagation by means of *diffraction*, *refraction*, and *reflection*
- **Thin lens model**: a simple geometric model of image formation that considers only refraction
- **Pinhole model**: a thin lens model in which the lens aperture is decreased to zero and all rays are forced to go through the optical center and remain undeflected (diffraction becomes dominant).

Pinhole Camera Model



- **Focal plane**: perpendicular to the **optical axis** with a circular aperture at the **optical center**

- **Image plane**: parallel to the focal plane and a distance f (**focal length**) in **meters** from the optical center

- The pinhole camera model is described in an **optical frame** centered at the optical center with the optical axis as the z-axis:
  - **optical frame**: x = right, y = down, z = forward
  - **regular frame**: x = forward, y = left, z = up

- **Image flip**: the object appears upside down on the image plane. To eliminate this effect, we can simply flip the image $(x, y) \to (-x, -y)$, which corresponds to placing the image plane $\{z = -f\}$ in front of the optical center instead of behind $\{z = f\}$.

Pinhole Camera Model

- **Field of view**: the angle subtended by the spatial extend of the image plane as seen from the optical center. If s is the side of the image plane in meters, then the field of view is $\theta = 2\arctan\left(\frac{s}{2f}\right)$.
  - For a flat image plane: $\theta < 180^\circ$.
  - For a spherical or ellipsoidal imaging surface, common in omnidirectional cameras, θ can exceed 180°.

- **Ray tracing**: assuming a pinhole model and Lambertian surfaces, image formation can be reduced to tracing rays from points on objects to pixels. A mathematical model associating 3-D points in the world frame to 2-D points in the image frame must account for:
  1. **Extrinsics**: world-to-camera frame transformation
  2. **Projection**: 3D-to-2D coordinate projection
  3. **Intrinsics**: scaling and translation of the image coordinate frame

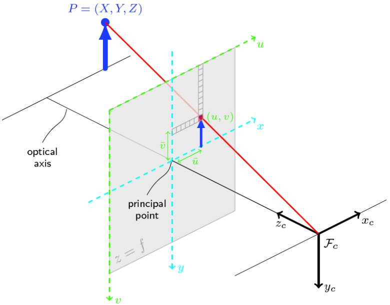
Extrinsics

- Let ${}_w\mathbf{p}_r \in \mathbb{R}^3$ and ${}_wR_r \in SO(3)$ be the position and orientation of the (regular) camera frame in the world frame
- **Rotation from regular to optical frame**: ${}_oR_r := \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$
- Let (X_w, Y_w, Z_w) be the coordinates of point $\mathbf{m}$ in the world frame. The coordinates of $\mathbf{m}$ in the optical frame are then:

$$\begin{pmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{pmatrix} = \begin{bmatrix} {}_oR_r & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} {}_wR_r & {}_w\mathbf{p}_r \\ \mathbf{0}^\top & 1 \end{bmatrix}^{-1} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix} = \begin{bmatrix} {}_oR_r {}_wR_r^\top & -{}_oR_r {}_wR_r^\top {}_w\mathbf{p}_r \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

Projection

- The 3D-to-2D perspective projection operation relates the optical-frame coordinates (X_o, Y_o, Z_o) of point $\mathbf{m}$ to its image coordinates (x, y) using similar triangles:



$$x = f\frac{X_o}{Z_o} \qquad y = f\frac{Y_o}{Z_o}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \frac{1}{Z_o}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{pmatrix}$$

- The above can be decomposed into:

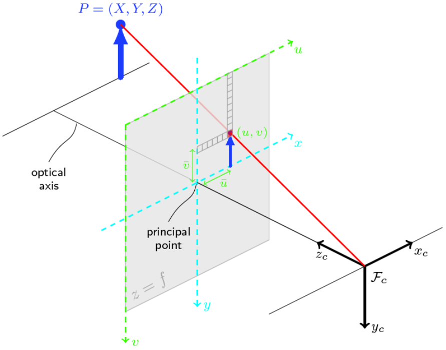
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{image flip: } F_f}\underbrace{\begin{bmatrix} -f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{focal scaling: } K_f}\underbrace{\frac{1}{Z_o}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{canonical projection: } \pi}\begin{pmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{pmatrix}$$

Intrinsics

- Images are obtained in terms of pixels (u, v) with the origin of the pixel array typically in the upper-left corner of the image
- The relationship between the image frame and the pixel array is specified via the following parameters:
  - (s_u, s_v) [pixels/meter]: define the **scaling** from meters to pixels and the **aspect ration** $\sigma = s_u/s_v$
  - (c_u, c_v) [pixels]: coordinates of the *principal point* used to translate the image frame origin, e.g., $(c_u, c_v) = (320.5, 240.5)$ for a 640×480 image
  - s_θ [pixels/meter]: **skew factor** that scales non-rectangular pixels and is proportional to $\cot(\alpha)$ where α is the angle between the coordinate axes of the pixel array
- Normalized coordinates in the image frame are converted to pixel coordinates in the pixel array using the **intrinsic parameter matrix**:

$$\underbrace{\begin{bmatrix} s_u & s_\theta & c_u \\ 0 & s_v & c_v \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pixel scaling: } K_s} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{image flip: } F_f} \underbrace{\begin{bmatrix} -f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{focal scaling: } K_f} = \underbrace{\begin{bmatrix} fs_u & fs_\theta & c_u \\ 0 & fs_v & c_v \\ 0 & 0 & 1 \end{bmatrix}}_{\text{calibration matrix: } K} \in \mathbb{R}^{3\times 3}$$

Pinhole Camera Model Summary



- **Extrinsics**:

$$\begin{pmatrix}X_o\\Y_o\\Z_o\\1\end{pmatrix} = \begin{bmatrix} {}_oR_{r\,w}R_r^\top & -{}_oR_{r\,w}R_r^\top\,{}_w\mathbf{p}_r \\ \mathbf{0}^\top & 1\end{bmatrix}\begin{pmatrix}X_w\\Y_w\\Z_w\\1\end{pmatrix}$$

- **Projection and Intrinsics**:

$$\underbrace{\begin{pmatrix}u\\v\\1\end{pmatrix}}_{\text{pixels}} = \underbrace{\begin{bmatrix} fs_u & fs_\theta & c_u \\ 0 & fs_v & c_v \\ 0 & 0 & 1\end{bmatrix}}_{\text{calibration: } K}\underbrace{\frac{1}{Z_o}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}_{\text{canonical projection: } \pi}\begin{pmatrix}X_o\\Y_o\\Z_o\\1\end{pmatrix}$$

Perspective Projection Camera Model

- The **canonical projection function** for vector $\mathbf{x} \in \mathbb{R}^3$ is:

$$\pi(\mathbf{x}) := \frac{1}{\mathbf{e}_3^\top \mathbf{x}} \mathbf{x}$$

- **Camera observation model**: state $\mathbf{x} = (\mathbf{p}, R)$ with position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$ of the optical frame, intrinsic camera matrix $K \in \mathbb{R}^{3\times3}$, observed point $\mathbf{m} \in \mathbb{R}^3$, pixel $\mathbf{z} \in \mathbb{R}^2$:

$$\mathbf{z} = h(\mathbf{x}, \mathbf{m}) = PK\pi(R^\top(\mathbf{m} - \mathbf{p})) \qquad P := \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2\times3}$$

- The camera model can be written directly in terms of the camera optical frame pose $T \in SE(3)$ using homogeneous coordinates:

$$\underline{\mathbf{z}} = K\pi(PT^{-1}\underline{\mathbf{m}}) \qquad \underline{\mathbf{m}} := \begin{bmatrix} \mathbf{m} \\ 1 \end{bmatrix} \qquad P := \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{3\times4}$$

Radial Distortion and Other Camera Models

- **Wide field of view camera**: in addition to linear distortions described by the intrinsic parameters K, one can observe distortion along radial directions

- The simplest effective **model for radial distortion** is:

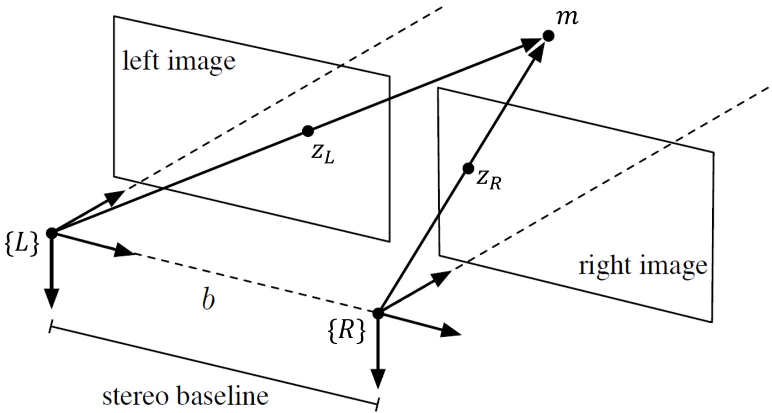
$$x = x_d(1 + a_1 r^2 + a_2 r^4)$$
$$y = y_d(1 + a_1 r^2 + a_2 r^4)$$

  where (x_d, y_d) are the pixel coordinates of distorted points and $r^2 = x_d^2 + y_d^2$ and a_1, a_2 are additional parameters modeling the amount of distortion

- **Spherical perspective projection**: if the imaging surface is a sphere $\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}$ (motivated by retina shapes in biological systems), we can define a spherical projection $\pi_s(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$ and use it in place of π in the perspective projection model

- **Catadioptric model**: uses an ellipsoidal imaging surface

Stereo Camera Model



Stereo Camera Model

- **Stereo camera**: two perspective cameras rigidly connected to one another with a known transformation
- Unlike a single camera, a stereo camera can determine the depth of a point from a single stereo observation
- **Stereo camera baseline**: the transformation between the two stereo cameras is only a displacement along the x-axis (optical frame) of size b
- The pixel coordinates $\mathbf{z}_L, \mathbf{z}_R \in \mathbb{R}^2$ of a point $\mathbf{m} \in \mathbb{R}^3$ in the world frame observed by a stereo camera at position $\mathbf{p} \in \mathbb{R}^3$ and orientation $R \in SO(3)$ with intrinsic parameters $K \in \mathbb{R}^{3\times 3}$ are:

$$\underline{\mathbf{z}}_L = K\pi\left(R^\top(\mathbf{m}-\mathbf{p})\right) \qquad \underline{\mathbf{z}}_R = K\pi\left(R^\top(\mathbf{m}-\mathbf{p}) - b\mathbf{e}_1\right)$$

Stereo Camera Model

- Stacking the two observations together gives the stereo camera model:

$$\begin{bmatrix} u_L \\ v_L \\ u_R \\ v_R \end{bmatrix} = \underbrace{\begin{bmatrix} fs_u & 0 & c_u & 0 \\ 0 & fs_v & c_v & 0 \\ fs_u & 0 & c_u & -fs_u b \\ 0 & fs_v & c_v & 0 \end{bmatrix}}_{M} \frac{1}{z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^\top(\mathbf{m} - \mathbf{p})$$

- Because of the stereo setup, two rows of M are identical. The vertical coordinates of the two pixel observations are always the same because the epipolar lines in the stereo configuation are horizontal.

- The v_R equation may be dropped, while the u_R equation is replaced with a **disparity** measurement $d = u_L - u_R = \frac{1}{z} fs_u b$ leading to:

$$\begin{bmatrix} u_L \\ v_L \\ d \end{bmatrix} = \begin{bmatrix} fs_u & 0 & c_u & 0 \\ 0 & fs_v & c_v & 0 \\ 0 & 0 & 0 & fs_u b \end{bmatrix} \frac{1}{z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^\top(\mathbf{m} - \mathbf{p})$$