Finding Eigenvalues of Matrices: Part I

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January 6, 2022

Finding Eigenvalues

Given that we have a matrix $A \in \mathbb{C}^{m \times m}$, we want to find the eigenvalues λ and associated eigenvectors v of A such that:

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0 \Rightarrow det(A - \lambda I) = 0.$$

This turns the problem to solve the equation:

$$p(x) = 0$$

where p is the characteristic polynomial of A.

However, this problem seems to be hard if p is greater than degree 5, since there is no general solution for polynomials with degree 5 or greater.

What should we do then?

Strategies

Since we cannot find general solutions for all p with degree greater than 5, so the basic method we proposed cannot find eigenvalues for all A.

Therefore, we could either

- Find some special matrices $\hat{A}$ which could form a p such that we could compute roots directly from it.

OR

- Find some way to transform A to structure of $\hat{A}$, while preserving properties like characteristic polynomials, eigenvlaues etc.

Case 1: Diagonal Matrices

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & m-1 & 0 \\ 0 & 0 & 0 & \dots & 0 & m \end{pmatrix}.$$

The characteristic polynomial is

$$p(x) =$$

The eigenvalues are

$$\lambda =$$

Case 2: The Eigenvalue Decomposition

if we could write A in the following form:

$$A = X\Lambda X^{-1}$$

where X non-singular, Λ diagonal.

We could then read-off eigenvalues from Λ as we mentioned above.

- **Why could we read off eigenvalues from Λ?**
- Since $A \to \Lambda$ is a similarity transformation, two similar matrices has same characteristic polynomial and eigenvectors.
- But note not every matrices could do the eigen value decomposition, since not all matrices are diagonalizable.

Case 3: Schur Factorisation

We firstly introduce **Schur Factorisation** where A could be written in:

$$A = QTQ^*$$

where Q is unitary and T is upper triangular.

And note that, **every square matrix has a Schur Factorisation.**

But how do we do that?

- Just like what we did in QR Factorisation, but a bit different:

$$A \to Q_1^* A Q_1 \to Q_2^* Q_1^* A Q_1 Q_2 \to \underbrace{Q_k^* \dots Q_2^* Q_1^*}_{=Q^*} A \underbrace{Q_1 Q_2 \dots Q_k}_{=Q} = T$$

- Note that this process stops untils the transformed matrix converges to an upper triangular matrix. **(seems complicated!)**

Similarity Transformation to Upper Hessenberg Form

Algorithm 1 Similarity Transformation to Upper Hessenberg Form

Require: $A \in \mathbb{C}^{m\times m}$

for $k = 1 \dots m-2$ **do**

$x \leftarrow A_{k+1:m,k}$

$v_k \leftarrow sign(x_1)\|x\|e_1 + x$

$v_k \leftarrow v_k/\|v_k\|$

$A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$

$A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m}v_k)v_k^*$

end for

The End