Cosmology 2018 - Tutorial 7

June 5, 2018

1 Comments on stiff equations

In the next exercise (the fourth) you will solve BBN (Big Bang Nucleosynthesis) numerically (under some simplifying assumptions). The equations of BBN are designated as "stiff" (which is hard to precisely define). Essentially, given a system of equations and a solution method ("non-stiff" one), the equations are stiff if in order to solve them one has to make step sizes unacceptably small[1].

The cure to non-stiffness lies in changing the solution method. Given a system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \;, \tag{1}$$

the simplest solution one can write

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_n) \;, \tag{2}$$

where h is a step size. This is basically the traditional Euler method which will probably not work for a stiff equation. A smarter thing we may do is an "implicit method"

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_{n+1}) \;, \tag{3}$$

which means that every step becomes much more difficult, because solving this algebraic equation is not generally easy. Expanding can help a bit,

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\mathbf{f}(\mathbf{y}_n) + h\left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{y}_n} \cdot (\mathbf{y}_{n+1} - \mathbf{y}_n)\right] \;, \tag{4}$$

thus, rearranging

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[1 - h\left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{y}_n}\right]^{-1} \cdot \mathbf{f}(\mathbf{y}_n) \;, \tag{5}$$

so we only have to invert $[1 - h\partial\mathbf{f}/\partial\mathbf{y}]$ in each iteration, which is not terribly difficult with Newton's method when h is small enough. Under this approximation the method is "semi-implicit".

In the exercise you are not expected to implement a non-stiff method from scratch (but you can, if you want). MATLAB offers prebuilt methods to deal with these equations. Also, Mathematica's NDSolve is quite powerful in dealing with stiff equations without external user interference.

[1] For example, one can show that for $\mathbf{y}' = -\mathbf{C}\cdot\mathbf{y}$, with $\mathbf{C}$ being a positive-definite matrix having maximal eigenvalue λ_{max}, the step size must be $h < 2/\lambda_{max}$.

More details on stiff equations can be found in Press, Teukolsky, Vetterling & Flannery, *Numerical recipes in C - the art of scientific computing* (1997) (not only for C programming), Mathworks resources and many more resources.

Note added: physically, there is a reason why BBN equations are stiff. When there are very fast reaction rates, perhaps close to equilibrium, the smallest time scales must be resolved when one uses an explicit solution method. This could be unreasonable. The implicit methods bypass this problem, working out the equilibrium abundances.

2 Axions

There are quite a few reasons to think there is a substantial amount of dark matter in the universe. One way that we saw it so far in the course is by fitting supernovae data to a $\Omega_m \sim 0.3$ and $\Omega_\Lambda \sim 0.7$ today, and finding from BBN $\Omega_b \sim \Omega_m/6$ (you will check that in the next exercise), implying that about 26% of the energy density of the present universe is in an unknown and unseen form of cold matter.

In the lectures the dark matter class of WIMPs (Weakly Interacting Massive Particles) was covered. In this class one may consider some fermions which are coupled to the standard model via $X + \bar{X} \leftrightarrow p + \bar{p}$ where X is the DM and p is the particle. Early enough they decouple and in order to get the relic abundance right, a constraint on their interaction is set.

A second popular class of particle dark matter[2] is called axions. Axions usually refer to light neutral spinless particles that appear due to a broken symmetry. For our present purposes they can be effectively described by scalar field dynamics. Consider the action for a minimally coupled scalar field in GR,

$$S[\phi] = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) . \tag{6}$$

If we tried to couple classical electromagnetism, for example, we would replace the term in parentheses by $(-1/4) F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta}$.

Varying the action with respect to ϕ yields a generalization of the Klein-Gordon equation

$$\Box \phi - \frac{\partial V}{\partial \phi} = 0 \quad , \quad \text{where } \Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) . \tag{7}$$

For a flat FRW metric it becomes, assuming homogeneousness and $V = \frac{1}{2} m^2 \phi^2$,[3]

$$\ddot{\phi} + 3H\dot{\phi} + m^2 \phi = 0 . \tag{8}$$

The energy momentum tensor can be found by applying Noether's theorem,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right) . \tag{9}$$

In the limit that the field behaves like a fluid in a flat FRW universe, we have then the density and the pressure

$$\rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \quad , \quad p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 . \tag{10}$$

[2]There are ideas of non-particle dark matter, e.g. primordial black holes.

[3]In popular models the mass is a time-dependent variable. We will treat it as constant here.

The ratio $w \equiv p/\rho$ tells us how this field behaves. Recall that $w = 0$ corresponds cold matter, $w = -1$ corresponds a cosmological constant and $w = 1/3$ corresponds radiation. We solve the equation of motion with the initial conditions

$$\phi(t_i) = \phi_0 \ \ , \ \ \dot{\phi}(t_i) = 0 \ . \tag{11}$$

This is essentially "mis-alignment". When the whatever-underlying-mechanism produces the field, around the time when $H \gg m$, fixing $\dot{\phi}$ initially to zero (assumed due to the Hubble friction term), and giving some non-zero value to the field (contrasted to 0, the minimum of the potential). The process in which the field relaxes to the potential minimum is called "vacuum-realignment". The mis-alignment mechanism is non-thermal, unlike the standard WIMP picture for example.

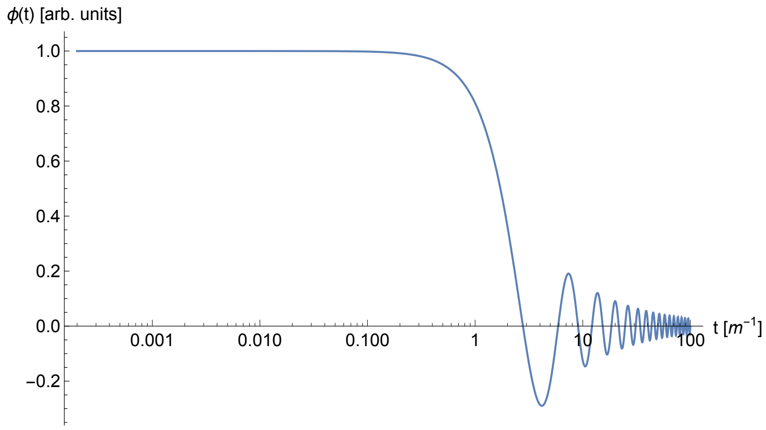

Figure 1: The solution for $\phi(t)$ when $\phi(t_i) = 1$ and t is measured in m^{-1} units (or simply set $m = 1$).

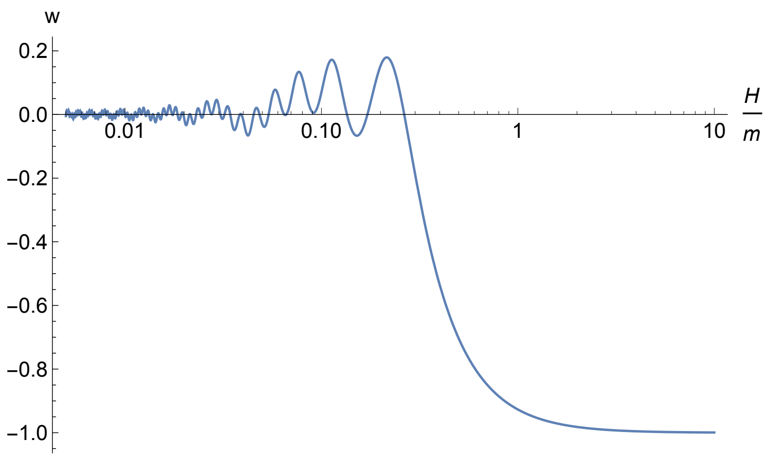

Figure 2: The solution for $w = p/\rho$ with the specified initial condition in a radiation-dominated universe averaged over a Hubble time ($\sim 1/H$) vs. H/m, meaning direction of time is to the left.

Denote t_1 as approximately the time in which the field starts oscillating, which is given by $H(t_1) \sim m$, when we can apply the WKB approximation which gives

$$\phi(t) \approx \phi_1 \left(\frac{a(t_1)}{a(t)} \right)^{3/2} \cos\left(mt + \alpha\right) \ , \tag{12}$$

where $\phi_1 = \phi(t_1) \sim \phi_0 \sim M$, where M is some characteristic scale of the underlying mechanism the produces the field. In the regime $m \gg H$ we can safely average $\cos^2 mt$ and leave the scale factor intact, yielding

$$\rho(t) \approx \frac{1}{2} m^2 \phi_1^2 \left(\frac{a(t_1)}{a(t)} \right)^3 , \tag{13}$$

which has the familiar a^{-3} scaling of non-relativistic matter. We wish to relate this Ω_{DM}. As usual, we follow the radiation as it dominated when the axions became dark matter, and they tell us the scale factors:

$$T_{\gamma,0} = (g/2)^{1/3} T(t_1) a(t_1)/a(t_0) , \tag{14}$$

and

$$m \approx H(t_1) = 1.66 g^{1/2} G^{1/2} T_1^2 , \tag{15}$$

yielding today (at t_0)

$$\rho(t_0) \approx m^{1/2} g^{-1/4} \phi_0^2 \left(\frac{4\pi^3 G}{45} \right)^{3/4} T_{\gamma,0}^3 . \tag{16}$$

Setting for the QCD axion[4] $mM \sim F_\pi m_\pi = 184 \times 139 \, MeV^2$ and ignoring factors of order unity

$$\rho \approx \frac{F_\pi^2 m_\pi^2 G^{3/4} T_{\gamma,0}^3}{m^{3/2}} , \tag{17}$$

yielding

$$\Omega_{DM} h^2 \approx \left(\frac{m}{10^{-5} eV} \right)^{-3/2} , \tag{18}$$

corresponding to $M \approx 10^{12} \, GeV$.

Another possibility is the ultra-light axion for which

$$\Omega_{DM} \sim 0.1 \left(\frac{\phi_0}{10^{17} \, GeV} \right)^2 \left(\frac{m}{10^{-22} \, eV} \right)^{1/2} , \tag{19}$$

where the initial value for ϕ is very close to the Planck scale, $M_{Pl} = 1/\sqrt{8\pi G} = 2.4 \times 10^{18} \, GeV$ (with $\hbar = c = 1$).

Note that in this analysis I have not made justice with the vast literature of axions. Among the many omitted things: for the QCD axion I have not told you what's the particle physics motivation (strong CP problem), the particle physics mechanism, the many constraints on M from astrophysics and interplay with inflation (which we have not studied yet). Similarly I have not elaborated on the ultra-light axions which are very popular nowadays. I have largely followed Weinberg's Cosmology (2008) §3.4B and arXiv:1610.08297.

[4] The correct expression is $m = \frac{F_\pi m_\pi}{M} \frac{\sqrt{m_d m_u}}{m_d + m_u}$, where F_π is the pion decay amplitude, m_π is the mass of the pion, M is the energy scale at which the Peccei-Quinn symmetry is broken and m_u and m_d are the up and down quarks which give an $\mathcal{O}(1)$ factor. This expression is true for temperatures well below the QCD scale, $\Lambda_{QCD} \sim 150 \, MeV \sim 10^{12} \, K$.