Legendre Transform

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If we specify the slope $s_0 = g(x_0) = f'(x_0)$ and the y-intercept $b(x_0)$ at x_0, then $f(x_0)$ can be uniquely determined

$$f(x_0) = f'(x_0)x_0 + b(x_0) \tag{1}$$

Since it is valid for all x_o, it can be written generally in terms of x as

$$f(x) = f'(x)x + b(x) \tag{2}$$

Recognizing $x = g^{-1}(s)$ and assuming $s = g(x)$ exits and is one-to-one-mapping, it's clear that the function $b(g^{-1}(s))$, given by

$$b(g^{-1}(s)) = f(g^{-1}(x)) - sg^{-1}(s) \tag{3}$$

contains the same information as the original $f(x)$ but expressed as a function of s instead of x.
We call the function $\tilde{f}(s) = b(g^{-1}(s))$ the **Lefendre transform** of $f(x)$.
$\tilde{f}(s)$ can be written compactly as

$$\tilde{f}(s) = f(x(s)) - sx(s) \tag{4}$$

And for a function of n valuables the Legendre transform of f is

$$\tilde{f}(s_1, ..., s_n) = f(x_1(s_1, ..., s_n), ..., x_n(s_1, ..., s_n)) - \sum_{i=1}^{n} s_i x_i(s_1, ..., s_n) \tag{5}$$