Isobaric phase space distribution and partition function

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Assume two system coupled to a common thermal reservoir so that each system is described by a canonical distribution at temperature T. System 2 act as a barostat whose number of particles and volume is much larger than system 1, respectively.
If the volume of each system were fixed, the total canonical partition function would be

$$\begin{aligned} Q(N,V,T) &= C_N \int dx_1 dx_2 e^{-\beta \mathscr{H}_1(x_1)+\mathscr{H}_2(x_2)} \\ &= g(N,N_1,N_2) C_{N_1} \int dx_1 d e^{-\beta \mathscr{H}_1(x_1)} C_{N_2} \int dx_2 d e^{-\beta \mathscr{H}_2(x_2)} \end{aligned} \tag{1}$$

Where $g(N,N_1,N_2)$ is an overall normalization constant.
The canonical phase space distribution function $f(x)$ of combined system 1 and 2 is

$$f(x) = \frac{C_N e^{-\beta \mathscr{H}}}{Q(N,V,T)} \tag{2}$$

To determine the distribution function of system 1, we need to integrate over the phase space of system 2

$$\begin{aligned} f_1(x_1,V_1) &= \frac{g(N,N_1,N_2)}{Q(N,V,T)} C_{N_1} d e^{-\beta \mathscr{H}_1(x_1)} C_{N_2} \int dx_2 d e^{-\beta \mathscr{H}_2(x_2)} \\ &= \frac{Q_2(N-N_1,V-V_1,T)}{Q(N,V,T)} C_{N_1} d e^{-\beta \mathscr{H}_1(x_1)} \end{aligned} \tag{3}$$

Express the partition function in terms of Helmholtz free energies according to $Q(N,V,T) = e^{\beta A(N,V,T)}$,

$$\frac{Q_2(N_2,V-V_1,T)}{Q(N,V,T)} = e^{\beta[A(N-N_1,V-V_1,T)-A(N,V,T)]} \tag{4}$$

Expand $A(N-N_1,V-V_1,T)$ to first order

$$\begin{aligned} A(N-N_1,V-V_1,T) &\approx A(N,V,T) - N_1 \Big(\frac{\partial A}{\partial N}\Big)\Big|_{N_1=0,V_1=0} - V_1 \Big(\frac{\partial A}{\partial V}\Big)\Big|_{N_1=0,V_1=0} \\ &= A(N,V,T) - \mu N_1 + P V_1 \end{aligned} \tag{5}$$

So

$$f_1(x_1, V_1) = g(N, N_1, N_2)e^{\beta\mu N_1}e^{-\beta PV_1}e^{-\beta\mathscr{H}_1} \tag{6}$$

which means the distribution can of the system can be obtained. If we focus on the system and drop the extraneous "1" subscript, the equation can be rearranged as

$$e^{-\beta\mu N}\int_0^\infty dV\int dxf(x,V) = I_N\int_0^\infty dV\int dxe^{-\beta[\mathscr{H}+pV]} \tag{7}$$

which defines the partition function of the isothermal-isobaric ensemble as

$$\Delta(N, P, T) == I_N\int_0^\infty dV\int dxe^{-\beta[\mathscr{H}+pV]} \tag{8}$$