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Line-Of-Sight Based Spacecraft Attitude and Position Tracking Control

[Rakesh R Warier](#)^{a,1,*}, [Arpita Sinha](#)^{a,2}, [Srikant Sukumar](#)^{a,2}

^a*Systems and Control Engineering, Indian Institute of Technology Bombay, Mumbai, Maharashtra, India, 400076*

Abstract

This paper considers the problem of formation control of three spacecraft consisting of one leader and two followers. The leader spacecraft controls its attitude and position to track a desired attitude and position trajectory in the Earth Centered Inertial (ECI) frame. Each follower spacecraft tracks a desired relative attitude and relative position trajectory with respect to the leader spacecraft. Absolute attitude control law for the leader and relative attitude control laws for the followers are obtained in terms of line-of-sight vectors between the spacecraft. A relative attitude determination scheme using line-of-sight vectors is also proposed. The state feedback laws proposed in this work guarantee almost global asymptotic stability of the desired closed-loop equilibrium.

1. Introduction

Spacecraft formation flying has been extensively studied in recent years. The idea that a group of spacecraft flying as a formation can act as one large virtual instrument that is more efficient, more robust and less costly than a large monolithic spacecraft of similar capabilities has significant appeal. Simbol-X [1], MAXIM [2], NEAT [3] are among several spacecraft formation flying interferometry missions that were planned in the last decade. For a spacecraft formation to work as a virtual instrument, it is essential that the spacecraft in formation maintain precise relative attitudes and positions with respect to each other. From the control theory perspective, this poses a combined attitude and position control problem.

There have been several articles dealing with precise relative position control of spacecraft in formation, primarily utilizing navigation data from GPS [4]. The set of all possible attitudes of a rigid body is the set of 3×3 real orthogonal matrices with determinant 1, commonly referred to as the special orthogonal group $SO(3)$. The special orthogonal group $SO(3)$ is not a Euclidean space.

*Corresponding author

Email addresses: rakeshwarier@sc.iitb.ac.in (Rakesh R Warier), asinha@sc.iitb.ac.in (Arpita Sinha), srikant@sc.iitb.ac.in (Srikant Sukumar)

¹PhD Scholar, Systems and Control Engineering, Indian Institute of Technology Bombay

²Assistant Professor Systems and Control Engineering, Indian Institute of Technology Bombay

However classical approaches to attitude control make use of local representations like Euler angles and non unique representations like quaternions. It was only recently that rigid body attitude control problems were analysed in $SO(3)$. Some examples of rigid body attitude control in $SO(3)$ are given in [5] and [6]. In this article, the attitude control system is constructed in $SO(3)$, thus avoiding singularities associated with local parametrizations and ambiguity of quaternions.

Most of the existing relative attitude control laws (for example [7]) follow a common framework. It is assumed that absolute attitude of each spacecraft is measured independently with respect to a common inertial frame and are communicated to each other so as to calculate relative attitude. As relative attitudes are calculated indirectly, the accuracy is limited by attitude sensors of either spacecraft. Line-of-sight (LOS) vectors are a convenient way to measure relative attitudes. Line-of-sight observations between spacecraft can be found by standard light-beam focal-plane detector technology or by laser communication hardware as given in [8]. Cyclic formation control of spacecraft using inertial line-of-sight was proposed in [9]. The notion of attitude determination of a spacecraft from three independent vector observations in the spacecraft's own frame was proposed in [10], where the authors gave TRIAD and QUEST algorithms for attitude determination. More recently, authors in [11] suggest a scheme for determining relative attitude between two spacecraft using line-of-sight measurements to each other and to a reference body. Authors in [12] proposed a relative navigation scheme based on line-of-sight vectors. However their contribution did not include a control design. Both [13] and [14] propose attitude control laws for a single spacecraft making use of inertial line-of-sight measurements. Attitude stabilization of a single spacecraft with internal actuation using vector observations is solved in [13], while angular velocity free attitude control of single spacecraft using inertial line-of-sight measurements was proposed in [14]. Attitude synchronization in $SO(n)$ of multiple agents sharing only a common link dependent vector is shown in [15]. Lee in [16] proposes a control law to asymptotically stabilize relative attitude between two spacecraft making use of line-of-sight direction observations between them, and line-of-sight direction observations to a common object. In [16] control torques are obtained in terms of line-of-sight vectors. Authors in [17] extend the work of [16] to achieve tracking control of relative attitudes between multiple spacecraft. However in all these articles, spacecraft positions were assumed to be fixed during attitude maneuvers. The Lyapunov analysis used in several earlier contributions like [14],[16] and [17] could not allow translation since it would vary the line-of-sight vectors. Hence, the assumption that spacecraft positions are fixed was essential to prove closed loop stability in aforementioned results. This is restrictive as spacecraft formation applications require simultaneous control of attitude and position of the spacecraft. In this paper, we propose combined position and attitude control for a three spacecraft formation using line-of-sight measurements while assuming a serial communication architecture.

In [18] and [19], the authors proposed a novel line-of-sight based relative attitude control law

that allowed translational motion of spacecraft during attitude manoeuvres. However control of position dynamics was not explicitly considered in [18]. This paper extends the concept to a three spacecraft formation with serial network architecture under gravity. The leader spacecraft controls its absolute position and absolute attitude with respect to an inertial frame so as to track a desired attitude and position trajectory. Two follower spacecraft control their relative position and relative attitude with respect to the leader, to track desired relative position and attitude trajectories. All attitude control laws are obtained in terms of line-of-sight unit vectors between spacecraft without using any external source. Leader spacecraft measures line-of-sight unit vectors to the follower spacecraft in its own body frame and communicates the same. The resulting closed loop system is shown to be almost globally asymptotically stable. A preliminary version of these results with a two spacecraft leader-follower architecture is given in [20].

In [21], simultaneous attitude and translational tracking control is considered assuming translational dynamics of the spacecraft to be double integrator. While the problem formulation is similar, this work provides a different control scheme and stability analysis. Compared to [21], the spacecraft translational dynamics is considered to be Keplerian instead of double integrator and the work here makes no assumption about dynamics of line-of-sight unit vectors. In addition, this paper provides a novel framework to help reformulate several geometric attitude control schemes based on trace like error functions to line-of-sight control schemes. This last part is discussed further in section 6

Trace and modified trace attitude error functions are widely used in geometric control literature ([22], [5] and [23] among many others). In this work, a trace like attitude error function is obtained in terms of line-of-sight vectors. Thus, the attitude control torques obtained by our proposed scheme are in the same spirit as aforementioned geometric control schemes. However, contrary to standard geometric control schemes using the trace error function, estimation of absolute or relative attitudes to implement the attitude control scheme is not required.

The paper is organized as follows, section 2 mathematically formulates the problem, section 3 describes the proposed attitude determination scheme, section 4 describes error functions that are used later in the paper, section 5 develops the control law and provides stability results for the closed loop system. Discussion of the results are provided in section 6, numerical simulations are given in section 7, and the conclusions are summarized in section 8.

2. Problem Formulation

2.1. Mathematical Preliminaries

Spacecraft attitude dynamics are modeled as rigid body dynamics. Spacecraft attitude and translation dynamics are assumed to be decoupled. The rigid body attitude evolves over $SO(3)$, a

compact manifold given by

$$\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^\top R = RR^\top = I, \det(R) = 1\}. \quad (1)$$

Because of the topological properties of $\text{SO}(3)$, no globally asymptotically stable equilibrium exists under continuous control ([24]). $\text{SO}(3)$ forms a Lie group under the matrix multiplication operation. The lie algebra (tangent space at identity) of $\text{SO}(3)$ is denoted as $\text{so}(3)$ [25] and defined as,

$$\text{so}(3) = \{S \in \mathbb{R}^{3 \times 3} \mid S = -S^\top\}. \quad (2)$$

The map $\wedge : \mathbb{R}^3 \rightarrow \text{so}(3)$, often called the “hat map”, denotes the isomorphism from \mathbb{R}^3 to $\text{so}(3)$. If $x = [x_1, x_2, x_3]^\top$, $x \in \mathbb{R}^3$

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \quad (3)$$

Further \hat{x} is a skew symmetric matrix satisfying, $\hat{x}y = x \times y$, $\forall y$. The inverse of the hat map is denoted as $(\cdot)^\vee : \text{so}(3) \rightarrow \mathbb{R}^3$ and can be implicitly defined as $(\hat{x})^\vee = x$. The map $\text{skew} : \mathbb{R}^{3 \times 3} \rightarrow \text{so}(3)$ is defined as $\text{skew}(A) := \frac{A - A^\top}{2}$. It is evident from above that for all $A \in \mathbb{R}^{3 \times 3}$, $(\text{skew}(A))^\vee$ is well defined. Also, let $\text{sym}(A) := \frac{A + A^\top}{2}$ and notice that $A = \text{sym}(A) + \text{skew}(A)$. Let $\text{tr}(\cdot)$ be the trace of a square matrix, defined as the sum of its diagonal elements.

2.2. Dynamics

Equations of motion of the i -th (for $i = 1, 2, 3$) spacecraft orbiting earth under gravity, are given by

$$\dot{r}_i = v_i \quad (4)$$

$$\dot{v}_i = -\frac{\mu r_i}{\|r_i\|^3} + u_i \quad (5)$$

where $r_i \in \mathbb{R}^3$ is the position vector and $v_i \in \mathbb{R}^3$ is the velocity vector of the centre of mass of spacecraft i in ECI frame, $u_i \in \mathbb{R}^3$ is the force applied per unit mass and $\mu = 3.98658366 \times 10^{14} \text{m}^3 \text{s}^{-2}$ is the gravitational parameter of Earth.

Let the attitude of spacecraft i be given by the rotation matrix R_i representing the transformation from body fixed frame of spacecraft i to a common inertial frame. Attitude dynamics of the i -th spacecraft, for $i = 1, 2, 3$ are given by

$$\dot{R}_i = R_i \hat{\Omega}_i \quad (6)$$

$$J_i \dot{\Omega}_i = J_i \Omega_i \times \Omega_i + \tau_i \quad (7)$$

where $J_i \in \mathbb{R}^{3 \times 3}$ is the moment of inertia, $\Omega_i \in \mathbb{R}^3$ is the angular velocity, and $\tau_i \in \mathbb{R}^3$ is the control torque of i -th spacecraft. Ω_i and τ_i are represented in the i -th spacecraft's body fixed frame.

2.3. Formation Specifications

Consider a spacecraft as illustrated in the figure Fig.1. (x, y, z) is the Earth Centered Inertial reference frame. Axes (X', Y', Z') , (X'', Y'', Z'') and (X''', Y''', Z''') are the body fixed frames of spacecraft one, two and three respectively.

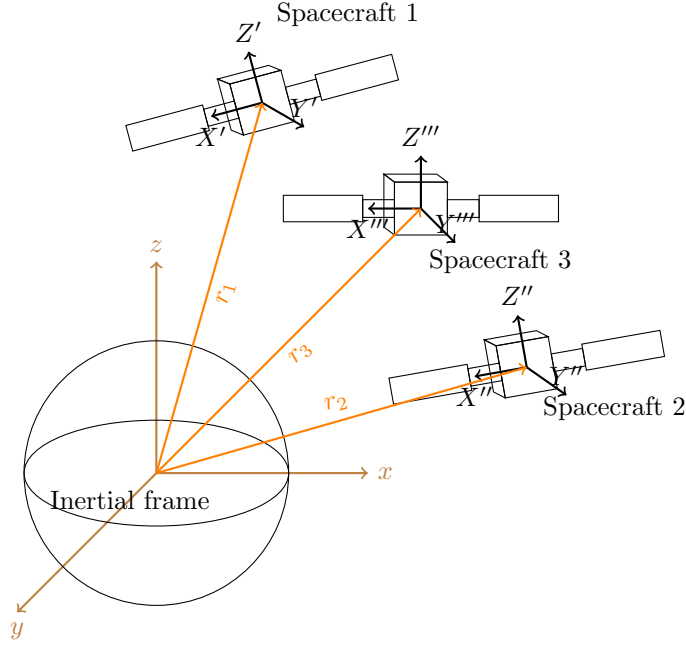


Figure 1: Spacecraft in formation. r_1 , r_2 and r_3 are the position vectors to the center of mass.

Here spacecraft 1 is the leader while spacecraft 2 and 3 are the followers. It is required that the leader spacecraft should track a desired absolute attitude and position trajectory. Let $r_1^d(t)$ and $v_1^d(t)$ be the desired position and velocity of the leader spacecraft satisfying,

$$\dot{r}_1^d = v_1^d. \quad (8)$$

Let the desired time varying attitude of the leader be given by $R_1^d(t)$. Then it satisfies the following kinematic relation,

$$\dot{R}_1^d = R_1^d \hat{\Omega}_1^d \quad (9)$$

where $\Omega_1^d(t)$ is the desired angular velocity. The follower spacecraft 2 and 3 are required to track a desired relative attitude and relative position trajectory with respect to the leader spacecraft. Let Q_{k1} denote the relative attitude of the follower spacecraft k with respect to the leader spacecraft 1.

$$Q_{k1} := R_1^\top R_k, \quad k = 2, 3. \quad (10)$$

The time derivative of relative attitude Q_{k1} is evaluated as,

$$\begin{aligned}\dot{Q}_{k1} &= -R_1^\top R_1 \widehat{\Omega}_1 R_1^\top R_k + R_1^\top R_k \widehat{\Omega}_k \\ &= -\widehat{\Omega}_1 Q_{k1} + Q_{k1} \widehat{\Omega}_k.\end{aligned}\quad (11)$$

Also, let r_{k1} and v_{k1} be the relative position and relative velocity of follower spacecraft k with respect to the leader spacecraft, satisfying

$$\dot{r}_{k1}^d = v_{k1}^d. \quad (12)$$

Let $Q_{k1}^d(t)$ be the desired relative attitude trajectory of spacecraft k with respect to the attitude of spacecraft 1. $Q_{k1}^d(t)$ is assumed to be a smooth function of time satisfying the following kinematic relation,

$$\dot{Q}_{k1}^d = Q_{k1}^d \widehat{\Omega}_{k1}^d \quad (13)$$

where $\Omega_{k1}^d(t)$ is the desired relative angular velocity of follower spacecraft k with respect to the leader spacecraft. Using (12), the desired position and velocity trajectories for the k -th ($k = 2, 3$) spacecraft are defined as,

$$r_k^d(t) := r_1(t) + r_{k1}^d(t) \quad (14)$$

$$v_k^d(t) := v_1(t) + v_{k1}^d(t). \quad (15)$$

Similarly, the desired attitude $R_k^d(t)$ and angular velocity $\Omega_k^d(t)$ trajectories of follower spacecraft k can be defined in terms of desired relative trajectories and the leader's attitude and angular velocity as,

$$R_k^d = R_1 Q_{k1}^d \quad (16)$$

$$\Omega_k^d = \Omega_{k1}^d + (Q_{k1}^d)^\top \Omega_1. \quad (17)$$

The control objectives are,

1. Achieving desired position trajectory tracking for the leader spacecraft $\lim_{t \rightarrow \infty} r_1(t) = r_1^d(t)$ and $\lim_{t \rightarrow \infty} v_1(t) = v_1^d(t)$,
2. Achieving desired relative position trajectories between the leader and two follower spacecraft $\lim_{t \rightarrow \infty} r_{21}(t) = r_{21}^d(t)$, $\lim_{t \rightarrow \infty} v_{21}(t) = v_{21}^d(t)$ and $\lim_{t \rightarrow \infty} r_{31}(t) = r_{31}^d(t)$ and $\lim_{t \rightarrow \infty} v_{31}(t) = v_{31}^d(t)$,
3. Achieving desired absolute attitude of the leader spacecraft $\lim_{t \rightarrow \infty} R_1(t) = R_1^d(t)$,
4. Achieving desired relative attitude between the leader and two follower spacecraft, $\lim_{t \rightarrow \infty} (R_1^\top(t) R_2(t)) = Q_{21}^d(t)$, and $\lim_{t \rightarrow \infty} (R_1^\top(t) R_3(t)) = Q_{31}^d(t)$,
5. Achieving desired angular velocities, $\lim_{t \rightarrow \infty} \Omega_i(t) = \Omega_i^d(t)$ for $i = 1, 2, 3$.

Additionally, the following assumption is introduced and will be employed in the sequel.

Assumption 1. *It is assumed that the center of masses of the three spacecraft never become collinear or coinciding, mathematically this can be written as, $(r_2(t) - r_1(t)) \times (r_3(t) - r_2(t)) \neq [0 \ 0 \ 0]^\top$, $\forall t \geq 0$.*

2.4. Error Variables

The control problem is reformulated in error variables for ease of notation and tractability. The attitude error of spacecraft $i = 1, 2, 3$ is defined to be

$$X_i := (R_i^d)^\top R_i. \quad (18)$$

For the follower spacecraft $k = 2, 3$, X_k can be written as the error between the actual relative attitude of follower spacecraft Q_{k1} and the desired relative w.r.t the leader spacecraft attitude Q_{k1}^d .

$$X_k := (Q_{k1}^d)^\top Q_{k1} = (R_k^d)^\top R_k. \quad (19)$$

Similar to (11), the time derivative of X_1 can be obtained as,

$$\dot{X}_1 = \frac{d}{dt}((R_1^d)^\top R_1) = -\hat{\Omega}_1^d X_1 + X_1 \hat{\Omega}_1. \quad (20)$$

The derivatives of attitude errors of the follower spacecraft k is given by

$$\begin{aligned} \dot{X}_k &= \frac{d}{dt}((Q_{k1}^d)^\top Q_{k1}) \\ &= -(Q_{k1}^d)^\top Q_{k1}^d \hat{\Omega}_{k1}^d (Q_{k1}^d)^\top Q_{k1} + (Q_{k1}^d)^\top (-\hat{\Omega}_1 Q_{k1} + Q_{k1} \hat{\Omega}_k) \\ &= -\hat{\Omega}_{k1}^d X_{k1} - (Q_{k1}^d)^\top \hat{\Omega}_1 Q_{k1} + (Q_{k1}^d)^\top Q_{k1} \hat{\Omega}_k \\ &= -\hat{\Omega}_{k1}^d X_{k1} - (Q_{k1}^d)^\top \hat{\Omega}_1 Q_{k1}^d (Q_{k1}^d)^\top Q_{k1} + X_{k1} \hat{\Omega}_k \\ &= -\hat{\Omega}_{k1}^d X_{k1} - (Q_{k1}^d)^\top \hat{\Omega}_1 X_{k1} + X_{k1} \hat{\Omega}_k. \end{aligned} \quad (21)$$

Angular velocity errors e_{Ω_i} , $i = 1, 2, 3$ are defined to be

$$e_{\Omega_i} := \Omega_i - \Omega_i^d. \quad (22)$$

Note that when $X_i = I$ ($i = 1, 2, 3$), e_{Ω_i} needs to go to zero for achieving tracking.

The position and velocity error variables ζ_i , ν_i for spacecraft $i = 1, 2, 3$ are constructed as below,

$$\zeta_i := r_i - r_i^d \quad (23)$$

$$\nu_i := v_i - v_i^d. \quad (24)$$

2.5. Measurement Strategies

Let $(i, j) \in \{(1, 2), (2, 1), (1, 3), (2, 3), (3, 1), (3, 2)\}$. The line-of-sight unit vector observed from the i -th spacecraft to the j -th spacecraft with respect to the inertial frame is denoted as s_{ij} and defined as,

$$s_{ij} := \frac{(r_j - r_i)}{\|(r_j - r_i)\|}. \quad (25)$$

Define l_{ij} to be the line-of-sight unit vector observed from the i -th spacecraft to the j -th spacecraft, with respect to the i -th body fixed frame. This is given by

$$l_{ij} = R_i^\top s_{ij} = R_i^\top \frac{(r_j - r_i)}{\|(r_j - r_i)\|}. \quad (26)$$

For each follower spacecraft k , the leader spacecraft measures relative position and velocity of the follower spacecraft, i.e. $r_1 - r_k$ and $v_1 - v_k$ with respect to itself. The leader spacecraft communicates line-of-sight vectors l_{12} , l_{13} , angular velocity Ω_1 and angular acceleration $\dot{\Omega}_1$ to the two follower spacecraft. For position control, the respective relative velocities and positions are also communicated.

3. Relative Attitude Determination

Relative attitude between two coordinate frames can be found using three linearly independent vectors represented in the two frames (section 2.2 of chapter 12, [26]). Line-of-sight vectors measured in the respective body frames can be used to create the aforementioned linearly independent (in fact orthonormal) unit-vectors for relative attitude determination as illustrated below.

Let $(i, j, k) \in \{(1, 2, 3), (2, 1, 3), (3, 1, 2), (1, 3, 2)\}$, Define

$$s_{ijk} := \frac{(s_{ij} \times s_{ik})}{\|s_{ij} \times s_{ik}\|} = \frac{((r_j - r_i) \times (r_k - r_i))}{\|(r_j - r_i) \times (r_k - r_i)\|} \quad (27)$$

$$s_{ijjk} := s_{ij} \times s_{ijk} \quad (28)$$

$$l_{ijk} := \frac{(l_{ij} \times l_{ik})}{\|l_{ij} \times l_{ik}\|} = \frac{R_1^\top ((r_j - r_i) \times (r_k - r_i))}{\|(r_j - r_i) \times (r_k - r_i)\|} \quad (29)$$

$$l_{ijjk} := l_{ij} \times l_{ijk}. \quad (30)$$

It is clear that both $\{s_{ij}, s_{ijk}, s_{ijjk}\}$ and $\{l_{ij}, l_{ijk}, l_{ijjk}\}$ form orthonormal basis in \mathbb{R}^3 . Additionally following equations illustrate that assumption 1 is enough to ensure existence of such bases.

$$(r_2 - r_1) \times (r_3 - r_1) = (r_2 - r_3 + r_3 - r_1) \times (r_3 - r_1) = (r_3 - r_1) \times (r_3 - r_2)$$

$$(r_1 - r_2) \times (r_3 - r_2) = (r_1 - r_3 + r_3 - r_2) \times (r_3 - r_2) = -(r_3 - r_1) \times (r_3 - r_2).$$

Due to assumption 1, $(r_3 - r_1) \times (r_3 - r_2) \neq [0 \ 0 \ 0]^\top$ and l_{ijk} and l_{ijjk} are well defined. The relative attitude determination technique is illustrated by evaluating the attitude of spacecraft 2 with respect to the leader spacecraft 1. The following matrix with unit vectors l_{12} , l_{123} and l_{1223} as the columns is constructed,

$$\begin{aligned} P_1 &:= [l_{12} \ l_{123} \ l_{1223}] = [R_1^\top s_{12} \ R_1^\top (s_{123}) \ R_1^\top (s_{1223})] \\ &= R_1^\top [s_{12} \ s_{123} \ s_{1223}]. \end{aligned} \quad (31)$$

Similarly, a matrix with unit vectors l_{21} , l_{213} and $(-l_{2113})$ as its columns is constructed below.

$$\begin{aligned} P_2 &:= [l_{21} \ l_{213} \ -l_{2113}] = -[R_2^\top s_{12} \ R_2^\top(s_{123}) \ R_2^\top(s_{1223})] \\ &= -R_2^\top [s_{12} \ s_{123} \ s_{1223}]. \end{aligned} \quad (32)$$

Note that the columns of both P_1 and P_2 form orthogonal bases, therefore implying that P_1 and P_2 are orthogonal matrices. Specifically, $P_1^{-1} = P_1^\top$ and $P_2^{-1} = P_2^\top$. From (31) and (32),

$$[s_{12} \ s_{123} \ s_{1223}] = -R_2 P_2 = R_1 P_1. \quad (33)$$

Using (33), the relative attitude Q_{21} is obtained to be

$$Q_{21} = R_1^\top R_2 = -P_1 P_2^\top. \quad (34)$$

Similarly R_1 and Q_{31} can be determined as,

$$R_1 = [s_{12} \ s_{123} \ s_{1223}] [l_{12} \ l_{123} \ l_{1223}]^\top = P_0 P_1^\top \quad (35)$$

$$Q_{31} = -[l_{13} \ l_{132} \ -l_{1332}] [l_{31} \ l_{312} \ -l_{3112}]^\top = -P_3 P_4^\top \quad (36)$$

where $P_0 = [s_{12} \ s_{123} \ s_{1223}]$, $P_3 = [l_{13} \ l_{132} \ -l_{1332}]$, and $P_4 = [l_{31} \ l_{312} \ -l_{3112}]$. The attitude determination scheme is not explicitly used in the attitude control law. However, the identities described above will be used to obtain control torques in terms of line-of-sight vectors.

4. Control Lyapunov Function

We make use of a control Lyapunov function like approach to design the control law and prove stability results. Consider the error variables defined in section 2.4, i.e. X_i , e_{Ω_i} , ζ_i , ν_i for $i = 1, 2, 3$. Now consider a positive definite function of the form given by

$$\mathcal{V} = \frac{1}{2} \sum_{i=1}^3 \alpha_i \text{tr}(I - X_i) + \frac{1}{2} \sum_{i=1}^3 e_{\Omega_i}^\top J_i e_{\Omega_i} + \frac{1}{2} \sum_{i=1}^3 (\zeta_i^\top K_i \zeta_i + \nu_i^\top \nu_i) \quad (37)$$

where $\alpha_i > 0$, $K_i = K_i^\top \in \mathbb{R}^{3 \times 3}$ and $K_i > 0$.

In the following section we expand and compute the derivatives of the three components that make up the control Lyapunov function. It is worth noting that the three components can be considered error functions namely, attitude error function, angular velocity error function and translation error function.

4.1. Attitude Error Function

Trace and modified trace functions are very commonly used in geometric attitude control design. For $Q_1, Q_2 \in \text{SO}(3)$, $\text{tr}(Q_1^\top Q_2) = 1 + 2 \cos(\theta)$, where θ is the angle of single axis rotation between Q_1 and Q_2 . The function $\text{tr}(Q_1^\top Q_2)$ attains maximum value when $Q_1 = Q_2$. Let

$$\Psi_R = \sum_{i=1}^3 \alpha_i \Psi_{R_i} = \sum_{i=1}^3 (\alpha_i \text{tr}(I - X_i)) \quad (38)$$

where $\Psi_{R_i} = \text{tr}(I - X_i)$ and $\alpha_i > 0$. It can be seen that $\Psi_{R_i} \geq 0$ and equal to zero only when $R_i^d = R_i$. Taking derivatives (from lemma A.1, [27])

$$\dot{\Psi}_R = - \sum_{i=1}^3 \text{tr}(\dot{X}_i). \quad (39)$$

For spacecraft 1, using equation (20) obtain,

$$\dot{\Psi}_{R_1} = -\text{tr}(-\hat{\Omega}_1^d X_1 + X_1 \hat{\Omega}_1) = -\text{tr}(X_1(\Omega_1 - \Omega_1^d)^\wedge) = -\text{tr}(X_1 \hat{e}_{\Omega_1}). \quad (40)$$

Similarly for $k = 2, 3$

$$\begin{aligned} \dot{\Psi}_{R_k} &= -\text{tr}\left(-\hat{\Omega}_{k1}^d X_{k1} - ((Q_{k1}^d)^\top \Omega_1)^\wedge X_{k1} + X_{k1} \hat{\Omega}_k\right) \\ &= -\text{tr}(X_k(\Omega_k - (Q_{k1}^d)^\top \Omega_1 - \Omega_{k1}^d)^\wedge) = -\text{tr}(X_k \hat{e}_{\Omega_k}). \end{aligned} \quad (41)$$

From the second part of lemma 1, (40) and (41) can be rewritten as

$$\dot{\Psi}_{R_i} = -2(\text{skew}(X_i))^\vee \cdot e_{\Omega_i}, \quad i = 1, 2, 3. \quad (42)$$

By the attitude determination scheme (35),

$$\dot{\Psi}_{R_1} = 2(\text{skew}((R_1^d)^\top P_0 P_1^\top))^\vee \cdot e_{\Omega_1} \quad (43)$$

$$= -((R_1^d)^\top s_{12} \times l_{12} + (R_1^d)^\top s_{123} \times l_{123} + (R_1^d)^\top s_{1223} \times l_{1223}) \cdot e_{\Omega_1}. \quad (44)$$

The last step is obtained by making use of lemma 2 with $A = (R_1^d)^\top P_0$ and $B = P_l$. Similar steps can be used to obtain

$$\dot{\Psi}_{R_k} = -((Q_{k1}^d)^\top l_{k1} \times l_{1k} + (Q_{k1}^d)^\top l_{k1j} \times l_{1kj} - (Q_{k1}^d)^\top l_{k11j} \times l_{1kkj}) \cdot e_{\Omega_k}. \quad (45)$$

The critical points of function $\text{tr}(I - X_i)$ are given by (from [27], Proposition 11.31), $X_i \in \{I\} \cup \mathcal{N}$, where $\mathcal{N} = \{\text{expm}(\pi \hat{a}) : a \in \mathbb{R}^3, \|a\| = 1\}$. Also at $X_i = I$, Ψ_{R_i} is at its minimum and at other critical points it achieves the maximum value of 4. Each Ψ_{R_i} can be written in terms of line-of-sight unit vectors using the relative attitude determination scheme and equation (A.2). For example Ψ_{R_2} can be written as,

$$\begin{aligned} \Psi_{R_2} &= \text{tr}(I - X_2) = \text{tr}(I + P_1 P_2^\top) \\ &= 3 + \left((Q_{21}^d)^\top l_{21}\right) \cdot l_{12} + \left((Q_{21}^d)^\top l_{213}\right) \cdot l_{123} - \left((Q_{21}^d)^\top l_{2113}\right) \cdot l_{1223}. \end{aligned}$$

It can be noted that this is similar to the attitude error function used in [16] and [17] except for the last term.

4.2. Angular Velocity Error Function

An angular velocity error function of the following form is chosen

$$\Psi_\Omega(e_{\Omega_1}, e_{\Omega_2}, e_{\Omega_3}) = \frac{1}{2} \sum_{i=1}^3 (e_{\Omega_i})^\top J_i e_{\Omega_i} = \frac{1}{2} \sum_{i=1}^3 (\Omega_i - \Omega_i^d)^\top J_i (\Omega_i - \Omega_i^d). \quad (46)$$

Clearly Ψ_Ω is non-negative and attains zero only when the error variables $e_{\Omega_1}, e_{\Omega_2}$ and e_{Ω_3} are zero. Now,

$$\begin{aligned}\frac{d}{dt} \left(\frac{1}{2} e_{\Omega_i}^\top J_i e_{\Omega_i} \right) &= (e_{\Omega_i})^\top \left((J_i \Omega_i) \times \Omega_i + \tau_i - J_i \dot{\Omega}_i^d \right) \\ &= e_{\Omega_i}^\top \left((J_i (e_{\Omega_i} + \Omega_i^d)) \times (e_{\Omega_i} + \Omega_i^d) \right) + e_{\Omega_i}^\top (\tau_i - J_i \dot{\Omega}_i^d) \\ &= e_{\Omega_i}^\top \left(J_i (e_{\Omega_i} + \Omega_i^d) \times (\Omega_i^d) + \tau_i - J_i \dot{\Omega}_i^d \right).\end{aligned}$$

By (22), $e_{\Omega_i} + \Omega_i^d = \Omega_i$, hence

$$\dot{\Psi}_\Omega = \sum_{i=1}^3 e_{\Omega_i}^\top \left(J_i (\Omega_i) \times (\Omega_i^d) + \tau_i - J_i \dot{\Omega}_i^d \right). \quad (47)$$

4.3. Translation Error Function

The translation error function is chosen to be

$$\begin{aligned}\Psi_r &= \frac{1}{2} \sum_{i=1}^3 \left((r_i - r_i^d)^\top K_i (r_i - r_i^d) + \frac{1}{2} (v_i - v_i^d)^\top (v_i - v_i^d) \right) \\ &= \frac{1}{2} \sum_{i=1}^3 \left(\zeta_i^\top K_i \zeta_i + \frac{1}{2} \nu_i^\top \nu_i \right)\end{aligned}$$

where K_1, K_2, K_3 are symmetric positive definite matrices.

$$\dot{\Psi}_r = \sum_{i=1}^3 \left((v_i - v_i^d)^\top \left(K_i (r_i - r_i^d) - \frac{\mu}{\|r_i\|^3} r_i + u_i - \dot{v}_i^d \right) \right). \quad (48)$$

5. Formation Control

The control law is described and asymptotic stability results are proven in this section.

5.1. Control Law

The translation and rotational feedback laws are given as follows,

$$u_1 = -\alpha_{v_1}(v_1 - v_1^d) - K_1(r_1 - r_1^d) + \frac{\mu}{\|r_1\|^3}r_1 + \dot{v}_1^d \quad (49)$$

$$u_2 = -\alpha_{v_2}(v_2 - v_2^d) - K_2(r_2 - r_2^d) + \frac{\mu}{\|r_2\|^3}r_2 + \dot{v}_2^d \quad (50)$$

$$u_3 = -\alpha_{v_3}(v_3 - v_3^d) - K_3(r_3 - r_3^d) + \frac{\mu}{\|r_3\|^3}r_3 + \dot{v}_3^d \quad (51)$$

$$\begin{aligned} \tau_1 = & \alpha_1((R_1^d)^\top s_{12} \times l_{12} + (R_1^d)^\top s_{123} \times l_{123} + (R_1^d)^\top s_{1223} \times l_{1223}) \\ & - \alpha_{\Omega_1} e_{\Omega_1} - (J_1(\Omega_1)) \times \Omega_1^d + J_1 \dot{\Omega}_1^d \end{aligned} \quad (52)$$

$$\begin{aligned} \tau_2 = & \alpha_2((Q_{21}^d)^\top l_{12}) \times l_{21} + \alpha_2((Q_{21}^d)^\top l_{123}) \times l_{213} \\ & - \alpha_2((Q_{21}^d)^\top (l_{1223})) \times l_{2113} - \alpha_{\Omega_2} e_{\Omega_2} - (J_2(\Omega_2)) \times \Omega_2^d + J_2 \dot{\Omega}_2^d \end{aligned} \quad (53)$$

$$\begin{aligned} \tau_3 = & \alpha_3((Q_{31}^d)^\top l_{13}) \times l_{31} + \alpha_3((Q_{31}^d)^\top l_{312}) \times l_{321} \\ & - \alpha_3((Q_{31}^d)^\top (l_{3112})) \times l_{3221} - \alpha_{\Omega_3} e_{\Omega_3} - (J_3(\Omega_3)) \times \Omega_3^d + J_3 \dot{\Omega}_3^d \end{aligned} \quad (54)$$

where the control gains $\alpha_i, \alpha_{v_i}, \alpha_{\Omega_i} > 0$ and K_i are positive definite symmetric matrices for $i = 1, 2, 3$. In each position control law, u_1, u_2 and u_3 , the first term represents is derivative, second term is proportional, and the rest feedforward.. In case of attitude control, such an explanation is not straight forward. However, from a geometrical viewpoint as detailed in works like [28], we can consider the first term in attitude control to represent the proportional part, the next term (containing e_{Ω_i}) the derivative term and the last term the feedforward component.

5.2. Stability Results

Theorem 1. *Consider the transformed system with states,*

$$(X_1, e_{\Omega_1}, X_2, e_{\Omega_2}, X_3, e_{\Omega_3}, \zeta_1, \nu_1, \zeta_2, \nu_2, \zeta_3, \nu_3) \quad (55)$$

which are defined in (18),(22)-(24) with dynamics given by (4)-(7), and (12)-(13). In addition, let the assumption 1 be satisfied. Under the control law given by (49)-(54), following properties hold:

(i) *The positive limit set of the closed loop dynamics is given by*

$$\begin{aligned} \bar{\mathcal{M}} = & \{(N_1, e_{\Omega_1}, N_2, e_{\Omega_2}, N_3, e_{\Omega_3}, \zeta_1, \nu_1, \zeta_2, \nu_2, \zeta_3, \nu_3) \mid N_i \in \{I\} \cup \mathcal{N}, \\ & e_{\Omega_i} = \nu_i = \zeta_i = 0, i = 1, 2, 3\} \end{aligned} \quad (56)$$

where $\mathcal{N} = \{\exp(\pi \hat{a}) : a \in S^2\}$.

(ii) *The desired equilibrium configuration \mathcal{M}_1 given by*

$$\mathcal{M}_1 = (I, 0, I, 0, I, 0, 0, 0, 0, 0, 0, 0) \quad (57)$$

is almost globally asymptotically stable.

Proof. La Salle's invariance principle and Chetaev's instability theorem are used for completing the proof.

(i) Consider the Lyapunov function

$$\mathcal{V} = \Psi_\Omega + \Psi_r + \Psi_R.$$

Clearly $\mathcal{V} \geq 0$, and $\mathcal{V} = 0$ only when the desired control objectives are satisfied. \mathcal{V} is positive definite with respect to variables $I - X_i$, e_{Ω_i} , ζ_i and ν_i , for $i = 1, 2, 3$. Now taking derivative of \mathcal{V} along closed-loop trajectories yields,

$$\begin{aligned}\dot{\mathcal{V}} &= \dot{\Psi}_R + \dot{\Psi}_\Omega + \dot{\Psi}_r \\ &= \alpha_1 \left((-R_1^d)^\top s_{12} \times l_{12} - (R_1^d)^\top s_{123} \times l_{123} - (R_1^d)^\top s_{1223} \times (l_{1223}) + \tau_1 \right) \cdot e_{\Omega_1} \\ &\quad + \alpha_2 \left(-(Q_{21}^d)^\top l_{12} \times l_{21} - (Q_{21}^d)^\top l_{123} \times l_{213} + (Q_{21}^d)^\top l_{1223} \times l_{2113} + \tau_2 \right) \cdot e_{\Omega_2} \\ &\quad + \alpha_3 \left(-(Q_{31}^d)^\top l_{13} \times l_{31} - (Q_{31}^d)^\top l_{132} \times l_{312} + (Q_{31}^d)^\top l_{1332} \times l_{3112} + \tau_3 \right) \cdot e_{\Omega_3} \\ &\quad + \sum_{i=1}^3 e_{\Omega_i}^\top \left(J_i(e_{\Omega_i} + \Omega_i^d) \times (\Omega_i^d) + \tau_i - J_i \dot{\Omega}_i^d \right) \\ &\quad + \sum_{i=1}^3 (v_i - v_i^d) \cdot \left(K_i(r_i - r_i^d) - \frac{\mu}{\|r_i\|^3} r_i + u_i - \dot{v}_i^d \right).\end{aligned}$$

Substituting the control law described in (49)-(54) reduces above to,

$$\dot{\mathcal{V}} = - \sum_{i=1}^3 (\alpha_{\Omega_i} \|e_{\Omega_i}\|^2 + \alpha_{\nu_i} \|\nu_i\|^2).$$

Thus \mathcal{V} is bounded from below and $\dot{\mathcal{V}} \leq 0$, which implies that $\lim_{t \rightarrow \infty} \mathcal{V}(t)$ exists by monotonicity. Now by La Salle's invariance principle (theorem 6.19, [29]), the states of the system converge asymptotically to the largest positively invariant set in $\mathcal{V}^{-1}(0)$. To apply La Salle's invariance principle, a compact invariant set is constructed as follows

$$\begin{aligned}\mathcal{K} &= \left\{ (X_1, e_{\Omega_1}, X_2, e_{\Omega_2}, X_3, e_{\Omega_3}, \zeta_1, \nu_1, \zeta_2, \nu_2, \zeta_3, \nu_3) \right. \\ &\quad \left. \in \text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^{18} \mid \mathcal{V}(t) \leq \mathcal{V}(0) \right\}.\end{aligned}$$

\mathcal{K} forms an invariant set in $\text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^{18}$ since $\dot{\mathcal{V}} \leq 0$. Further, we have,

$$\dot{\mathcal{V}}^{-1}(0) = \left\{ (N_1, 0, N_2, 0, N_3, 0, \zeta_1, 0, \zeta_2, 0, \zeta_3, 0) \in \mathcal{K} \right\}. \quad (58)$$

It is easy to verify that the largest invariant set in $\dot{\mathcal{V}}^{-1}(0)$, is the following,

$$\begin{aligned}\bar{\mathcal{M}} &= \{ (N_1, e_{\Omega_1}, N_2, e_{\Omega_2}, N_3, e_{\Omega_3}, \zeta_1, \nu_1, \zeta_2, \nu_2, \zeta_3, \nu_3) \mid N_1, N_2, N_3 \in \{I\} \cup \mathcal{N}, \\ &\quad e_{\Omega_i} = \nu_i = \zeta_i = 0, i = 1, 2, 3 \}.\end{aligned} \quad (59)$$

Thus $\bar{\mathcal{M}}$ forms the positive limit set.

- (ii) The desired equilibrium configuration is shown to be almost globally stable in two steps. First, \mathcal{M}_1 is shown to be locally asymptotically stable. Second, the equilibrium configurations $\mathcal{M} \in \bar{\mathcal{M}} \setminus \mathcal{M}_1$ are shown to be unstable.

\mathcal{V} is positive definite and zero only at \mathcal{M}_1 and $\dot{\mathcal{V}} \leq 0$, which gives us Lyapunov stability of desired equilibrium. Now consider a conservative region of attraction of \mathcal{M}_1 is given by

$$\Psi_R(0) < \min(4\alpha_1, 4\alpha_2, 4\alpha_3) \quad (60)$$

$$\sum_{i=1}^3 \lambda_{\max}(J_i) \|e_{\Omega_i}\|^2 + \Psi_r < \min(4\alpha_1, 4\alpha_2, 4\alpha_3) - \Psi_R(0) \quad (61)$$

where λ_{\max} represents the largest eigenvalue. For all the undesired equilibrium configurations, $\Psi_{R_i} = 4$ for at least one i . Under (60) and (61),

$$\mathcal{V}(0) < \min(4\alpha_1, 4\alpha_2, 4\alpha_3). \quad (62)$$

Since $\dot{\mathcal{V}} \leq 0$,

$$0 \leq \mathcal{V}(t) < \mathcal{V}(0) < \min(4\alpha_1, 4\alpha_2, 4\alpha_3). \quad (63)$$

This guarantees that when initial conditions satisfy (60) and (61) the undesired equilibrium configuration is avoided and system dynamics converge to desired configuration \mathcal{M}_1 . This gives local asymptotic stability of \mathcal{M}_1 .

Consider $\mathcal{M} \in \bar{\mathcal{M}} \setminus \mathcal{M}_1$. Let the value of \mathcal{V} at \mathcal{M} be ϵ . It is known that $\epsilon \geq \min\{4\alpha_1, 4\alpha_2, 4\alpha_3\}$. Define

$$\mathcal{W} = \epsilon - \mathcal{V}. \quad (64)$$

At the undesired configuration \mathcal{M} , $\mathcal{W} = 0$. In arbitrarily close neighbourhood of \mathcal{M} it is possible to find points where $\mathcal{W} > 0$ with e_{Ω_1} , e_{Ω_2} , e_{Ω_3} , ν_1 , ν_2 or ν_3 non zero. This implies that the function $\mathcal{W} > 0$, and $\dot{\mathcal{W}} = -\dot{\mathcal{V}} > 0$ (strictly greater than zero because e_{Ω_1} , e_{Ω_2} , e_{Ω_3} , ν_1 , ν_2 , or $\nu_3 \neq 0$). Thus by Chetaev's theorem ([30], Theorem 3.3) in an arbitrarily small neighbourhood around the undesired equilibrium, there exist a solution trajectory that will escape, rendering the undesired equilibrium unstable.

Such a neighbourhood can be illustrated by following example. Let \mathcal{M}_2 be an undesired equilibrium, given by

$$\mathcal{M}_2 = (N_1, 0, I, 0, I, 0, 0, 0, 0, 0) \quad (65)$$

where $N_1 = \text{diag}([1 \ -1 \ -1])$. The value of the Lyapunov function becomes $\mathcal{V} = 4\alpha_1$ and $\mathcal{W} = 4\alpha_1 - \mathcal{V}$. Consider a small variation $\omega_1 \in \text{so}(3)$ such that for arbitrary small $\delta > 0$, the following condition is satisfied,

$$\text{tr}(I - N_1) - \text{tr}(I - N_1 \expm(\omega_1)) > \delta \quad (66)$$

where \expm is the matrix exponential. There exist e'_{Ω_1} , e'_{Ω_2} , e'_{Ω_3} , ν'_1 , ν'_2 and ν'_3 not all of them zero such that

$$\sum_{i=1}^3 \left(\alpha_{\Omega_i} \|e'_{\Omega_i}\|^2 + \alpha_{\nu_i} \|\nu'_i\|^2 \right) < \delta. \quad (67)$$

Thus set $\mathcal{M}_\delta = \{(N_1 \expm(\omega_1), e'_{\Omega_1}, I, e'_{\Omega_2}, I, e'_{\Omega_3}, 0, \nu'_1, 0, \nu'_2, 0, \nu'_3)\}$ where (66) and (67) are satisfied forms the neighbourhood where $\mathcal{W} > 0$ and $\dot{\mathcal{W}} > 0$.

Since any equilibrium $\mathcal{M} \in \bar{\mathcal{M}} \setminus \mathcal{M}_1$ is unstable, from theorem 3.2.1 of [31] each of them have non trivial unstable manifolds. Let W^s and W^c be the stable and center manifold of the undesired configuration \mathcal{M} , respectively. Then $M = W^s \cup A$, where $A \subset W^c$ denotes the set of initial conditions from which system dynamics can converge to the undesired equilibrium configuration \mathcal{M} . Since each undesired equilibrium has non trivial unstable manifolds, the set M is lower dimensional than the tangent space $TSO(3) \times \mathbb{R}^3 \times TSO(3) \times \mathbb{R}^3 \times TSO(3) \times \mathbb{R}^3 \times \mathbb{R}^{18}$, and therefore, its measure is zero. Any solution starting from outside of this zero-measure set asymptotically converges to the desired equilibrium configuration \mathcal{M}_1 . Thus \mathcal{M}_1 is almost globally asymptotically stable. Similar arguments have been used in [23].

□

6. Discussion

The feedback law proposed in the current work in order to achieve relative attitude tracking along with formation keeping does not require attitude estimation, which is essential for many existing control schemes. It is worthwhile to note that the control torques obtained here are similar in spirit to geometric control schemes with trace error candidate Lyapunov functions. Therefore the current framework can be used to rewrite many existing geometric control schemes with trace error candidate Lyapunov functions that require knowledge of relative attitudes to line-of-sight based control laws not needing relative attitude estimation.

For example, consider the problem of n rigid bodies trying to achieve consensus in attitudes. This is a problem considered in two different contexts in [5] and [32]. Connection topology is assumed to be an undirected connected spanning tree graph. Both these articles make use of trace error functions as part of the control Lyapunov function in the following form,

$$\mathcal{V} = \sum_{i=1}^n \left(\sum_{j \rightsquigarrow i} \text{tr} (I - R_i^\top R_j) + \Omega_i^\top J_i \Omega_i \right) \quad (68)$$

where $j \rightsquigarrow i$ indicate that agent j is connected by a communication architecture to agent i . The standard analysis based on Lyapunov theory and La Salle's invariance principle gives almost global asymptotic stability of the control law given by,

$$\tau_i = -k_{\Omega_i} \Omega_i - 2 \left(\sum_{j \rightsquigarrow i} \text{skew} (R_i^\top R_j)^\wedge \right) \quad (69)$$

where $k_{\Omega_i} > 0$. It is evident that the above control law assumes availability of relative attitude information between each spacecraft pair connected via a communication link.

Now let there be a non-collinear agent with index $p(i, j)$ for any two spacecraft i, j that are connected via a communication link. Let the line-of-sight unit vectors between each connected pair of spacecraft and to the common agent be measured in the respective body fixed frames. With our framework, the same control scheme as (69) can be implemented using appropriate line-of-sight unit vector measurements and angular velocity. The resulting controller given below will not require measurement/estimation of relative attitudes.

$$\tau_i^c = -k_{\Omega_i} \Omega_i - \sum_{j \rightsquigarrow i} (b_{ij} \times b_{ji} + b_{ijp(i,j)} \times b_{jip(i,j)} - b_{ijjp(i,j)} \times b_{jiip(i,j)}). \quad (70)$$

Here unit vectors $b_{ijp(i,j)}$ and $b_{jip(i,j)}$ are defined as in (29) and (30). Note that we do not make any restrictive assumptions on the line of sight dynamics other than non-collinearity of i, j and $p(i, j)$.

We have control τ_i^c which is equal to τ_i (from Lemma 2), but written in terms of LOS vectors. Results from [21] can be applied to the above case but would require reconstructing the proof entirely as the control law will vary in magnitude from (69), whose stability is already proven in literature.

Also note that in this example the communication graph considered between agents is a spanning tree which is more general than the daisy chain architecture used in [21]. Similarly, the consensus control law for rigid body agents under general undirected graph, developed in [15] can also be rewritten in terms of LOS vectors by utilizing our framework.

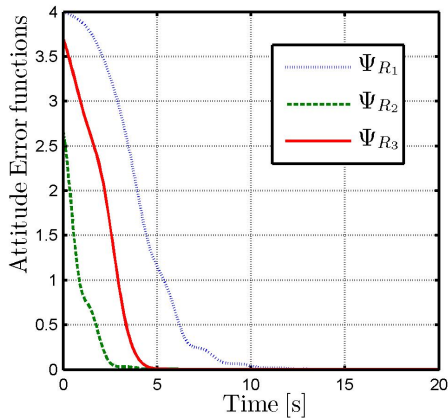
7. Simulation Results

For the purpose of simulations it is assumed that, (i) the leader follows a lower earth circular orbit 700 km above the surface with $\frac{\pi}{9}$ radians inclination (ii) spacecraft 2 trails the leader in the same orbit maintaining along track formation (iii) spacecraft 3 maintains 10 km separation from leader in inertial (ECI) z direction. Let $\rho = 7078.1\text{km}$ be the distance of spacecraft from centre of earth, $\beta = \sqrt{\frac{\mu}{\rho^3}}$ be the angular velocity of the desired orbit and $\phi = \frac{\pi}{9}$ radians be the inclination of the desired orbit. Let $\theta = 0.001$ radians, be the angle by which follower trails the leader. Mathematically, the desired position trajectories are given by,

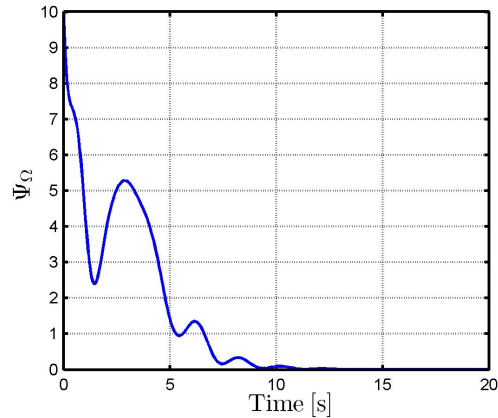
$$r_1^d(t) = \rho([\cos(\beta t) \sin(\beta t) \sin \phi \sin(\beta t) \cos \phi]^\top) \quad (71)$$

$$r_{21}^d(t) = r_1^d(t + \theta) - r_1^d(t) \quad (72)$$

$$r_{31}^d(t) = [0 \ 0 \ 10]^\top. \quad (73)$$



(a) Variation of attitude error functions Ψ_{R_i} , $i = 1, 2, 3$ with time



(b) Variation of angular velocity tracking error function Ψ_Ω , with time

Figure 2: Error functions with respect to time

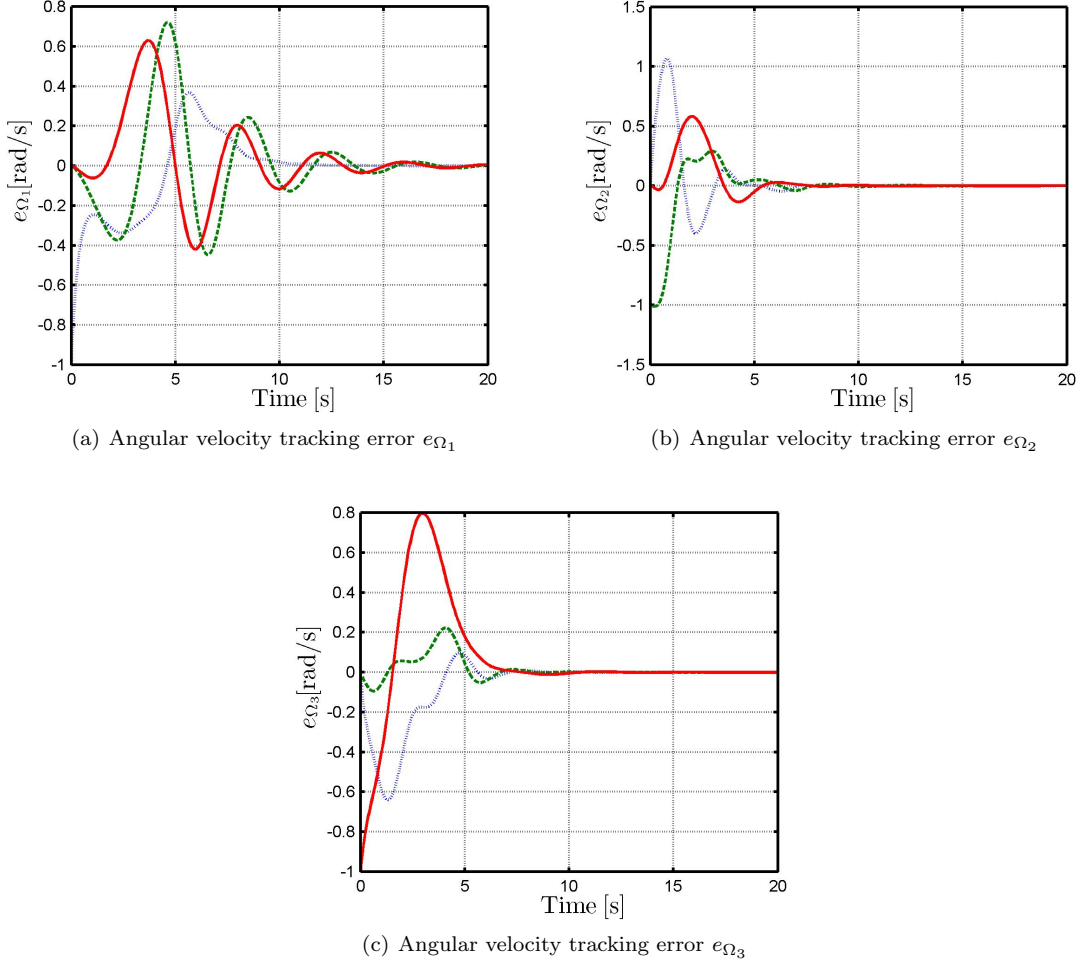


Figure 3: Angular velocity tracking errors of the spacecraft. (First component: dotted, second component: dashed, third component: solid)

The desired attitude of the leader is prescribed as,

$$R_1^d(0) = I; \quad \Omega_1^d(t) = [1 \ 0 \ 0]^\top. \quad (74)$$

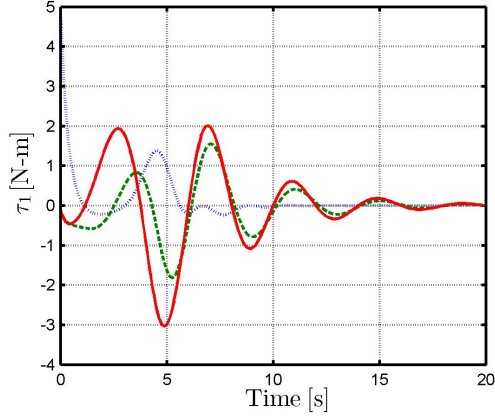
Let $a_1 = \frac{1}{\sqrt{3}}[1 \ 1 \ 1]^\top$ and $a_2 = [0 \ 0 \ 1]^\top$. The desired relative attitudes are chosen to be,

$$Q_{21}^d(0) = I; \quad \Omega_{21}^d(t) = [0 \ 1 \ 0]^\top \quad (75)$$

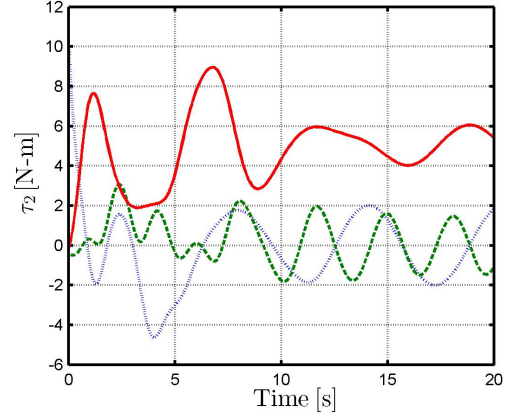
$$Q_{31}^d(0) = \expm(\frac{\pi}{3}\hat{a}_2); \quad \Omega_{31}^d(t) = [0 \ 0 \ 1]^\top \quad (76)$$

where \expm is the matrix exponential. Moments of inertia of all three spacecraft are taken to be identical, i.e. $J_i = \text{diag}[2, 3, 5]\text{Nm}^2$, for $i = 1, 2, 3$. Initial orientation of spacecraft 1 was chosen to be $R_1(0) = \exp(0.99\pi\hat{a}_1)$. Initial orientation of the follower spacecraft is chosen to be $R_2(0) = \exp(\pi\hat{a}_2)$ and $R_3(0) = I$. Initial angular velocities of all spacecraft were set to zero.

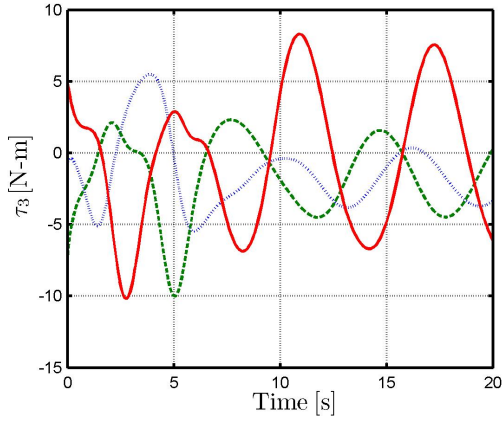
The initial positions of spacecraft 1, 2 and 3 are assumed to be $r_1(0) = r_1^d(0)$, $r_2(0) = r_1^d(0.001) + [1 \ 0 \ 1]^\top$ and $r_3(0) = r_1^d(0) + [0 \ 0 \ 11]^\top$. Initial velocities of spacecraft are chosen as



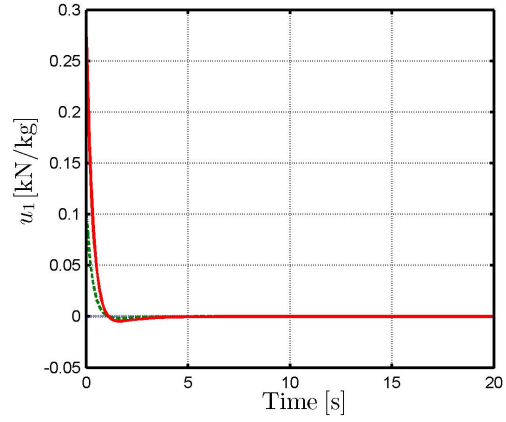
(a) Control torque applied to spacecraft 1



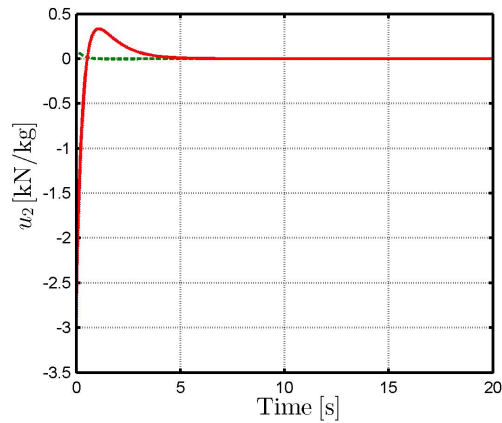
(b) Control torque applied to spacecraft 2



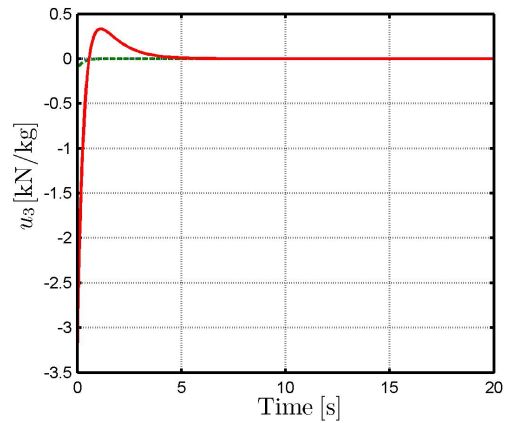
(c) Control torque applied to spacecraft 3



(d) Control force per unit mass applied to spacecraft 1



(e) Control force per unit mass applied to spacecraft 2



(f) Control force per unit mass applied to spacecraft 3

Figure 4: Torque and force inputs applied to the spacecraft. (First component: dotted, second component: dashed, third component: solid)

$v_1(0) = [0 \ 2.541 \ 6.981] km/s$, $v_2(0) = [0 \ 2.54 \ 6.981] km/s$ and $v_3(0) = [0 \ 2.592 \ 7.122] km/s$. Initial spacecraft velocities are chosen to reflect a 1% deviation from the desired values. The control gains

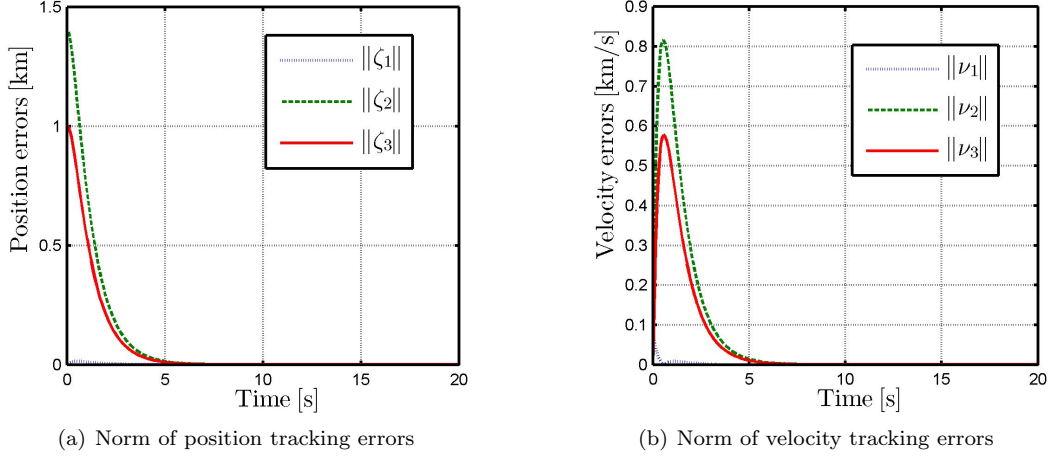


Figure 5: Position and velocity tracking errors

are chosen to be $\alpha_{\Omega_1} = \alpha_{\Omega_2} = 5$, $\alpha_{\Omega_3} = 7$, $\alpha_1 = \alpha_2 = 4$, $\alpha_3 = 3.6$, $K_1 = K_2 = K_3 = \text{diag}[3 \ 3 \ 3]$ and $\alpha_{v_1} = \alpha_{v_2} = \alpha_{v_3} = 4$.

Figure (2 a) shows Ψ_{R_i} , $i = 1, 2, 3$ converging to zero. From section 4.1 we know the maximum value of the attitude error function to be 4. As can be seen from figure (2 a), the attitude error function of leader spacecraft Ψ_{R_1} starts very close to 4, indicating the initial attitude of the leader to be almost 180 degree apart from the desired attitude at $t = 0$. Figure (2 b) shows variation of the combined angular velocity error Ψ_{Ω} with time. From (2) we can see that all attitude errors Ψ_{R_i} and angular velocity errors Ψ_{Ω} converge to zero by 10 seconds. The evolution of individual components of angular velocity errors are shown in figure (3).

The torque and force inputs are plotted in figures (4). Note that the control torques τ_2 and τ_3 do not decay to zero. This is because while desired relative angular velocities Ω_{21}^d and Ω_{31}^d are constants, the actual desired angular velocities Ω_2^d and Ω_3^d , given by equation (17), are time-varying. Figure (5) shows the norm of position and velocity tracking errors going to zero. The control gains can be modulated to ensure achievable control torques and forces from available actuators.

8. Concluding remarks

A novel control law for absolute attitude tracking for leader and relative attitude tracking for two follower while maintaining serial formation is proposed in this work. The attitude control laws are obtained in terms of line-of-sight vectors and guarantee asymptotic tracking of desired attitude trajectories even in the presence of spacecraft translation dynamics under gravity. No observations to external objects such as stars are used, which typically require expensive sensor equipment and complex on-board computation algorithms. The attitude control law proposed is coordinate free and therefore does not suffer from singularity or result in the winding phenomenon.

The desired equilibrium configuration is shown to be almost globally asymptotically stable under the proposed control law. The numerical example verifies our theoretical claims.

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Appendix A. Some Useful Results

The following results are useful. $\forall A_1, A_2 \in R^{n \times n}$, $a_1, a_2 \in \mathbb{R}^3$, and $R \in SO(3)$.

$$R^\top \widehat{a_1} R = \widehat{R^\top a_1} \quad (\text{A.1})$$

$$tr(A_1^\top A_2) = \sum_{i=1}^3 col_i(A_1) \cdot col_i(A_2) \quad (\text{A.2})$$

$$tr(\widehat{a_1} \widehat{a_2}) = 2(a_1 \cdot a_2) \quad (\text{A.3})$$

$$tr(\text{sym}(A_1) \widehat{a_1}) = 0 \quad (\text{A.4})$$

Here $col_i(A_1)$ is the i -th column of matrix A_1 . Also 3×3 identity matrix is denoted by I .

Lemma 1. *Let $tr()$ be the trace of a square matrix, defined as sum of its diagonal elements, then $\forall R_1 \in SO(3)$ and $\forall a_1 \in \mathbb{R}^3$,*

$$tr(R_1(R_1^\top a_1)^\wedge) = tr(R_1 \widehat{a_1}) \quad (\text{A.5})$$

$$= 2(\text{skew}(R_1))^\vee \cdot a_1 \quad (\text{A.6})$$

Proof. First part of lemma is easily proved using (A.1) and changing the order of the two matrices inside the trace function

$$tr(R_1(R_1^\top a_1)^\wedge) = tr(R_1 R_1^\top \hat{a}_1 R_1) = tr(R_1 \hat{a}_1) \quad (\text{A.7})$$

Now making use of the definition of skew and sym functions,

$$tr(R_1 \hat{a}_1) = tr((\text{skew}(R_1) + \text{sym}(R_1)) \hat{a}_1) \quad (\text{A.8})$$

$$= 2(\text{skew}(R_1))^\vee \cdot a_1 \quad (\text{A.9})$$

Last part is obtained by making use of linearity of the trace function, (A.4) and (A.3). \square

Lemma 2. Let $A, B \in \mathbb{R}^{3 \times 3}$, Then

$$\begin{aligned} (2\text{skew}(AB^\top))^\vee &= (AB^\top - BA^\top)^\vee \\ &= -(a \times x) - (b \times y) - (c \times z) \end{aligned} \quad (\text{A.10})$$

where a, b, c are columns of A and x, y, z are columns of B .

Proof. Let $\Gamma = (AB^\top - BA^\top)$. Clearly Γ is skew symmetric, thus has diagonal elements zero. Non-diagonal elements of Γ are obtained to be

$$(\Gamma)_{12} = (a_1 x_2 - a_2 x_1) + (b_1 y_2 - b_2 y_1) + (c_1 z_2 - c_2 z_1) \quad (\text{A.11})$$

$$(\Gamma)_{13} = (a_1 x_3 - a_3 x_1) + (b_1 y_3 - b_3 y_1) + (c_1 z_3 - c_3 z_1) \quad (\text{A.12})$$

$$(\Gamma)_{23} = (a_2 x_3 - a_3 x_2) + (b_2 y_3 - b_3 y_2) + (c_2 z_3 - c_3 z_2) \quad (\text{A.13})$$

By definition of $(\cdot)^\vee$ map,

$$(\Gamma)^\vee = \begin{bmatrix} -(\Gamma)_{23} & (\Gamma)_{13} & -(\Gamma)_{12} \end{bmatrix}^\top \quad (\text{A.14})$$

The standard expression of the vector product is given by,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_2 x_3 - a_3 x_2 \\ -(a_1 x_3 - a_3 x_1) \\ a_1 x_2 - a_2 x_1 \end{bmatrix} \quad (\text{A.15})$$

Substituting (A.11)-(A.13) in (A.14) and comparing terms with (A.15), the identity (A.10) is established. \square