Textbook (3.4)

$$X_t = 0.8 X_{t-2} + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

$\because E(X_t) = 0.8 E(X_t) + 0, \rightarrow E(X_t) = 0$

$$\therefore \gamma(h) = Cov(X_{t+h}, X_t) = E(X_{t+h} X_t)$$

$$= E[(0.8 X_{t+h-2} + Z_{t+h}) X_t]$$

$$= 0.8 E(X_{t+h-2} X_t) + E(Z_{t+h} X_t)$$

$$= 0.8 \gamma(h-2)$$

for $h \geq 1$,

$$\gamma(1) = 0.8 \gamma(-1) \longrightarrow \gamma(1) = 0$$

$$\gamma(2) = 0.8 \gamma(0)$$

$$\gamma(3) = 0.8 \gamma(1) = 0 \quad (\gamma(1) = 0)$$

$$\gamma(4) = 0.8 \gamma(2) = 0.8^2 \gamma(0)$$

$$\gamma(5) = 0.8 \gamma(3) = 0$$

$$\vdots$$

$$\therefore \gamma(h) = \begin{cases} 0.8^{|\frac{h}{2}|} \gamma(0), & h \text{ even}, h \geq 1 \\ 0, & h \text{ odd}, h \geq 1 \end{cases}$$

$$\therefore \rho(h) = \begin{cases} 0.8^{|\frac{h}{2}|}, & h \text{ even} \\ 0, & h \text{ odd} \end{cases}$$

$$\therefore \begin{cases} \phi_{11} = \rho(1) = 0 \\ \phi_{22} = \rho(2) - \rho(1)^2 / 1 - \rho(1)^2 = \rho(2) = 0.8 \\ \phi_{hh} = 0, \quad h \geq 3 \ \text{(by defination)} \end{cases}$$

3、 $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = e_t$, $e_t \sim WN(0, \sigma^2)$

a、 $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + e_t = \phi_1(\phi_1 X_{t-2} + \phi_2 X_{t-3} + e_{t-1}) + \phi_2(\phi_1 X_{t-3} + \phi_2 X_{t-4} + e_{t-2}) + e_t = \cdots$

$= e_t + \phi_1 e_{t-1} + \phi_2 e_{t-2} + \phi_1^2 e_{t-2} + \phi_1\phi_2 e_{t-3} + \phi_1^3 e_{t-3} + \cdots$

Derive ψ_j :

$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1 \leftarrow \psi(B) e_t = X_t$

$1 + (\psi_1 - \phi_1)B + (\psi_2 - \phi_1\psi_1 - \phi_2)B^2 + (\psi_3 - \phi_1\psi_2 - \phi_2\psi_1)B^3 + \cdots = 1$

$\therefore \psi_1 = \phi_1$, $\psi_2 = \phi_1\psi_1 + \phi_2$, $\psi_3 = \phi_1\psi_2 + \phi_2\psi_1$ $\cdots$

$\therefore \boxed{\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}}$

b、 $X_t - 1.1 X_{t-1} + 0.3 X_{t-2} = e_t$ $\qquad \phi_1 = 1.1 \quad \phi_2 = -0.3$

$\boxed{\psi_j = 1.1\psi_{j-1} - 0.3\psi_{j-2}}$

$\psi_0 = 1$, $\psi_1 = 1.1$ $\psi_2 = 1.1^2 - 0.3 = 0.91$

$\psi_3 = 0.91 \times 1.1 - 0.3 \times 1.1 = 0.671$

$\psi_4 = 1.1 \times 0.671 - 0.3 \times 0.91 = 0.7381 - 0.273 = 0.465$

$\psi_5 = 1.1 \times 0.465 - 0.3 \times 0.671 = 0.31$

c、 ACF: $\gamma(h) = E(X_{t+h} X_t) = E[(1.1 X_{t+h-1} - 0.3 X_{t+h-2} + e_t) X_t]$

$= 1.1 E(X_{t+h-1} X_t) - 0.3 E(X_{t+h-2} X_t) + E(e_t X_t)$

$\therefore \gamma(h) - 1.1\gamma(h-1) + 0.3\gamma(h-2) = 0$

$\therefore \rho(h) - 1.1\rho(h-1) + 0.3\rho(h-2) = 0$

$\rho(0) = 1 \qquad \rho(1) - 1.1 + 0.3\rho(-1) = 0$

$\rho(1) = 0.846 \leftarrow (1+0.3)\rho(1) = 1.1$

$\rho(2) - 1.1 \times 0.846 + 0.3 = 0 \rightarrow \rho(2) = 0.631$

$\rho(3) - 1.1 \times 0.631 + 0.3 \times 0.846 = 0 \rightarrow \rho(3) = 0.44$

$\rho(4) - 1.1 \times 0.44 + 0.3 \times 0.631 = 0 \rightarrow \rho(4) = 0.295$

$\rho(5) - 1.1 \times 0.295 + 0.3 \times 0.44 = 0 \rightarrow \rho(5) = 0.192$

d. PACF: $\phi_{11} = \rho(1) = 0.846$

$$\phi_{22} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \frac{0.631 - 0.846^2}{1 - 0.846^2} = -0.3$$

$\phi_{hh} = 0,\ h \geq 3.$ (by defination

4. $X_t - \phi X_{t-1} = e_t + \theta e_{t-1}$

a. $X_t = \phi X_{t-1} + e_t + \theta e_{t-1} = \phi(\phi X_{t-2} + e_{t-1} + \theta e_{t-2}) + e_t + \theta e_{t-1} = \cdots$

$= e_t + \theta e_{t-1} + \phi e_{t-1} + \phi\theta e_{t-2} + \phi^2 e_{t-2} + \phi^2 \theta e_{t-3} + \cdots$

$\because X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} \quad \therefore X_t = (1 + \psi_1 B + \psi_2 B^2 + \cdots) e_t = \psi(B) e_t$

$\therefore (1 - \phi B)\psi(B) = (1 + \theta B)$

$\therefore (1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1 + \theta B$

$1 + (\psi_1 - \phi)B + (\psi_2 - \phi\psi_1)B^2 + (\psi_3 - \phi\psi_2)B^3 + \cdots - \theta B = 1$

$1 + (\psi_1 - \phi - \theta)B + (\psi_2 - \phi\psi_1)B^2 + (\psi_3 - \phi\psi_2)B^3 + \cdots = 1$

$\therefore \psi_1 = \phi + \theta,\ \psi_2 = \phi\psi_1 = \phi(\phi + \theta), \cdots$

Thus $\boxed{\psi_j = \phi^{j-1}(\phi + \theta)}$, for $j \geq 1$

6. $X_t - 0.6 X_{t-1} = e_t - 0.2 e_{t-1}$

$\phi = 0.6, \quad \theta = -0.2$

$\psi_1 = 0.6 - 0.2 = 0.4$

$\psi_2 = 0.6 \times 0.4 = 0.24$

$\psi_3 = 0.6^2 \times 0.4 = 0.144$

$\psi_4 = 0.6^3 \times 0.4 = 0.086$

$\psi_5 = 0.6^4 \times 0.4 = 0.052$

5. $X_t = e_t + 0.8 e_{t-1} - 0.15 e_{t-2}$ $\quad \theta_1 = 0.8 \quad \theta_2 = -0.15$

a. $$\rho(h) = \begin{cases} 1 & h=0 \\ \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} = 0.409 & h=1 \\ \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2} = -0.09 & h=2 \\ 0, & h \geq 3 \end{cases}$$

b. $\phi_{11} = \rho(1) = 0.409$

$$\phi_{22} = \frac{\rho(2) - \rho(1)^2}{1-\rho(1)^2} = \frac{-0.09 - 0.409^2}{1-0.409^2} = -0.309$$

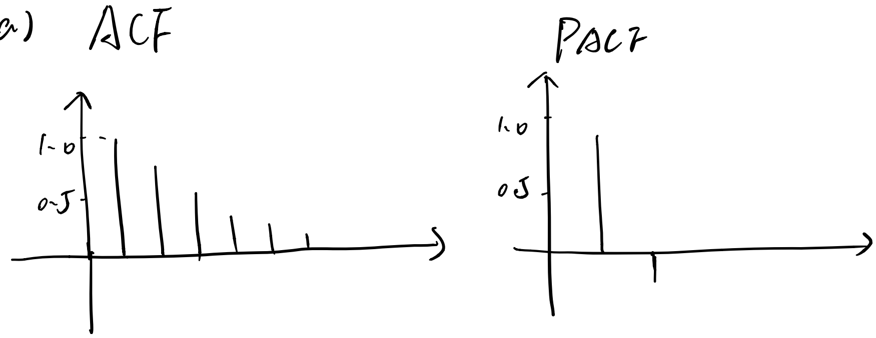
$$\phi_{33} = \frac{\rho(3) - \phi_{21}\rho(2) - \phi_{22}\rho(1)}{1 - \phi_{21}\rho(1) - \phi_{22}\rho(2)} = 0.232$$

$$\phi_{44} = \frac{\rho(4) - \phi_{31}\rho(3) - \phi_{32}\rho(2) - \phi_{33}\rho(1)}{1 - \phi_{31}\rho(1) - \phi_{32}\rho(2) - \phi_{33}\rho(3)} = -0.188$$

$\phi_{55} = \ldots = 0.157$

$$\phi_{hh} = \frac{\rho(h) - \sum_{j=1}^{h-1} \phi_{h-1,j}\rho(h-j)}{1 - \sum_{j=1}^{h-1} \phi_{h-1,j}\rho(j)}, \quad h \geq 2$$

6. (a)



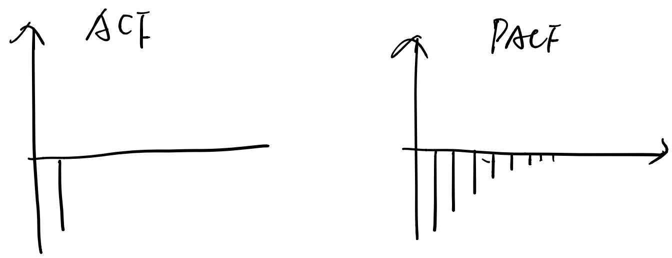
It's a AR(2) process

$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = e_t$

$\phi_{11} = \frac{\phi_1}{1-\phi_1} \approx 0.75 \Rightarrow \phi_1 = (1+0.2) \times 0.75 = 0.9$

$\phi_{22} = \phi_2 \approx -0.2$

(b)



MA(1) process

$X_t = e_t + \theta e_{t-1}$

$\rho(1) \approx 0.4 = \frac{\theta}{1+\theta^2}$

$\rightarrow \theta = -0.5$ or $\theta = -2$

MA to be causal.

$\theta = -0.5$

7. (Discussion)

Consider the ts process

$X_t = -2t + e_t + 0.5 e_{t-1}, \quad e_t \sim WN(0, \sigma^2)$

(a) Compute mean & ACF of X_t. Is the process stationary?

(b) Define a new process. $W_t - X_t - X_{t-1}$, where is defined in part (a). Is the process W_t stationary? Justify your answer.

$E(X_t) = -2t$

$\gamma(h) = \mathrm{Cov}(X_t, X_{t+h})$

$h=0.\ \ \mathrm{Var}(X_t) = (1+0.5^2)\sigma^2$

$h=1\ \ \gamma(h) = \mathrm{Cov}(-2t + e_t + 0.5 e_{t-1}$ [illegible]

$h=1, \quad \gamma(h) = \mathrm{Cov}(e_t + 0.5 e_{t-1}, e_{t+1} + 0.5 e_t)$

$$= 0.5\,\sigma^2$$

$$\therefore \gamma(h) = \begin{cases} (1+0.5^2)\,\sigma^2, & h=0 \\ 0.5\,\sigma^2 & h=1 \\ 0 & h\geq 2 \end{cases}$$