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Coulomb Logarithm Formulae for Collisions between Species with Different Temperatures

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This brief note proposes the practically useful Coulomb logarithm formulae that can be applied to collisions between any particles in a plasma consisting of various species with different temperatures. The classical and quantum-mechanical formulae of the Coulomb logarithm are both derived and are seamlessly connected. Their wide and flexible applicability is suitable for implementation in numerical codes. The formulae proposed can recover the previously-known ones that are valid for specific cases.

The Coulomb logarithm appears in whatever collisional processes. Even for high-temperature fusion plasma, collisions still play an important role because they give rise to the neoclassical bootstrap current and resistivity of a plasma and they govern the slowing-down process of fast particles due to neutral beam injection (NBI) and fusion reaction. In considering these processes, the accurate calculation of the Coulomb logarithm is of significant importance. Nevertheless, a numerical constant of, say, 17[1] has been sometimes employed as an estimate of the Coulomb logarithm in wide ranges of density and temperature of about 10^{19}–10^{21} m^{-3} and 10^2–10^4 eV, respectively, because the dependence of the value on density and temperature is weak due to its logarithmic nature; otherwise rather simple formulae are typically exploited given in Refs. 1–3. Each of these formulae is valid solely for a specific type of collisions such as electron–electron collisions, and many of them assume that all particle species are in thermal equilibrium, i.e., they share common temperature. In general, however, this assumption does not hold. Especially when considering collisions between thermal particles and fast-beam particles, their mean temperatures are crucially different. Such simplified and specific formulae may not always be suitable from the practical point of view, because in general wide and flexible applicability is required for numerical codes. In this brief note, we therefore propose handy and comprehensive formulae of the Coulomb logarithm applicable to collisions between any species with different temperatures. The formulae will be beneficial not only to the fusion plasma research but also to the fundamental and applied plasma research.

Revisiting the derivation of the Coulomb logarithm based on Ref. 3, we then derive the formulae valid for collisions between any particles, regardless of their species and temperature. Deriving the Coulomb logarithm, we consider a two-body problem in classical mechanics to relate the deflection angle, θ, of a test particle to the impact parameter. Owing to the Debye shielding, the upper bound of the impact parameter, what is called the cutoff, can be set to the Debye length, λ_D. The minimum value, $r_\perp$, corresponding to the $\pi/2$ deflection, is defined as

$$r_\perp = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 \mu u^2}, \tag{1}$$

where the subscript, 1 or 2, identifies the colliding particle, μ is the reduced mass expressed as $\mu = m_1 m_2/(m_1 + m_2)$ and u is the relative velocity of the colliding particles before the collision. Note that the charge state Z is defined as an absolute value. Other variables follow the conventional notation. We, therefore, define the classical Coulomb logarithm as[1,3]

$$L^{\rm cl} = \ln\left(\frac{\lambda_D}{r_\perp}\right), \tag{2}$$

representing the cumulative effects of all Coulomb collisions within a Debye sphere of a field particle, for impact parameters ranging from $r_\perp$ to λ_D. Details of the two-body problem and the derivation of Eq. (2) are described in Ref. 3.

When considering a plasma consisting of electrons and multiple ion species with different temperatures, we have to express the Debye length as

$$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \sum_j \frac{1}{\lambda_{Dj}^2},$$

where j denotes the ion species. Substituting the definition of the species-dependent Debye length into the above yields

$$\lambda_D = \left[\frac{\epsilon_0}{e^2}\left(\frac{n_e}{T_e} + \sum_j \frac{Z_j^2 n_j}{T_j}\right)^{-1}\right]^{1/2} \equiv \left(\frac{\epsilon_0}{e^2}\frac{T_*}{n_*}\right)^{1/2}, \tag{3}$$

where the definition n_*/T_* has been introduced for the sake of convenience. Often the ion terms are not included in the definition of the Debye length, but it can be justified only for a plasma with the ion temperature much higher than the electron temperature.

The relative velocity u is replaced by the equivalent mean temperature according to the relation $mu^2/2 = 3T/2$, which is appropriate for a thermal species in a Maxwellian distribution, to obtain

$$u^2 = 3\,\frac{m_2 T_1 + m_1 T_2}{m_1 m_2}. \tag{4}$$

We finally have

$$\mu u^2 = 3\,\frac{m_2 T_1 + m_1 T_2}{m_1 + m_2} = 3\,\frac{A_2 T_1 + A_1 T_2}{A_1 + A_2}, \tag{5}$$

where A is the mass number with respect to the proton mass.

Substituting Eqs. (1) and (3) into Eq. (2) gives

$$\begin{aligned} L^{\rm cl} &= \ln\left[12\pi\left(\frac{\epsilon_0}{e}\right)^{3/2}\right] - \ln\left[\frac{Z_1 Z_2 (A_1 + A_2)}{A_2 T_1 + A_1 T_2}\left(\frac{n_*}{T_*}\right)^{1/2}\right] \\ &= 30.37 - \ln(Z_1 Z_2) - \ln\left[\frac{A_1 + A_2}{A_2 T_1 + A_1 T_2}\left(\frac{n_*}{T_*}\right)^{1/2}\right], \end{aligned} \tag{6}$$

which is the general form of the classical Coulomb logarithm. Here and hereafter numerical constants are computed using density in m^{-3} and temperature in eV.

Sometimes the ions including impurities are assumed to be in thermal equilibrium among themselves due to their similar mass and Z effect in the scattering angle.[3)] For this case, we regard that the ion temperatures are identical to one another and thus we can simply reduce n_*/T_* to $n_*/T_* = n_\mathrm{e}(T_\mathrm{e}^{-1} + Z_\mathrm{eff}/T_\mathrm{i})$, where T_i is the common ion temperature and Z_eff, the effective charge number defined by $Z_\mathrm{eff} \equiv \sum_j Z_j^2 n_j/n_\mathrm{e}$.

The above classical-mechanical analysis can apply if $r_\perp$ is longer than $\lambda_\mathrm{qm}/2\pi$, where λ_qm is the de Broglie wavelength given by $\lambda_\mathrm{qm} = h/(\mu u)$. Here, h denotes the Planck constant. For this case, the classical Coulomb logarithm given in Eq. (6) is valid. Conversely, when the relative velocity increases and thus $r_\perp$ decreases so that $r_\perp$ is comparable to or shorter than $\lambda_\mathrm{qm}/2\pi$, the quantum-mechanical effect has to be taken into account. For deuterium plasmas, the threshold temperatures above which the quantum-mechanical effect is significant are 6.65 eV for electron–electron collisions, 13.3 eV for electron–ion collisions, and 24.5 keV for ion–ion collisions,[3)] respectively: These threshold values will be examined later. In general, the quantum-mechanical Coulomb logarithm formula should be exploited for collisions involving electrons in fusion-oriented plasmas. However, if the fast ions typically having the equivalent temperature above this threshold take part in collisions against thermal ions, the quantum-mechanical correction should be required.

Using the fine-structure constant defined by $\alpha = e^2/(2\epsilon_0 hc) \simeq 1/137$, the criterion for the quantum-mechanical effect being significant is rewritten as

$$\frac{2\pi}{\lambda_\mathrm{qm}} r_\perp = Z_1 Z_2 \frac{\alpha c}{u} < 1.$$

When $u \gg \alpha c$, the quantum-mechanical scattering problem can be solved relatively easily by virtue of the Born approximation. The quantum-mechanical effect slightly modifies the differential cross section of Coulomb collisions.[3)] The quantum-mechanical Coulomb logarithm can be derived in a manner similar to the derivation of the classical Coulomb logarithm to obtain

$$\begin{aligned} L^\mathrm{qu} &= \frac{1}{4}\int_0^\pi \frac{\sin^2(\theta/2)\sin\theta}{[\sin^2(\theta/2)+\epsilon^2]^2}\,d\theta \\ &= \frac{1}{2}\ln\left(\frac{1+\epsilon^2}{\epsilon^2}\right) - \frac{1}{2(1+\epsilon^2)} \approx \ln\left(\frac{1}{\epsilon}\right) - \frac{1}{2}, \end{aligned} \tag{7}$$

where the definition $\epsilon = \lambda_\mathrm{qm}/(4\pi\lambda_\mathrm{D})$ has been introduced and $\epsilon \ll 1$ has been used in the final equality. This expression differs from the classical one in that ϵ is included and the lower limit of the integral is set to zero, the latter of which is because the divergence of the definite integral is avoidable due to the finite ϵ. Further details regarding the derivation of Eq. (7) are described in Ref. 3. Using the relationship

$$\ln\left(\frac{1}{\epsilon}\right) = \ln\left(\frac{\lambda_\mathrm{D}}{r_\perp}\frac{4\pi r_\perp}{\lambda_\mathrm{qm}}\right) = L^\mathrm{cl} + \ln\left(Z_1 Z_2 \frac{2\alpha c}{u}\right),$$

we find

$$L^\mathrm{qu} = L^\mathrm{cl} + \ln\left(Z_1 Z_2 \frac{2\alpha c}{u}\right) - \frac{1}{2}, \tag{8}$$

indicating that the quantum-mechanical Coulomb logarithm is related to the classical Coulomb logarithm via a correction term. The correction terms can be reduced to $5.053 + \ln(Z_1 Z_2) + 0.5\ln[A_1A_2/(A_2T_1 + A_1T_2)]$. We finally obtain the quantum-mechanical Coulomb logarithm in the form

$$\begin{aligned} L^\mathrm{qu} &= 35.42 - \ln\left[\frac{A_1+A_2}{A_2T_1+A_1T_2}\left(\frac{n_*}{T_*}\right)^{1/2}\right] \\ &\quad + \frac{1}{2}\ln\left(\frac{A_1A_2}{A_2T_1+A_1T_2}\right), \end{aligned} \tag{9}$$

in which the dependence on the charge number vanishes in contrast to the classical one. It is evident that the classical and quantum-mechanical expressions show the same dependence on density, but the different dependence on temperature due to the quantum correction term.

We subsequently have to evaluate the threshold temperature above which the quantum-mechanical Coulomb logarithm should be used. From Eq. (8), the threshold value of the relative velocity, u_thr, is determined from the requirement

$$\ln\left(Z_1 Z_2 \frac{2\alpha c}{u_\mathrm{thr}}\right) = \frac{1}{2}. \tag{10}$$

When $u \gg u_\mathrm{thr}$, the quantum-mechanical formula is used. Rewriting Eq. (10) with respect to the temperature yields

$$\begin{aligned} \frac{A_2T_1^\mathrm{thr} + A_1T_2^\mathrm{thr}}{A_1+A_2} &= \frac{4\alpha^2c^2m_\mathrm{p}}{3ee}(Z_1Z_2)^2\frac{A_1A_2}{A_1+A_2} \\ &= 2.451\times10^4(Z_1Z_2)^2\frac{A_1A_2}{A_1+A_2}, \end{aligned} \tag{11}$$

where m_p denotes the proton mass and e, Napier's constant. As understood in Eq. (5), the term on the left-hand side is the definition of a mean temperature; hence, this equation enables us to evaluate the threshold temperature.

We show in the following typical threshold values, assuming $T_1 = T_2$ and deuterium plasmas. For electron–electron collisions, given $A_1 = A_2 \simeq 1/1836$ and $Z_1 = Z_2 = 1$, we have $T_\mathrm{ee}^\mathrm{thr} \approx 6.67$ eV. For electron–ion collisions, given $A_1 \simeq 1/1836 \ll A_2 = 2$ and $Z_1 = Z_2 = 1$, we have $T_\mathrm{ei}^\mathrm{thr} \approx 13.3$ eV. For ion–ion collisions, given $A_1 = A_2 = 2$ and $Z_1 = Z_2 = 1$, we have $T_\mathrm{ii}^\mathrm{thr} \approx 24.5$ keV. These threshold values are consistent with those previously known.

The classical and quantum-mechanical formulae are valid in the region of $u < \alpha c$ and in the limit of $u \gg \alpha c$, respectively. The quantum-mechanical calculation becomes extremely complicated in the intermediate region, i.e., $u \approx \alpha c$:[3)] The underlying binary-collision theory and the assumptions that have been used to derive the formulae would be violated in this region. Fortunately, because it is physically obvious that the slow variation of the Coulomb logarithm with u also obtains even in this region,[3)] we can extrapolate the classical and quantum-mechanical formulae into the intermediate region up to the threshold value, at which both expressions coincide.

Even though Eqs. (6) and (9) are sufficient as the Coulomb logarithm formulae for any collisions, it is still meaningful to reduce them to the simplified ones valid for electron–electron, electron–ion, and ion–ion collisions, respectively, for comparison with those given in Refs. 1 and 2.

For this specific purpose, a plasma consisting of electrons and single hydrogenic ion species with identical temperatures is assumed, i.e., $n_e = n_i$ and $n_*/T_* = 2n_e/T_e$.

For electron–electron and electron–ion collisions we can simplify them as

$$L_{ee}^{cl} = L_{ei}^{cl} = 30.0 - \ln(n_e^{1/2} T_e^{-3/2}),$$
$$L_{ee}^{qu} = 30.9 - \ln(n_e^{1/2} T_e^{-1}),$$
$$L_{ei}^{qu} = 31.3 - \ln(n_e^{1/2} T_e^{-1}).$$

L_{ee}^{qu} is nearly identical to that given in Refs. 1, 3 and 4. Due to the disparate mass difference between electrons and ions, in a typical fusion plasma terms related to T_i vanish in the formula without requiring the thermal equilibrium assumption. It is found that L_{ee}^{qu} differs by a factor of $\ln(2^{-1/2})$ from L_{ei}^{qu}. Note that L_{ei}^{qu} is identical to that given in Refs. 1 and 3, whereas it is slightly different from the NRL formula.[2)] Recalling the assumption of $T_e = T_i$, we can replace T_i by T_e in the following. For ion–ion collisions, the Coulomb logarithms are reduced to

$$L_{ii}^{cl} = 30.0 - \ln(n_e^{1/2} T_i^{-3/2}),$$
$$L_{ii}^{qu} = 35.1 - \ln(n_e^{1/2} T_i^{-1}).$$

L_{ii}^{qu} is identical to that given in Ref. 3. For ion–ion collisions, Refs. 1 and 2 solely show the classical formulae. The numerical coefficient given in Ref. 1 is identical to that shown here.

When applying Eq. (6) to classical ion–ion collisions, we find that it is different from the NRL mixed ion–ion formula[2)] in two ways despite their overall similarity. One is due to the definition of n_*/T_*: Eq. (3) includes the electron contribution, whereas the NRL formula does not. We believe that electrons must always contribute to the Debye shielding in a quasi-neutral plasma, even for ion–ion collisions. The other is the difference in the numerical coefficient: 29.9 in the NRL formula, while 30.4 in Eq. (6). The difference cannot be explained by $\ln(2^{-1/2})$ because this factor arises from the thermal equilibrium assumption.

It is worth mentioning collisions between thermal and fast particles. In tokamak experiments, there exist fast ions due typically to NBI. Often they form a large part of the total plasma energy. Even though the distribution of beam ions is not Maxwellian, their equivalent temperature, T_b, may be roughly defined by $T_b = (2/3)E_b$ as one choice. Here, E_b denotes the injection energy of NBI. Once T_b is defined, the classical and quantum-mechanical formulae given in Eqs. (6) and (9) can be used to calculate the Coulomb logarithm even for collisions between thermal and fast particles.

The beam ions may take part in the estimate of the Debye length, but, as seen in Eq. (3), particles with lower temperature (i.e., slow) have more influence on the Debye length compared to those with high temperature (i.e., fast). A small fraction of beam-ion density, $n_i > n_b$, also dilutes their contribution. The modification to the Debye length stemming from beam ions is thus negligible.[5)]

The large mass discrepancy forces us to always use the quantum-mechanical formula for electron-beam ion collisions. L_{eb}^{qu} may be identical to L_{ei}^{qu} due to $A_b \simeq A_i \gg A_e$. For thermal ion-beam ion collisions, due to $T_b \gg T_i$, we obtain the threshold energy given by

$$E_b^{thr} \simeq 3.68 \times 10^4 (Z_i Z_b)^2 A_b,$$

applying Eq. (11). Substituting $A_b = 2$ and $Z_i = Z_b = 1$ yields $E_b \simeq 73.6\,\mathrm{keV}$, which is comparable to or higher than that of typical NBI. To our knowledge, only one paper has made specific mention of the Coulomb logarithm for collisions involving beam ions.[6)] Equations (6) and (9) can be reduced to L_{ib}^{cl}, L_{ib}^{qu} and E_b^{thr} shown in Ref. 6 when assuming $E_b = T_b \gg T_i = T_e$, apart from the slight difference in numerical constants. Given that their equations are specific to the beam-thermal interaction and require the aforementioned assumptions, our formulae are much more useful in the sense that can be applied to any collisions.

In summary, the Coulomb logarithm formulae for collisions between particles with different temperatures are derived. The classical and quantum-mechanical formulae are given in Eqs. (6) and (9) and are connected at the threshold value given in Eq. (11). Given the mean temperature of fast ions, they are applicable to collisions involving fast ions. From the numerical point of view, it is straightforward to implement Eqs. (6) and (9) in numerical codes and their calculation is not costly compared to the calculation of the simplified formulae.

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