CCAPP Research Summary: Modeling of EoR Ionization Front

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Chenxiao Zeng

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2 Introduction

Please refer to the introduction section in [1].

3 Analytic Analysis

We are considering interaction between five species: electron (e^-), hydrogen ion (H^+), helium ion (He^+), neutral hydrogen (H), and neutral helium (He). Some properties are listed in Table.1.

In the process we are discussing, the interactions are sorted in five groups: electrons and ions, ions and ions, electrons and neutrals, ions and neutrals, and between neutrals and neutrals. We

Table 1: Useful properties of species.

Parameters	e^-	H^+	He^+	H	He
Mass (kg)	9.109×10^{-31}	1.673×10^{-27}	6.647×10^{-27}	1.673×10^{-27}	6.647×10^{-27}
Mass (MeV/c^2)	0.511	938.272	3727.911	938.896	3728.422
$\ln \Lambda$					

aim to generate a 5 matrix of energy transferring rate between species. For all species we adopt Maxwellian distributions $f(v)$ for calculations of speeds.

3.1 Basic Principles of Plasma Physics

The average kinetic energy of a particle in three dimensions is

$$E_{av} = \frac{3}{2}kT = 3 \cdot \frac{1}{2}mv^2 \tag{1}$$

Accordingly, its average thermal speed is

$$v = \left(\frac{kT}{m}\right)^{1/2} \tag{2}$$

In addition, quasi-neutrality demands that

$$n_i(\text{total ion species}) \simeq n_e \equiv n \tag{3}$$

The strategy I adopt is illustrated as following:

relative velocity distribution $\times$ energy loss rate for single collision $\rightarrow$ distributional energy loss

3.2 Electron $\Longleftrightarrow$ Ions

In this section we aim to find the momentum and energy loss rate when electrons collide ions. To do so, we need two components: the thermal distribution of electrons and the collision rate as a function of velocity.

In the stationary reference frame of ions, the electrons obey the velocity distributions of [2]:

$$f_e(\boldsymbol{v}) = n_e\Big(\frac{m_e}{2\pi kT_e}\Big)^{3/2} \exp\Big[-\frac{m_e(\boldsymbol{v}-\boldsymbol{v_d})^2}{2kT_e}\Big] \tag{4}$$

where $\boldsymbol{v_d}$ is the drift velocity of the electrons ensemble, and $\boldsymbol{v}$ is the velocity of electrons **relative** to that of ions. We do not include the thermal motion of the ions because they are moving slowly compared with electrons.

Consider a single electron collides with an ion, the momentum collision frequency $\nu_{ei,p}$ (p denotes momentum) is [3]:

$$\nu_{ei,p} = n_i \frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2} \frac{4\pi(m_e + m_i)}{m_i m_e^2 v^3} \ln \Lambda_e \tag{5}$$

where $\ln\Lambda$ is the Coulomb Logarithm , and according to [4],

$$\ln\Lambda_e = 22.1 + \ln\Big[\Big(\frac{E_e}{kT_e}\Big)\Big(\frac{T_e}{10^4\mathrm{K}}\Big)\Big(\frac{\mathrm{cm}^{-3}}{n_e}\Big)\Big] \tag{6}$$

By introducing the Coulomb Logarithm factor, we constrain the lower and upper cutoffs when integrating over all relevant impact parameters b.

Therefore, combining Eq.4, 5, and 6, we obtain the total momentum loss rate per unit volume, assuming that $\ln\Lambda$ is a constant instead of function of v. In addition, we assume that $|\boldsymbol{v}_d|$ is the velocity of the ionization front:

$$\begin{aligned}
-\frac{d\boldsymbol{p}_e}{dt} &= \int d^3v\ f_e(\boldsymbol{v})\nu_{ei,p}(v)m_e\boldsymbol{v} \\
&= \int d^3v\ m_e\boldsymbol{v}n_e\Big(\frac{m_e}{2\pi kT_e}\Big)^{3/2}\exp\Big[-\frac{m_e(\boldsymbol{v}-\boldsymbol{v_d})^2}{2kT_e}\Big]\cdot n_i\frac{q_e^2q_i^2}{(4\pi\epsilon_0)^2}\frac{4\pi(m_e+m_i)}{m_im_e^2v^3}\ln\Lambda_e \\
&= m_en_e\Big(\frac{1}{2\pi v_e^2}\Big)^{3/2}\cdot n_i\frac{q_e^2q_i^2}{(4\pi\epsilon_0)^2}\frac{4\pi(m_e+m_i)}{m_im_e^2}\ln\Lambda_e\int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e(\boldsymbol{v}-\boldsymbol{v_d})^2}{2kT_e}\Big]
\end{aligned} \tag{7}$$

where $v_e = (kT_e/m)^{1/2}$ is the average thermal speed at temperature T_e. In addition, we can define the the integral as

$$\begin{aligned}
I &\equiv \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e(\boldsymbol{v}-\boldsymbol{v}_d)^2}{2kT_e}\Big] \\
&= \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e}{2kT_e}(\boldsymbol{v}-\boldsymbol{v}_d)^2\Big] \\
&= \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e}{2kT_e}(\boldsymbol{v}\cdot\boldsymbol{v}-2\boldsymbol{v}\cdot\boldsymbol{v}_d+\boldsymbol{v}_d\cdot\boldsymbol{v}_d)\Big] \\
&= \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e}{2kT_e}(\boldsymbol{v}\cdot\boldsymbol{v})\Big]\exp\Big[\frac{m_e}{T_e}(\boldsymbol{v}\cdot\boldsymbol{v}_d)\Big]\exp\Big[-\frac{m_e}{2kT_e}(\boldsymbol{v}_d\cdot\boldsymbol{v}_d)\Big] \\
&= \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{m_e}{2m_ev_e^2}(\boldsymbol{v}\cdot\boldsymbol{v})\Big]\exp\Big[\frac{m_e}{m_ev_e^2}(\boldsymbol{v}\cdot\boldsymbol{v}_d)\Big]\exp\Big[-\frac{m_e}{2m_ev_e^2}(\boldsymbol{v}_d\cdot\boldsymbol{v}_d)\Big] \\
&= \int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{1}{2v_e^2}(\boldsymbol{v}\cdot\boldsymbol{v})\Big]\exp\Big[\frac{1}{v_e^2}(\boldsymbol{v}\cdot\boldsymbol{v}_d)\Big]\exp\Big[-\frac{1}{2v_e^2}(\boldsymbol{v}_d\cdot\boldsymbol{v}_d)\Big] \\
&= \exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int d^3v\ \frac{\boldsymbol{v}}{v^3}\exp\Big[-\frac{v^2}{2v_e^2}\Big]\exp\Big[\frac{1}{v_e^2}(\boldsymbol{v}\cdot\boldsymbol{v}_d)\Big]
\end{aligned} \tag{8}$$

For one component of the momentum vector $\boldsymbol{p}_e$, e.g. p_{ex}, we can align the corresponding axis to the drift velocity $\boldsymbol{v}_d$. Therefore, the integral I becomes

$$I_x = \exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int d^3v\ \frac{v_x}{v^3}\exp\Big[-\frac{v^2}{2v_e^2}\Big]\exp\Big[\frac{1}{v_e^2}(v_xv_d)\Big] \tag{9}$$

Assuming the Maxwellian distribution is isotropic, we can establish the relation such that $3v_x^2 = v^2$, or $v_x = v/\sqrt{3}$. Therefore,

$$\begin{aligned}
I_x &= \exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int_0^\infty d^3v\ \frac{v}{\sqrt{3}v^3}\exp\Big[-\frac{1}{2v_e^2}v^2\Big]\exp\Big[\frac{v_d}{\sqrt{3}v_e^2}v\Big] \\
&= \exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int_0^\infty d^3v\ \frac{1}{\sqrt{3}v^2}\exp\Big[-\frac{1}{2v_e^2}v^2\Big]\exp\Big[\frac{v_d}{\sqrt{3}v_e^2}v\Big] \\
&= \exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int_0^\infty dv\ 4\pi v^2\frac{1}{\sqrt{3}v^2}\exp\Big[-\frac{1}{2v_e^2}v^2\Big]\exp\Big[\frac{v_d}{\sqrt{3}v_e^2}v\Big] \\
&= \frac{4\pi}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\int_0^\infty dv\ \exp\Big[-\frac{1}{2v_e^2}v^2\Big]\exp\Big[\frac{v_d}{\sqrt{3}v_e^2}v\Big] \\
&= \frac{4\pi}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\exp\Big[\frac{v_d^2}{6v_e^2}\Big]\sqrt{\frac{\pi}{2}}v_e\Big(\mathrm{Erf}(\frac{v_d}{\sqrt{6}v_e})+\lim_{a\to\infty}\mathrm{Erf}(\frac{3a-\sqrt{3}v_d}{3\sqrt{2}v_e})\Big) \\
&= \frac{4\pi}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\exp\Big[\frac{v_d^2}{6v_e^2}\Big]\sqrt{\frac{\pi}{2}}v_e\Big(\mathrm{Erf}(\frac{v_d}{\sqrt{6}v_e})+1\Big)
\end{aligned} \tag{10}$$

Therefore, in x direction, the momentum loss rate per unit volume is

$$-\frac{dp_{ex}}{dt} = m_e v_e n_e \nu_{ei,p,\mathrm{maxwell}} \tag{11}$$

where $\nu_{ei,\mathrm{maxwell}}$ is the momentum collision frequency averaged over Maxwellian velocity distribution:

$$\begin{aligned}
\nu_{ei,p,\mathrm{maxwell}} \equiv& \Big(\frac{1}{2\pi v_e^2}\Big)^{3/2}\cdot n_i\frac{q_e^2q_i^2}{(4\pi\epsilon_0)^2}\frac{4\pi(m_e+m_i)}{m_i m_e^2}\ln\Lambda_e \\
&\cdot\frac{4\pi}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{2v_e^2}\Big]\exp\Big[\frac{v_d^2}{6v_e^2}\Big]\sqrt{\frac{\pi}{2}}v_e\Big(\mathrm{Erf}(\frac{v_d}{\sqrt{6}v_e})+1\Big) \\
=& \Big(\frac{1}{2\pi v_e^2}\Big)^{3/2}\cdot n_i\frac{q_e^2q_i^2}{(4\pi\epsilon_0)^2}\frac{4\pi(m_e+m_i)}{m_i m_e^2}\ln\Lambda_e \\
&\cdot\frac{2\pi\sqrt{\pi}}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{3v_e^2}\Big]\Big(\mathrm{Erf}(\frac{v_d}{\sqrt{6}v_e})+1\Big)
\end{aligned} \tag{12}$$

According to [3], the energy transfer rate per unit volume is proportional to that of momentum transfer and mass species mass fraction:

$$\nu_{ei,E,maxwell} = \frac{m_e+m_i}{2m_e}\nu_{ei,p,\mathrm{maxwell}} \simeq \frac{m_i}{2m_e}\nu_{ei,p,\mathrm{maxwell}} \tag{13}$$

Therefore, the energy loss rate is

$$\begin{aligned}
-\frac{dE_{ex}}{dt} &= \frac{1}{2}m_e v_e^2 n_e \nu_{ei,E,\mathrm{maxwell}} \\
&= \frac{1}{2}m_e v_e^2 n_e\cdot\frac{m_i}{2m_e} \\
&\quad\cdot\Big(\frac{1}{2\pi v_e^2}\Big)^{3/2}\cdot n_i\frac{q_e^2q_i^2}{(4\pi\epsilon_0)^2}\frac{4\pi(m_e+m_i)}{m_i m_e^2}\ln\Lambda_e \\
&\quad\cdot\frac{2\pi\sqrt{\pi}}{\sqrt{3}}\exp\Big[-\frac{v_d^2}{3v_e^2}\Big]\Big(\mathrm{Erf}(\frac{v_d}{\sqrt{6}v_e})+1\Big)
\end{aligned} \tag{14}$$

(Refer to Dosledge 2. try to directly calculate dE/dt, assuming two maxwellian distributions)

Above derivation requires updates, which is shown below.

Energy should be in the unit of Rydberg

$$
\begin{aligned}
\Delta t' &= 2.5 \times 10^{15}\text{cm}^{-2} \\
t &= t'/F \\
F &= v_i(n_H(1+f_e)) \\
\dot{E_{ei}} &= \frac{3}{2}k_B\ {}^K\bar{\nu}_{ei}(T_e - T_i) \rightarrow \text{energy rate gained by ion} \\
\text{where,}\quad {}^K\bar{\nu}_{ei} &= 2\frac{m_e}{m_i}{}^P\bar{\nu}_{ei} = \frac{2^{1/2}m_e^{1/2}n_eZe^4\ln\Lambda_{ei}}{6\pi^{3/2}\epsilon_0^2 m_i T_e^{3/2}}
\end{aligned}
\tag{15}
$$

Therefore, in the rescaled time, the energy rate that electron transfer to ions is

$$
\frac{dE_{ei}}{dt'} = \dot{E_{ei}}\frac{dt}{dt'} = \frac{\dot{E_{ei}}}{F} = \frac{3}{2F}k_B\frac{2^{1/2}m_e^{1/2}n_eZe^4\ln\Lambda_{ei}}{6\pi^{3/2}\epsilon_0^2 m_i T_e^{3/2}}(T_e - T_i) \tag{16}
$$

in the unit of [energy]/[cm^{-2}] which we can substitute back to Eqn.A8 in the paper.

In SI units,

$$
\begin{aligned}
k_B &= 1.38064852 \times 10^{-23} \\
m_e &= 9.10938356 \times 10^{-31} \\
Z &= 1 \text{ for HII} \\
e &= 1.602176634 \times 10^{-19} \\
\ln\Lambda_{ei} &\approx 20?\ \text{(subjected to change)} \\
\epsilon_0 &= 8.854187817 \times 10^{-12} \\
m_i &= 1.6726219 \times 10^{-27}
\end{aligned}
\tag{17}
$$

Therefore, the energy transfer rate in the rescaled time should be the function of n_e, T_e, and T_i. For each cell j, $n_{e,j} = n_H \cdot x_{e,j} = 1.9 \times 10^{-4} a_{-1}^{-3}\Delta \cdot x_{e,j}$. (How should we define Δ in this case, and can we approximate a_{-1}^{-3} to be a constant since the process happens rapidly? Yes from Chris.)

Taking $\Delta \approx 1$, and $a_{-1} = 10/(1+z) \approx 10/(1+9) = 1$ (subjected to change) Therefore,

$$
n_{e,j} = n_H \cdot x_{e,j} = 1.9 \times 10^{-4} x_{e,j}\ \text{cm}^{-3} = 1.9 \times 10^{2} x_{e,j}\ \text{m}^{-3}[1] \tag{18}
$$

On the other hand, $v_i = F/(n_H(1+f_e))$, $t' = Ft$. Therefore,

$$
\begin{aligned}
F &= v_i * (n_H(1+f_e)) \\
f_e &= 0.079
\end{aligned}
\tag{19}
$$

3.3 Ions $\Longleftrightarrow$ Ions

3.4 Electron $\Longleftrightarrow$ Neutrals

3.5 Ions $\Longleftrightarrow$ Neutrals

3.6 Neutrals $\Longleftrightarrow$ Neutrals

4 Numerical Calculation

References

[1] C. Hirata, MNRAS **474**, 2173 (2018)

[2] R. Fitzpatrick, *Plasma Physics: An Introduction*, CRC Press (2014)

[3] F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, Springer International Publishing (2016)

[4] B. Draine, *Physics of the Interstellar and Intergalactic Medium*, Princeton University Press (2011)

[5] M. McQuinn, Annu. Rev. Astron. Astrophys. **54**, 313 (2016)