Quantum Turbulence: An Exercise in Modeling?

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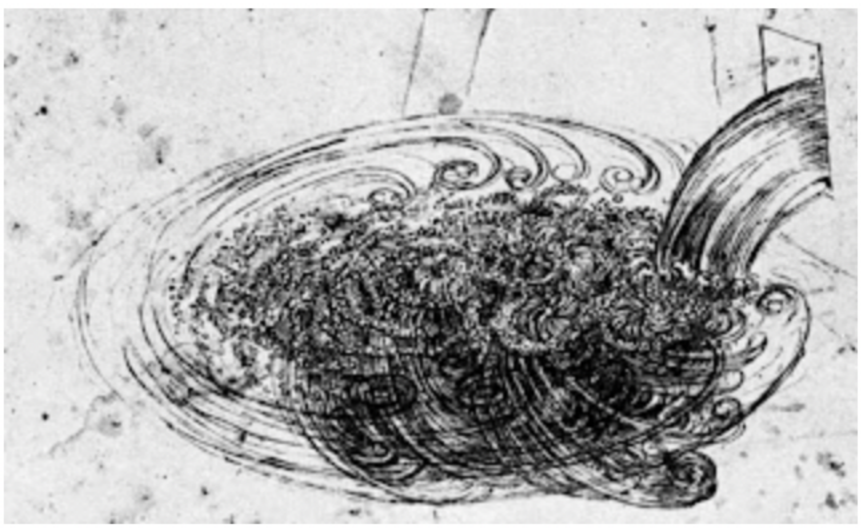
1 Abstract

Quantum turbulence, characterized by the chaotic motion of quantized vortices in Superfluid systems, represents a frontier in understanding non-classical fluid dynamics. We will explore the distinct behaviors of quantum turbulence and discuss the various models used to model those behaviors. Following the route of diving into the dynamics, we discuss how they are seen in experiment. Finally, we uncover the Quantum/Superfluid Turbulent steady state, and view its dynamics.

2 Introduction

Quantum Turbulence sits at the intersection of two important unsolved Physics problems. Understanding Turbulence has been a very long sought achievement which is still unsolved, but that has **vast** implications for not only fluid models in physicists labs, but large-scale models like climate modeling, and entire industries of engineering. On the other hand, understanding the quantum effects that lurk within an understanding of Quantum Turbulence would give powerful insight into similar phenomena in superconductors, or even our conception of how these quantum effects may act on a larger scale i.e. in neutron stars.

2.1 Classical Turbulence



(Fig.1, DaVinci's Drawing of turbulence in an artificial waterfall)

Classical Turbulence is a problem that has experienced centuries of scientific research. It also has wonderful large-scale effects that are particularly apparent if you look at the weather or Jupiter's Great Red Spot. Classical Turbulence shares many qualities with Quantum/Superfluid Turbulence. For one, they both exhibit energy cascades from large to small wavenumbers. Secondly, they both display a chaotic nature, and finally, they-given the right conditions-beget themselves. What begins as a small "puff" may end up a large cascade, or not. It is all dependent on whether that next domino falls.

3 Background

The idea of quantized circulation was first proposed by Onsager for a series of annular rings in a rotating superfluid. Feynman considered that a vortex in a superfluid can take the form of a vortex filament with quantized circulation κ and a core of atomic dimensions. Early experimental studies on superfluid hydrodynamics focused primarily on thermal counterflow. The flow is driven by an injected hear current, and the normal fluid and superfluid flow in opposite directions. The superflow was found to become dissipative when the relative velocity between the two fluids exceeds a critical value. Gorter and Mellink attributed the dissipation to mutual friction between two fluids and considered the possibility of superfluid turbulence. Feynman proposed a turbulent superfluid state consisting of a tangle of quantized vortices. Hall and Vinen performed experiments of second sound attenuation in rotating ^4He, where second sound refers to the entropy wave in which superfluid and normal fluid oscillate oppositely, and its propagation and attenuation give information on the vortex density in the fluid. They found that mutual friction arises from the interaction between the normal fluid and quantized vortices. Vinen confirmed Feynman's findings experimentally by showing that the dissipation in thermal counterflow arises from mutal friction between vortices and the normal flow. Vinen also succeeded in observing quantized circulation using vibrating wires in rotating superfluid ^4He[11]. Subsequently, many experimental studies have examined superfluid turbulence (ST) in thermal counterflow systems and have revealed a variety of physical phenomena. Since the dynamics of quantized vortices are nonlinear and non-local, it has not been easy to quantitatively understand these observations on the basis of vortex dynamics. Schwarz clarified the picture of ST based on tangled vortices by numerical simulation of the quantized vortex filament model in the thermal counterflow. However since the thermal counterflow has no analogy in conventional fluid dynamics, this study was not helpful in clarifying the relationship between ST and classical turbulence. ST is often called quantum turbulence (QT), which emphasizes quantum effects. QHD, including QT, is reduced to the motion of quantized vortices. Hence, understanding the dynamics of quantized vortices is a key issue in QHD.[9] While there has been some success in modeling Superfluid Turbulence, much of the problem comes from trying to model it appropriately at different temperatures and length scales.

4 Models

Since there has not yet been a theory in Quantum Turbulence that is able to consistently model the system across all length scales and temperatures, I will present a variety.

4.1 2-Fluid Model

At $T = 0$ we assume that all of our fluid is Superfluid. However, getting to $T = 0$ in order to measure properties of pure Superfluids is like going to Sweden to get a funny necklace. In order to better understand real Superfluids, often there is a need to also deal with the parts of the overall fluid that are not Superfluid. In this case, we are able to model the overall fluid as a pair of some vector fields v_n, v_s that correspond to the "normal" fluid and Superfluid respectively. These normally obey the limits $\frac{\rho_s}{\rho} \to 0$ for $T >> T_c$ and $\frac{\rho_s}{\rho} \to 1$ for $T \to 0$. Note that v_s, v_n are **independent** velocity fields.[1] The independent nature of v_n and v_s becomes apparent in a phenomenon called Counterflow.

4.1.1 Counterflow

The term Counterflow refers to a situation in a two-fluid Superfluid, i.e. $\rho_n, \rho_s > 0$ such that these two parts have equal an opposite flow rates. For example, if there is a heating element introduced to one end of a container of fluid, the normal fluid will-as usual-expand, creating a decrease in local fluid density, which the Superfluid part rushes to even out. Counterflow can be used experimentally to create Superfluid turbulence. Thermal Counterflow, being a relatively apparent phenomenon, was one of the first to gain attention experimentally in Superfluidity and Superfluid Turbulence.

4.2 HVBK

The HBVK, or Hall-Vinen-Bekarevich-Khalatnikov method for modeling Superfluid turbulence are expressed as

$$\partial_t \rho = -\nabla \cdot J \tag{1}$$

$$\partial_t J_i = -\partial_k T_i^k \tag{2}$$

$$\partial_t S = -\nabla \cdot (S v_n) + \frac{R}{T} \tag{3}$$

$$\rho_s \partial_t v_z + \rho (v_s \cdot \nabla) v_s = -\rho \nabla (\mu - \frac{1}{2} |v_s - v_n|^2) + \rho_s f \tag{4}$$

This describes the velocity fields of the two fluids v_s, v_n in the Galilean frame of the normal fluid, ignoring thermal diffusivity and viscosity. We also have the definitions of the Stress Tensor:

$$T_i^k = \rho_s v_{si} v_s^k + \rho_n v_{ni} v_n^k + (P + \lambda \cdot \omega) \delta_i^k - \lambda_i \omega^k \tag{5}$$

$$\tag{6}$$

Euler's Pressure Law:

$$P = -\varepsilon_0 + TS + \rho\mu \tag{7}$$

and the Superfluid First Law:

$$d\varepsilon_0 = \mu d\rho + T + (J - \rho v_n) \cdot d(v_s - v_n) + \lambda \cdot d\omega, \omega = \nabla \times v_s \tag{8}$$

The HVBK method gives the largest overview on ST, as it doesn't really resolve dynamics at the level that either the Gross-Pitaevskii Equation nor even the Vortex Filament Models do.

4.3 Vortex Filament Models

Vortex Filament Models act from the assumption that the quantum vortices are "filaments", or $1D$ strands in $3D$ space, which ends up meaning that they exhibit none of the vortex-vortex interactions that are predicted by the Gross-Pitaevskii Equation and observed experimentally. However, in some cases, they can allow for simulation of ST, particularly in higher temperatures as the Gross-Pitaevskii equation assumes $T = 0$, while these models do not.

4.3.1 Schwarz Model

In a Schwarz model, local Superfluid velocity $u_s(s,t)$ is evaluated as the sum of the background flow velocity u_{s0} and the velocity u_{in} induced at s by all the vortices, which can be calculated using the full Biot-Savart integral

$$u_{in} = \frac{\kappa}{4\pi} \int \frac{(s_1 - s) \times ds_1}{|s_1 - s|^3}$$

[2] This model can fall short of giving thorough predictions as it depends on the velocity field of the normal fluid u_n as a parameter, rather than a value to be calculated in the model itself, which leads to a flaw in description that the normal fluid flow does not need to evolve with the system, which is untrue. This is backed up by the result that numerical modelings of Schwarz model which use a local induction approximation or LIA may be unsuitable due to the absence of interactions between vortices.[]

4.3.2 2W Model

Attempting to resolve the most prominent issue of the Schwarz model, the use of u_n as a parameter, In the 2W model, u_n is calculated by solving the classical Navier-Stokes equation with an added mutual friction term.[2]

$$\frac{\partial u_n}{\partial t} + (u_n \cdot \nabla) u_n = -\frac{1}{\rho_{He}} \nabla P + \nu_n \nabla^2 u_n + \frac{F_{ns}}{\rho_n}$$

Where ρ_n and ρ_{He} are, respectively, the normal-fluid density and the total density of He II, P is the pressure, ν_n is the He II kinematic viscosity, and F_{ns} is the mutual friction per unit volume which can be calculated as

$$F_{ns} = \frac{1}{\Delta\Omega(r)} \int_{\mathcal{L}(r)} (-\frac{f_{sn}}{\Delta\xi}) d\xi$$

where $\mathcal{L}(r)$ denotes that the integration is performed along all the vortex lines in the computational cell $\Delta\Omega(r) = \Delta x \times \Delta y \times \Delta z$ located at r.[2]

4.3.3 S2W Model

In the S2W model, the mutual friction force that acts on a vortex segment $\Delta\xi$ is given by

$$f_{sn} = [-Ds' \times (s' \times (u_n - u_L)) - \rho_n \kappa s' \times (u_n - u_L)]\Delta\xi$$

where the only friction coefficient D can be calculated as

$$D = -\frac{4\pi\rho_n\nu_n}{[0.0772 + \ln(|u_n^{\perp} - u_L|\frac{a_0}{4\nu_n})]}$$

Here, $a_0 \approx 1\mathring{A}$ is the vortex-core radius and $u_n^{\perp}$ denotes the local normal-fluid velocity at the vortex-segment location that is projected in the plane perpendicular to the segment. By balancing the Magnus force f_M and the revised mutual friction force, the equation of motion for the vortex segment is now given by

$$\frac{ds}{dt} = u_s + \beta s' \times (u_n - u_s) - \beta' s' \times [s' \times (u_n - u_s)]$$

where the coefficients β and β' depend on D. Note that the hydrodynamic description of the normal fluid is applicable only when the roton mean free path in He II is much smaller than the relevant length scales. This path is about 30 nm at 1K and decreases with increasing the temperature. Therefore the hydrodynamic description of the normal fluid should be reasonable for a vortex ring of tens to hundreds of microns in diameter at temperatures above 1.6K.[2]

4.3.4 Vortex Filament Models: Overview

For a quantized vortex ring with a radius R moving in quiescent He II, the self-induced superfluid velocity at the ring's location is given by $u_s = \frac{\kappa}{4\pi R}[\ln(8\frac{R}{a_0}) - \frac{1}{2}]\hat{n}$, which is the same in all three vortex filament models. However, the local u_n is different, which leads to the different mutual friction dissipation rate. In the Schwarz model, $u_n = 0$ and therefore the highest mutual friction dissipation is expected. In both the 2W model and the S2W model, the back action of the mutual friction in the normal fluid generates two oppositely polarized normal-fluid vortex rings. In the 2W model, the two normal-fluid vortex rings are concentrically located nearly in the same plane as the quantized vortex ring, whereas in the S2W model, the two formal fluid rings are slightly shifted to above and below the quantized-ring plane. This shift changes the direction of the local u_n. Nonetheless, the induced local u_n in both models has a component in the same direction as the local u_s, which effectively reduces the mutual friction dissipation as compared to that in the

Schwarz model.[2]

Although these Vortex Filament Models can provide some perspective on of the mesoscale dynamics of Superfluid with sparse vortices, they ultimately fail due primarily to the fact that they do not deal with the phenomena of vortex reconnections, i.e. in these models, the vortices are largely fixed, making it impossible to model the effects of vortex reconnections, a key phenomenon in Superfluid Turbulence.

5 Gross-Pitaevskii Equation

Idea: Hamilton for interacting bosons in the pseudopotential approximation.
Note: Use Heisenberg equation of motion for field operators

$$\mathcal{H} = \sum_k \frac{\hbar^2 k^2}{2m} a_k^+ a_k + \frac{u_0}{2v} \sum_{k_1,k_2,k_3,k_4} a_{k_1}^+ a_{k_2}^+ a_{k_3} a_{k_4} \cdot \delta_{k_1+k_2+k_3+k_4} \tag{9}$$

Write in terms of field operators

$$\hbar\vec{k} = -i\hbar\vec{\nabla} \tag{10}$$

$$\mathcal{H} = \int d^3\vec{r} \left[\frac{\hbar^2}{2m} \nabla\Psi^* \cdot \nabla\Psi + \frac{u_0}{2} \Psi^*(r)\Psi^*(r) \cdot \Psi(r)\Psi(r) \right] \tag{11}$$

$$i\hbar\partial_t \hat{\Psi}(r,t) = \left[\hat{\Psi}(r,t), \hat{\mathcal{H}} \right] \tag{12}$$

$$\left[\hat{\Psi}(r,t), \hat{\mathcal{H}} \right] = -\frac{\hbar^2}{2m} \nabla^2 \hat{\Psi} + u_0 \hat{\Psi}^*(r,t)\hat{\Psi}(r,t)\hat{\Psi}(r,t) \tag{13}$$

$\Psi = \Psi_0(r,t) + \hat{\Psi}(r,t)$

$$\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 \right) \Psi = u_0 |\Psi_0|^2 \Psi_0 \tag{14}$$

Where $V(r) = u_0|\Psi_0|^2$ is the "potential".[12]

The Gross-Pitaevskii Equation can describe quantum vortex line in Bose-Einstein Condensates with environments including external potentials etc...

However, the GPE doesn't work for $T > 0$, making it hard to made accurate, experimentally verifiable, predictions with it. It does the best out of any of the models at describing vortex-vortex interactions, however, its lack of generalization beyond $T = 0$ makes it hard to use experimentally.

6 Experiments

6.1 Second Sound Attenuation

[10] Second sound is an entropy wave that is predicted by how two-fluid hydrodynamics couple to the superfluid order parameter. "First sound" exists both above and below the superfluid transition. However, "second sound" only propagates in Superfluid. This "second sound" is characterized by the shear viscosity and the thermal conductivity.

$$\chi_{nn}(k,\omega) = \frac{nk^2}{m}\frac{\omega^2 - v^2k^2 - iD_sk^2\omega}{(\omega^2 - c_1^2k^2 + iD_1k^2\omega)(\omega^2 - c_2^2k^2 + iD_2k^2\omega)} \tag{15}$$

This equation, deduced from momentum and energy conservation, describes the "density response function" at a wavenumber k and frequency ω. Here, c_1, c_2 and D_1, D_2 are, respectively, the speed and diffusivity of first and second sound[10]. Since we are able to measure both the first and second sound, with their primary difference being that the second sound not being able to propagate thorough normal fluid, we are able to estimate the density of Superfluid as we know that the second sound must attenuate when it travels through a region with no Superfluid. This information is important, because there are parts of the Superfluid that have distinct and sharp changes in their density(quantum vortices), and so we find that the attenuation of second sound in the Superfluid gives us a good way to estimate the vortex density in the path of our sound.

6.2 Reaching a Steady State

In order to better understand the mechanisms that regulate the dynamics of ST/QT, we will examine a Superfluid as it approaches and achieves a Steady State.

In the first stage ($0 \leq t \leq 0.4s$), the critical radius R_c determines the vortex destiny. Vortex ring sections in which the radius of curvature exceeds R_c expand in the direction perpendicular to v_{ns} through mutual friction, while small vortex rings shrink. Thus, vortices evolve and become anisotropic. At the end of this stage, large vortices appear that are comparable to the system size under periodic boundary conditions. These vortices survive with a large radius of curvature, and continuously generate small vortices by reconnections in the subsequent stages so that they function as "vortex mills". In the second stage ($0.4 < t \leq 2.0s$), vortex tangles undergo continuous evolution despite the decreasing anisotropy. As vortex rings expand, reconnections between vortices occur frequently. Reconnections generate vortices with various curvatures, resulting in them shrinking and expanding as discussed for the first stage. Local sections with a small radius of curvature formed by reconnections have an almost isotropic self-induced velocity, which prevents the vortices from lying perpendicular to v_{ns}. In addition, as the VLD increases, vortex expansion becomes slower than in the first stage because the reconnection distorts vortices, which prevents a vortex from smoothly expanding. In the third stage ($t > 2.0s$), the statistically steady state is realized by the competition between the growth and decay of a vortex tangle. The growth mechanism is still vortex expansion through mutual friction. The decay mechanism either creates vortices with local radii of curvature smaller than R_c or vortices with the self-induced velocity oriented in the opposite direction to v_{ns} after the reconnections. The increasing VLD causes more reconnections so that the decay mechanism becomes effective. When the VLD has increased sufficiently, the two mechanism begin to compete so that the vortex tangle enters that statistically steady state. The LIA calculation cannot realize this competition, which shows that vortex interaction is essential for creating a steady state.[9](presuming laminar normal flow)

7 Conclusion

I find these dynamics interesting, as they fly in the face of the original models of energy dissipation that were created when originally trying to model counterflow. Similarly, these mechanisms appear to be extremely dependent on vortex-vortex interactions, something that Vortex Filament Models completely miss. In a perfect world, the goal would be to be able to extend the Gross-Pitaevskii Equation to be able to completely cover all of the dynamics of a realistic superfluid system. However, trying to modify Vortex Filament models in unphysical ways to try to fit appropriate vortex-vortex dynamics is doable, but may lead to a more accurate model which we may use to learn almost nothing about the mechanisms at play in Quantum/Superfluid Turbulence. While this seems relatively obvious, this is the direction that I believe the field of Quantum/Superfluid Turbulence will take, especially given increased focus on AI probably leading to an uptick in trying to use computer/AI modeling to solve complex problems.

8 References

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