MATH40011 Calculus for JMC
Revision Note

Feifan Fan

April 2, 2020

Contents

Chapter 1

Differentiation

1.1 Basic Differentiation

1.1.1 Analysis of Parametised Variables

Given that a trajectory $f(x, y) = 0$, and the variables x, y are functions of time, t, how do we analyse the change of x, y respect to t, at several specific points? (for example, what is the value of $\frac{dy}{dt}$ when $x = 0$?)

Sometimes the trajectory have several distinct patterns that we are familiar with. We can use them to find the parametric equations of x, y first.
For example:

- The equation of a circle:

  $$f(x, y) = (x - a)^2 + (y - b)^2 - r^2 = 0$$

  We can easily find that $x - a = rcos\theta$ and $y - b = rsin\theta$.

  Therefore, the parametric equations of x, y are:

  $$\begin{cases} x = a + rcost \\ y = b + rsint \end{cases}$$

- The equation of an ellipse:

  $$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Similarly, we can get the parametric equations:

$$\begin{cases} x = acost \\ y = bsint \end{cases}$$

Obviously we can just find the value of t from the given value of x or y, and substitute the value into the formula of derivatives of y (or x) to find the rate of change of the required variable. However...

We do have a general method of find the rate of change of the parametised variable(s) from a given value:

- We have the expression of the trajectory, $f(x, y) = 0$. Therefore, the value of $\frac{dy}{dx}$ is really easy to get by implicit differenitation.
- Since we have:
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = g(x, y)$$
Therefore, from a given value of x (or y), the relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be confirmed, in the form $\frac{dy}{dt} = k\frac{dx}{dt}$ where k is a constant.
- Using implicit differentiation again, but respect to t, we can get the relationship between x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in the form of $h(x, y, \frac{dy}{dt}, \frac{dx}{dt}) = 0$.
- Since we know the relationship betwen $\frac{dy}{dt}$ and $\frac{dx}{dt}$, and the value of x (or y) is given, so find the rate of change of the other variable at some specific point is just trivial.

1.1.2 Parametric Graph Sketching

In this section, we are going to sketch a parametric curve $x = f(t)$, $y = g(t)$.

Some basic rules:

- Try to find the function $y = h(x)$ between y and x

  and try to determine the symmetry of the curve:

  i.e. determine whether $h(x) = h(-x)$

  (even function, curve is symmetrical about y axis)

or $h(x) + h(-x) = 0$

(odd function, curve is symmetrical about the origin)

or if we have $y^2 = t(x) \implies y = \pm\sqrt{t(x)}$

(not a function, but curve is symmetrical about the x axis

since a value of x points to two values with opposite signs)

- Find the zero points of the curve:

  Calculate the value of t such that $y = g(t) = 0$, and calculate the correspnding value of x to get the zero point $(x, 0)$

Some special rules for the parametric curves:

- Determine when the tangent line to the curve is *horizontal* and *vertical*: *Horizontal:* find the value of t such that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 0$$

  Hence, the tangent is horizontal if

$$\frac{dy}{dt} = g'(t) = 0 \text{ and } \frac{dx}{dt} = f'(t) \neq 0$$

  Then we could calculate the value of t such that $h(t) = \frac{dy}{dx} = 0$, and find the corresponding value of $\frac{d^2y}{dx^2}$, which is expressed as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dt}(h(t))\frac{dt}{dx}$$
$$= \frac{h'(t)}{\frac{dx}{dt}} = \frac{h'(t)}{f'(t)}$$

  We substitute the value of the valud of t into this formula,

  and we can determine:

$$\begin{array}{l} \frac{d^2y}{dx^2} = \frac{h'(t)}{f'(t)} < 0 \implies \text{local maximum} \\ \frac{d^2y}{dx^2} = \frac{h'(t)}{f'(t)} = 0 \implies \text{inflection point} \\ \frac{d^2y}{dx^2} = \frac{h'(t)}{f'(t)} > 0 \implies \text{local minimum} \end{array}$$

Vertical: find the value of t such that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \pm\infty$$

Hence, the tangent is horizontal if

$$\frac{dy}{dt} = g'(t) \neq 0 \text{ and } \frac{dx}{dt} = f'(t) = 0$$

- Determine at which points is the tangent line to the curve parallel to $y = x$: Since for the line $y = x$, $\frac{dy}{dx} = 1$,

  Therefore, if the tangent line is parallel to y = x,

  $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 1$$

  Hence, we have

  $$g'(t) = \frac{dy}{dt} = \frac{dx}{dt} = f'(t) \text{ where } g'(t) = f'(t) \neq 0$$

- Find the limit of the gradient when t is close to 0:

  $$\lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ and } \lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Similarly, find the limit of the gradinet when t tends to $\pm\infty$:

  $$\lim_{t\to\pm\infty} \frac{dy}{dx} = \lim_{t\to\pm\infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

  In conclusion, we can combine the elements above and find the "dirction" of t to sketch the parametric curve.

Example: Sketch the curve

$$\begin{cases} x = t^2 \\ y = t^3 - t \end{cases}$$

in the cartesian plane.

1. Firstly we know that $y = t^3 - t = t(t^2 - 1) = t(x - 1)$.
   And we squared y to get:

$$y^2 = t^2(x-1)^2 = x(x-1)^2$$

   Therefore,

$$y = \pm\sqrt{x(x-1)^2}$$

   Clearly this curve is symmetrical about x axis. *(symmetry ✓)*

2. Then we let $y = 0$ to find the zero point(s), i.e.

$$y = t^3 - t = t(t^2 - 1) = 0$$

   So we have:

$$t = 0 \text{ or } t^2 = 1$$

   Therefore,

$$t = 0 \text{ or } \pm 1$$

   We substitute the value of t in the expression of x, and get:

$$\begin{array}{lll} t = 0, & x = 0 & y = 0 \\ t = 1, & x = 1 & y = 0 \\ t = -1, & x = 1 & y = 0 \end{array}$$

   Therefore, the zero points of this parametric curve is $(0, 0)$ and $(1, 0)$.

   (zero points ✓)

3. Next we need to determine when the tangent of the curve is horizontal (also determine the local extrema) and when it is vertical:

   Horizontal: We can easily calculate that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 1$.

   Therefore, let $\frac{dy}{dt} = 3t^2 - 1 = 0$:

   We then have:

$$\begin{array}{l} t^2 = \frac{1}{3} \\ t = \pm\frac{\sqrt{3}}{3} \end{array}$$

   We confirmed that when $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$.

   Therefore, the points are at:

$$\begin{array}{lll} t = \frac{\sqrt{3}}{3}, & x = \frac{1}{3} & y = -\frac{2\sqrt{3}}{9} \\ t = -\frac{\sqrt{3}}{3}, & x = \frac{1}{3} & y = \frac{2\sqrt{3}}{9} \end{array}$$

i.e. The (local extrema) points are $(\frac{1}{3}, -\frac{2\sqrt{3}}{9})$ and $(\frac{1}{3}, \frac{2\sqrt{3}}{9})$.

Since

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{3t^2-1}{2t})$$

$$= \frac{d}{dt}(\frac{3}{2}t - \frac{1}{2t})\frac{dt}{dx} = \frac{1}{2t}(\frac{3}{2} + \frac{1}{2t^2}) = \frac{3}{4t} + \frac{1}{8t^3}$$

Clearly, we could see that when t is negative, so is the value of $\frac{d^2y}{dx^2}$.

In this case, $(\frac{1}{3}, \frac{2\sqrt{3}}{9})$ is the local maximum point.

Similarly, $(\frac{1}{3}, -\frac{2\sqrt{3}}{9})$ is the local minimum point.

Vertical:

Just like how we deal with the points with horizontal tangents, let $\frac{dx}{dt} = 2t = 0$:

Clearly, the only answer that satisfies the equation is $t = 0$.

Therefore, $(0, 0)$ is the point where the tangent is vertical.

(special points with horizontal/vertical tangents ✓)

4. Also we need to determine when the tangent is parallel to y = x:

   Let $\frac{dy}{dt} = \frac{dx}{dt}$, i.e. $2t = 3t^2 - 1 \implies 3t^2 - 2t - 1 = 0$:

   Then we have

$$\begin{aligned} &(t-1)(3t+1) = 0 \\ &t_1 = 1 \text{ or } t_2 = -\frac{1}{3} \end{aligned}$$

   Therefore, the points we required are:

$$\begin{aligned} t &= 1, \quad x = 1 \quad y = 0 \implies (1, 0) \\ t &= -\frac{1}{3} \quad x = \frac{1}{9} \quad y = \frac{8}{27} \implies (\frac{1}{9}, \frac{8}{27}) \end{aligned}$$

5. Finally we find the behaviour of the gradient, when it is close to 0 and/or it tends to $\pm\infty$:

$$\lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{3t^2-1}{2t} = -\infty$$

$$\lim_{t\to 0^-} \frac{dy}{dx} = \lim_{t\to 0^-} \frac{3t^2-1}{2t} = \infty$$

$$\lim_{t\to \infty} \frac{dy}{dx} = \lim_{t\to \infty} \frac{3t^2-1}{2t} = \infty$$

$$\lim_{t\to-\infty}\frac{dy}{dx}=\lim_{t\to-\infty}\frac{3t^2-1}{2t}=-\infty$$

Also since $x=t^2$, therefore,

$$\lim_{t\to 0}x=\lim_{t\to 0}t^2=0$$

$$lim_{t\to\pm\infty}x=\lim_{t\to\pm\infty}t^2=\infty$$

6. In conclusion, we combine the elements together and plot the graph below:

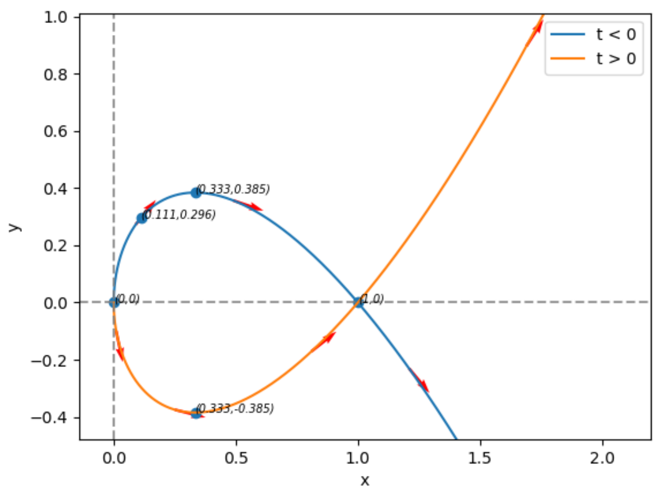

Figure 1.1: the parametric curve with direction of gradients

Remark. The direction of the red arrows on the graph indicates the direction of $\frac{dy}{dx}$, the gradient of the curve. These are good indcates in the graph sketching.

1.2 Differentiability & Continuity of a function

Appendix A

Code for Graph Sketching

This chapter contains all the code for demonstrative graph sketching in the notes. Most of the code is written in Python, a few lines of code are in Matlab, maybe there are some code in R as well.

A.1 Parametric Curve Sketching

This is the code for sketch the parametric curve in chapter():

$$\begin{cases} x = t^2 \\ y = t^3 - t \end{cases}$$

in the cartesian plane, written in Python:

```python
# this is the code for plotting the parametric graph:
import matplotlib.pyplot as plt
import numpy as np
import math

fig = plt.figure()
ax = fig.add_subplot(1, 1, 1)

# plot a graph for a given interval
def graph_plotting(start, end, interval, marking):
    x_value = []
    y_value = []
    for t in np.arange(start, end, interval):
        x = t ** 2
        y = t ** 3 - t
        x_value.append(x)
```

```python
        y_value.append(y)
    plt.plot(x_value, y_value, '-', label = marking)

graph_plotting(-1.5, 0, 0.0005, "t < 0")
graph_plotting(0, 1.5, 0.0005, "t > 0")

# plot arrows to demonstrate the direction of the change of t
def arrow_ploting(start, end, interval):
    for t in np.arange(start, end, interval):
        x = t ** 2
        y = t ** 3 - t
        if t >= 0:
            dx = (t + 0.0000001) ** 2 - x
            dy = (t + 0.0000001) ** 3 - (t + 0.0000001) - y
        else:
            dx = (t - 0.0000001) ** 2 - x
            dy = (t - 0.0000001) ** 3 - (t - 0.0000001) - y

        plt.quiver(x, y, dx, dy, color = "r", width = 0.005)

arrow_ploting(-1.5, 1.5, 0.4)

# plot reference lines to the graph
plt.axvline(0, color = 'gray', linestyle = '--', alpha = 0.8)
plt.axhline(0, color = 'gray', linestyle = '--', alpha = 0.8)

# plot special points on the graph
special_points = [(0, 0), (1, 0), (1 / 3, -2 * math.sqrt(3) / 9),
(1 / 3, 2 * math.sqrt(3) / 9), (1 / 9, 8 / 27)]

plt.scatter(*zip(*special_points))

for point in special_points:
    x0 = point[0]
    y0 = point[-1]
    ax.text(x0, y0,
    "(" + str(round(x0, 3)) + "," + str(round(y0, 3)) + ")",
    fontsize = 7, color = "black", style = "italic",
    weight = "light")

plt.legend()
plt.xlabel("x")
plt.ylabel("y")
plt.show()
```

A.2 False Position

A.3 Starting the Appendices

Actually, using appendices is quite simple. Immediately after the end of the last chapter and before the start of the first appendix, simply enter the command `\appendix`. This will tell LaTeX to change how it interprets the commands `\chapter`, `\section`, *etc.*

Each appendix is actually a chapter, so once the `\appendix` command has been called, start a new appendix by simply using the `\chapter` command.

Note that the `\appendix` command should be called only once–not before the start of each appendix.

All the fancy referencing and tools still work. You only need to add the appendix command and all will be as it should be.

Appendix B

Another Appendix