HW5

Haozhe Chen

1. $\bar{X} = 0.157 \quad \theta = -0.6 \quad \sigma^2 = 1$

For MA(1): $X_t = \mu + e_t - 0.6 e_{t-1}$

$$\begin{cases} \gamma(0) = \sigma^2 + 0.36\sigma^2 = 1.36 \\ \gamma(1) = -0.6\sigma^2 = -0.6 \\ \gamma(h) = 0, \quad h \geq 2 \end{cases}$$

$\therefore Var(\bar{X}) = \frac{1}{n} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma(h)$

$= \frac{1}{100}\left(1.36 + 2\left(1 - \frac{1}{100}\right)(-0.6)\right)$

$= 0.00172$

$\therefore$ 95% CI for $\bar{X}$: $0.157 \pm 1.96\sqrt{0.00172} = 0.157 \pm 0.081$

$\therefore \mu = 0$ does not $\in$ this interval. $\therefore$ not compatible

2. $\hat{\rho}(1) = 0.432, \quad \hat{\rho}(2) = 0.145, \quad n = 100, \quad \theta = 0.6$

The conservative 95% interval is $\pm 1.96/\sqrt{n} \approx (-0.2, 0.2)$

$H_0: \rho(1) = 0$

$\hat{\rho}(1) = 0.432$ is outside the interval.

$\therefore \rho(1)$ significantly different from 0.

$H_0: \rho(2) = 0$

$\hat{\rho}(2) = 0.145$ is inside the interval.

$\therefore \rho(2)$ cannot reject H_0.

Compatible with MA(1)

$\theta = 0.6, \ \rho(1) = \frac{0.6}{1+0.6^2} \approx 0.44$

$\therefore \theta = 0.6$ is consistent

- Alternative Solution,

For MA(1): $W_{ii} = \begin{cases} 1 - 3\rho^2(1) + 4\rho^4(1), & i = 1 \\ 1 + 2\rho^2(1), & i > 1 \end{cases}$ $\quad \rho(1) = 0.44$, when $\theta = 0.6$

plug in

$Var\,\hat{\rho}(i) = \frac{W_{ii}}{n}$

$Var\,\hat{\rho}(1) = 0.00576 \quad Var\,\hat{\rho}(2) = 0.00139$

$\therefore$ 95% CI for $\rho(1)$: $0.432 \pm 1.96\sqrt{0.00576} \approx (0.28, 0.58)$

95% CI for $\rho(2)$: $0.145 \pm 1.96\sqrt{0.00139} \approx (-0.086, 0.38)$

$\therefore \rho(1) = 0.44, \ \rho(2) = 0$ fall into their intervals respectively,

$\therefore$ The data is compatible with MA(1) process with $\theta = 0.6$

3. In R Session

4. (a). ACF $\to 0$ $\to$ Could be AR(1)

PACF Cuts off after lag 1

$\underbrace{\phi_{11} = \phi_1 = -0.6}$

$\therefore$ The candidate model is $X_t + 0.6 X_{t-1} = e_t$

(b) ACF cuts-off after lag 1

PACF $\to 0$, like a damped sine wave

$\therefore$ It Might be MA(1)

$\therefore \rho(1) = \frac{\theta}{1+\theta^2} = -0.45$

$\to \theta = -1.6$ or $\theta = -0.63$

when $\theta = -1.6$ $\theta(B) = 1 - 1.6B = 0$ $\to |B| < 1$ $\therefore$ Not invertible

when $\theta = -0.63$ $\theta(B) = 1 - 0.63B = 0$ $\to |B| > 1$ $\therefore$ invertible

$\therefore$ The model may be $X_t = e_t - 0.63 X_{t-1}$

(c) ACF $\to 0$ (sine wave)

PACF Cuts-off after lag 2

$\therefore$ AR(2) Could be a potential Candidate

$$\begin{cases} \phi_{11} = \frac{\phi_1}{1-\phi_2} = 0.8 \\ \phi_{22} = \phi_2 = -0.6 \end{cases} \Rightarrow \begin{cases} \phi_1 = 1.28 \\ \phi_2 = -0.6 \end{cases}$$

$\therefore X_t - 1.28 X_{t-1} + 0.6 X_{t-2} = e_t$

(d) Both ACF and PACF cut-off after lag 1.

It could potentially be ARMA(1,1), AR(1), or MA(1)

Take MA(1) as an example

$$\rho(1) = \frac{\theta}{1+\theta^2} = -0.25 \to \begin{cases} \theta = 0.27 \checkmark \to \text{invertible} \\ \theta = 3.73 \; \cancel{\checkmark \to \text{invertible}} \end{cases}$$

$\therefore$ It Could be $\cancel{X_t = e_t + 3.73 X_{t-1}}$ $X_t = e_t + 0.27 X_{t-1}$

5. AR(2): $X_t - X_{t-1} + 0.25 X_{t-2} = e_t$, $\phi_1 = +1$ $\phi_2 = -0.25$

$\rho(1) = \frac{\phi_1}{1-\phi_2} = 0.8$ $\quad \leftarrow \rho(h) - \phi_1 \rho(h-1) - \phi_2 \rho(h-2) = 0$

$\rho(2) = \phi_1 \rho(1) + \phi_2 \rho(0) = 0.55$

$\rho(3) = \phi_1 \rho(2) + \phi_2 \rho(1) = 0.35$

$\rho(4) = \phi_1 \rho(3) + \phi_2 \rho(2) = 0.213$

$\rho(5) = \phi_1 \rho(4) + \phi_2 \rho(3) = 0.125$

$\hat{\rho}(1)=0.41$ $\hat{\rho}(2)=0.32$, $\hat{\rho}(3)=0.26$, $\hat{\rho}(4)=0.21$, $\hat{\rho}(5)=0.16$

6. The ACF $\longrightarrow 0$

① Assume it is a AR(1) process. $X_t-\phi X_{t-1}=e_t$

$\because \rho(h)=\phi^h$ for AR(1)

$\therefore \hat{\rho}(1)=0.41$

$\hat{\rho}(2)=0.41^2=0.17$

$\hat{\rho}(3)=0.41^3=0.07$

$\hat{\rho}(4)=0.41^4=0.03$

$\hat{\rho}(5)=0.41^5=0.012$

$\phi^h/\phi^{h-1}=\rho(h)/\rho(h-1)=\phi'=\hat{\rho}(1)=0.41$

plug in $\hat{\rho}(1)-\hat{\rho}(5)$

$\hat{\rho}(1)'=\hat{\rho}(2)/\hat{\rho}(1)=0.78$

" $\hat{\rho}(3)/\hat{\rho}(2)=0.8125$

" $\hat{\rho}(4)/\hat{\rho}(3)=0.876$ ......

So, it seems not like a AR(1) process.

② Assume it is a ARMA(1,1) process.

$\rho(h)=\phi^{h-1}\rho(1)$, for $h\geq 2$

$\therefore \rho(2)=\phi\rho(1) \rightarrow \rho(2)/\rho(1)=\phi$

... $\rho(3)/\rho(2)=\phi$

$\rho(4)/\rho(3)=\phi$

...

$\rho(h)/\rho(h-1)=\phi$

plug in $\hat{\rho}(1)\sim\hat{\rho}(5)$

$\hat{\rho}(2)/\hat{\rho}(1)=0.32/0.41=0.78$

$\hat{\rho}(3)/\hat{\rho}(2)=0.8125$

$\hat{\rho}(4)/\hat{\rho}(3)=0.876$

$\hat{\rho}(5)/\hat{\rho}(4)=0.762$

$\therefore$ It is more like a ARMA(1,1) process! $\phi\simeq 0.8$

7. ϕ_{11}, ϕ_{22} are lie outside the 95% CI.

$\therefore$ PACF cutoff after lag 2.

$\therefore$ It might be an AR(2) model.

try $\begin{cases}\phi_{11}=\dfrac{\phi_1}{1-\phi_2}=0.8\\ \phi_{22}=\phi_2=-0.6\end{cases} \rightarrow \begin{cases}\phi_1=1.28\\ \phi_2=-0.6\end{cases}$

$\therefore$ The candidate model could be $X_t-1.28X_{t-1}+0.6X_{t-2}=e_t$

HW 5

Haozhe Chen

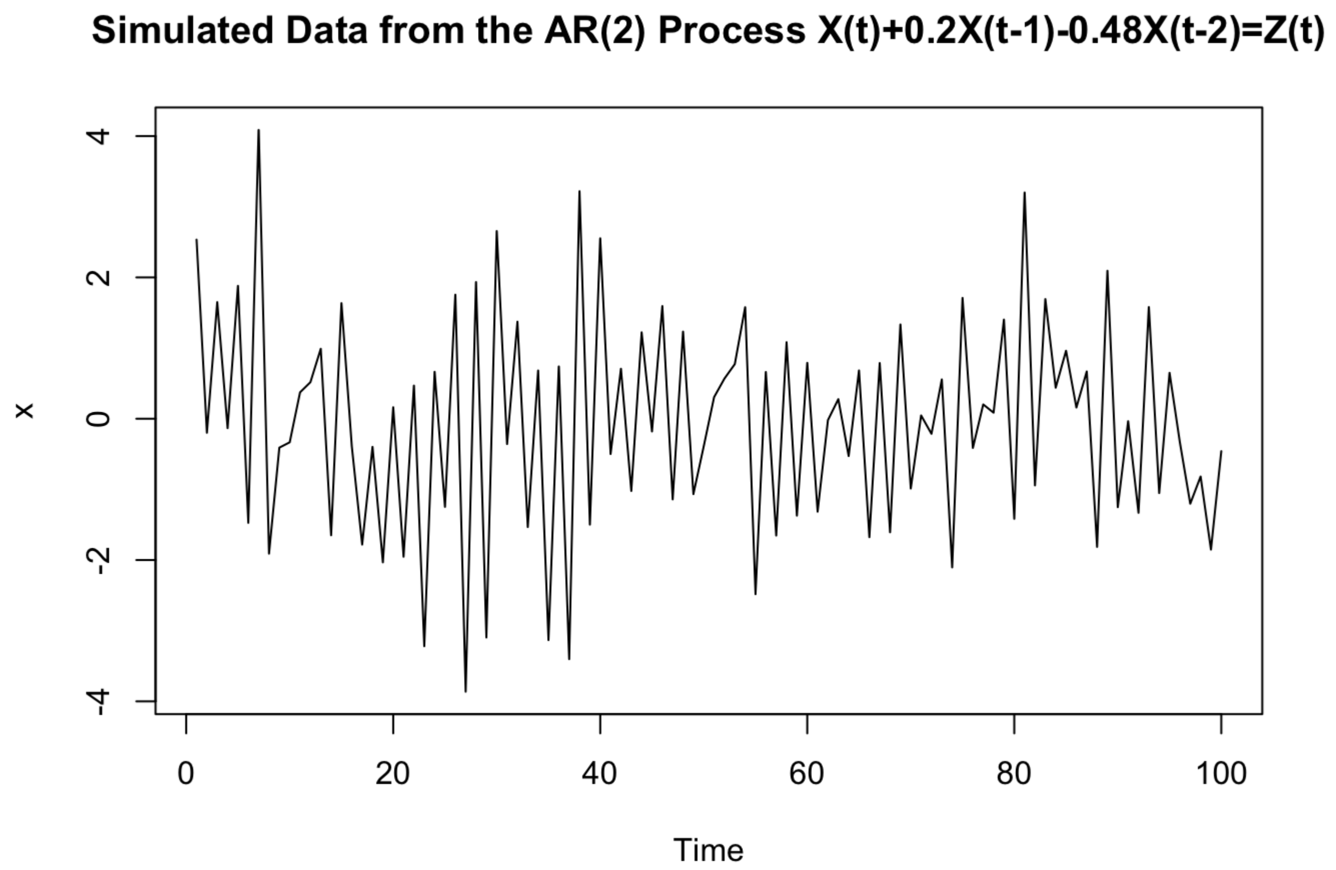
3. For each one of the following ARMA processes, choose parameters such that the process is causal and invertible. In each case, use the arima.sim function in R to generate a sample realization of size 100: Generate a time series plot of the simulated series, and in each case plot both population and sample ACF and PACF.

i. AR(2) : $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$

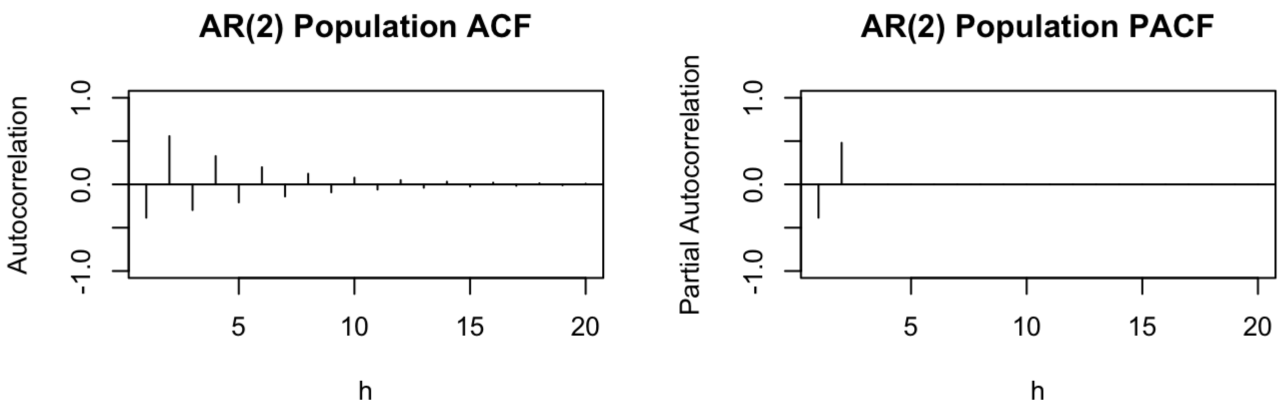
```
polyroot(c(1,0.2,-0.48))
```

```
## [1]  1.666667+0i -1.250000-0i
```

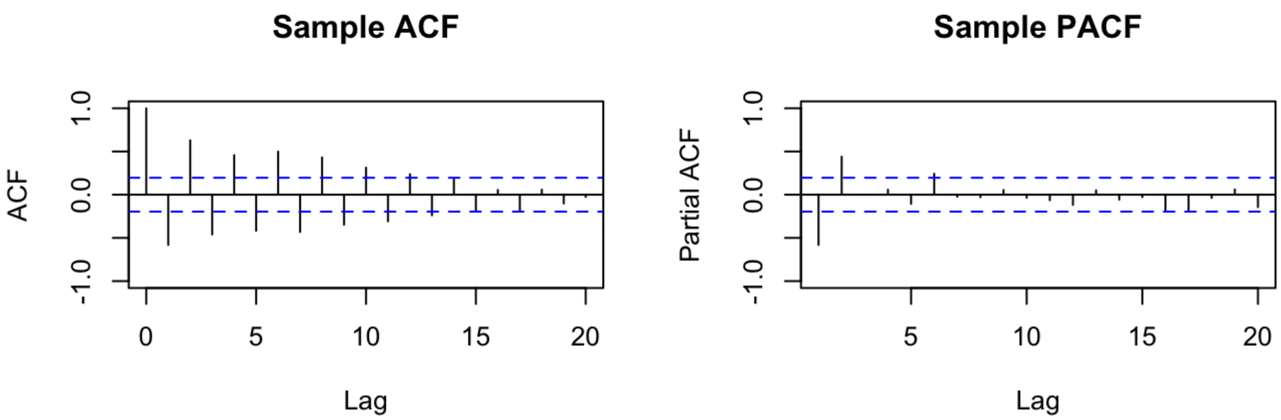
```
# the roots lie outside the unit circle, hence it is causal.
# The process is also invertible becasue the MA part can be expressed as an AR model.

x=arima.sim(n=100, list(ar=c(-0.2,0.48)))
plot.ts(x)
title(main="Simulated Data from the AR(2) Process X(t)+0.2X(t-1)-0.48X(t-2)=Z(t)")
```



```
par(mfrow=c(2,2))
y = ARMAacf(ar=c(-0.2,0.48),lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Autocorrelation", main = "AR(2) Population ACF")
abline(h = 0)
y = ARMAacf(ar=c(-0.2,0.48),lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "AR(2) Population PACF")
abline(h = 0)
acf(x,main="Sample ACF", ylim = c(-1,1))
pacf(x,main="Sample PACF", ylim = c(-1,1))
```



(ii)



ARMA(1,1) : $X_t - 0.5X_{t-1} = Z_t + 0.3Z_t$

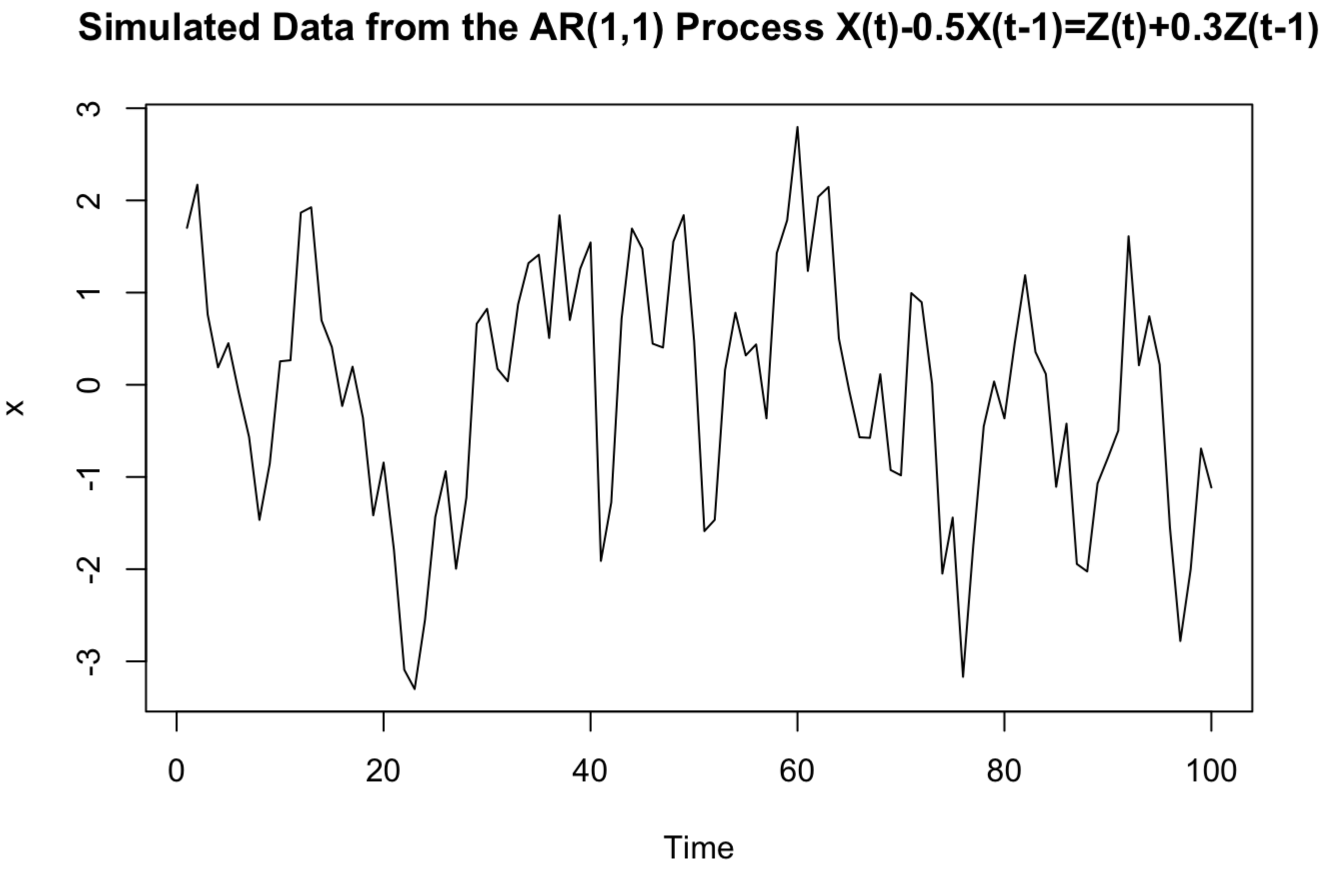
```
polyroot(c(1,-0.5))
```

```
## [1] 2+0i
```

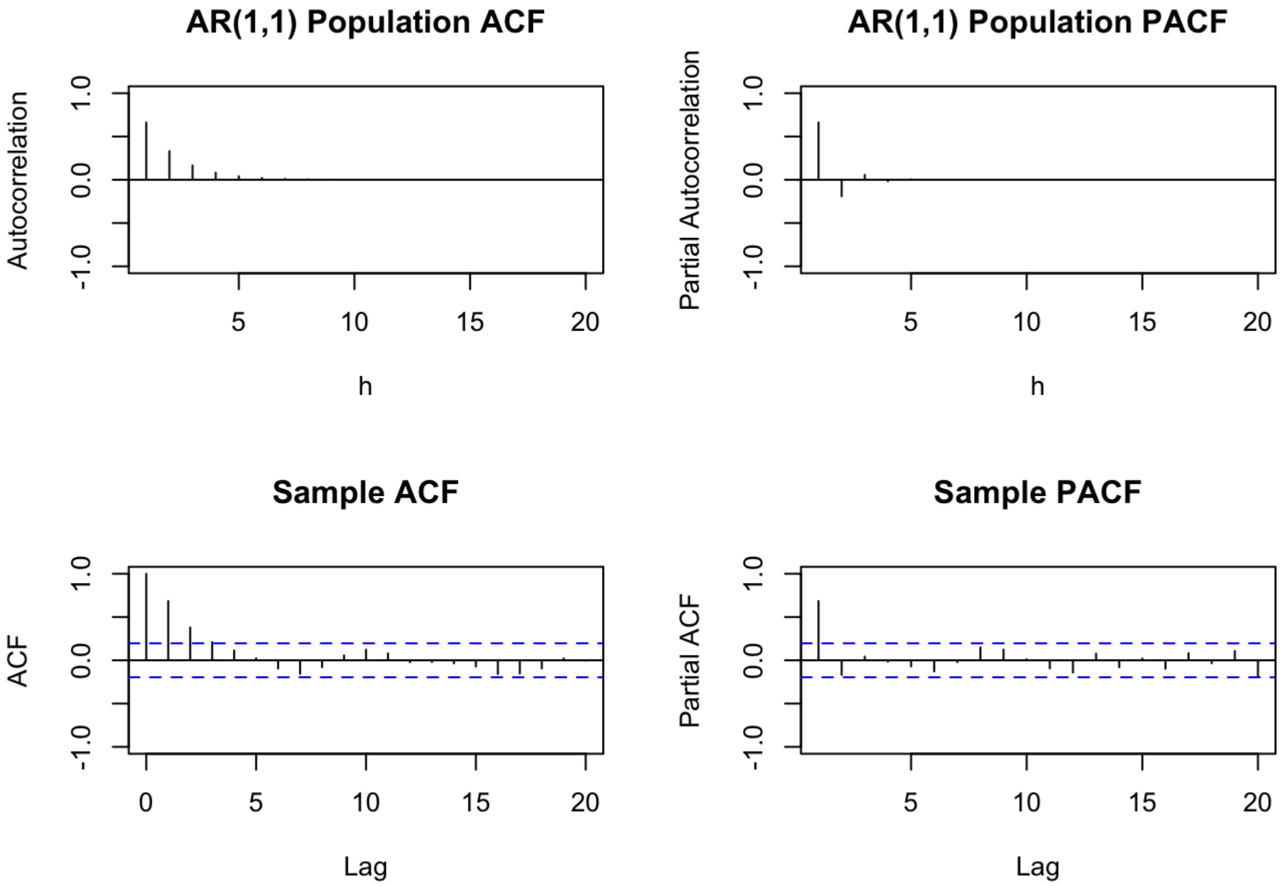
```
polyroot(c(1,0.3))
```

```
## [1] -3.333333+0i
```

```
# Both the root of phi(B) and theta(B) are lie outside the unit circle
# So it is causal and invertible

x=arima.sim(n=100, list(ar=c(0.5), ma=c(0.3)))
plot.ts(x)
title(main="Simulated Data from the AR(1,1) Process X(t)-0.5X(t-1)=Z(t)+0.3Z(t-1)")
```




```
par(mfrow=c(2,2))
y = ARMAacf(ar=c(0.5), ma=c(0.3),lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Autocorrelation", main = "AR(1,1) Population ACF")
abline(h = 0)
y = ARMAacf(ar=c(0.5), ma=c(0.3),lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "AR(1,1) Population PACF")
abline(h = 0)
acf(x,main="Sample ACF", ylim = c(-1,1))
pacf(x,main="Sample PACF", ylim = c(-1,1))
```

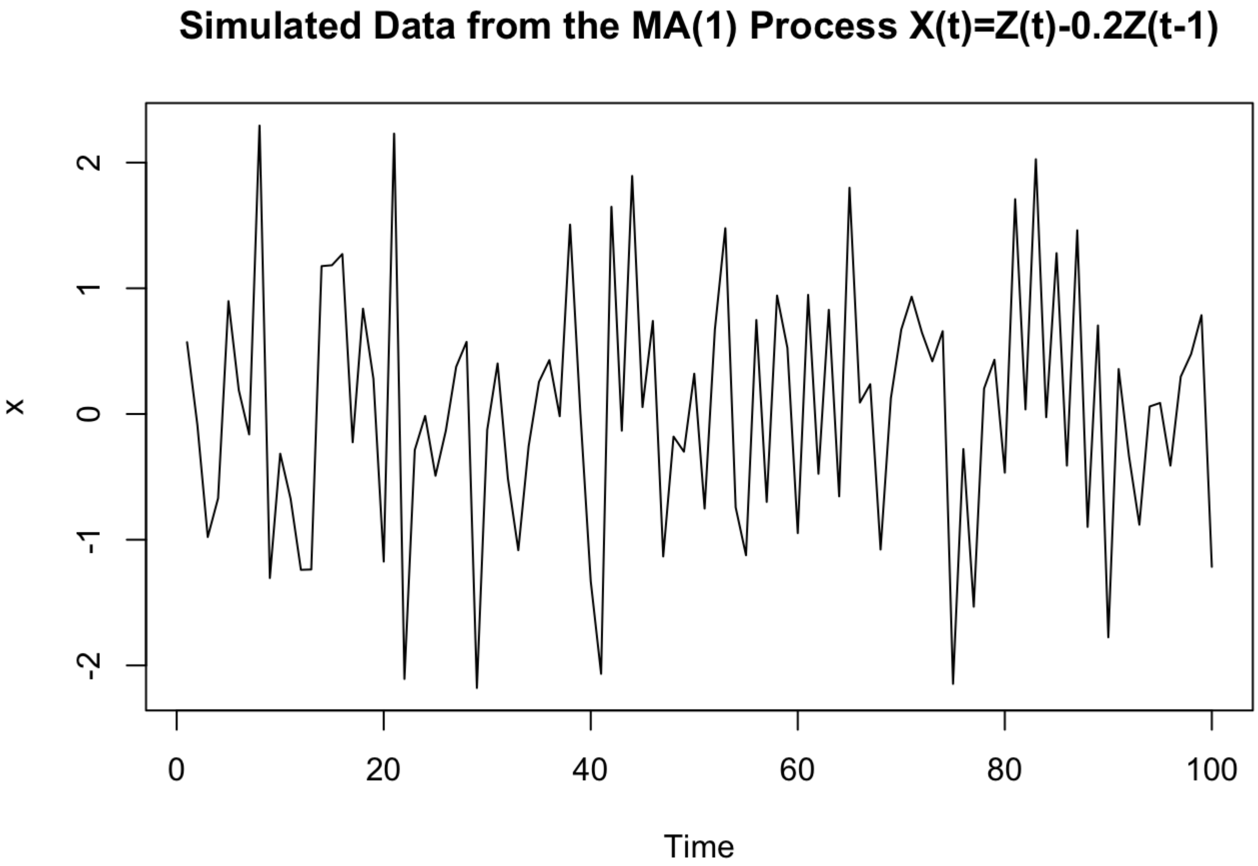


(iii) MA(1) : $X_t = Z_t - 0.2Z_{t-1}$

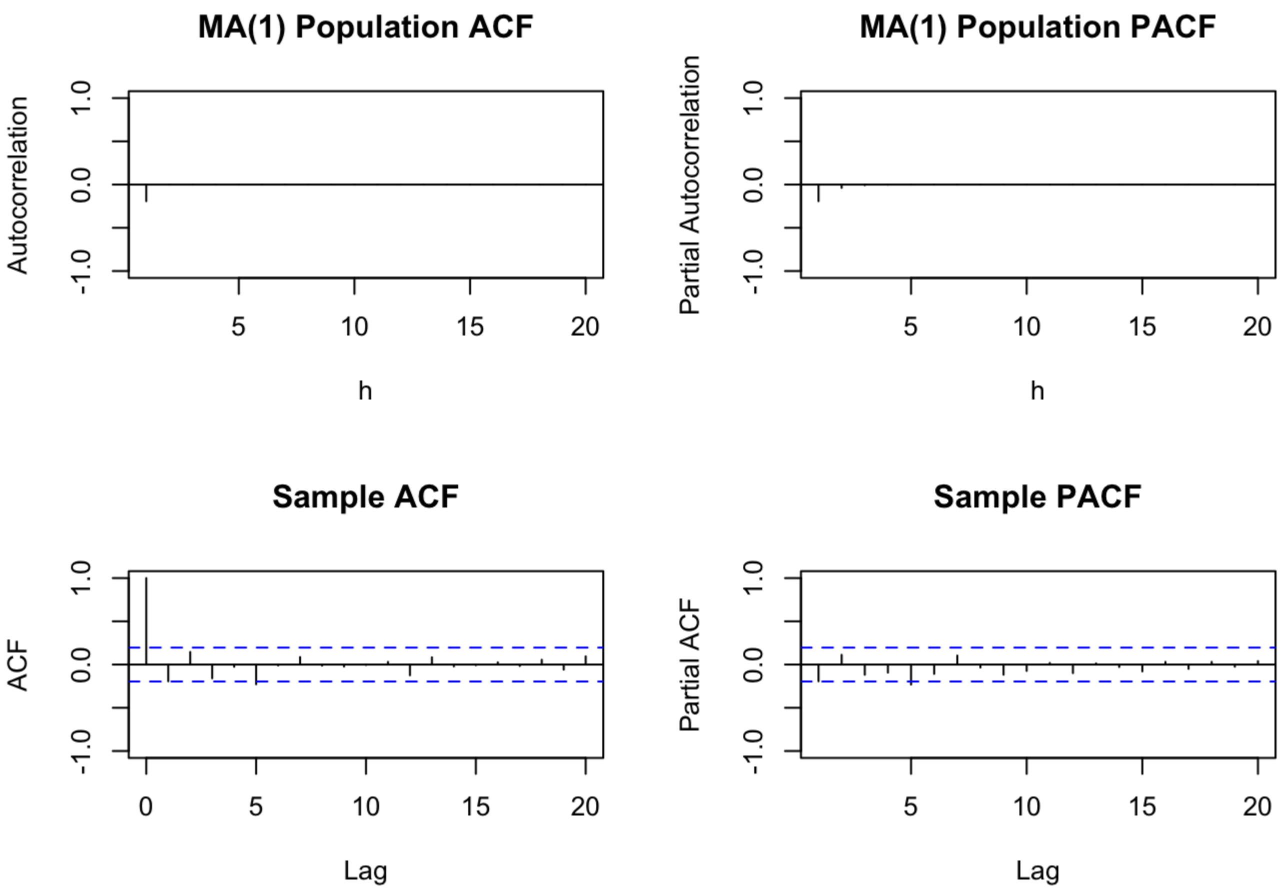
```
polyroot(c(1,-0.2))
```

```
## [1] 5+0i
```

```
# the root lie outside the unit circle

x=arima.sim(n=100, list(ma=c(-0.2)))
plot.ts(x)
title(main="Simulated Data from the MA(1) Process X(t)=Z(t)-0.2Z(t-1)")
```




```
par(mfrow=c(2,2))
y = ARMAacf(ma=c(-0.2),lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Autocorrelation", main = "MA(1) Population ACF")
abline(h = 0)
y = ARMAacf(ma=c(-0.2),lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "MA(1) Population PACF")
abline(h = 0)
acf(x,main="Sample ACF", ylim = c(-1,1))
pacf(x,main="Sample PACF", ylim = c(-1,1))
```



(iv) ARMA(1,2) : $X_t - 0.3X_{t-1} = Z_t + 0.2Z_{t-1} - 0.48Z_{t-2}$

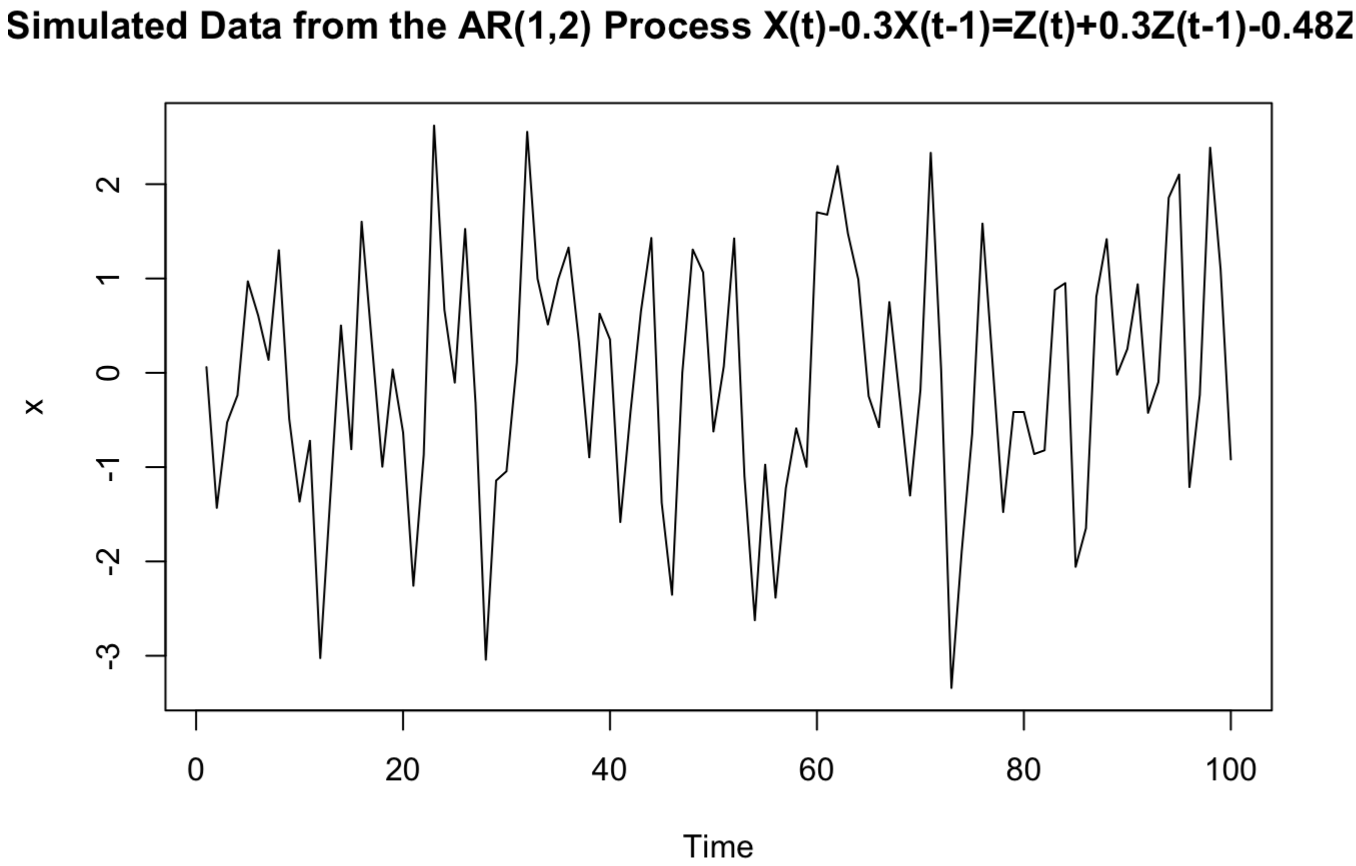
```
polyroot(c(1,-0.3))
```

```
## [1] 3.333333+0i
```

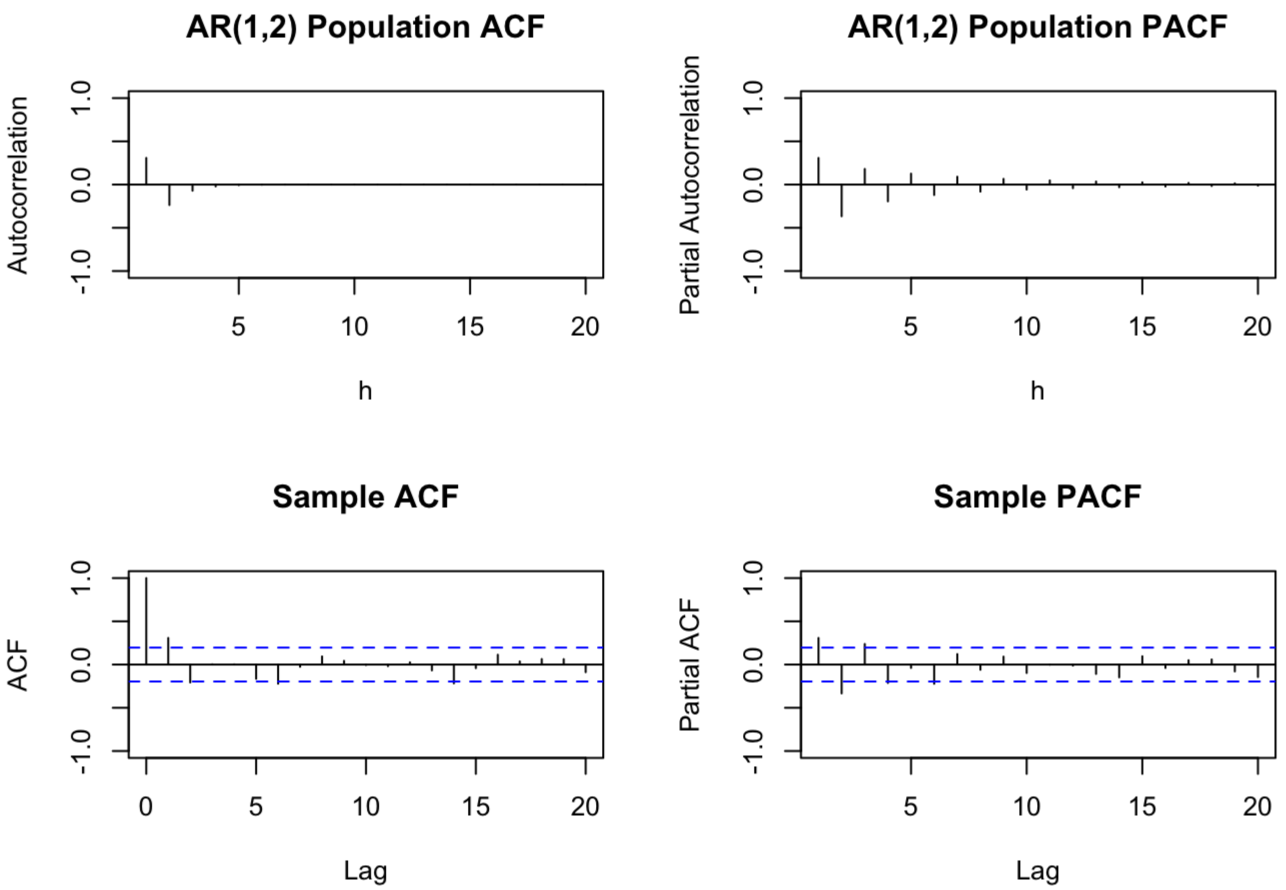
```
polyroot(c(1,0.2,-0.48))
```

```
## [1]  1.666667+0i -1.250000-0i
```

```
# Both the root of phi(B) and theta(B) are lie outside the unit circle
# So it is causal and invertible

x=arima.sim(n=100, list(ar=c(0.3), ma=c(0.3,-0.48)))
plot.ts(x)
title(main="Simulated Data from the AR(1,2) Process X(t)-0.3X(t-1)=Z(t)+0.3Z(t-1)-0.4
8Z(t-2)")
```




```
par(mfrow=c(2,2))
y = ARMAacf(ar=c(0.3), ma=c(0.3,-0.48),lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Autocorrelation", main = "AR(1,2) Population ACF")
abline(h = 0)
y = ARMAacf(ar=c(0.3), ma=c(0.3,-0.48),lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "AR(1,2) Population PACF")
abline(h = 0)
acf(x,main="Sample ACF", ylim = c(-1,1))
pacf(x,main="Sample PACF", ylim = c(-1,1))
```



5.Verification of ACF's of AR(2): $X_t - X_{t-1} + 0.25X_{t-2} = e_t$

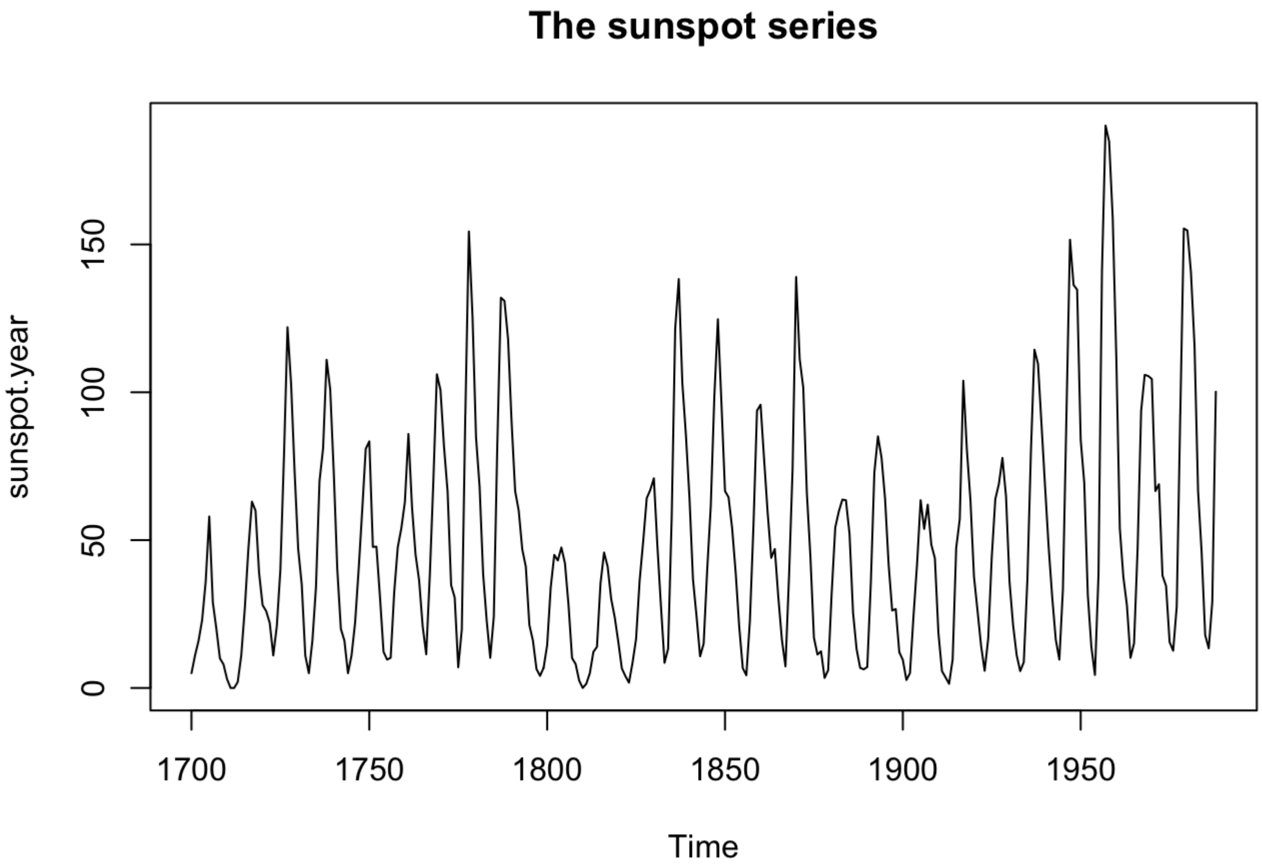
```
rho_h = ARMAacf(ar=c(1,-0.25),lag.max=5)
round(rho_h,3)
```

```
##     0     1     2     3     4     5
## 1.000 0.800 0.550 0.350 0.213 0.125
```

8. Consider the annual sunspots data in R given in data file sunspot.year (yearly numbers of sunspots from 1700 to 1988).

i. Plot the time series and describe the features of the data.

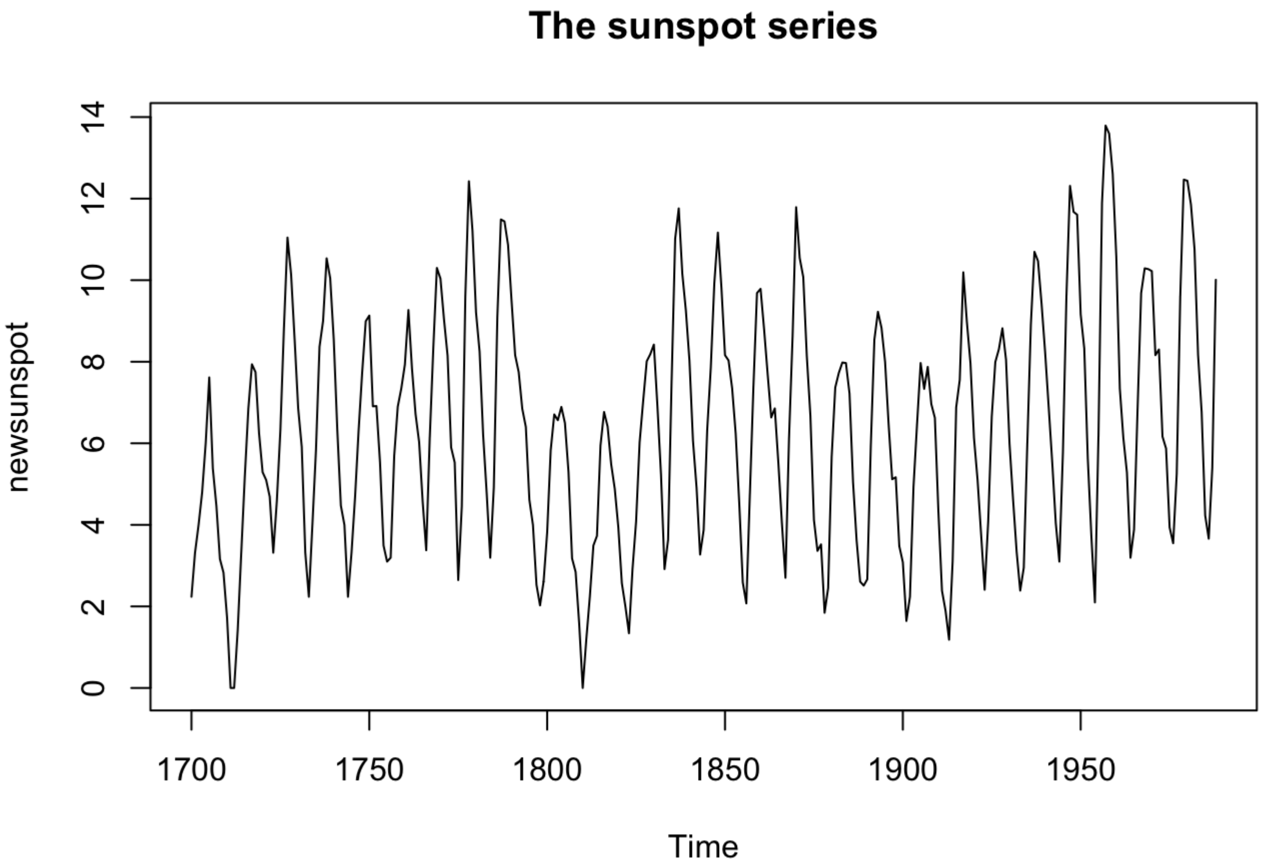
```
plot.ts(sunspot.year,main = "The sunspot series")
```




```
# the series always goes up first and then goes down, it has certain seansonal trend.
```

ii. Generate a new time series by transforming the data as newsunspot=sqrt(sunspot.year).Plot the new time series. Why is the square-root transformation necessary?

```
newsunspot = sqrt(sunspot.year)
plot.ts(newsunspot, main = "The sunspot series")
```



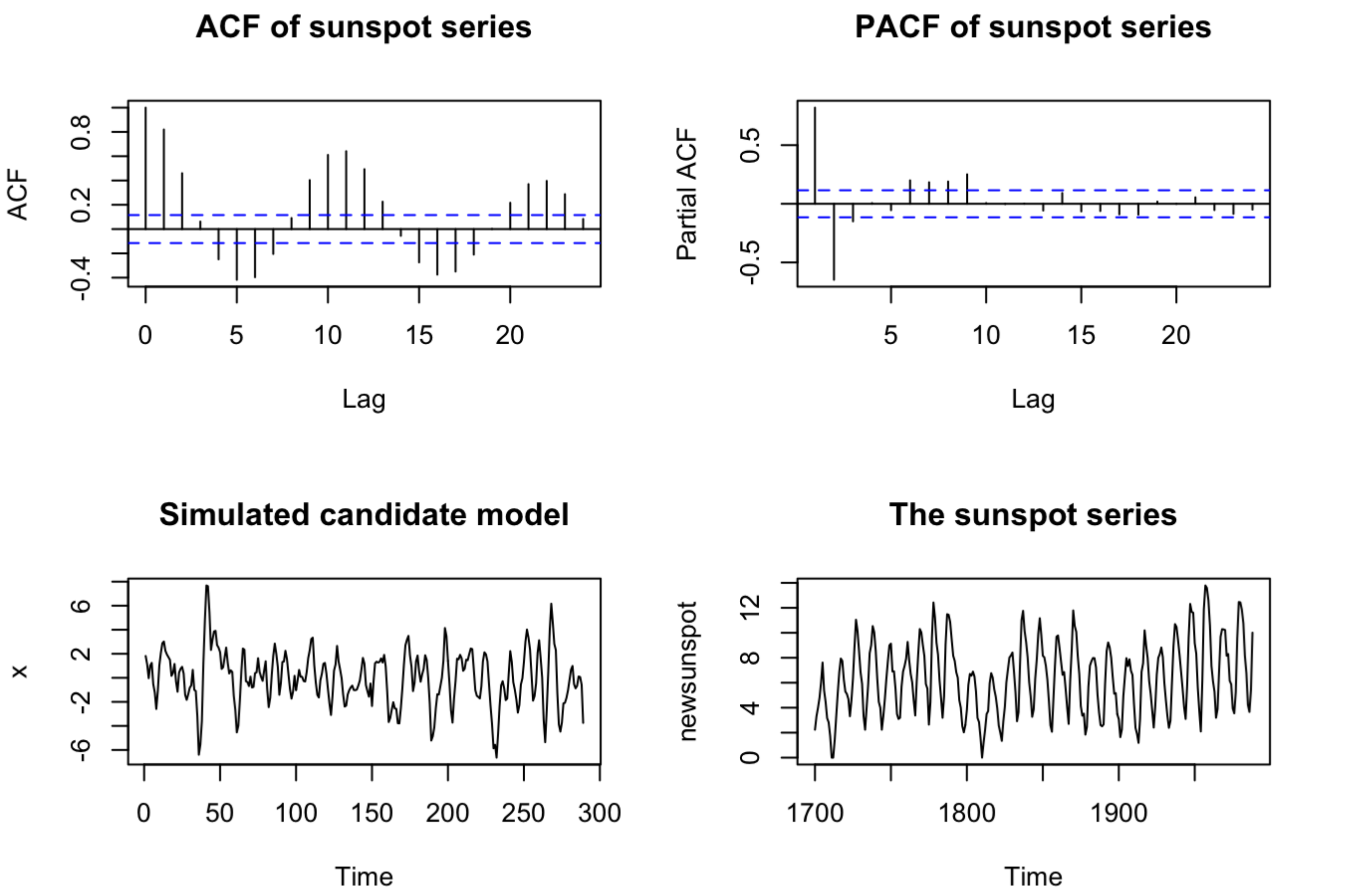

```
# We use transformations because we want to stablize the variance across time, and im
prove the signal.
```

ii. Plot ACF and PACF of the transformed data. Based on these plots,propose a plausible model and justify your answer

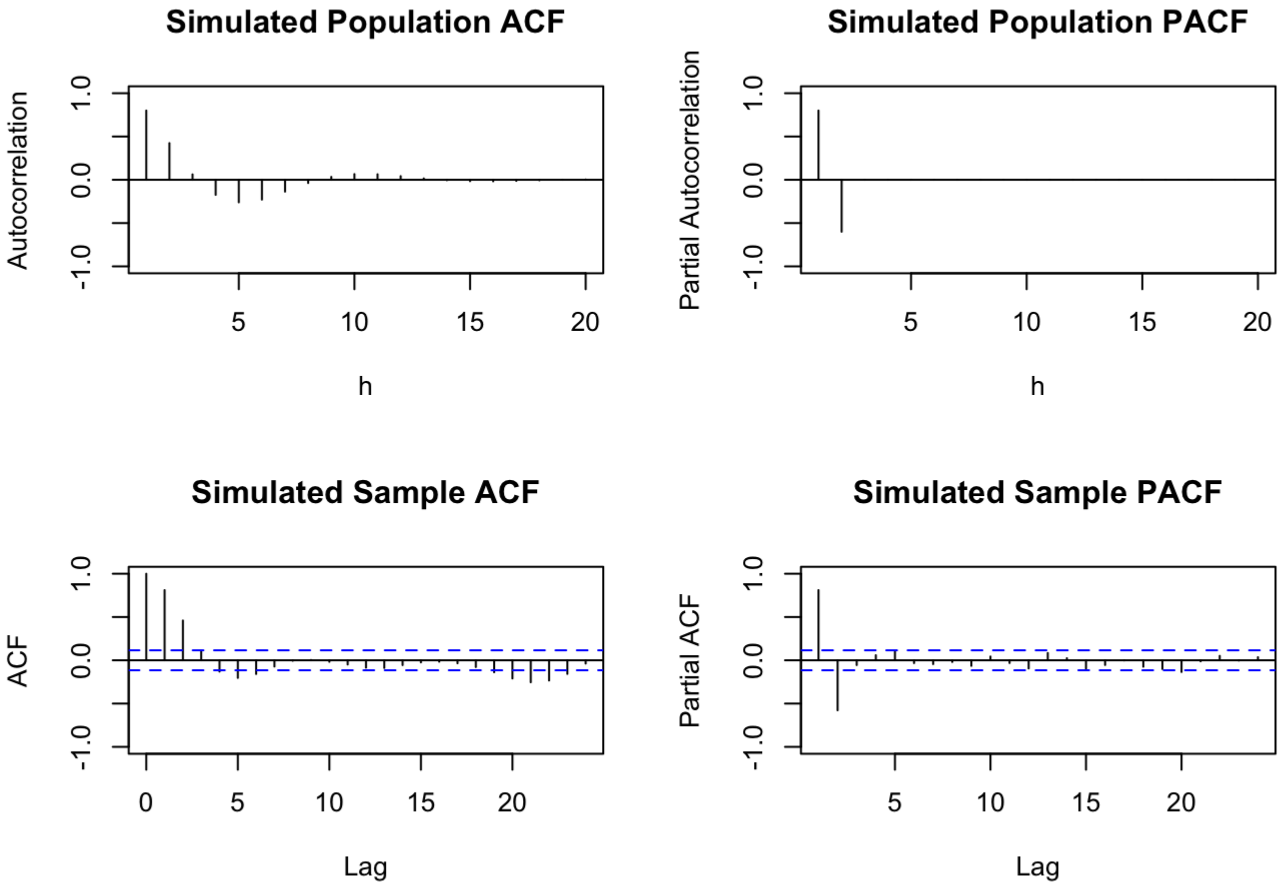
```
par(mfrow=c(2,2))
acf(newsunspot, main = "ACF of sunspot series")
pacf(newsunspot, main = "PACF of sunspot series")
# The ACF decays to 0, and PACF seems like cut-off after lag 2
# First aussuming this is an AR(2) model,caculate phi_1 and phi_2 based on phi_11 and
phi_22,
# We have: X(t)-1.28X(t-1)+0.6X(t-2)=e(t)

# The model is causal and invertible
polyroot(c(1,-1.28,0.6))
```

```
## [1] 1.066667+0.727247i 1.066667-0.727247i
```

```
# plot the ACF and PACF
x=arima.sim(n=length(newsunspot), list(ar=c(1.28,-0.6)))
plot.ts(x,main="Simulated candidate model")
plot.ts(newsunspot, main = "The sunspot series")
```




```
par(mfrow=c(2,2))
y = ARMAacf(ar=c(1.28,-0.6),lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Autocorrelation", main = "Simulated Population ACF")
abline(h = 0)
y = ARMAacf(ar=c(1.28,-0.6),lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "Simulated Population PACF")
abline(h = 0)
acf(x,main="Simulated Sample ACF", ylim = c(-1,1))
pacf(x,main="Simulated Sample PACF", ylim = c(-1,1))
# It captures some of the trend of sunpot series, but not looks very similar.
# Therefore it might be an AR(2,1) model based on the eacf.
TSA::eacf(newsunspot)
```




```
## AR/MA
##   0 1 2 3 4 5 6 7 8 9 10 11 12 13
## 0 x x o x x x x o x x x  x  x  o
## 1 x x o x x x x o x x x  x  x  o
## 2 x o o x x o x o o o o  x  o  o
## 3 x o o x o o x o o o o  x  o  o
## 4 x o o x o o o o o o o  x  o  o
## 5 x o x x x o o x o o o  x  o  o
## 6 x x x x x o o x o o o  x  o  o
## 7 x x o x x o o o o o o  o  o  o
```