MATH40011 Calculus for JMC
Revision Note

Feifan Fan

March 30, 2020

Chapter 1

Differentiation

1.1 Basic Differentiation

1.1.1 Parametric Graph Sketching

We are going to sketch a parametric curve $x = f(t)$, $y = g(t)$.

Some basic rules:

- Try to find the function $y = h(x)$ between y and x

  and try to determine the symmetry of the curve:

  i.e. determine whether $h(x) = h(-x)$

  (even function, curve is symmetrical about y axis)

  or $h(x) + h(-x) = 0$

  (odd function, curve is symmetrical about the origin)

  or if we have $y^2 = t(x) \implies y = \pm\sqrt{t(x)}$

  (not a function, but curve is symmetrical about the x axis

  since a value of x points to two values with opposite signs)

- Find the zero points of the curve:

  Calculate the value of t such that $y = g(t) = 0$, and calculate the correspnding value of x to get the zero point $(x, 0)$

Some special rules for the parametric curves:

- Determine when the tangent line to the curve is *horizontal* and *vertical*:

Horizontal: find the value of t such that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 0$$

Hence, the tangent is horizontal if

$$\frac{dy}{dt} = g'(t) = 0 \text{ and } \frac{dx}{dt} = f'(t) \neq 0$$

Vertical: find the value of t such that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \pm\infty$$

Hence, the tangent is horizontal if

$$\frac{dy}{dt} = g'(t) \neq 0 \text{ and } \frac{dx}{dt} = f'(t) = 0$$

- Determine at which points is the tangent line to the curve parallel to $y = x$: Since for the line $y = x$, $\frac{dy}{dx} = 1$,

  Therefore, if the tangent line is parallel to y = x,

  $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 1$$

  Hence, we have

  $$g'(t) = \frac{dy}{dt} = \frac{dx}{dt} = f'(t) \text{ where } g'(t) = f'(t) \neq 0$$

- Find the limit of the gradient when t is close to 0:

  $$\lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ and } \lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Similarly, find the limit of the gradinet when t tends to $\pm\infty$:

  $$\lim_{t\to\pm\infty} \frac{dy}{dx} = \lim_{t\to\pm\infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

  In conclusion, we can combine the elements above and find the "dirction" of t to sketch the parametric curve.

Example: Sketch the curve

$$\begin{cases} x = t^2 \\ y = t^3 - t \end{cases}$$

in the cartesian plane.

1. Firstly we know that $y = t^3 - t = t(t^2 - 1) = t(x - 1)$.
   And we squared y to get:

   $$y^2 = t^2(x-1)^2 = x(x-1)^2$$

   Therefore,

   $$y = \pm\sqrt{x(x-1)^2}$$

   Clearly this curve is symmetrical about x axis. (symmetry ✓)

2.

1.2 Differentiability & Continuity of a function