Hw 6

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MA585
Homework 6

1. Consider the Johnson and Johnson quarterly earnings data from January 1960 to December 1980 (R data *JohnsonJohnson*)

R

a. Plot the data. Describe the features of the data. Do the data look stationary? Explain your answer.

b. Apply an appropriate variance stabilizing transformation, if necessary.

c. Carry out classical decomposition of the data, plot the transformed series along with the ACF and PACF.

d. Identify an ARMA model for the transformed data.

e. Repeat c and d, but instead of classical decomposition, use differencing to make the data stationary.

R

2. Consider the time series of the numbers of users connected to the Internet through a server every minute (R data WWWusage). Carry out a test for unit root. Apply necessary transformation and identify plausible ARMA models

3. Identify each of the models below as $ARIMA(p, d, q)$. Specify the order of the models (p, d, q) and the model parameters ϕ and θ.

a. $X_t = 10 + X_{t-1} + e_t + 0.6e_{t-1}$

b. $X_t = 3 + 1.25X_{t-1} - 0.25X_{t-2} + e_t - 0.2e_{t-2}$

c. $X_t - 1.7X_{t-1} + 0.7X_{t-2} = -8 + e_t$

a. $X_t - X_{t-1} = 10 + e_t + 0.6e_{t-1}$

$\phi_p(B) = 1 - B = 0 \rightarrow B = 1.$

$\therefore$ Unit Root $\rightarrow$ need differencing $\xrightarrow{d=1}$ AR(0)

$\therefore \nabla X_t = 10 + e_t + 0.6e_{t-1}$

$\therefore$ It is ARIMA(0, 1, 1). with $\theta = 0.6$

b. $X_t - 1.25X_{t-1} + 0.25X_{t-2} = 3 + e_t - 0.2e_{t-2}$

$\phi_p(B) = (1 - 1.25B + 0.25B^2) = 0$

$(1 - B)(1 - 0.25B) = 0.$

$\therefore B = 1 \rightarrow$ Unit Root $\xrightarrow{d=1}$ AR(1) diff..

$X_t - X_{t-1} - 0.25X_{t-1} + 0.25X_{t-2} = 3 + e_t - 0.2e_{t-2}$

$\nabla X_t - 0.25\nabla X_{t-1} = 3 + e_t - 0.2e_{t-2}$

$\therefore$ ARIMA(1,1,1). $\phi = 0.25$, $\theta = -0.2$

c. $1 - 1.7B + 0.7B^2 = \phi_p(B) = 0$

$\rightarrow (1-0.7B)(1-B) = 0$

$\therefore$ There is unit root $\xrightarrow[\text{diff}]{d=1}$ AR(1)

$\therefore X_t - X_{t-1} - 0.7X_{t-1} + 0.7X_{t-2} = -8 + e_t$

$\therefore \nabla X_t - 0.7\nabla X_{t-1} = -8 + e_t$

$\therefore$ ARIMA(1,1,0) with $\phi = 0.7$.

4. Consider the two models

$$X_t = 0.9X_{t-1} + 0.09X_{t-2} + e_t$$

and

$$X_t = X_{t-1} + e_t - 0.1e_{t-1}$$

a. Identify both models as $ARIMA(p,d,q)$. Specify (p,d,q) and ARMA parameters ϕ and θ.

b. In what way the two models are different?

c. In what way the two models are similar? What does this tell you about model selection for time series data?

a. 1 $X_t = 0.9X_{t-1} + 0.09X_{t-2} + e_t$

$1 - 0.9B - 0.09B^2 = 0$

$B = 1.009252$ or $B = -11.009252$

$\therefore$ No need to differencing

$\therefore$ ARIMA(2,0,0) $\phi_1 = 0.9$ $\phi_2 = 0.09$

② $X_t = X_{t-1} + e_t - a_1 e_{t-1}$

$1 - B = 0 \quad \rightarrow B = 1 \quad \therefore$ difference $(d=1)$

$\therefore \nabla X_t = e_t - a_1 e_{t-1}$

$\therefore$ ARIMA$(0, 1, 1)$. $\theta = -a_1$

b. ① is stationary

② is not stationary before differencing

After differencing: ② is composed of MA process whereas ① is composed of AR process; $\therefore$ they are different in terms of compositions

c. Both ① and integrated ② are invertible and causal

This tells us we should make sure that the models we have eventually integreted should have these properties

5. Let X_t be a stationary process with autocovariance function $\gamma(h)$.

a. Show that the process ∇X_t is stationary and find its autocovariance function.

b. Show that the process $\nabla^2 X_t$ is also stationary.

a. $\nabla X_t = X_t - X_{t-1}$

$E(\nabla X_t) = E(X_t) - E(X_{t-1}) = 0$

$$\begin{cases} \gamma_{\nabla X_t}(0) = Cov(X_t - X_{t-1}, X_t - X_{t-1}) = Var X_t - Var X_{t-1} = 0 \\ \gamma_{\nabla X_t}(1) = Cov(X_t - X_{t-1}, X_{t+1} - X_t) = -\sigma^2_{X_t} \\ \gamma_{\nabla X_t}(h) = 0, \quad h \geq 2 \end{cases}$$

$\therefore \nabla X_t$ is stationary, because $\gamma_{\nabla X_t}(h)$ depends on h, not t.

b. $\nabla^2 X_t = \nabla\nabla X_t = X_t - 2X_{t-1} + X_{t-2}$

$E(\nabla^2 X_t) = EX_t - 2EX_{t-1} + EX_{t-2} = 0$

$$\begin{cases} \gamma_{\nabla^2 X_t}(0) = Var X_t + 4 Var X_{t-1} + Var X_{t-2} = 6\sigma_{X_t}^2 \\ \gamma_{\nabla^2 X_t}(1) = Cov(X_t - 2X_{t-1} + X_{t-2},\ X_{t+1} - 2X_t + X_{t-1}) \\ \quad = -2Cov(X_{t-1}, X_{t-1}) - 2Cov(X_t, X_t) \\ \quad = -4\sigma_{X_t}^2 \\ \gamma_{\nabla^2 X_t}(2) = Cov(X_t - 2X_{t-1} + X_{t-2},\ X_{t+2} - 2X_{t+1} + X_t) \\ \quad = Cov(X_t, X_t) = \sigma_{X_t}^2 \\ \gamma_{\nabla^2 X_t}(h) = 0, \quad h \geq 3 \end{cases}$$

$\because X_t$ is stationary

$\therefore \sigma_{X_t}$ is not depending on t

$\therefore \gamma_{\nabla^2 X_t}(h)$ only depends on h, not t,

$\therefore \nabla^2 X_t$ is stationary

6. Let e_t be zero mean Gaussian white noise process with variance σ^2 and let $|\phi| < 1$ be a constant. Consider the process, starting at X_1,

$$X_1 = e_1$$
$$X_t = \phi X_{t-1} + e_t, \qquad t = 2, 3, ...$$

a. Express X_t as a linear combination of the white noise process e_t.

$$X_t = \phi(\phi X_{t-2} + e_{t-1}) + e_t$$

$$\therefore X_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1$$

$$X_t = \sum_{j=0}^{t-1} \phi^j e_{t-j}$$

b. Use the result in (a) to compute the mean and the variance of the process X_t. Is the process X_t stationary? $e_t \sim WN(0, \sigma^2)$

$$E X_t = E e_t + \phi E e_{t-1} + \phi^2 E e_{t-2} + \cdots = 0$$

$$Var X_t = (1 + \phi^2 + \phi^4 + \cdots \phi^{2(t-1)}) \sigma^2$$

$$= \frac{1 - \phi^{2(t-1)}}{1 - \phi^2} \sigma^2$$

$\because$ $Var X_t$ depends on t, $\therefore$ not stationary

c. Show

$$Correlation(X_t, X_{t-h}) = \phi^h \left[\frac{Var(X_{t-h})}{Var(X_t)}\right]^{1/2}$$

for $h \geqslant 0$.

$$Cov(X_t, X_{t-h}) = Cov\left(\sum_{j=0}^{t-1} \phi^j e_{t-j}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$$

$$= Cov\left(\sum_{j=0}^{h-1} \phi^j e_{t-j} + \sum_{j=h}^{t-1} \phi^j e_{t-j}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$$

$$= Cov\left(\sum_{j=h}^{t-1} \phi^j e_{t-j}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$$

$$= Cov\left(\phi^h \sum_{j=h}^{t-1} \phi^{j-h} e_{t-j}, X_{t-h}\right)$$

$$= Cov(\phi^h X_{t-h}, X_{t-h})$$

$$= \phi^h Var(X_{t-h})$$

$$Corr(X_t, X_{t-h}) = \frac{Cov(X_t, X_{t-h})}{\sqrt{Var X_t \, Var X_{t-h}}}$$
$$= \phi^h \left(\frac{Var(X_{t-h})}{Var X_t}\right)^{1/2}$$

d. Argue that for large t,

$$Var(X_t) \approx \frac{\sigma^2}{1-\phi^2}$$

and

$$Correlation(X_t, X_{t-h}) \approx \phi^h, \quad h \geqslant 0,$$

so in a sense, X_t is "asymptotically stationary."

$$Var X_t = \frac{1-\phi^{2(t-1)}}{1-\phi^2} \sigma^2$$

$$\phi^{2(t-1)} \xrightarrow{t\to\infty} 0 \quad (|\phi|<1)$$

$$\therefore 1-\phi^{2(t-1)} \xrightarrow{t\to\infty} 1$$

$$\therefore Var X_t \xrightarrow{t\to\infty} \frac{\sigma^2}{1-\phi^2}$$

$$Corr(X_t, X_{t-h}) = \phi^h \sqrt{\frac{Var X_{t-h}}{Var X_t}}$$

$\because$ for $t\to\infty$, $Var X_{t-h} = Var X_t$

$$\therefore Corr(X_t, X_{t-h}) \approx \phi^h$$

$\therefore \rho(h) = \phi^h$, it is an ACF for AR(1),

$\therefore X_t$ is asymptotically stationary.

e. This result can be used to simulate observations from a stationary Gaussian AR(1) model. Explain how this can be done.

The resultes shows that a process generated based on the algorithm may not be stationary for small t, but eventually will become a stationary sequence from an AR(1) process. Thus, to generate a sample path of size n from a stationary process, we have to allow a large burn-in period before retaining the n observations.

f. Write a R code generate a random sample of size 500 from the AR(1) process with $\phi = 0.6$ and $\sigma^2 = 0.8$. Plot the simulated series along with the sample ACF and PACF of the series. Is the sample ACF and PACF consistent with AR(1)?

R

R part

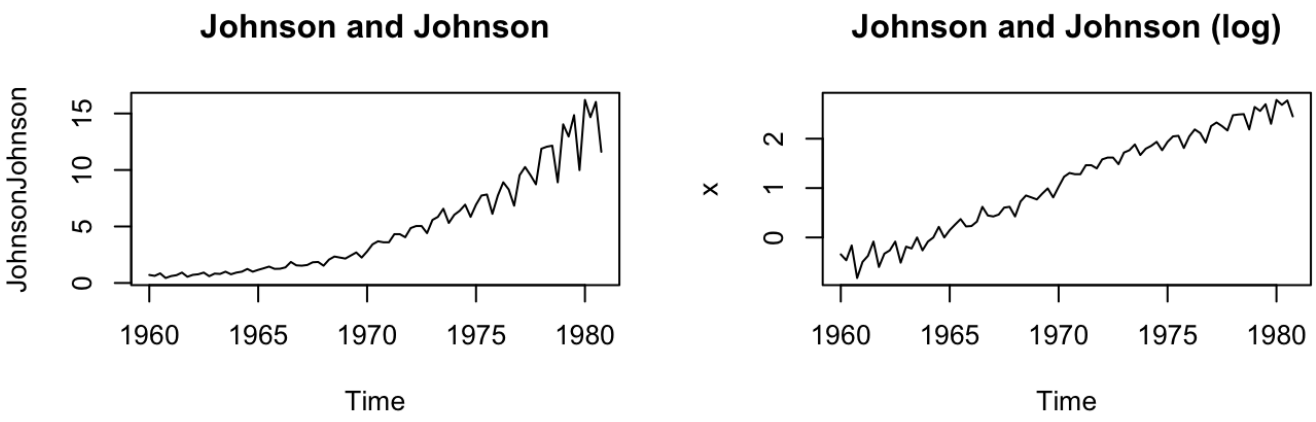
HW 6

Haozhe Chen

1. Consider the Johnson and Johnson quarterly earnings data from January 1960 to December 1980 (R data JohnsonJohnson)

a. Plot the data. Describe the features of the data. Do the data look stationary? Explain your answer.
b. Apply an appropriate variance stabilizing transformation, if necessary.
c. Carry out classical decomposition of the data, plot the transformed series along with the ACF and PACF.
d. Identify an ARMA model for the transformed data.
e. Repeat c and d, but instead of classical decomposition, use differencing to make the data stationary.

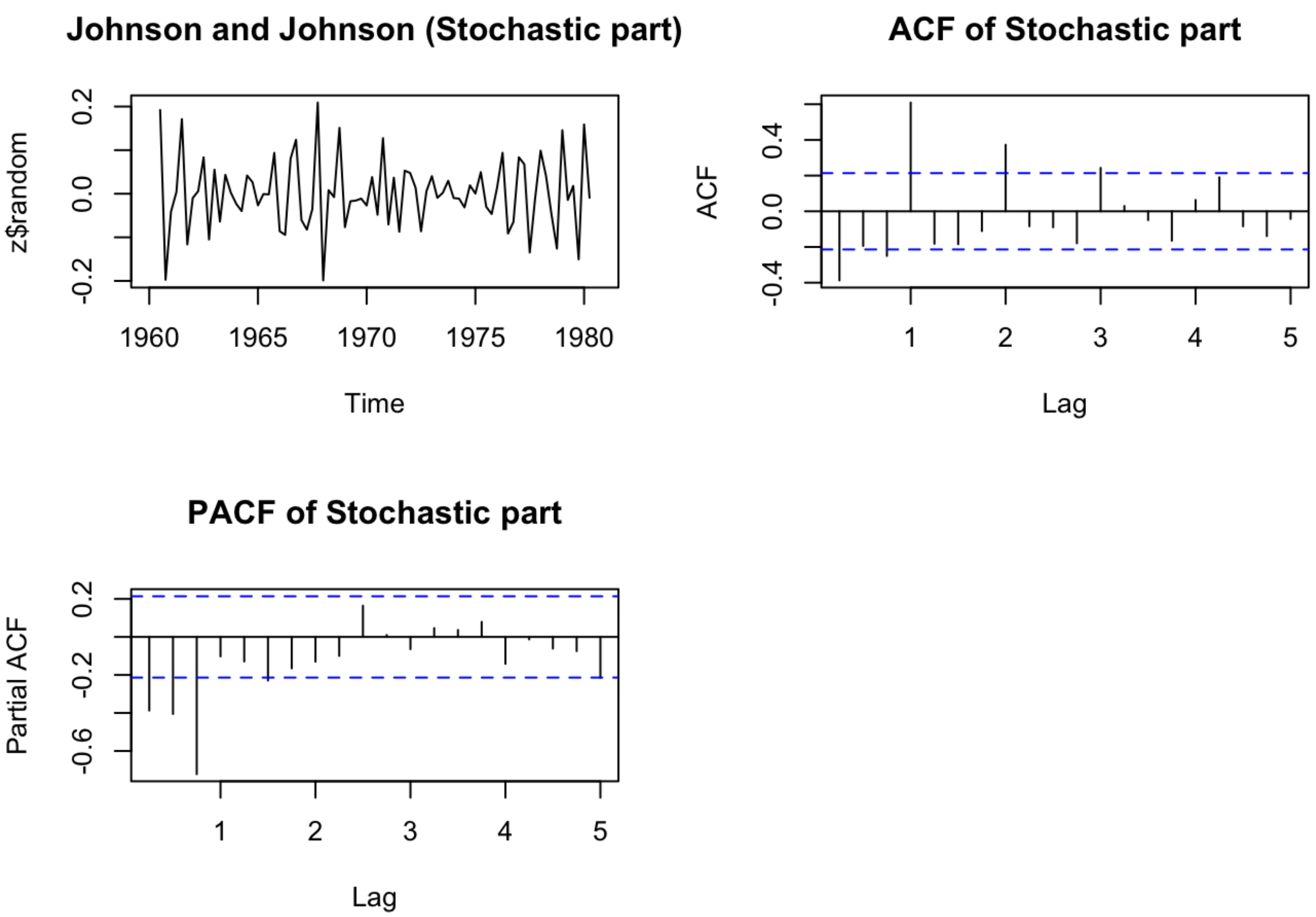
```
# JohnsonJohnson
# plot the series
par(mfrow=c(2,2))
plot.ts(JohnsonJohnson, main='Johnson and Johnson')
# The series look like a random walk, not stationary. The variance looks at depending on t.
# apply log transformation
x <- log(JohnsonJohnson)
plot.ts(x, main = 'Johnson and Johnson (log)')

par(mfrow=c(2,2))
```



```
# apply decomposition
z <- decompose(x, type="additive")
# plot the stochastic part
plot.ts(z$random, main = 'Johnson and Johnson (Stochastic part)' )
# plot acf and pacf
acf(z$random, na.action = na.pass, lag.max=20, main = 'ACF of Stochastic part')
pacf(z$random, na.action = na.pass, lag.max=20, main = 'PACF of Stochastic part')
# ACF cuts-off after lag 1; PACF cuts-off after lag 3
# The stationary part might be MA(1), AR(3), or ARMA(3,1)
# Apply eacf function
TSA::eacf(na.omit(z$random))
```

```
## AR/MA
##   0 1 2 3 4 5 6 7 8 9 10 11 12 13
## 0 x o x x o o o x o o o  x  o  o
## 1 x o o x x o o x o o o  o  o  o
## 2 x o x x x o o x o o x  x  o  o
## 3 o o x o o o x o o o o  o  o  o
## 4 x o o o o o x o o o o  o  o  o
## 5 x o o o o o o o o o o  o  o  o
## 6 x o o o o o x o o o o  o  o  o
## 7 x x o o o o o o o o o  o  o  o
```

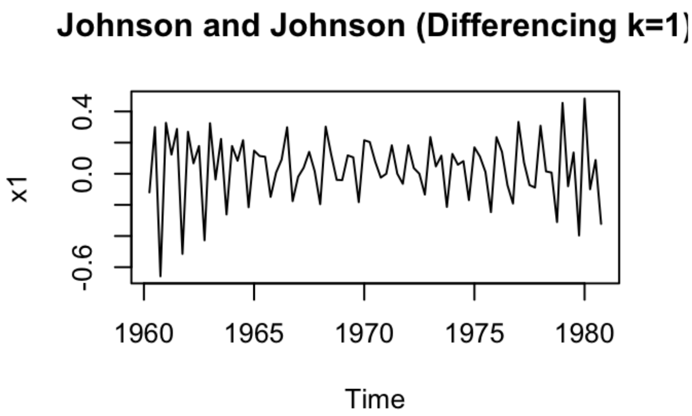
```
# The stationary part is identified as a MA(1) process

par(mfrow=c(2,2))
```

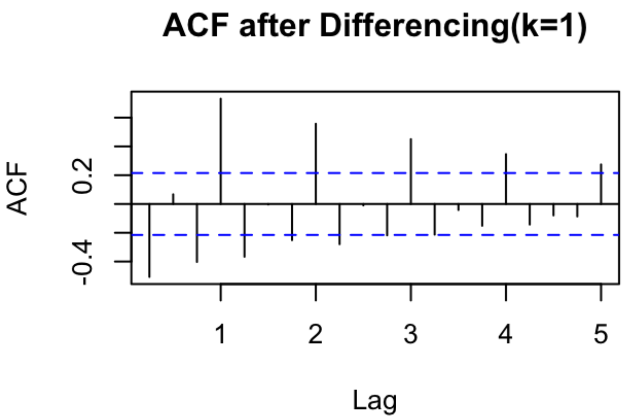


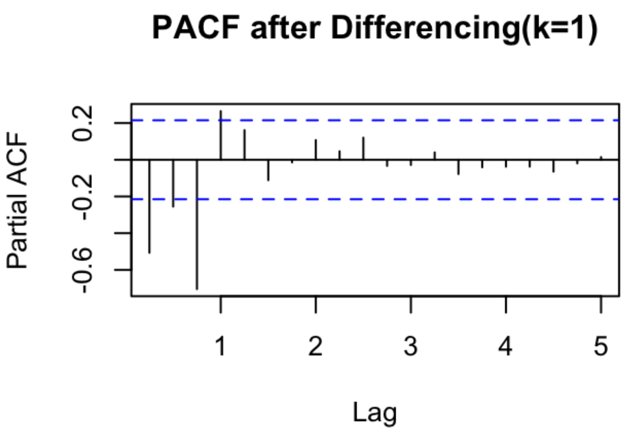
```
# Apply differencing, k=1
x1 <- diff(x)
# plot the series
plot.ts(x1, main = 'Johnson and Johnson (Differencing k=1)')
# plot acf and pacf
acf(x1, na.action = na.pass, lag.max=20, main = 'ACF after Differencing(k=1)')
pacf(x1, na.action = na.pass, lag.max=20, main = 'PACF after Differencing(k=1)')
# ACF cuts-off after lag 1; PACF cuts-off after lag 4
# The stationary part might be MA(1), AR(4), or ARMA(4,1)
# Apply eacf function
TSA::eacf(x1)
```

```
## AR/MA
##   0 1 2 3 4 5 6 7 8 9 10 11 12 13
## 0 x o x x x o x x x o o  x  o  o
## 1 x o o x x o o x x o o  x  o  o
## 2 x x x x x x o x o x o  x  o  o
## 3 x o o o o o x o o o o  o  o  o
## 4 x x o o o o x o o o o  o  o  o
## 5 x x o o o o x o o o o  o  o  o
## 6 x o x x o o x o o o o  o  o  o
## 7 o x x x o o x o o o o  o  o  o
```

```
# The stationary part is identified as a MA(1) process
```







2. Consider the time series of the numbers of users connected to the Internet through a server every minute (R data WWWusage). Carry out a test for unit root. Apply necessary transformation and identify plausible ARMA models.

```
x <- WWWusage
adf.test(x)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  x
## Dickey-Fuller = -2.6421, Lag order = 4, p-value = 0.3107
## alternative hypothesis: stationary
```

```
# fail to reject H0, apply transformation

x1 <- log(WWWusage)
adf.test(x1)
```

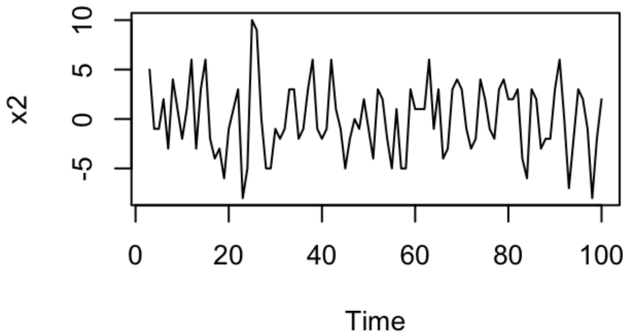
```
##
##  Augmented Dickey-Fuller Test
##
## data:  x1
## Dickey-Fuller = -2.7591, Lag order = 4, p-value = 0.2622
## alternative hypothesis: stationary
```

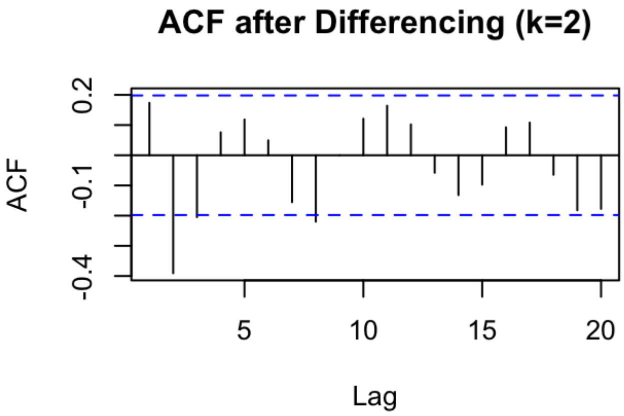
```
# still fail to reject H0, try differencing

# apply differencing to eliminate the trend, k=2
x2 <- diff(diff(WWWusage))
adf.test(x2)
```

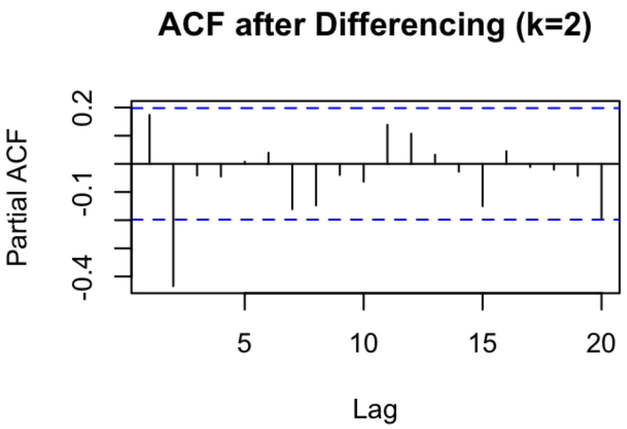
```
## Warning in adf.test(x2): p-value smaller than printed p-value
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  x2
## Dickey-Fuller = -4.828, Lag order = 4, p-value = 0.01
## alternative hypothesis: stationary
```

```
# reject the null, x2 is stationary

# identify the model
par(mfrow=c(2,2))
plot.ts(x2)
acf(x2, lag.max=20, main = 'ACF after Differencing (k=2)')
pacf(x2, lag.max=20, main = 'ACF after Differencing (k=2)')
# according to the ACF and PACF plots, x2 might be an ARMA(0,0) process,
# which means it might be a white noise process
```

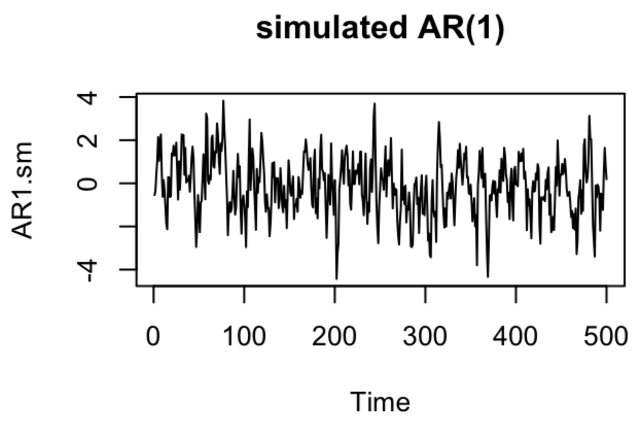


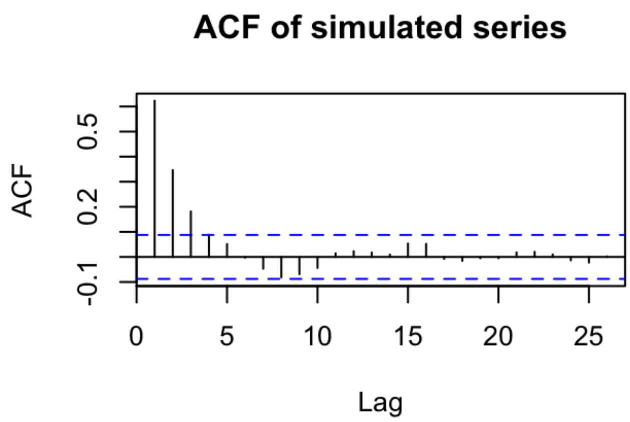


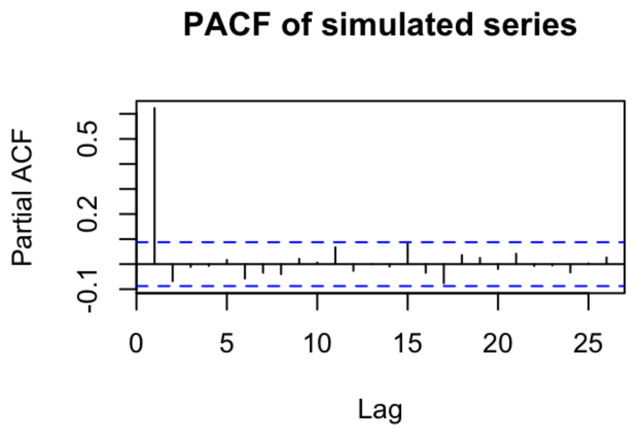


6f. Write a R code generate a random sample of size 500 from the AR(1) process with phi = 0.6 and sigma^2 = 0.8, Plot the simulated series along with the sample ACF and PACF of the series. Is the sample ACF and PACF consistent with AR(1)?

```
# simulate the series
AR.sm <- list(order = c(1,0,0), ar = 0.6, sd = sqrt(0.8))
AR1.sm <- arima.sim(n=500, AR.sm)
par(mfrow=c(2,2))
plot.ts(AR1.sm, main = 'simulated AR(1)')
acf(AR1.sm, main = 'ACF of simulated series')
pacf(AR1.sm, main = 'PACF of simulated series')
# the ACF dampens to zero and PACF cuts-off at lag 1,
# so it is consistent with AR(1)
```



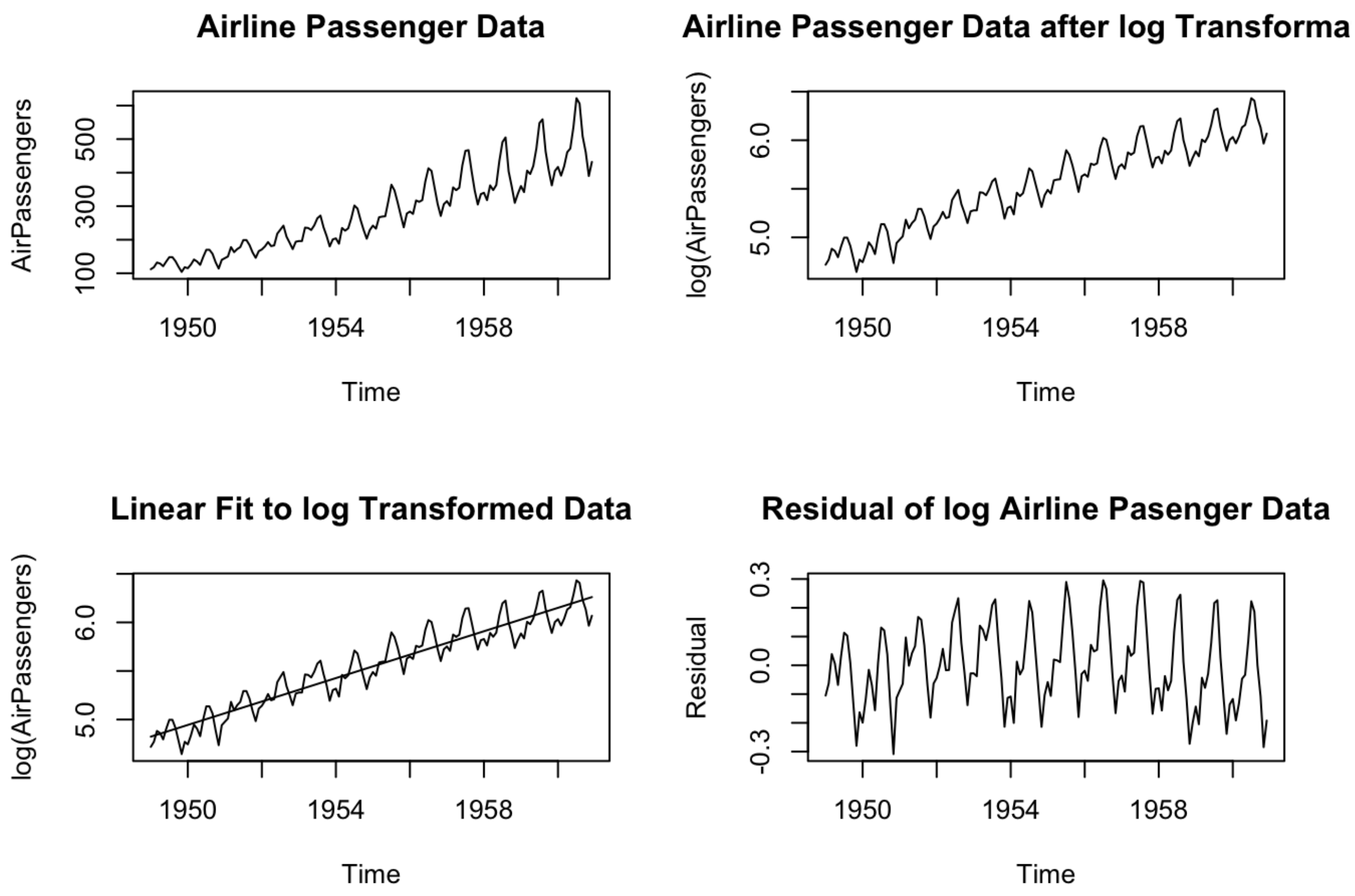




R code for Lecture Notes 4

Estimation of a Trend: Least Square Approach

```
par(mfrow=c(2,2))
plot.ts(AirPassengers, main="Airline Passenger Data")
plot.ts(log(AirPassengers), main="Airline Passenger Data after log Transformation")
# length(AirPassengers)
fit <- lm(log(AirPassengers)~c(1:144))
plot(log(AirPassengers), main="Linear Fit to log Transformed Data")
lines(ts(fit$fitted.values, frequency=12, start=c(1949,1)))
plot(ts(fit$residuals,frequency=12,start=c(1949,1)),ylab="Residual")
title(main="Residual of log Airline Pasenger Data")
```

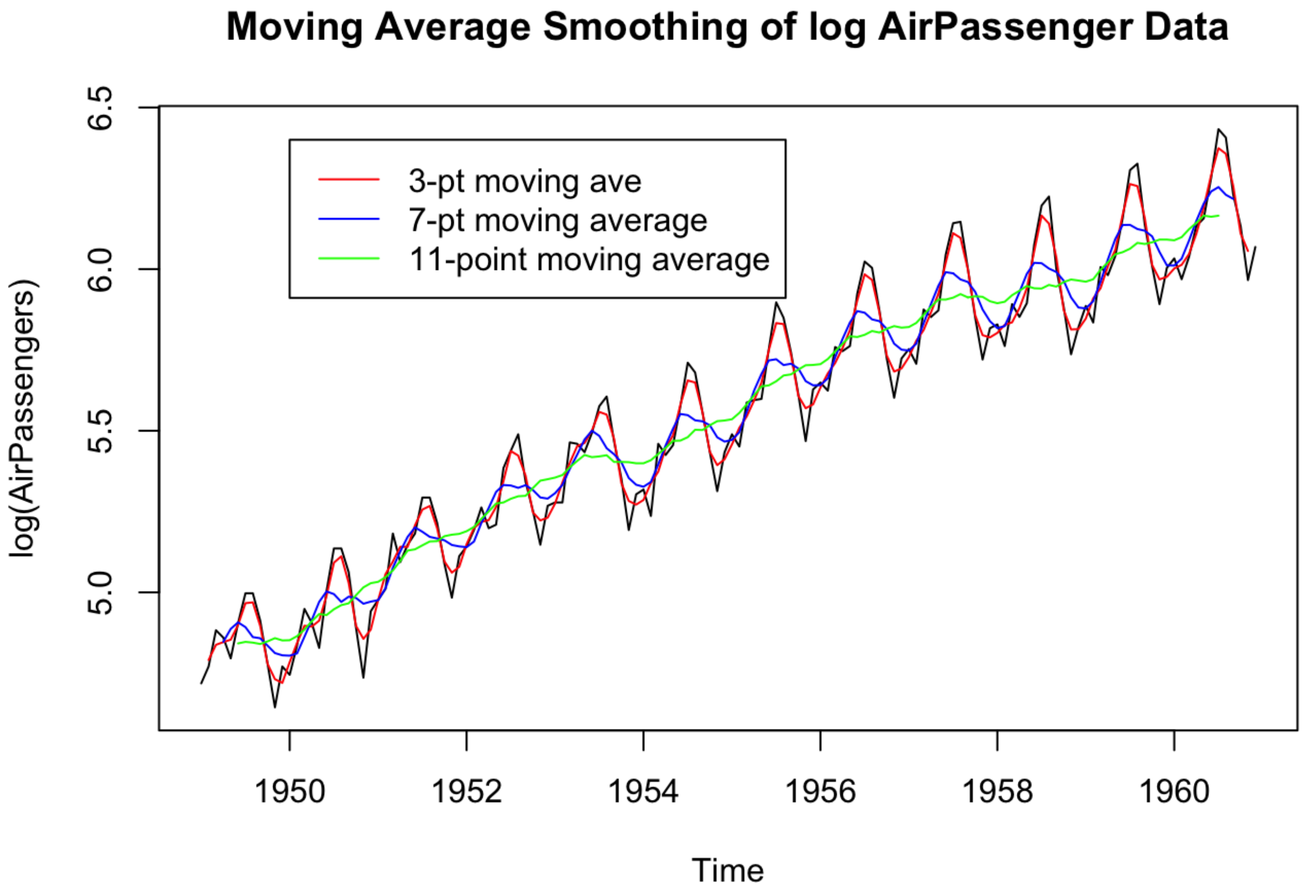


Estimation of a Trend: Moving Average Filter

```
library(forecast)
```

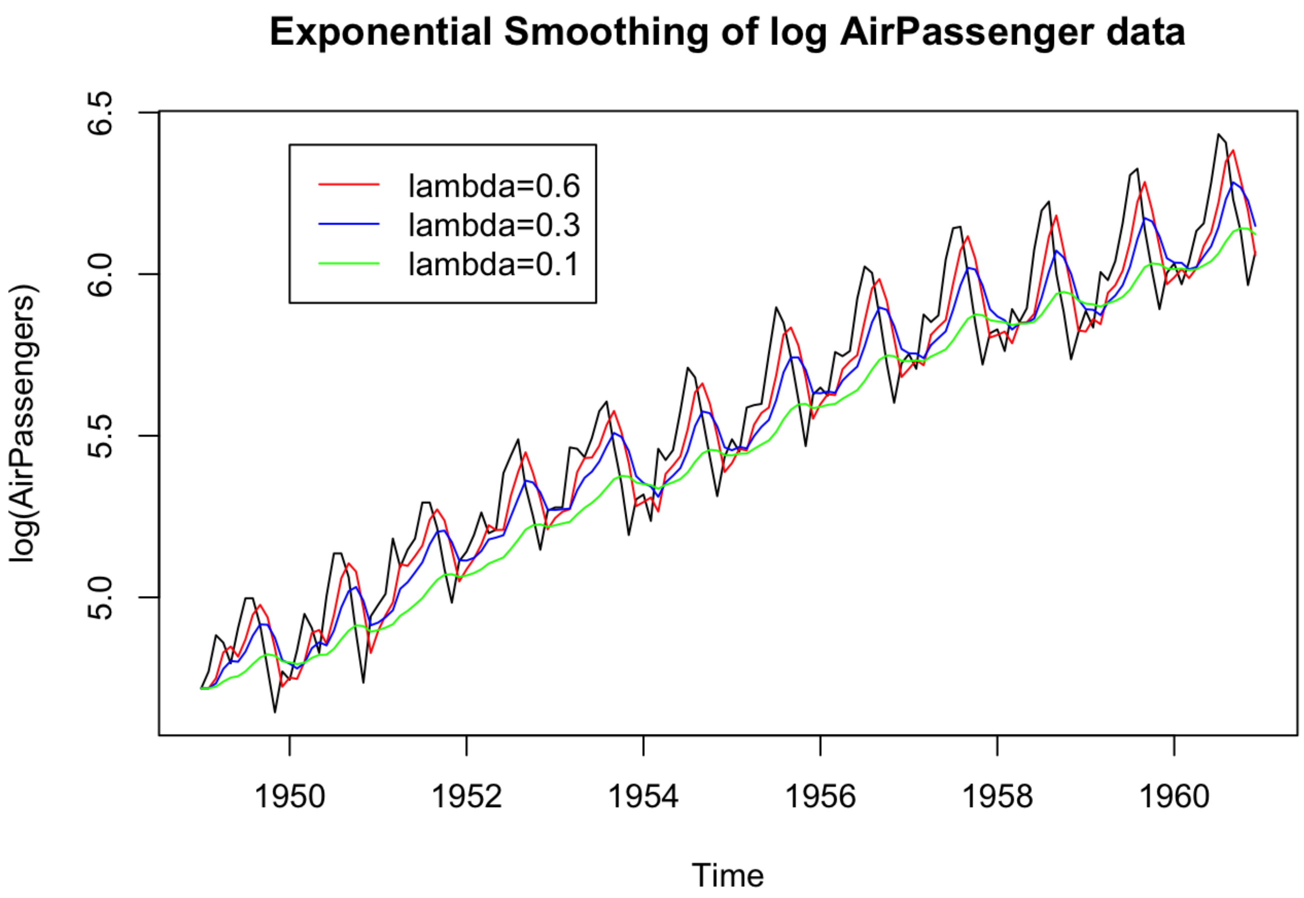
```
## Registered S3 methods overwritten by 'forecast':
##   method             from
##   fitted.Arima       TSA
##   fitted.fracdiff    fracdiff
##   plot.Arima         TSA
##   residuals.fracdiff fracdiff
```

```
plot(log(AirPassengers), main="Moving Average Smoothing of log AirPassenger Data")
lines(ma(log(AirPassengers),order=3),col="red")
lines(ma(log(AirPassengers),order=7),col="blue")
lines(ma(log(AirPassengers),order=11),col="green")
legend(1950,6.4,c("3-pt moving ave","7-pt moving average","11-point moving average"),
lty=c(1,1),
col=c("red","blue","green"))
```



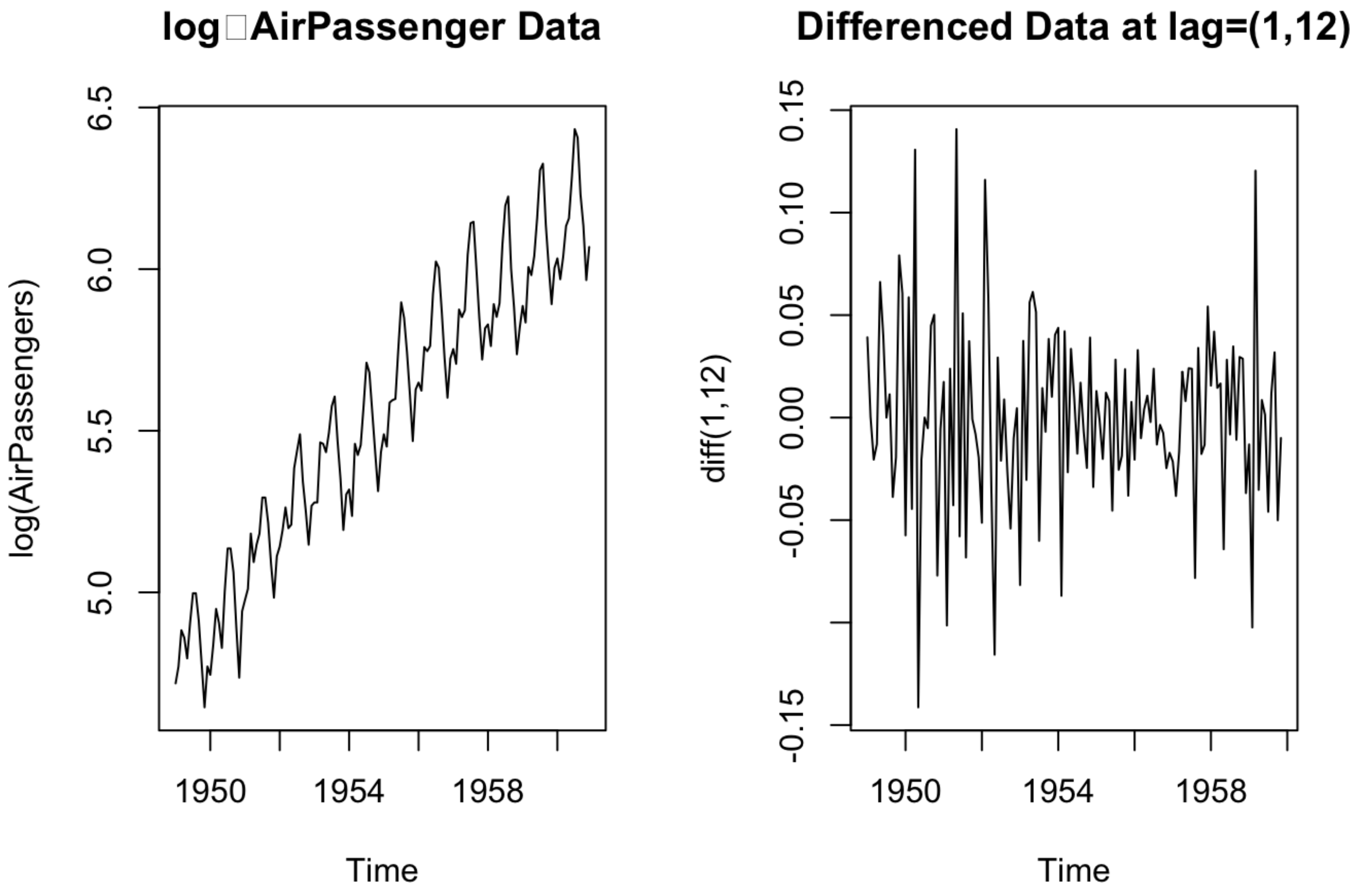
Estimation of a Trend: Exponential Smoothing

```
library(forecast)
fit1 <- ses(log(AirPassengers),alpha=0.6,initial="simple",h=3)
fit2 <- ses(log(AirPassengers),alpha=0.3,initial="simple",h=3)
fit3 <- ses(log(AirPassengers),alpha=0.1, initial="simple",h=3)
plot(log(AirPassengers), main="Exponential Smoothing of log AirPassenger data")
lines(fitted(fit1),col="red")
lines(fitted(fit2),col="blue")
lines(fitted(fit3),col="green")
legend(1950,6.4,c("lambda=0.6","lambda=0.3","lambda=0.1"), lty=c(1,1),col=c("red","bl
ue","green"))
```



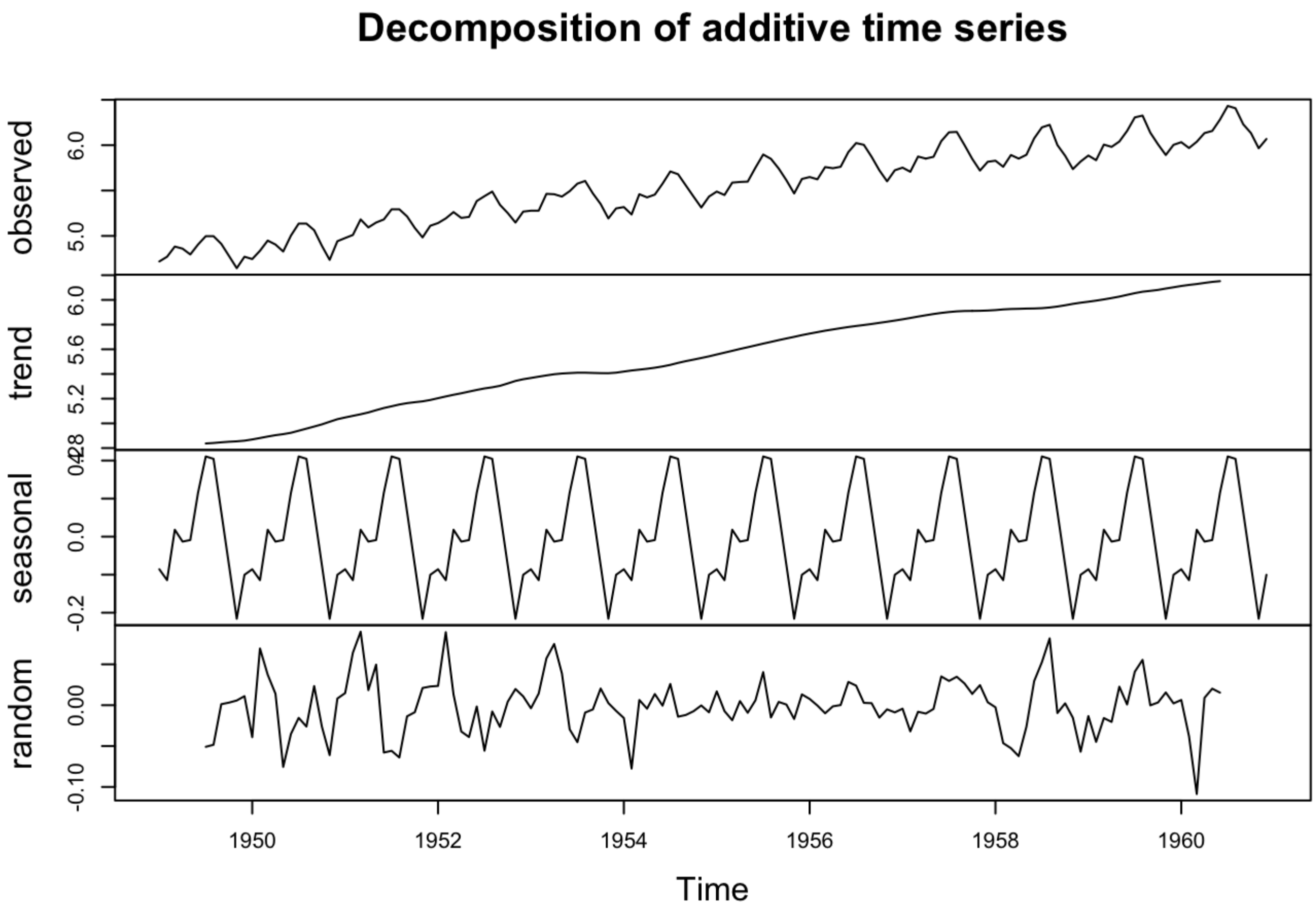
Trend via Differencing

```
par(mfrow=c(1,2))
plot(log(AirPassengers), main = "log-AirPassenger Data")
plot(ts(diff(diff(log(AirPassengers)), lag=12),frequency=12, start=c(1949,1)),
        ylab="diff(1,12)",main="Differenced Data at lag=(1,12)")
```

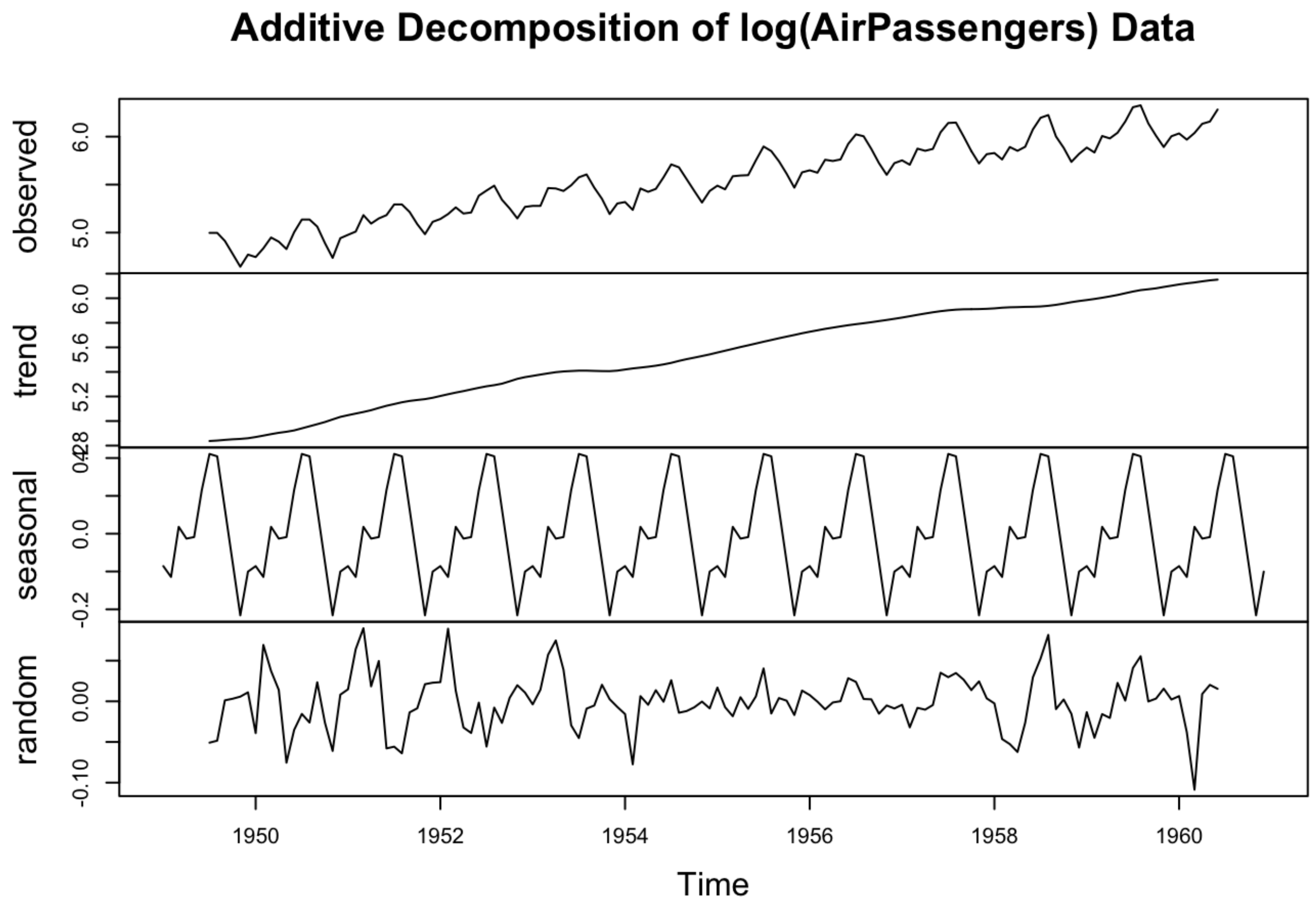


Additive and Multiplicative Decomposition

```
z <- decompose(log(AirPassengers), type="additive")
plot(z)
```




```
# Changing type to "multiplicative" gives multiplicative decomposition.
# The function decomp.plot allows the title of decompose to be changed.
decomp.plot <- function(x, main = NULL, ...)
{
  if(is.null(main))
    main <- paste("Decomposition of", x$type, "time series")
    plot(cbind(observed = x$random +
                  if (x$type == "additive") x$trend + x$seasonal
               else x$trend * x$seasonal,
               trend = x$trend, seasonal = x$seasonal,
    random = x$random), main = main, ...)
}
z <- decompose(log(AirPassengers), type="additive")
decomp.plot(z, main="Additive Decomposition of log(AirPassengers) Data")
```



Harmonic Regression Example

```
library(itsmr)
```

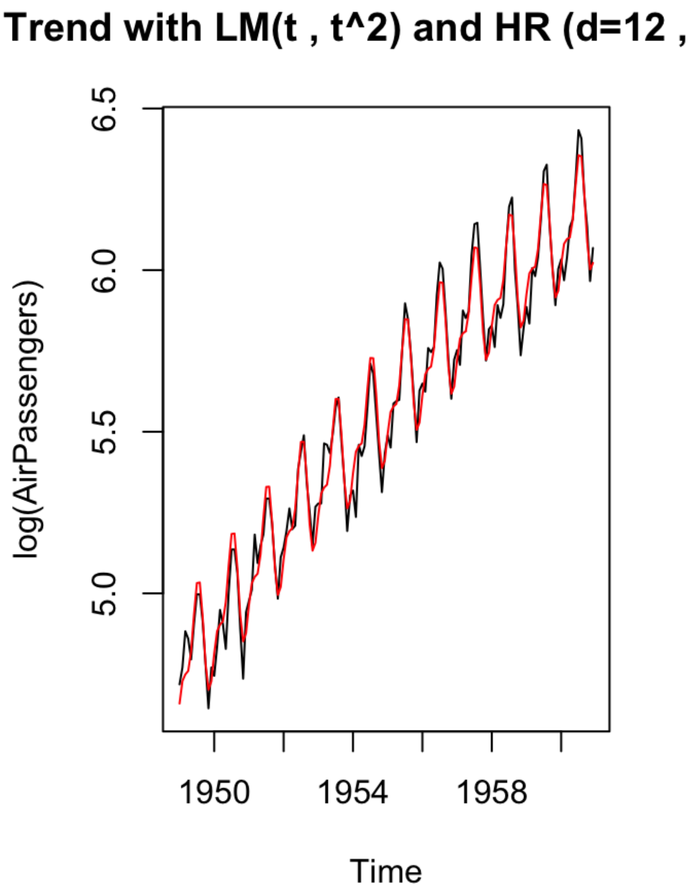
```
##
## Attaching package: 'itsmr'
```

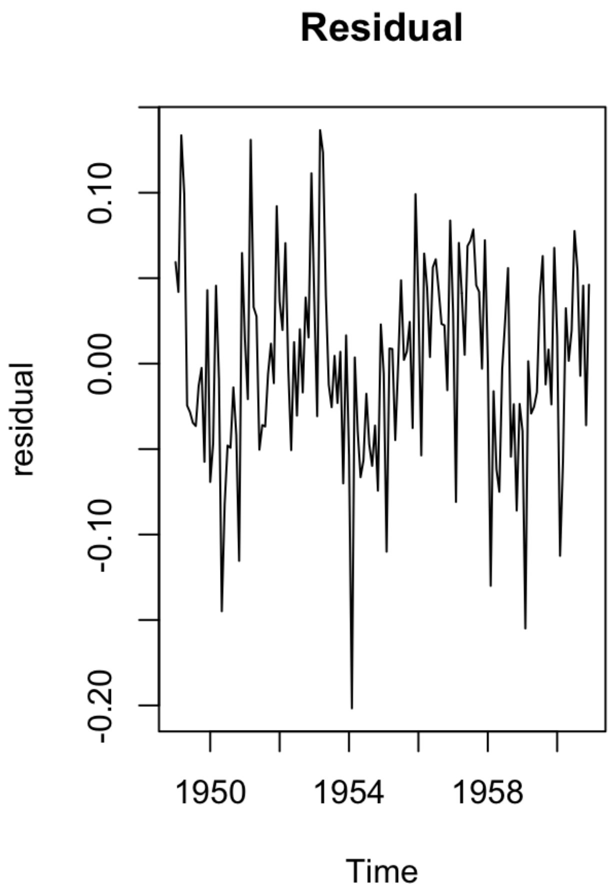
```
## The following object is masked from 'package:forecast':
##
##     forecast
```

```
## The following object is masked from 'package:tseries':
##
##     arma
```

```
## The following objects are masked from 'package:TSA':
##
##     periodogram, season
```

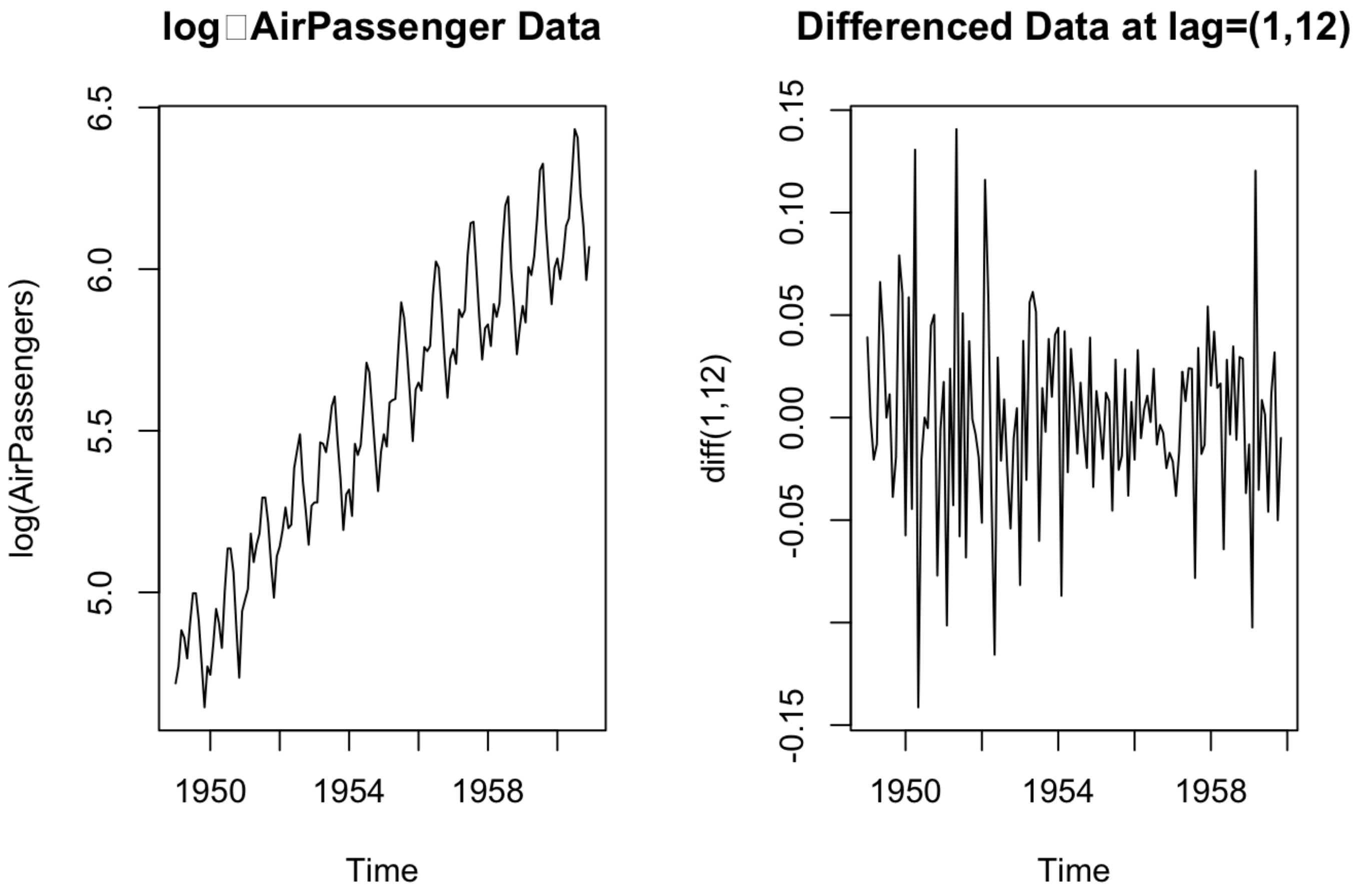
```
x <- trend(log(AirPassengers), 2)
dlogair <- log(AirPassengers) - x
y <- hr(dlogair,c(12,6))
fit <- x + y
res <- log(AirPassengers) - fit
par(mfrow=c(1,2))
plot.ts(log(AirPassengers),main="Trend with LM(t , t^2) and HR (d=12 , l=2)")
lines(ts(fit,frequency=12, star=c(1949,1)),col="red")
plot.ts(res, ylab="residual", main="Residual")
```





Seasonal Differencing

```
par(mfrow=c(1,2))
plot(log(AirPassengers),main="log-AirPassenger Data")
plot(ts(diff(diff(log(AirPassengers)),lag=12),frequency=12,start=c(1949,1)),
        ylab="diff(1,12)",main="Differenced Data at lag=(1,12)")
```


log AirPassenger Data
log(AirPassengers)
6.5
6.0
5.5
5.0
1950
1954
1958
Time
Differenced Data at lag=(1,12)
diff(1,12)
0.15
0.10
0.05
0.00
-0.05
-0.15
1950
1954
1958
Time