Extended EVSI problem

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Expected Utility formulation of the VOI problem: Expected Value of Partial Perfect Information (EVPXI) extension

Consider the following type of an Value of Information problem where a decision-maker determines the value of having access to a sample of information that allows them to resolve some uncertainty about the problem.

A decision-maker is interested in choosing a management program that will maximise the total size of the population. The population follows a binomial distribution:

$$M \sim \text{Binom}(n, p)$$

where M is a random variable that is distributed according to the binomial distribution with parameters n, the initial population size and p, the probability that an individual in the population will persist in the habitat after a defined period of time. Here, we assume that n is fixed in each problem. Perhaps this can represent the initial number of individuals in the population, or the number of individuals that are reintroduced to the environment as part of a threatened species reintroduction program in Canessa et al. (2015).

Specifically the parameter p depends on the management action the decision-maker chooses (action a), and the state of the world s that can be revealed through investing into a monitoring programme.

Consider the following structure of the problem:

	State 1	State 2
Action 1	$p_{1,1}$	$p_{1,2}$
Action 2	$p_{2,1}$	$p_{2,2}$

Here, $p_{a,s}$ is a random variable of the outcomes (based on a measured value) in action a and s state of the world. The sample of information can resolve the uncertainty over which state of the world s the world is in. Thus, the sample of information can enable the decision-maker to pinpoint which distribution of outcome is associated with each action.

We further assume that the decision-maker hopes to maximise the utility they derive, which is a function of the population size under management.

$$\max_a \mathbb{E}_s[U(M|p_{a,s})]$$

where the utility function is defined with a parameter z as follows:

$$U(m) = m^z, \quad z > 0$$

From this, we know that the expected utility of the system before resolving s can be expressed as follows:

$$EU_{\text{uncertainty}} = \max_a \mathbb{E}_s[U(M|p_{a,s})]$$

where $EU_{uncertainty}$ is the value of the system under uncertainty.

Likewise, we know that the expected utility of the system after resolving s can be written as follows:

$$EU_{\text{less uncertainty}} = \mathbb{E}_s[\max_a U(M|p_{a,s})]$$

where $EU_{\text{less uncertainty}}$ is the value of the system under uncertainty.

The Expected Utility of Partial Perfect Information (in terms of increases in utility) arising from resolving the state of the world s, thus knowing p, can thus be calculated as follows:

$$\text{EUPXI} = \mathbb{E}_s[\max_a U(M|p_{a,s})] - \max_a \mathbb{E}_s[U(M|p_{a,s})]$$

Because resolving s only resolves uncertainty over the distribution of M, and does not resolve the actual value of M, resolving s does not lead to knowledge of M for certain. Therefore, resolving s only leads to partial perfect information.

The problem with expressing the value of information in terms of expected utility is that it is not comparable across risk aversion levels of z, hence not comparable across decision-makers with different levels of risk aversion.

We can re-express the EUPXI problem to an Expected Value of Partial Perfect Information by expressing the changes in utility in terms of certainty-equivalents. Certainty equivalents can be interpreted as the certain number of population that produces the same level of utility as a random distribution, defined as follows:

$$U(\text{CE}) = \mathbb{E}[U(M)]\text{CE} = U^{-1}(E[(U(M)])$$

where CE is a certain value with no randomness that generates the same utility as the expected utility of M.

The change in the expected value caused by the partial perfect information can be given as follows:

$$\text{EVPXI} = U^{-1}(\mathbb{E}_s[\max_a U(M|p_{a,s})]) - U^{-1}(\max_a \mathbb{E}_s[U(M|p_{a,s})])$$

EVPXI when a "safe" action exists

We can work through an example based loosely on the Chytrid example in Canessa et al. (2015).

The number of frogs M follows a binomial distribution with an initial population n and probability of survival p. From the binomial distribution, we know that the mean μ and sigma σ^2 changes with respect to n and p:

$$\mu = np\sigma^2 = np(1-p)$$

The decision-maker is trying to examine whether or not to translocate frogs to a new site with potential Chytrid, with the probability of survival of an individual p dependent on whether or not Chytrid is present, and whether translocation is conducted:

	State 1 (No Chytrid) $\pi(s_1) = 0.5$	State 2 (with Chytrid) $\pi(s_2) = 0.5$
Action 1 (do nothing)	0.5	0.5
Action 2 (translocate)	0.675	0.275

where π is the probability of the state being true.

In this example, if the decision-maker does nothing, the expected number of frogs that will survive after the period is $0.5n$. The decision-maker can choose to translocate; if he translocates, then the expected number of frogs that will survive is $0.675n$ if the site is Chytrid-free and $0.275n$ if the site has Chytrid. For example, if $n = 200$, the number of frogs that will survive will follow the following distributions in terms of mean and variance, with the mean of the distributions matching those in Canessa et al. (2015):

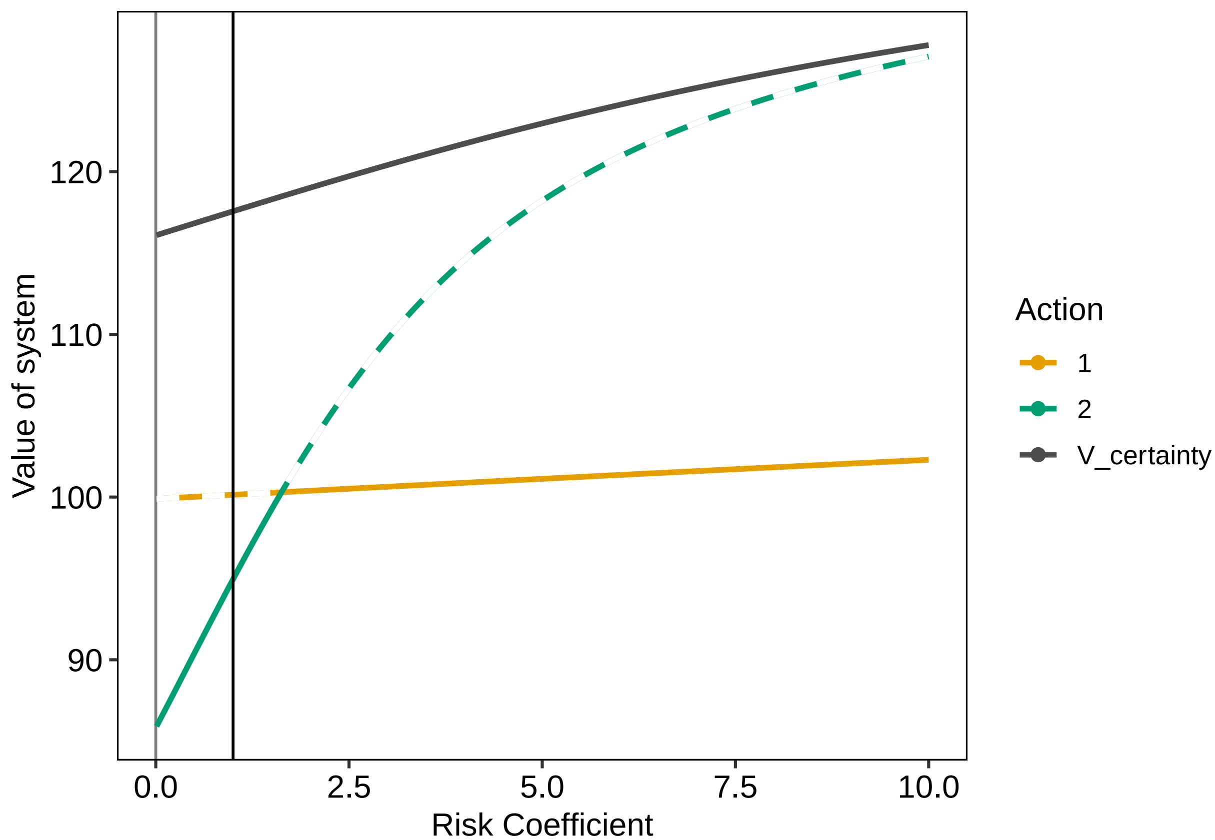
	State 1 (No Chytrid)	State 2 (with Chytrid)
Action 1 (do nothing)	$\mu = 100, \sigma^2 = 25$	$\mu = 100, \sigma^2 = 25$
Action 2 (translocate)	$\mu = 135, \sigma^2 = 29.6$	$\mu = 55, \sigma^2 = 19.9$

Here, if the decision-maker decides not to obtain partial perfect information, the optimal action would be to do nothing. If the decision-maker decides to do the test for Chytrid, they will translocate if no Chytrid is detected, and do nothing if Chytrid is detected.

We can then compute the EVPXI of this particular problem:

```r
set.seed(103905)

S <- 1000 # Number of draws from binomial distribution

gamma_seq = seq(0.01, 10, 0.01)
#gamma_seq[1] <- gamma_seq[1] + 1e-10
pref <- 'CE'

# Variation of Canessa (2015) Example 1 of Chytrid,
# with number of frogs modelled as binomial distribution
# with the same mean as Canessa
n <- 200

n_states <- S*2
n_actions <- 2

a11 <- rbinom(S, n, 0.5)
a21 <- rbinom(S, n, 0.5)
a12 <- rbinom(S, n, 0.675)
a22 <- rbinom(S, n, 0.275)

action_state <- matrix(c(a11, a21, a12, a22), byrow = T, ncol = S*2)
colnames(action_state) <- paste0('s', 1:n_states)
rownames(action_state) <- paste0('a', 1:n_actions)

Y <- list(c(1:S), c((S+1):(S*2)))
pY <- c(0.5, 0.5)

e1 <- list()
```

```
e1$action_state <- action_state
e1$p <- rep(1/(S*2), S*2)
e1$Y <- Y
e1$pY <- c(0.5, 0.5)
e1$evsi <- TRUE

e1_voi <- fcn_VOI_simulation(e1, pref = 'CE', gamma_seq = gamma_seq)

e1_ce <- lapply(gamma_seq, function(gamma) apply(e1$action_state, 1, pref_define(pref), lambda=gamma, p
  bind_rows()
e1_ce$V_certainty <- e1_voi$V_certainty
e1_ce$gamma <- gamma_seq
e1_ce_plt <- fcn_plt_ce(e1_ce, n_actions)+
  geom_line(data = e1_voi, aes(y = V_uncertainty, x = gamma), color = 'white', linetype = 2, linewidth =
  geom_vline(xintercept = 1)
e1_ce_plt
```



Here, we observe how the certainty-equivalent value of the systems change with respect to the risk coefficient (z) in the utility function. We focus on the changes in the value of adopting Action 1 under uncertainty (yellow), adopting Action 2 under uncertainty (green), and adopting the optimal action under partial perfect information. The black line shows the value of the system under partial perfect information, and the dashed line shows the value of the optimal action under complete uncertainty. The gap between the solid gray line and the dashed white line is thus is the value of partial perfect information.

We see the following:

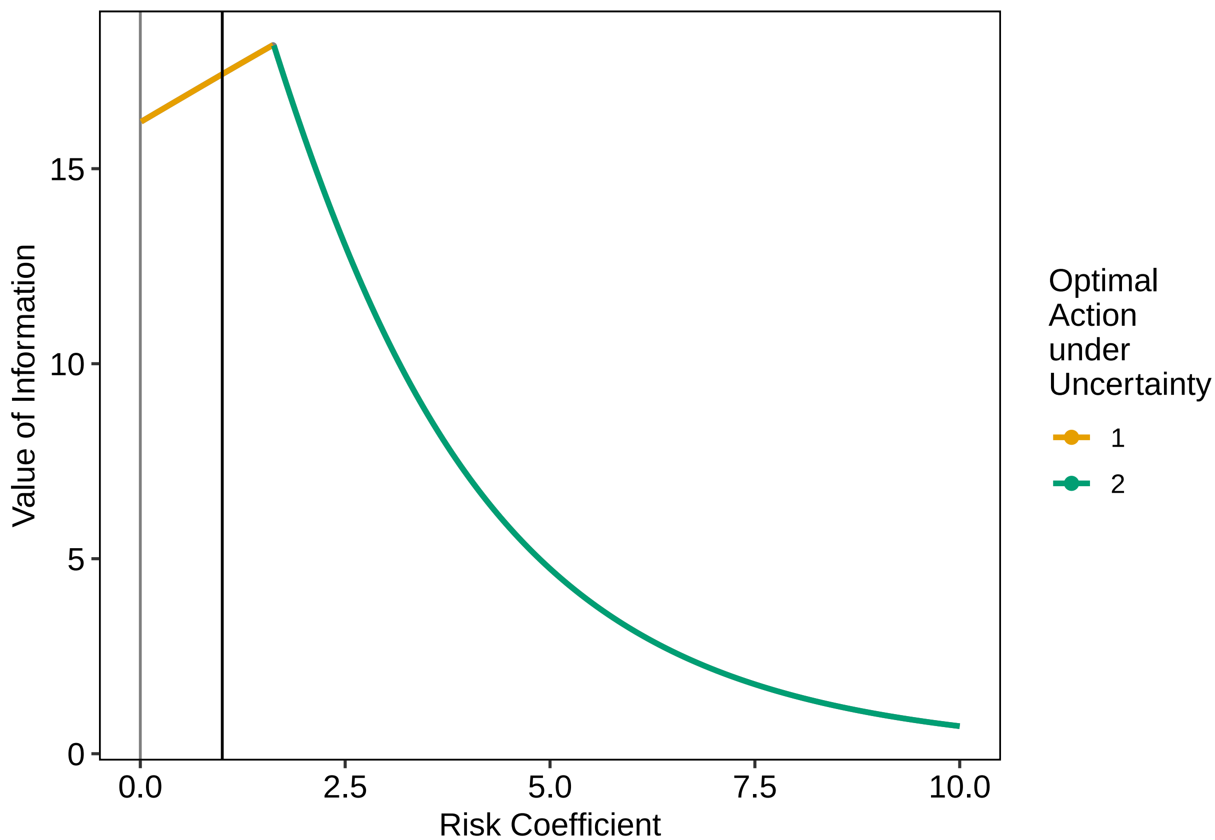
1. Action 1 is preferred under complete uncertainty for risk-averse (z<1) cases, as well as mild risk-loving

attitudes (z>1 and z<1.5). But a decision-maker who is strongly risk-loving (z>2) will act to choose Action 2 even under complete uncertainty, and has a high value for the random gamble by choosing Action 2 under complete uncertainty. The high value under complete uncertainty means that the partial perfect information has limited value.

2. Action 2 already provides a relatively certain outcome under complete uncertainty and protects the decision-maker from downsides (extremely low values of M). As the upside provided by the partial information becomes less valuable, the value of information thus declines as the decision-maker becomes more risk-averse.

We can then plot the value of partial perfect information as follows, colored by the optimal action chosen under uncertainty:

```
e1_vpi_plt <- fcn_plt_vpi(e1_voi, n_actions)+ geom_vline(xintercept = 1)
e1_vpi_plt
```

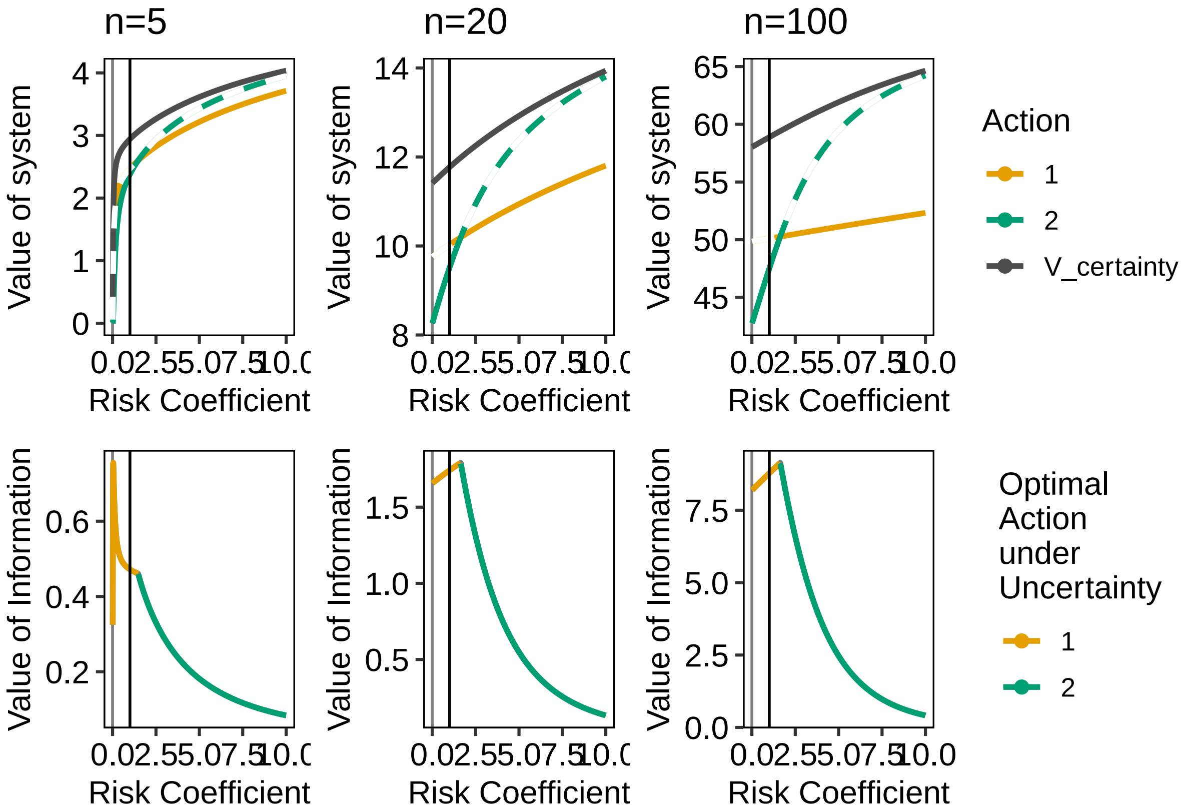


Here, we see that the value of information, in this case, declines with respect to stronger risk aversion (lower risk coefficient).

However, these results change if n changes. Changes to n particularly has an effect over whether or not M will reach really low levels close to 0, which is what a risk-averse decision-maker is particularly averse to.

```
N <- c(5, 20, 100)
plt_out <- list()
for (i in 1:length(N)) {
  n <- N[i]
```

```
  a11 <- rbinom(S, n, 0.5)
  a21 <- rbinom(S, n, 0.5)
  a12 <- rbinom(S, n, 0.675)
  a22 <- rbinom(S, n, 0.275)

  action_state <- matrix(c(a11, a21, a12, a22), byrow = T, ncol = S*2)
  colnames(action_state) <- paste0('s', 1:n_states)
  rownames(action_state) <- paste0('a', 1:n_actions)

  Y <- list(c(1:S), c((S+1):(S*2)))
  pY <- c(0.5, 0.5)

  e1 <- list()
  e1$action_state <- action_state
  e1$p <- rep(1/(S*2), S*2)
  e1$Y <- Y
  e1$pY <- c(0.5, 0.5)
  e1$evsi <- TRUE

  e1_voi <- fcn_VOI_simulation(e1, pref = 'CE', gamma_seq = gamma_seq)

  e1_ce <- lapply(gamma_seq, function(gamma) apply(e1$action_state, 1, pref_define(pref), lambda=gamma,
    bind_rows()
  e1_ce$V_certainty <- e1_voi$V_certainty
  e1_ce$gamma <- gamma_seq
  e1_ce_plt <- fcn_plt_ce(e1_ce, n_actions)+
    geom_line(data = e1_voi, aes(y = V_uncertainty, x = gamma), color = 'white', linetype = 2, linewidth
    geom_vline(xintercept = 1)
  e1_vpi_plt <- fcn_plt_vpi(e1_voi, n_actions)+ geom_vline(xintercept = 1)

  if (i != length(N)) {
    e1_ce_plt <- e1_ce_plt + theme(legend.position = 'none')
    e1_vpi_plt <- e1_vpi_plt + theme(legend.position = 'none')
  }
  e1_ce_plt <-e1_ce_plt + ggtitle(paste0("n=",n))

  plt_out[[i]] <- wrap_plots(e1_ce_plt / e1_vpi_plt)

}

plt_out[[1]] | plt_out[[2]] | plt_out[[3]]
```



Interestingly, we observe that specifically for the n=5 case, the value of information is actually highest in the risk-averse case. This is because when n=5, there is a significant chance (`dbinom(0,5,0.5)=0.03`) where choosing no action (Action 1) will cause population levels to reach 0. These situations generate very low utility for the risk-averse decision-maker. Framed in another way, the act of choosing no action itself is a risky outcome. In this case, the decision-maker would be much more willing to invest in partial information, because it reduces the chances of M reaching 0 to be `dbinom(0,5,0.5)*0.5 + dbinom(0,5,0.675)*0.5=0.01`. Thus, Action 1 is no longer a "safe" action. But if n is large enough, the probability that $M = 0$ is too small, such that for the risk-averse decision-maker, choosing Action 2 can already guarantee populations to persist in the habitat, making it a "safe" action.

EVPXI when there is no obvious "safe" action

We proceed to analyse the second example in Canessa, where there is no obvious "safe" action to take that can avoid really bad outcomes. In this example, the researchers are uncertain about a probability of survival parameter associated with the reintroduction of the threatened turtle. They can choose turtles of 3, 4 or 5 year old to release (Actions 1,2 and 3 respectively), but face competing hypotheses about which ones are true (State 1, 2 and 3). Which age group to release, and which hypothesis is true, jointly determines the survival rate of turtles that are reintroduced.

	State 1 $\pi(s_1) = 0.4$	State 2 $\pi(s_2) = 0.2$	State 3 $\pi(s_3) = 0.4$
Action 1	0.689	0.582	0.547
Action 2	0.729	0.674	0.484
Action 3	0.745	0.710	0.332

where the numbers represent the uncertain p parameter of the binomial distribution, and π represents the probability of each state.

Again, we assume that the number of turtles surviving after a defined time period M to follow a binomial distribution of n, the number of individuals released into the wild, and $p_{a,s}$, the probability of survival of each individual.

We first illustrate the results when $n = 10$, which gives us some significant probability for M to be equal to 0.

[to be cont'd]