Chapter 6
Nonlinear Systems of ODEs

So far in our development of numerical methods and analysis tools, we have been working with the scalar ODE $u_t = f(u,t)$. In this chapter, we will extend to systems of ODEs where the state $\mathbf{u}$ is now a d-dimensional vector.

22 Self-Assessment

Before reading this chapter, you may wish to review...

- Linear systems and matrices [18.03 Lecture Notes]
- Eigenvalues and vectors [18.03 Lecture Notes]
- Complex and repeated eigenvalues [18.03 Lecture Notes]
- Linear phase portraits [Mathlet]
- Matrix-vector products [Mathlet]

After reading this chapter you should be able to...

- identify nonlinear systems of ODEs by distinguishing them from linear systems of ODEs
- generate a system of ODEs from higher-order scalar ODEs by reduction of order (aka, variation of parameters)
- linearize a nonlinear system of ODEs about a given state
- calculate the Jacobian matrix for a nonlinear system of ODEs

23 Nonlinear Systems

Until this point we have studied first-order scalar ODEs of the form $u_t = f(u,t)$ where $u_t = du/dt$ is the time-derivative. In this unit we will extend this concept to systems of ODEs $\mathbf{u}_t = \mathbf{f}(\mathbf{u},t)$ where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{u},t) = \begin{bmatrix} f_1(\mathbf{u},t) \\ f_2(\mathbf{u},t) \\ \vdots \\ f_d(\mathbf{u},t) \end{bmatrix} \tag{38}$$

are now vectors with d components. We denote by $\mathbf{u}_t$ the componentwise time-derivative; that is, $\mathbf{u}_t = \mathbf{f}(\mathbf{u},t)$ can be written out explicitly

$$\mathbf{u}_t = \begin{bmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_d/dt \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{u},t) \\ f_2(\mathbf{u},t) \\ \vdots \\ f_d(\mathbf{u},t) \end{bmatrix} = \mathbf{f}(\mathbf{u},t). \tag{39}$$

Systems of ODEs can arise in several ways. Later in the class, these systems will come from the spatial discretization of partial differential equations. However, they can also arise from reduction of order (aka variation of parameters) of a higher-order ODE as we demonstrate for a nonlinear pendulum in Example 4.1.

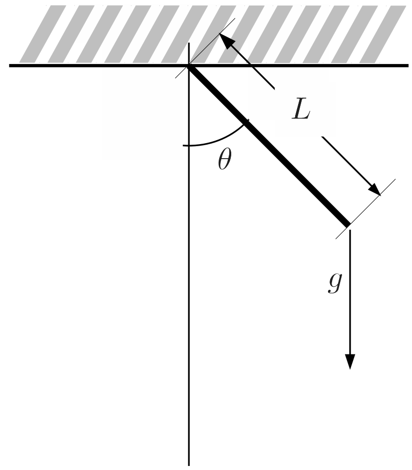

Fig. 4 Nonlinear pendulum.

Example 4.1 *Nonlinear Pendulum*
A nonlinear pendulum is an example of a second-order oscillator. Let $\theta(t)$ denote the angle of the pendulum with respect to the vertical line through its hinge point (see Figure 4) as a function of time t. The nonlinear dynamics of an ideal pendulum (no damping) of length L released from rest at $\theta = \pi/3$ are given by the second-order ODE

$$\theta_{tt} + \frac{g}{L}\sin\theta = 0, \qquad \theta(0) = \frac{\pi}{3},\ \theta_t(0) = 0. \tag{40}$$

We will now rewrite this second-order scalar equation as a first-order *system* of equations. Let

$$\omega(t) = \theta_t \tag{41}$$

be the angular rate of the pendulum. Then, (40) becomes

$$\omega_t + \frac{g}{L}\sin\theta = 0. \tag{42}$$

We now have two first-order ODEs, (41) and (42).

Exercise 1. Write (41) and (42) as a system of the form $\mathbf{u}_t = \mathbf{f}(\mathbf{u},t)$ by defining u_1, u_2, f_1, and f_2. Which of the following is the system you found?

(a) $u_1 = \theta, u_2 = \omega,\ f_1 = \omega,\ f_2 = \frac{-g}{L}\theta$
(b) $u_1 = \theta, u_2 = \omega,\ f_1 = \theta,\ f_2 = \frac{-g}{L}\theta$
(c) $u_1 = \theta, u_2 = \omega,\ f_1 = \omega,\ f_2 = \frac{g}{L}\sin(\theta)$
(d) $u_1 = \theta, u_2 = \omega,\ f_1 = \omega,\ f_2 = \frac{-g}{L}\sin(\theta)$

We will revisit this model of a nonlinear pendulum in the class session.

We have shown that a second-order scalar ODE can be transformed into a first-order system of ODEs. The nonlinear pendulum system as well as many other systems are nonlinear systems. When performing analysis we will often linearize these systems.

24 Linearization of Nonlinear Systems

It is often challenging to analyze nonlinear systems. On the other hand, there are many analysis tools that are applicable to linear systems. We can understand the local behavior of nonlinear systems through the linear approximation. In the limiting case, we could consider linearizing a nonlinear system at every point on a given trajectory. Then we may be able to make claims about the behavior of the nonlinear system.

For a function of two variables $h(x,y)$, we can write a linear approximation for $h(x^*+\tilde{x}, y^*+\tilde{y})$ about the point (x^*,y^*)

$$h(x^*+\tilde{x}, y^*+\tilde{y}) = h(x^*,y^*) + \left.\frac{\partial h}{\partial x}\right|_{(x^*,y^*)} \tilde{x} + \left.\frac{\partial h}{\partial y}\right|_{(x^*,y^*)} \tilde{y} + \mathscr{O}(\tilde{x}^2, \tilde{y}^2, \tilde{x}\tilde{y}) \tag{43}$$

by retaining only the linear terms in the Taylor series expansion.

We will now perform the analogous linear approximation for the system of ODEs $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$. For simplicity, we have assumed an autonomous system; i.e., $\mathbf{f}$ does not depend explicitly on time t. In this case, the terms we truncate will be on the order of the length of the perturbation which we denote by the Euclidean norm $\|\tilde{\mathbf{u}}\| = \left(\sum_{i=1}^{d} u_i^2\right)^{1/2}$. If we linearize about $\mathbf{u} = \mathbf{u}^*$, for the ith component f_i of $\mathbf{f}$ we have

$$f_i(\mathbf{u}^* + \tilde{\mathbf{u}}) = f_i(\mathbf{u}^*) + \sum_{j=1}^{d} \left.\frac{\partial f_i}{\partial u_j}\right|_{\mathbf{u}^*} \tilde{u}_j + \mathscr{O}(\|\tilde{\mathbf{u}}\|^2). \tag{44}$$

We can write this equation more compactly by recognizing the sum as a dot-product

$$f_i(\mathbf{u}^* + \tilde{\mathbf{u}}) = f_i(\mathbf{u}^*) + \begin{bmatrix} \left.\frac{\partial f_i}{\partial u_1}\right|_{\mathbf{u}^*} & \cdots & \left.\frac{\partial f_i}{\partial u_d}\right|_{\mathbf{u}^*} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_d \end{bmatrix} + \mathscr{O}(\|\tilde{\mathbf{u}}\|^2) \tag{45}$$

$$= f_i(\mathbf{u}^*) + \left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_{\mathbf{u}^*}^{T} \tilde{\mathbf{u}} + \mathscr{O}(\|\tilde{\mathbf{u}}\|^2) \tag{46}$$

where we have introduced the notation

$$\left.\frac{\partial f_i}{\partial \mathbf{u}}\right|_{\mathbf{u}^*} = \begin{bmatrix} \left.\frac{\partial f_i}{\partial u_1}\right|_{\mathbf{u}^*} \\ \vdots \\ \left.\frac{\partial f_i}{\partial u_d}\right|_{\mathbf{u}^*} \end{bmatrix}. \tag{47}$$

Note here that the derivative of the scalar function f_i with respect to the vector $\mathbf{u}$ of inputs is itself a vector. By taking the function index i from 1 to d, we will generate linear approximations of $\tilde{f}_1(\mathbf{u}), \ldots, \tilde{f}_d(\mathbf{u})$. If we collect all of those into a vector, we have

$$\mathbf{f}(\mathbf{u}^* + \tilde{\mathbf{u}}) = \mathbf{f}(\mathbf{u}^*) + \begin{bmatrix} \left.\frac{\partial f_1}{\partial \mathbf{u}}\right|_{\mathbf{u}^*}^{T} \\ \vdots \\ \left.\frac{\partial f_d}{\partial \mathbf{u}}\right|_{\mathbf{u}^*}^{T} \end{bmatrix} \tilde{\mathbf{u}} + \mathscr{O}(\|\tilde{\mathbf{u}}\|^2) = \mathbf{f}(\mathbf{u}^*) + \left[\mathbf{J}_f\big|_{\mathbf{u}^*}\right]\tilde{\mathbf{u}} + \mathscr{O}(\|\tilde{\mathbf{u}}\|^2), \tag{48}$$

where we have introduced the *Jacobian* of $\mathbf{f}$ at $\mathbf{u}^*$

$$\left[\mathbf{J}_f\big|_{\mathbf{u}^*}\right] = \begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{\mathbf{u}^*} & \left.\frac{\partial f_1}{\partial u_2}\right|_{\mathbf{u}^*} & \cdots & \left.\frac{\partial f_1}{\partial u_d}\right|_{\mathbf{u}^*} \\ \left.\frac{\partial f_2}{\partial u_1}\right|_{\mathbf{u}^*} & \left.\frac{\partial f_2}{\partial u_2}\right|_{\mathbf{u}^*} & \cdots & \left.\frac{\partial f_2}{\partial u_d}\right|_{\mathbf{u}^*} \\ \vdots & & & \vdots \\ \left.\frac{\partial f_d}{\partial u_1}\right|_{\mathbf{u}^*} & \left.\frac{\partial f_d}{\partial u_2}\right|_{\mathbf{u}^*} & \cdots & \left.\frac{\partial f_d}{\partial u_d}\right|_{\mathbf{u}^*} \end{bmatrix}. \tag{49}$$

Note that the Jacobian is a d-by-d matrix of partial derivatives of each of the function components $f_i(\mathbf{u})$ with respect to each of the input vector components u_j.

The nonlinear system $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$ can therefore be linearized about the point $\mathbf{u}^*$ to derive a linear approximation of the dynamics of the perturbation $\tilde{\mathbf{u}}$,

$$\tilde{\mathbf{u}}_t = \mathbf{f}(\mathbf{u}^* + \tilde{\mathbf{u}}) = \mathbf{f}(\mathbf{u}^*) + \left[\mathbf{J}_f\big|_{\mathbf{u}^*}\right]\tilde{\mathbf{u}} + \mathcal{O}(\|\tilde{\mathbf{u}}\|^2) \approx \mathbf{f}(\mathbf{u}^*) + \left[\mathbf{J}_f\big|_{\mathbf{u}^*}\right]\tilde{\mathbf{u}} \tag{50}$$

Exercise 2. Consider the system of ODEs $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$ where $f_1(u_1, u_2) = u_1^2 + 3$ and $f_2(u_1, u_2) = u_1^2 u_2 + u_2^2$. Write the Jacobian matrix as a function of u_1 and u_2. What is the component $J_{2,1}$ (2nd row, 1st column) at the point $(u_1, u_2) = (3, -2)$.

In the next section, we will revisit the scalar ODE to introduce eigenvalue stability. In the following section, we will discuss eigenvalue stability for systems of ODEs.