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Exchange of Energy between Gases at Different Temperatures

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An expression for the energy exchange between two Maxwellian gases at different temperatures is derived. The resultant formula contains an integral in which the scattering cross section for momentum transfer appears. If the cross section for hard-sphere collisions is used, the expression reduces to an expression obtained by Cravath for this case. If the cross section for screened Coulomb interactions is used, the results agree with results for the same case obtained by Spitzer and others.

INTRODUCTION

In many applications of kinetic theory, it is necessary to know the rate at which energy is transferred from a gas at one temperature to a gas at a different temperature when the two gases are mixed. This information is important in ionospheric physics, astrophysics, controlled fusion research, and numerous other branches of physics and chemistry.

Cravath[1] has derived an expression for the rate of energy transfer between two Maxwellian gases interacting by means of hard-sphere collisions.

Dougal and Goldstein,[2] Landau,[3] Spitzer,[4] and Chandrasekhar[5] have derived expressions for the energy exchange when the two gases interact by means of a screened inverse-square force.

In the present paper a general expression for the energy exchange is derived which can be applied to a mixture of two gases in which the interaction between the two gases is by means of hard elastic collisions of arbitrary cross section. This expression is then applied to the two cases of hard-sphere and Coulomb interaction and the results are found to agree exactly with the results of Cravath[1] and Spitzer.[4]

ENERGY EXCHANGE

Let $W(\mathbf{v}', \mathbf{v})\, d\mathbf{v}\, dt$ be the probability that a particle of mass m and velocity $\mathbf{v}'$ which is passing through a gas of particles of mass M and density N undergoes a collision between t and $t + dt$ and ends up with its velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. The average loss of energy per unit time of such a particle will then be

$$\int \tfrac{1}{2}m(v'^2 - v^2)W(\mathbf{v}', \mathbf{v})\, d\mathbf{v}. \tag{1}$$

If we have a collection of particles of mass m, density n, and velocity distribution $f(\mathbf{v})$, where $f(\mathbf{v})$ is normalized to one, then the rate of exchange of energy per unit volume per unit time from this gas to the gas of particles of mass M is given by

$$Q = n \iint \tfrac{1}{2}m(v'^2 - v^2)W(\mathbf{v}', \mathbf{v})f(\mathbf{v}')\, d\mathbf{v}'\, d\mathbf{v}. \tag{2}$$

Desloge and Matthysse[6] have calculated an expression for $W(\mathbf{v}', \mathbf{v})$. They give

$$W(\mathbf{v}', \mathbf{v}) = [1 + (m/M)]^3 \int W_0[\mathbf{v}' - \mathbf{V}', \cdot \mathbf{v} - \mathbf{V}' - (m/M)(\mathbf{v}' - \mathbf{v})]F(\mathbf{V}')\, d\mathbf{V}', \tag{3}$$

where $F(\mathbf{V}')$ is the velocity distribution function for the gas of particles of mass M and

$$W_0(\mathbf{v}', \mathbf{v}) = Nv'q(v', \theta)[\delta(v - v')/v^2], \tag{4}$$

where δ is the Dirac delta function and $q(v', \theta)$ is the differential scattering cross section for an observer fixed on the scatterer.

If Eq. (3) is substituted in (2) we obtain

$$Q = n\left(1 + \frac{m}{M}\right)^3 \iiint [(\tfrac{1}{2}m)(v'^2 - v^2)] \cdot W_0\left[\mathbf{v}' - \mathbf{V}', \mathbf{v} - \mathbf{V}' - \left(\frac{m}{M}\right)(\mathbf{v}' - \mathbf{v})\right] \cdot F(\mathbf{V}')f(\mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}'\, d\mathbf{V}'. \tag{5}$$

We now make the following transformation of coordinates:

$$\mathbf{x} = \mathbf{v}' - \mathbf{V}', \tag{6}$$

[1] A. M. Cravath, Phys. Rev. **36**, 248 (1930).
[2] A. A. Dougal and L. Goldstein, Phys. Rev. **109**, 615 (1958).
[3] L. Landau, Physik. Z. Sowjetunion **10**, 154 (1936).
[4] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).
[5] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, Illinois, 1942).
[6] E. A. Desloge and S. W. Matthysse, Am. J. Phys. **28**, 1 (1960).

$$\mathbf{y} = \mathbf{v} - \mathbf{V}' - (m/M)(\mathbf{v}' - \mathbf{v}), \tag{7}$$

$$\mathbf{z} = [1/(m + M)](m\mathbf{v}' + M\mathbf{V}'). \tag{8}$$

The Jacobian of the transformation is

$$\left| J\left(\frac{\mathbf{v}, \mathbf{v}', \mathbf{V}'}{\mathbf{x}, \mathbf{y}, \mathbf{z}}\right) \right| = \left[1 + \left(\frac{m}{M}\right)\right]^{-3}. \tag{9}$$

We further assume

$$f(\mathbf{v}) = (m/2\pi kT)^{\frac{3}{2}} \exp(-mv^2/2kT), \tag{10}$$

$$F(\mathbf{V}) = (M/2\pi kT^*)^{\frac{3}{2}} \exp(-MV^2/2kT^*). \tag{11}$$

On carrying out the transformation we obtain

$$Q = \frac{(mM)^{\frac{3}{2}} n}{(TT^*)^{\frac{3}{2}} (2\pi k)^3} \iiint \left\{ \left(\frac{mM}{M + m}\right) \mathbf{z} \cdot (\mathbf{x} - \mathbf{y}) + \left[\frac{mM^2}{2(m + M)^2}\right](x^2 - y^2) \right\} \cdot \exp(-az^2 + b\mathbf{z} \cdot \mathbf{x} - cx^2) \cdot W_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \, d\mathbf{z}, \tag{12}$$

where

$$a = (m/2kT) + (M/2kT^*), \tag{13}$$

$$b = [mM/k(m + M)][(T - T^*)/TT^*], \tag{14}$$

$$c = [mM/2k(M + m)^2][(mT + MT^*)/TT^*]. \tag{15}$$

But $W_0(\mathbf{x}, \mathbf{y}) = 0$ unless $x = y$, and if $x = y$ then $(x^2 - y^2) = 0$. Therefore the term containing $(x^2 - y^2)$ in (12) will vanish on integration.

Equation (12) thus reduces to

$$Q = \frac{(mM)^{5/2} n}{(TT^*)^{\frac{3}{2}} (2\pi k)^3 (m + M)} \cdot \iiint [\mathbf{z} \cdot (\mathbf{x} - \mathbf{y})] W_0(\mathbf{x}, \mathbf{y}) \cdot \exp(-az^2 + b\mathbf{z} \cdot \mathbf{x} - cx^2) \, d\mathbf{x} \, d\mathbf{y} \, d\mathbf{z}. \tag{16}$$

Noting that

$$-az^2 + b\mathbf{z} \cdot \mathbf{x} - cx^2 = -a[\mathbf{z} - (b/2a)\mathbf{x}]^2 - [(4ac - b^2)/4a]x^2,$$

and carrying out the integration over $\mathbf{z}$ we obtain

$$Q = \frac{(mM)^{5/2} \pi^{\frac{3}{2}} bn}{(TT^*)^{\frac{3}{2}} (2\pi k)^3 (m + M) 2a^{5/2}} \cdot \iint W_0(\mathbf{x}, \mathbf{y})(x^2 - \mathbf{x} \cdot \mathbf{y}) \cdot \exp\left[-\left(\frac{4ac - b^2}{4a}\right)x^2\right] d\mathbf{x} \, d\mathbf{y}. \tag{17}$$

Substituting (4) in (17), replacing $d\mathbf{y}$ by $y^2 \sin\theta \, dy \, d\theta \, d\phi$, and integrating over y and ϕ, we obtain

$$Q = \frac{(mM)^{5/2} \pi^{\frac{3}{2}} bnN}{2(TT^*)^{\frac{3}{2}} (2\pi k)^3 (m + M) a^{5/2}} \cdot \int x^3 \exp\left[-\left(\frac{4ac - b^2}{4a}\right)x^2\right] q_m(x) \, d\mathbf{x}, \tag{18}$$

where

$$q_m(x) = \int q(x, \theta)(1 - \cos\theta) 2\pi \sin\theta \, d\theta, \tag{19}$$

is the cross section for momentum transfer.

Substituting (13), (14), and (15) in (18) and integrating over all directions, we obtain

$$Q = \frac{4\pi nN m^{7/2} M^{7/2} (T - T^*)}{(M + m)^2 (2\pi k)^{\frac{3}{2}} (mT^* + MT)^{5/2}} \cdot \int x^5 \exp(-Kx^2) q_m(x) \, dx, \tag{20}$$

where

$$K = [mM/2k(mT^* + MT)]. \tag{21}$$

HARD SPHERES

For the case of hard spheres $q_m(y) = \pi\sigma^2$, where σ is the sum of the radii of the two different type particles. Substituting this result in (20), we obtain

$$Q = \frac{8(2\pi)^{\frac{1}{2}} nN\sigma^2 k^{\frac{3}{2}} m^{\frac{1}{2}} M^{\frac{1}{2}} (mT^* + MT)^{\frac{1}{2}} (T - T^*)}{(M + m)^2}. \tag{22}$$

This agrees exactly with the result of Cravath.[1]

COULOMB INTERACTION

Let us assume that we have a gas of electrons of charge $-e$ and a gas of singly charged ions also of charge e. If we assume that the interaction is a screened Coulomb interaction with the screening distance the Debye length, then[7]

$$q_m(v) \approx \frac{e^4}{4\pi\epsilon_0^2 m^{*2} v^4} \ln\left[\frac{4\pi\epsilon_0 m^* v^2}{e^3} \left(\frac{\epsilon_0 kT}{n}\right)^{\frac{1}{2}}\right], \tag{23}$$

where m^* is the reduced mass. The term in the square brackets is usually very large, and approximations for its magnitude will not affect $q_{m^*}(v)$ appreciably. We therefore replace m^*v^2 by $3kT$ and obtain

$$q_m(v) \approx [e^4 \ln\Lambda / 4\pi\epsilon_0^2 m^{*2} v^4], \tag{24}$$

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[7] See, for example, D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion* (John Wiley & Sons, Inc., New York, 1961), p. 163.

where

$$\Lambda = [12\pi(\epsilon_0 kT)^{\frac{3}{2}}/e^3 n^{\frac{1}{2}}]. \tag{25}$$

Substituting (24) in (20), we have

$$Q = \frac{e^4 nNm \ln \Lambda[1 - (T^*/T)]}{2\pi\epsilon_0^2(2\pi mkT)^{\frac{1}{2}} M[1 + (mT^*/MT)]^{\frac{3}{2}}}. \tag{26}$$

Equation (26) agrees exactly with the result of Spitzer.[4] The results of Chandrasekhar[5] and Landau[3] agree closely but not exactly with the results of Spitzer. Dougal and Goldstein[2] have compared the different results, and have themselves presented an alternative expression for the case in which $T^* = 0$. If instead of calculating the average change of energy in a collision as in Eq. (2), we had calculated the average value of the relative change in energy and multiplied by the energy, then Eq. (2) would have been replaced by the following expression:

$$Q = n(\tfrac{3}{2}kT) \iint \left(\frac{v'^2 - v^2}{v'^2}\right) W(\mathbf{v}', \mathbf{v}) f(\mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}'. \tag{27}$$

If Eq. (27) is evaluated for the case in which $T^* = 0$ and if we use the accurate rather than the approximate value for the scattering cross section for a screened Coulomb interaction, then Eq. (27) can be shown to lead to Dougal and Goldstein's[2] result. It would seem however that Eq. (2) is a more realistic expression for Q than Eq. (27).

CONCLUSION

As far as the author knows, Eq. (20) has not been previously derived.[8] It is interesting that, although derived for hard collisions, it is able to be applied to the relatively soft Coulomb collisions. With Eq. (20) it should be possible, by proper choice of cross section, to fit the experimental data of Dougal and Goldstein[2] for the energy exchange between an electron gas and an ion gas. This calculation would give some insight into the nature of the actual cross section. There are numerous other possible uses of Eq. (20) which the author hopes to investigate and report on at a later date.

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[8] *Note added in proof.* The author was not aware when this article was submitted of the work of G. Boulègue, P. Chanson, R. Combe, M. Felix, and P. Strasman, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 242.