XJCO3910 – Combinatorial Optimisation

Coursework 1: Modelling; Simplex Method

Deadline: 2300 GMT on Wednesday 29 March 2023

Award: This piece of summative coursework is worth 10% of your grade

Submission:
1) Write or type your answers inside the boxes of the current document.
2) Scan **pages 1-4** using the MyPrint portal, including p. 1 (see instructions at https://it.leeds.ac.uk/it?id=kb_article&sysparm_article=KB0012731)
3) Upload one pdf-file containing **pages 1-4** to Gradescope (see instructions at https://help.gradescope.com/article/ccbpppziu9-student-submit-work#submitting_a_pdf)

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Question 1. **[10 marks]**

A manufacturer needs to produce a health-drink satisfying the following vitamin requirements: 1 ml of the drink should contain at least 6 mg of vitamin A, 7 mg of vitamin B and 10 mg of vitamin C.

There are two concentrates, concentrate X and concentrate Y, that can be mixed with water. The contents of vitamins A, B and C in concentrate X are 15, 14, 50 mg per ml, respectively. The contents of the vitamins in concentrate Y are 15, 28, 20 mg per ml, respectively. Concentrate X costs 0.4 pence per ml, and concentrate Y costs 0.5 pence per ml. We assume that water does not cost anything. See the table below for the summary of the input data.

	Concentrate X	Concentrate Y	Minimum quantity of vitamins required
Vitamin A	15 mg/ml	15 mg/ml	6 mg/ml
Vitamin B	14 mg/ml	28 mg/ml	7 mg/ml
Vitamin C	50 mg/ml	20 mg/ml	10 mg/ml
Cost	0.4 pence/ml	0.5 pence/ml	N/A

Model the problem of finding the quantities of concentrates X and Y and the quantity of water needed to produce 1 bottle of 1000 ml of the drink, so that the cost is minimised.

The variables introduced and their meaning:

x : the amount of concentrate X in ml to be mixed with water to produce 1 bottle of 1000ml of the drink.

y : the amount of concentrate Y in ml to be mixed with water to produce 1 bottle of 1000ml of the drink.

z : the amount of water in ml to be mixed with concentrates X and Y to produce 1 bottle of 1000ml of the drink.

C : the total cost.

The LP model: (*please present the model in the structured way*)

To minimize the total cost $C = 0.4x + 0.5y$

constraint in A : $(15x+15y)/1000 \geq 6$

constraint in B : $(14x+28y)/1000 \geq 7$

constraint in C : $(50x+20y)/1000 \geq 10$

total constraint $x+y+z = 1000$

and volume cannot be negative, so $x, y, z \geq 0$

minimize : $C = 0.4x + 0.5y$

subject to:
$$15x + 15y \geq 6000$$
$$14x + 28y \geq 7000$$
$$50x + 20y \geq 10000$$
$$x + y + z = 1000$$
$$x, y, z \geq 0$$

(do not solve the formulated LP)

Provide an explanation for the constraint which ensures that "1 ml of the drink should contain at least 6 mg of vitamin A"

To make sure that each ml beverage contains at lease 6 mg of vitamin A, the total amount of vitamin A should be divided by total volume of drink, 1000ml.

Therefore, the constraint can be formed : $(15x+15y)/1000 \geq 6/1000$

This can ensure that health-drink contains at least 6mg of vitamin A per ml.

Question 2 deals with a pair of LP problems A and B, primal and dual. Instructions are on p. **5**. **[20 marks]**

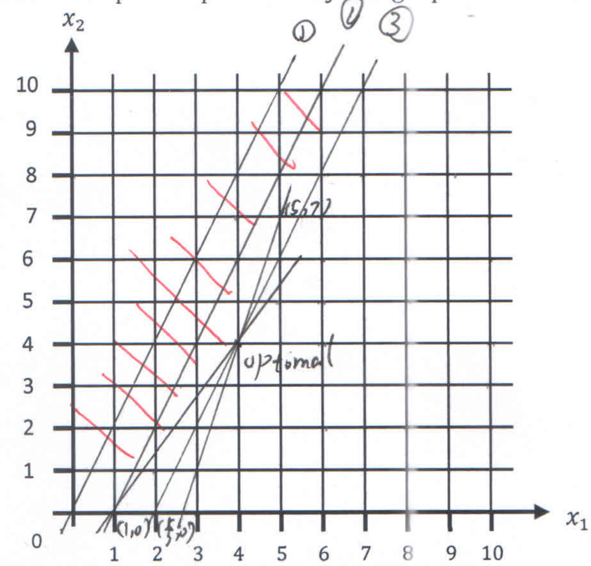
Problem A (primal):

max	$z =$	20_1	$-10x_2$		
s.t.		$3x_1$	$-x_2$	$\leq$	8
		$4x_1$	$-3x_2$	$\leq$	4
		$x_1,$	x_2	$\geq$	0

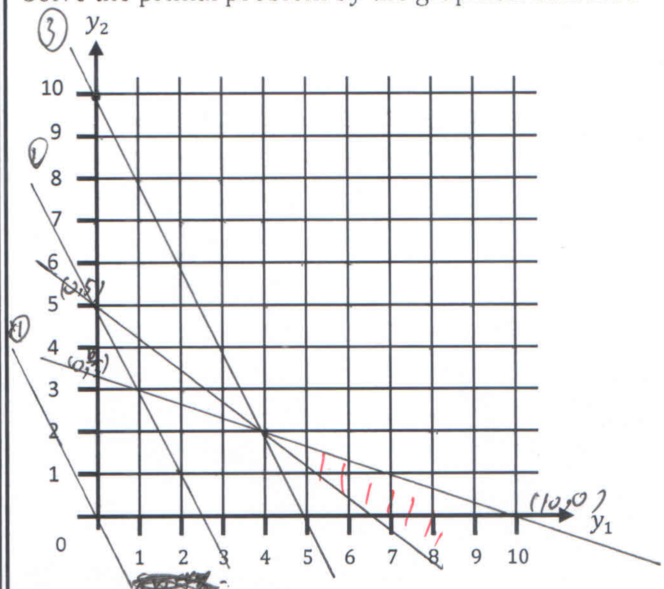
Problem B (dual):

min	$v =$	$8y_1$	$+4y_2$		
s.t.		$3y_1$	$+4y_2$	$\geq$	20
		$-y_1$	$-3y_2$	$\geq$	-10
		$y_1,$	y_2	$\geq$	0

Solve the primal problem by the graphical maethod[a,c]



Solve the primal problem by the graphical maethod[a,c]



Solve Problem A by the simplex method[b]

1. $z = 20x_1 - 10x_2 \Rightarrow -20x_1 + 10x_2 + z = 0$

$3x_1 - x_2 + x_3 = 8$

$4x_1 - 3x_2 + x_4 = 4$

$-20x_1 + 10x_2 + z = 0$

$x_1, x_2, x_3, x_4 \geq 0$

2.

1	x_1	x_2		Ratio
x_3	3	-1	8	$\frac{5}{3}$
x_4	4	-3	4	1 pivot row
z	-20	10	0	

2	x_4	x_2		Ratio
x_3	$-\frac{3}{4}$	$\frac{5}{4}$	5	4
x_1	$\frac{1}{4}$	$-\frac{3}{4}$	1	$-\frac{4}{3}$
z	5	-5	20	

3	x_4	x_3	
x_2	$-\frac{3}{5}$	$\frac{4}{5}$	4
x_1	$-\frac{1}{5}$	$\frac{3}{5}$	4
z	$\frac{5}{4}$	4	40

The optimal solution is $x_1 = 4$, $x_2 = 4$

The optimal value of the objective function is 40

Solve problem B by the dual simplex method[b]

1. $(-v) + 8y_1 + 4y_2 = 0$

$3y_1 + 4y_2 - t_1 = 20$

$-y_1 - 3y_2 - t_2 = -10$

$y_1, y_2, t_1, t_2 \geq 0$

$(-v) + 8y_1 + 4y_2 = 0$

$-3y_1 - 4y_2 + t_1 = -20$

$y_1 + 3y_2 + t_2 = 10$

$y_1, y_2, t_1, t_2 \geq 0$

2.

B_1	y_1	y_2		
t_1	-3	-4	-20	← pivot row
t_2	1	3	10	
-v	8	4	0	

Ratio $\frac{8}{3}$ pivot column

B_2	y_1	t_1	
y_2	$\frac{3}{4}$	$-\frac{1}{4}$	5
t_2	$-\frac{5}{4}$	$\frac{3}{4}$	-5
	5	1	-20

ratio 4 $-\frac{4}{3}$

B_3	t_2	t_1	
y_2	$\frac{3}{5}$	$\frac{1}{5}$	2
y_1	$-\frac{4}{5}$	$-\frac{3}{5}$	4
-v	4	4	-40

The optimal solution is $y_1 = 4$ $y_2 = 2$

The optimal value of the objective function is 40

[a,b,c] Consider the instructions on p. **5**

Question 3 **[10 marks]**

Consider again Problem B from Question 2:

Problem B

min	$v =$	$8y_1$	$+4y_2$		
s. t.		$3y_1$	$+4y_2$	$\geq$	20
		$-y_1$	$-3y_2$	$\geq$	-10
		$y_1,$	y_2	$\geq$	0

It can be solved by the two-phase simplex method.

Derive the first tableau of the 1st phase. Show all your working. *Do not solve the produced LP.*

Deriving the first tableau:

1. turn the min V into $\max(-V)$ $\quad v = 8y_1 + 4y_2 \Rightarrow (-v) = -8y_1 - 4y_2$

2. introduce the surplus variable y_3, y_4 to convert $\geq$ into $=$:

maximise $(-V) = -8y_1 - 4y_2$

subject to $\quad 3y_1 + 4y_2 - y_3 = 20$

$-y_1 - 3y_2 \quad - y_4 = -10$

$y_1, y_2, y_3, y_4 \geq 0$

3. introduce two artificial variables w_1, w_2, then:

minimise $w = w_1 + w_2$

$3y_1 + 4y_2 - y_3 \quad + w_1 \quad = 20$

$-y_1 - 3y_2 \quad - y_4 \quad + w_2 = -10$

$8y_1 + 4y_2 \quad + (-v) = 0$

$y_1, y_2, y_3, y_4, w_1, w_2 \geq 0$

4. we need to set tableau with w_1, w_2 as basic variables. To achieve this,

we need to use y_1, y_2, y_3, y_4 to represent w:

$w = w_1 + w_2$

$= (20 - 3y_1 - 4y_2 + y_3) + (-10 + y_1 + 3y_2 + y_4)$

$= 10 - 2y_1 - y_2 + y_3 + y_4$ $\quad$ as a result: $(-w) = -10 + 2y_1 + y_2 - y_3 - y_4$

The tableau:

1	y_1	y_2	y_3	y_4	
w_1	3	4	-1	0	20
w_2	-1	-3	0	-1	-10
$-V$	8	4	0	0	0
$-W$	-2	-1	1	1	-10

Instructions for **Question 2.**

[a] For the graphical solutions to Problem A and problem B,

- mark the solution region;
- for each line, specify two points it passes through;
- specify coordinates of the corner points of the feasible region;
- present the line for the objective function, together with the equation used to plot it, and indicate the direction in which the line should be moved;
- mark the point which defines the optimum and specify its coordinates.

[b] Use the primal simplex method for problem A in the tableau format and the dual method for problem B, also in the tableau format. Present all tableaux, indicating the choice of the pivot row, pivot column, and the pivot element. For each problem, state the final answer.

[c] After solving the problems analytically, reconsider the solutions found graphically.

- On the plot for the primal problem, mark the solutions obtained at all iterations of the primal simplex method and indicate the order in which they are obtained.
- On the plot for the dual problem, mark the solutions obtained at all iterations of the dual simplex method and indicate the order in which they are obtained.

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