Understanding Pseudo-similarity in Graphs: a Path to Proving the Reconstruction Conjecture

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1 Introduction

Two vertices u and v in a graph Γ are called *pseudo-similar* if the vertex-induced subgraphs $\Gamma - u$ and $\Gamma - v$ are isomorphic, but no automorphism of Γ maps u to v. Pseudo-similarity was discovered during attempts to prove the long-standing *Graph Reconstruction Conjecture*, of Kelly and Ulam [Kel57, Ula60]. Given a graph Γ, we define the *deck* of Γ, denoted $\mathrm{Deck}(\Gamma)$, to be the multiset of isomorphism classes of the graphs $\Gamma - u$ for each $u \in V\Gamma$. The Reconstruction Conjecture claims that every finite graph of order at least 3 is uniquely reconstructible from its deck; that is, given any two finite graphs Γ and Γ' on at least 3 vertices, $\mathrm{Deck}(\Gamma) \cong \mathrm{Deck}(\Gamma')$ if and only if $\Gamma \cong \Gamma'$. The Reconstruction Conjecture has resisted traditional algebraic methods for a considerable amount of time [NW78]. In particular, it is claimed in [HP66, Sto81] that a 'proof' of the Reconstruction Conjecture emerged in the 1960s which relied heavily on the non-existence of pseudo-similar vertices. This suggests that a valid proof of the Reconstruction Conjecture may be found through better understanding finite graphs which contain pseudo-similar vertices.

Alongside its study in connection with the Reconstruction Conjecture, pseudo-similarity has attracted interest for study in its own right, for example in [GK82, GK83, Lau97]. In particular, finding the size of the largest possible set of pseudo-similar vertices that a graph can contain has been the subject of intense research efforts [KP75, KJSS81], and the problem of determining the orders of the smallest graphs Γ_k each containing a set of mutually pseudo-similar vertices is still an open problem.

Both of these problems are closely related to the study of isomorphisms between induced subgraphs of a graph Γ. In [JJSS21], it is argued that the study of the *inverse monoid of partial automorphisms* of a graph Γ may be key to breakthroughs in problems like the Reconstruction Conjecture or understanding pseudo-similar vertices. A *partial automorphism* of a graph Γ is an isomorphism between its vertex-induced subgraphs, and the partial automorphisms of a graph Γ form an inverse monoid under composition of partial maps, denoted $\mathrm{PAut}(\Gamma)$. This monoid may be viewed as a generalisation of the automorphism group of Γ, as $\mathrm{PAut}(\Gamma)$ contains $\mathrm{Aut}(\Gamma)$ as a subgroup. $\mathrm{PAut}(\Gamma)$ is a rich source of additional information about the algebraic structure of Γ. For example, while many non-isomorphic graphs may have the same automorphism group (even more so, in [ER63] it is proven that almost all graphs have a trivial automorphism group), for any two finite graphs Γ and Γ', we have $\mathrm{PAut}(\Gamma) \cong \mathrm{PAut}(\Gamma')$ if and only if $\Gamma \cong \Gamma'$ or $\tilde{\Gamma} \cong \Gamma'$, where $\tilde{\Gamma}$ denotes the complement of Γ.

In this project, we study how the algebraic theory of inverse monoids of partial automorphisms of graphs began in [JJSS21] can be applied to understanding pseudo-similar vertices in graphs, with a particular emphasis on graphs with sets of mutually pseudo-similar vertices. These results are used to improve the best known upper bound on the size of a set of mutually pseudo-similar vertices in a graph Γ. Finally, we consider digraphs which contain pseudo-similar vertices. In general, it is easier to construct digraphs with pseudo-similar vertices than graphs. The falsity of the Reconstruction Conjecture for digraphs seems to support the connection between pseudo-similarity and graph reconstruction; indeed, the existence of certain families of digraphs with large sets of pseudo-similar vertices is key in the construction of infinite families of counterexamples to the digraph Reconstruction Conjecture [Sto77, Sto81]. We will find that the subtleties which exist for digraphs cause many of the results we prove for graphs to fail for digraphs.

1.1 Prerequisites from Graph Theory

A *graph* Γ is an ordered pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is the *vertex set* and $E\Gamma \subseteq [V\Gamma]^2$ is the *edge set*, which is a collection of 2-element subsets of $V\Gamma$. A *digraph* Γ is an ordered pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is the *vertex set* and $E\Gamma \subseteq V\Gamma \times V\Gamma$ is the *edge set*, which is a collection of ordered pairs of elements of $V\Gamma$. Digraphs may have loops, but have no multiple edges. All graphs and digraphs considered are finite. For more detail on general graph theory, see [Die24].

If Γ is a graph, and the vertices u and v are adjacent, then we write $u \sim v$. The *neighbourhood* of the vertex u in Γ is denoted by $N_\Gamma(u)$. The *degree* of the vertex u in Γ is written as $\deg_\Gamma(u)$. The *degree sequence* of Γ is the list of all the degrees of the vertices in Γ, with multiplicity, written in increasing order.

If Γ is a digraph, we write $u \to v$ to mean there is a directed edge from the vertex u to the vertex v. The *in-neighbourhood* of u in Γ is written as $N_{\Gamma\to}(u)$. Similarly, the *out-neighbourhood* of u in Γ is written as $N_{\Gamma\leftarrow}(u)$. If Γ is a digraph, we write $N_\Gamma(u)$ to mean $N_{\Gamma\to}(u) \cup N_{\Gamma\leftarrow}(u)$. The *extended neighbourhood* of u in Γ is the set $N'_\Gamma(u) = N_\Gamma(u) \cup \{u\}$.

An *isomorphism* φ between two graphs Γ, Γ' is a bijection

$$\varphi : V\Gamma \to V\Gamma'$$

which preserves both edges and non-edges. We denote isomorphism by $\Gamma \cong \Gamma'$. An *automorphism* is an isomorphism from a graph to itself. The group of automorphisms of Γ is denoted by $\mathrm{Aut}(\Gamma)$.

If Γ is a graph or a digraph, then the *induced subgraph* of Γ with vertex set $A \subseteq V\Gamma$ is denoted by $\Gamma[A]$. Given an isomorphism

$$\varphi : \Gamma[A] \to \Gamma[B]$$

between induced subgraphs of Γ, we abuse notation and write

$$\varphi : A \to B$$

where this will not cause confusion.

If $u \in V\Gamma$, the induced subgraph $\Gamma[V\Gamma \setminus \{u\}]$ is denoted by $\Gamma - u$. If $u, v \in V\Gamma$, the induced subgraph $\Gamma[V\Gamma \setminus \{u, v\}]$ is denoted by $\Gamma - u - v$.

We define the following two equivalence relations on $V\Gamma$.

1. Similarity: If $u, v \in V\Gamma$, then u and v are called *similar* if there is a $\varphi \in \mathrm{Aut}(\Gamma)$ such that $u\varphi = v$. This is denoted by $u \sim_S v$.

2. Removal-similarity: If $u, v \in V\Gamma$, then u and v are called *removal-similar* if $\Gamma - u \cong \Gamma - v$. This is denoted $u \sim_R v$.

Definition 1.1 (Pseudo-similarity)**.** Let Γ be a graph (or a digraph), and let $u, v \in V\Gamma$. Then u and v are called *pseudo-similar* if they are removal-similar but not similar. That is, if $u \sim_R v$ but not $u \sim_S v$.

While similarity and removal-similarity are both equivalence relations, pseudo-similarity is *not* an equivalence relation. It is irreflexive (since a vertex is always similar to itself, by the identity automorphism), and is not, in general, transitive. Therefore, in order to study sets of vertices which are mutually pseudo-similar, the following definition is made.

Definition 1.2 (Mutually pseudo-similar vertices)**.** Let Γ be a graph (or a digraph), and let $A = \{u_1, \ldots, u_k\} \subseteq V\Gamma$, with $k \geq 2$. The vertices in A are called *mutually* pseudo-similar if they are all pairwise pseudo-similar.

In [Lau97], it is observed that many of the most interesting questions about pseudo-similarity in graphs are concerned with sets of mutually pseudo-similar vertices, and these questions have attracted the attention of many authors. Mutually pseudo-similar vertices also appear to be the most enigmatic: while existence of pairs of pseudo-similar vertices has an explanation in terms of a kind of truncation of cyclic symmetry—see [GK82]— no such explanation exists for larger sets of mutually pseudo-similar vertices. Much of the work in this essay is concerned with finding necessary and sufficient conditions for sets of mutually pseudo-similar vertices to exist. In particular, we consider the following question, first asked in [KJSS81] and most recently in [JJSS21].

Problem 1.3. *For each k, what is the order of the smallest graph Γ_k containing a set of k mutually pseudo-similar vertices?*

1.2 Inverse Monoids of Partial Graph Automorphisms

A *partial automorphism* of a graph Γ is an isomorphism between two of its vertex-induced subgraphs. The partial automorphisms of Γ form an *inverse monoid* under the operation of composition of partial maps. This monoid is called the *inverse monoid of partial automorphisms* of Γ, or the *partial automorphism monoid* of Γ, and is denoted $\mathrm{PAut}(\Gamma)$. We denote the domain of a partial automorphism φ by $\mathrm{dom}(\varphi)$, and the image by $\mathrm{im}(\varphi)$. The *rank* of a partial automorphism is the size of its domain.

We are interested in two aspects of the algebraic structure of such monoids; namely, Green's relations and the natural partial order, which we shall review. For a good source on general semigroup theory, see [How95]. The theory we review here is developed in more detail in [JJSS21]. We use the cycle-and-path notation from [JJSS21] to represent partial permutations, however we write mappings on the *right* of their arguments, and compose maps from *left to right*. This, in particular, has the effect of swapping the roles of the $\mathscr{L}$ and $\mathscr{R}$-relations. We denote composition of partial maps by $\circ$.

For a monoid M, Green's relations are a collection of equivalence relations defined on M which describe divisibility in M. These are $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$; the final two of which coincide in the case M is finite. For $a, b \in M$, the $\mathscr{L}$-relation is defined by

$$a\mathscr{L}b \text{ if there exist } x, y \in M \text{ such that } xa = b \text{ and } yb = a.$$

Dually, the $\mathscr{R}$-relation is defined by

$$a\mathscr{R}b \text{ if there exist } x, y \in M \text{ such that } ax = b \text{ and } by = a.$$

The $\mathscr{H}$-relation is defined by $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. That is, $a\mathscr{H}b$ if and only if $a\mathscr{L}b$ and $a\mathscr{R}b$. The $\mathscr{D}$-relation is the largest equivalence relation containing both the $\mathscr{L}$ and $\mathscr{R}$-relations. For $a, b \in M$,

$$a\mathscr{D}b \text{ if there exists } x \in M \text{ such that } a\mathscr{L}x\mathscr{R}b.$$

The final relation, $\mathscr{J}$ is equal to $\mathscr{D}$ in the case when M is finite (Chapter 2, [How95]). We are interested in studying Green's relations on the partial automorphism monoid of a graph. When Γ is a finite graph, $\mathrm{PAut}(\Gamma)$ is also finite, therefore we will not need to consider the $\mathscr{J}$-relation.

To study pseudo-similarity in graphs, we need to understand which graph-theoretic properties are described by Green's relations on $\mathrm{PAut}(\Gamma)$. The following proposition is from [JJSS21].

Proposition 1.4. *Let Γ be a graph, and $\varphi, \psi \in \mathrm{PAut}(\Gamma)$. Then*

1. *$\varphi \mathscr{L} \psi$ if and only if $\mathrm{im}(\varphi) = \mathrm{im}(\psi)$.*
2. *$\varphi \mathscr{R} \psi$ if and only if $\mathrm{dom}(\varphi) = \mathrm{dom}(\psi)$.*
3. *$\varphi \mathscr{H} \psi$ if and only if $\mathrm{dom}(\varphi) = \mathrm{dom}(\psi)$ and $\mathrm{im}(\varphi) = \mathrm{im}(\psi)$.*
4. *$\varphi \mathscr{D} \psi$ if and only if $\mathrm{dom}(\varphi) \cong \mathrm{dom}(\psi)$.*

Proof. Let Γ be a graph, and let $\varphi, \psi \in \mathrm{PAut}(\Gamma)$.

1. Suppose $\varphi \mathscr{L} \psi$. Then, there exist $\alpha, \beta \in \mathrm{PAut}(\Gamma)$ such that $\alpha \circ \varphi = \psi$ and $\beta \circ \psi$. Since $\mathrm{im}(\alpha \circ \varphi) \subseteq \mathrm{im}(\varphi)$, it follows that $\mathrm{im}(\psi) \subseteq \mathrm{im}(\varphi)$. The other containment comes from the symmetric argument considering $\beta \circ \psi$. Now suppose $\mathrm{im}(\varphi) = \mathrm{im}(\psi)$. Then, $(\psi \circ \varphi^{-1}) \circ \varphi = \psi \circ (\varphi^{-1} \circ \varphi) = \psi \circ \mathrm{id}_{\mathrm{im}(\varphi)} = \psi$. The other product follows from swapping the roles of φ and ψ. Hence $\varphi \mathscr{L} \psi$.
2. This argument is dual to the argument in (1).
3. Since $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, it follows that $\varphi \mathscr{H} \psi$ if and only if $\varphi \mathscr{L} \psi$ and $\varphi \mathscr{R} \psi$. Therefore, $\varphi \mathscr{H} \psi$ if and only if $\mathrm{im}(\varphi) = \mathrm{im}(\psi)$ and $\mathrm{dom}(\varphi) = \mathrm{dom}(\psi)$.
4. $\varphi \mathscr{D} \psi$ if and only if there exists $\alpha \in \mathrm{PAut}(\Gamma)$ such that $\varphi \mathscr{L} \alpha \mathscr{R} \psi$. The first relation holds if and only if $\mathrm{im}(\varphi) = \mathrm{im}(\alpha)$, and the latter relation if and only if $\mathrm{dom}(\alpha) = \mathrm{dom}(\psi)$. Hence, $\varphi \mathscr{D} \psi$ if and only if there exists an isomorphism $\alpha : \mathrm{dom}(\psi) \to \mathrm{im}(\varphi)$. This is equivalent to $\mathrm{dom}(\psi) \cong \mathrm{im}(\varphi)$, or equivalently, $\mathrm{dom}(\psi) \cong \mathrm{dom}(\varphi)$.

□

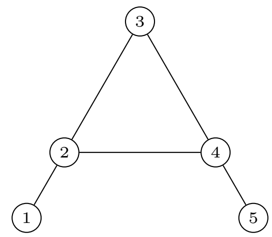
Remark. Proposition 1.4 is also true if we replace Γ with a digraph. The proof is the same.

Now we define the second aspect of the structure of $\mathrm{PAut}(\Gamma)$ we will need in order to study pseudo-similarity: the *natural partial order* on $\mathrm{PAut}(\Gamma)$. Given any induced subgraph $\Gamma[A]$ of Γ, the identity map on $\Gamma[A]$ is a partial automorphism, and is denoted $\mathrm{id}_{\Gamma[A]}$. The identity maps are *idempotents*, satisfying $(\mathrm{id}_{\Gamma[A]})^2 = \mathrm{id}_{\Gamma[A]}$, and are the only such idempotents of $\mathrm{PAut}(\Gamma)$. These idempotents can be used to define a partial order on $\mathrm{PAut}(\Gamma)$: given $\varphi, \psi \in \mathrm{PAut}(\Gamma)$, we define $\varphi \leq \psi$ if and only if there exists an idempotent e such that $\varphi = e \circ \psi$. This is the restriction ordering on $\mathrm{PAut}(\Gamma)$: for any $\varphi, \psi \in \mathrm{PAut}(\Gamma)$, the relation $\varphi \leq \psi$ is true if and only if φ is obtained by restricting the domain of ψ [JJSS21]. Where this is the case, ψ will be called an *extension* of φ, and φ is a partial automorphism *induced* by ψ. An element $\varphi \in \mathrm{PAut}(\Gamma)$ is called *maximal* in the partial order if there is no element $\psi \in \mathrm{PAut}(\Gamma)$ such that $\varphi \leq \psi$. That is, a partial automorphism is maximal if it cannot be extended to a partial automorphism with a larger domain.

2 Main Results

2.1 When does a graph contain a pair of pseudo-similar vertices?

It may first seem the case that vertices u, v are pseudo-similar in the graph Γ if there exists a partial automorphism $\varphi : \Gamma - u \to \Gamma - v$ which cannot be extended to an automorphism of Γ mapping u to v. This is, however, not sufficient, as shown in Example 2.1.



Γ

Figure 1: The graph Γ.

Example 2.1. Consider the graph Γ in Figure 1. The map

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & - \end{pmatrix}$$

is a partial automorphism of Γ of rank $|V\Gamma| - 1$ which maps the graph $\Gamma - 5$ to the graph $\Gamma - 1$. φ cannot be extended to an automorphism of Γ, as the only permutation which extends φ is

$$\varphi' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

which maps the edge $\{1, 2\}$ onto the non-edge $\{5, 3\}$. However, the vertices 1 and 2 are not pseudo-similar in Γ. The map

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

is an automorphism of Γ mapping 5 to 1, therefore $1 \sim_S 5$ in Γ.

The correct condition is stronger, and involves Green's $\mathscr{H}$-relation in $\mathrm{PAut}(\Gamma)$.

Theorem 2.2 (Pseudo-similarity)**.** *Let Γ be a graph with removal-similar vertices $u, v \in V\Gamma$. Then the following are equivalent:*

1. *The vertices u and v are pseudo-similar in Γ.*

2. *The $\mathscr{H}$-class H_φ of any partial automorphism $\varphi : \Gamma - u \to \Gamma - v$ has the property that every $\psi \in H_\varphi$ is maximal in the natural partial order on* $\mathrm{PAut}(\Gamma)$.

Proof. Let Γ, u and v be as given in the statement of the theorem. Suppose first that u and v are pseudo-similar in Γ. By assumption, $u \sim_R v$, so there exists a partial automorphism of Γ

$$\varphi : \Gamma - u \to \Gamma - v.$$

Suppose, for a contradiction, that there exists $\psi \in H_\varphi$ which is not maximal in the natural partial order on $\mathrm{PAut}(\Gamma)$. By Proposition 1.4, $\mathrm{dom}(\psi) = \Gamma - u$, and $\mathrm{im}(\psi) = \Gamma - v$, and so

$$\psi : \Gamma - u \to \Gamma - v.$$

Therefore, any extension ψ' of ψ must be of the form $\psi \vee (u\, v]$, and so maps u to v. Since ψ is not maximal in the natural partial order on $\mathrm{PAut}(\Gamma)$, it follows that $\psi' \in \mathrm{PAut}(\Gamma)$. Therefore, there is an automorphism ψ' of Γ which maps u to v. It follows that $u \sim_S v$ in Γ. This is a contradiction. Therefore, we must conclude that every $\psi \in H_\varphi$ is maximal in the partial order on $\mathrm{PAut}(\Gamma)$.

For the backward direction, let

$$\varphi : \Gamma - u \to \Gamma - v$$

be a partial automorphism. Such a φ exists by the assumption that $u \sim_R v$. Suppose that every $\psi \in H_\varphi$ is maximal in the partial order on $\mathrm{PAut}(\Gamma)$. We claim that no automorphism of Γ maps u to v. Suppose for a contradiction that $\theta \in \mathrm{Aut}(\Gamma)$ is such that $u\theta = v$. Then, θ induces a partial automorphism

$$\theta' : \Gamma - u \to \Gamma - v.$$

By Proposition 1.4, $\theta' \in H_\varphi$. However, $\theta' = \mathrm{id}_{\Gamma - u} \circ \theta$, and so $\theta' \leq \theta$ in the partial order on $\mathrm{PAut}(\Gamma)$. This contradicts the assumed maximality of every $\psi \in H_\varphi$. Therefore, if every $\psi \in H_\varphi$ is maximal in the natural partial order on $\mathrm{PAut}(\Gamma)$, then u and v are not similar. It follows that they are pseudo-similar. □

2.2 When does a graph contain a set of mutually pseudo-similar vertices?

Following Theorem 2.2, it seems appropriate to seek a characterisation of sets of mutually pseudo-similar vertices in terms of a graph's inverse monoid of partial automorphisms. Here we encounter the problem that pseudo-similarity is not an equivalence relation: it is irreflexive, and importantly, is not transitive. This is illustrated in examples 2.3 and 2.4. In contrast, Green's relations are all equivalence relations, which presents a difficulty in obtaining the desired characterisation. Given a $\mathscr{D}$-class D of rank $|V\Gamma| - 1$ elements in $\mathrm{PAut}(\Gamma)$, there may exist partial automorphisms in D (with their domain not equal to their image) which can be extended to automorphisms of Γ, and still, there may exist other partial automorphisms in D which cannot be extended to automorphisms of Γ. Two examples of this are given, one for digraphs and one for graphs; Figure 2 and Figure 3 respectively.

Example 2.3. Consider the digraph Γ in Figure 2. The vertex 1 is pseudo-similar to the vertex 5, and the vertex 5 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is not transitive in digraphs.

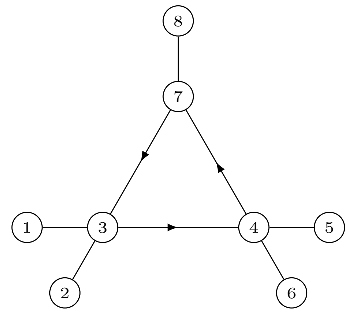


Figure 2: The digraph Γ.

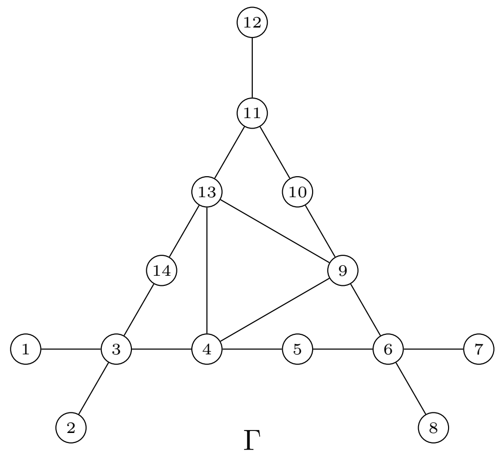


Figure 3: The graph Γ.

Example 2.4. Consider the graph Γ in Figure 3. The vertex 1 is pseudo-similar to the vertex 7, and the vertex 7 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is also not transitive in graphs.

For a set $A \subseteq V\Gamma$ to be mutually pseudo-similar, we must require that the conditions in Theorem 2.2 hold *pairwise* between the vertices in A. First, for concision, the following definition is made.

Definition 2.5. Given a graph Γ, and an $\mathscr{H}$-class H in $\mathrm{PAut}(\Gamma)$, then we say that H has the *element maximality property* if every $\varphi \in H$ is maximal in the natural partial order on $\mathrm{PAut}(\Gamma)$.

Theorem 2.6 (Mutually pseudo-similar vertices)**.** *Let Γ be a graph, and let $A = \{u_1, u_2, \ldots, u_k\} \subseteq V\Gamma$. Then the vertices in A are mutually pseudo-similar if and only if*

1. *The identity maps $\mathrm{id}_{\Gamma-u_1}$, $\mathrm{id}_{\Gamma-u_2}$, $\ldots$, $\mathrm{id}_{\Gamma-u_k}$ are all $\mathscr{D}$-related.*
2. *For any pair $u_i, u_j \in A$ of distinct vertices, the $\mathscr{H}$-class of any partial automorphism $\varphi : \Gamma - u_i \to \Gamma - u_j$ has the element maximality property.*

Proof. First suppose that Γ contains a set $A = \{u_1, u_2, \ldots, u_k\} \subseteq V\Gamma$ of vertices which are mutually pseudo-similar. Let $u_i, u_j \in A$, where $i \neq j$. By assumption, u_i and u_j are pseudo-similar, so are both removal-similar and not similar. Since u_i and u_j are removal-similar, $\Gamma - u_i \cong \Gamma - u_j$ and therefore $\mathrm{id}_{\Gamma - u_i} \mathscr{D} \mathrm{id}_{\Gamma - u_j}$ by Proposition 1.4. Moreover, since u_i and u_j are pseudo-similar, Theorem 2.2 implies that the $\mathscr{H}$-class of any partial automorphism

$$\varphi : \Gamma - u_i \to \Gamma - u_j$$

has the element maximality property. Since u_i and u_j were chosen arbitrarily, it follows that conditions (1) and (2) hold. This completes the forward direction.

Now suppose that Γ contains a set $A = \{u_1, u_2, \ldots, u_k\} \subseteq V\Gamma$ which satisfies conditions (1) and (2). Let $u_i, u_j \in A$, where $u_i \neq u_j$. By the assumption in condition (1), $\mathrm{id}_{\Gamma - u_i} \mathscr{D} \mathrm{id}_{\Gamma - u_j}$. Moreover, by Proposition 1.4,

$$\mathrm{dom}(\mathrm{id}_{\Gamma - u_i}) \cong \mathrm{dom}(\mathrm{id}_{\Gamma - u_j}),$$

so $\Gamma - u_i \cong \Gamma - u_j$. Therefore u_i and u_j are removal-similar.

By (2), the $\mathscr{H}$-class of any partial automorphism

$$\varphi : \Gamma - u_i \to \Gamma - u_j$$

has the element maximality property, and so, by Theorem 2.2, u_i and u_j are pseudo-similar. Since u_i and u_j were chosen arbitrarily, the vertices in A are pairwise pseudo-similar, hence are mutually pseudo-similar.

□

Remark. Theorems 2.2 and 2.6 are also true for digraphs. The proofs are identical.

Theorem 2.6 is limited in the sense that it is essentially Theorem 2.2 applied pairwise to the vertices in the set A. If these methods are to be used to provide a resolution to Question 1.3, then a stronger characterisation of mutually pseudo-similar vertices should be found. What is needed is a more complete theory of how Green's relations interact with the natural partial order on an inverse monoid M, in particular in the case when M is the partial automorphism monoid of a graph or a digraph.

2.3 When can a partial automorphism between vertex-deleted subgraphs of a graph Γ be extended to an automorphism of Γ?

This section considers when a partial automorphism of rank $|V\Gamma| - 1$ in a graph Γ can be extended to an automorphism of Γ.

Lemma 2.7. *Let Γ be a graph and $u, v \in V\Gamma$ be vertices. Let $\varphi : \Gamma - u \to \Gamma - v$ be a partial automorphism of Γ. Then φ can be extended to an automorphism of Γ if and only if φ induces a partial automorphism*

$$\psi : N_\Gamma(u) \to N_\Gamma(v).$$

Proof. Let Γ, u, v, and φ be as given in the statement of the lemma. Suppose φ induces a partial automorphism

$$\psi : N_\Gamma(u) \to N_\Gamma(v).$$

Since $\varphi : \Gamma - u \to \Gamma - v$, the only extension of φ is the function $\varphi' = \varphi \vee (u\,v]$ which maps u to v. A permutation of $V\Gamma$ is an automorphism of Γ if and only if it preserves both the edge and non-edge relations on $V\Gamma$; since φ' is an extension of the partial automorphism φ, it follows that φ' is an automorphism of Γ if and only if

$$u \sim w \text{ if and only if } v \sim w\varphi, \text{ for all } w \in V\Gamma.$$

Let $w \in V\Gamma$. Then $u \sim w$ if and only if $w \in N_\Gamma(u)$. Since φ induces the partial automorphism

$$\psi : N_\Gamma(u) \to N_\Gamma(v),$$

it follows that $w\varphi \in N_\Gamma(v)$ if and only if $w \in N_\Gamma(u)$. Hence, $u \sim w$ if and only if $v \sim w\varphi$. Therefore, φ' is an automorphism of Γ. This completes the forward direction.

For the backward direction, suppose that φ does not induce a partial automorphism

$$\psi : N_\Gamma(u) \to N_\Gamma(v).$$

Then, there exists $w \in N_\Gamma(u)$ such that $w\varphi \notin N_\Gamma(v)$. Therefore, w is such that $u \sim w$ but not $v \sim w\varphi'$, and so φ' is not an automorphism of Γ. This completes the backward direction. □

Theorem 2.8. *Let Γ be a graph with vertices $u, v \in V\Gamma$. Suppose that $\varphi : \Gamma - u \to \Gamma - v$ is a partial automorphism of Γ, and let A denote the image of $N_\Gamma(u)$ under φ. Then, u and v are similar in Γ if and only if $\Gamma - v$ has an automorphism mapping A to $N_\Gamma(v)$.*

Proof. Let Γ, u, v, φ and A be as given in the statement of the theorem. Suppose first that θ is an automorphism of $\Gamma - u$ such that $A\theta = N_\Gamma(v)$. We will show that $u \sim_S v$ in Γ. Consider the partial automorphism $\psi = \varphi \circ \theta$. Since

$$\varphi : \Gamma - u \to \Gamma - v,$$

and

$$\theta : \Gamma - v \to \Gamma - v,$$

it follows that $\mathrm{dom}(\psi) = \Gamma - u$, and $\mathrm{im}(\psi) = \Gamma - v$. Therefore,

$$\psi : \Gamma - u \to \Gamma - v.$$

Now, consider the image of $N_\Gamma(u)$ under ψ. From the definition of ψ,

$$\begin{aligned}(N_\Gamma(u))\psi &= (N_\Gamma(u))(\varphi\theta)\\ &= ((N_\Gamma(u))\varphi)\theta\\ &= A\theta\\ &= N_\Gamma(v).\end{aligned}$$

It follows that ψ induces a partial automorphism

$$\alpha : N_\Gamma(u) \to N_\Gamma(v),$$

and hence, by Lemma 2.7, ψ can be extended to an automorphism of Γ mapping u to v. Therefore, $u \sim_S v$ in Γ.

Now suppose that $u \sim_S v$ in Γ. We will show that $\Gamma - v$ has an automorphism mapping A to $N_\Gamma(v)$. Since u and v are similar in Γ, there exists an automorphism α of Γ such that $u\alpha = v$. Therefore, restricting α induces a partial automorphism

$$\psi : \Gamma - u \to \Gamma - v.$$

Moreover, it follows, since ψ is induced from by automorphism of Γ, that ψ induces a partial automorphism

$$\beta : N_\Gamma(u) \to N_\Gamma(v).$$

Consider the partial automorphism $\theta = \varphi^{-1}\psi$. Notice that, since

$$\varphi^{-1} : \Gamma - v \to \Gamma - u,$$

and

$$\psi : \Gamma - u \to \Gamma - v,$$

it follows that $\mathrm{dom}(\theta) = \mathrm{im}(\theta) = \Gamma - v$. Now consider the image of A under θ. From the definition of θ,

$$\begin{aligned} A\theta &= A(\varphi^{-1}\psi) \\ &= (A\varphi^{-1})\psi \\ &= (N_\Gamma(u))\psi \\ &= N_\Gamma(v). \end{aligned}$$

Therefore, θ is an automorphism of $\Gamma - v$ mapping A onto $N_\Gamma(v)$. This proves our claim.

□

2.4 If Γ is a graph on k vertices, what is the largest possible size of a set of mutually pseudo-similar in Γ?

Notation. If f, $g : \mathbb{N} \to \mathbb{R}$ are real sequences, then the notation $f(n) = \mathcal{O}(g(n))$ means there exists a real number $C > 0$ and $N \in \mathbb{N}$ such that $|f(n)| \leq C\,|g(n)|$ for all $n \geq N$.

The following lemma is an important observation in characterising graphs with sets of mutually pseudo-similar vertices, and follows directly from Lemma 2.7.

Lemma 2.9. *Let Γ be a regular graph. Then Γ does not contain a pair of pseudo-similar vertices.*

Proof. Suppose that the degree of any vertex in Γ is k. If $u \in V\Gamma$ then the set of neighbours $N_\Gamma(u)$ of u in Γ all have degree $k-1$ in the graph $\Gamma - u$. A similar observation is true for any other vertex $v \in V\Gamma$. By a degree argument, any partial automorphism $\varphi : \Gamma - u \to \Gamma - v$ must induce a partial automorphism

$$\psi : N_\Gamma(u) \to N_\Gamma(v).$$

Therefore, by Lemma 2.7, the partial automorphism φ can be extended to an automorphism φ' of Γ such that $u\varphi' = v$. It follows that if u, v are removal-similar in Γ, then they are similar. □

Theorem 2.10. *A graph Γ cannot have all its vertices mutually pseudo-similar.*

Proof. Let $u, v \in V\Gamma$ be such that $\Gamma - u \cong \Gamma - v$. Then $\deg_\Gamma(u) = \deg_\Gamma(v)$. It follows that if every vertex in Γ is removal-similar to every other vertex, then Γ is regular. Therefore, by Lemma 2.9, Γ does not contain any pseudo-similar vertices. □

We can do a little better.

Theorem 2.11. *Let Γ be a graph. Then Γ does not contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 1$.*

Proof. Let Γ be a graph. Let $A \subseteq V\Gamma$ be a set of mutually pseudo-similar vertices in Γ, and suppose for a contradiction that $|A| = |V\Gamma| - 1$.

Since all the vertices in A are removal-similar by assumption, $\deg_\Gamma(u) = k$ for all $u \in A$ and a fixed k. If $w \in V\Gamma \setminus A$ is the remaining vertex, then $\deg_\Gamma(w) \neq k$ since then Γ would be regular, so cannot contain pseudo-similar vertices by Lemma 2.9. Let $\deg_\Gamma(w) = r \neq k$.

We consider 3 cases, depending on the value of r.

- In the case $r = 0$, then w is an isolated vertex of Γ. It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma - w$. By Theorem 2.10, the vertices in A cannot be mutually pseudo-similar in $\Gamma - w$, and therefore cannot be mutually pseudo-similar in Γ. This is a contradiction.
- In the case $r = |V\Gamma| - 1$, consider the complement $\tilde{\Gamma}$ of Γ. By Corollary 6.5 in [JJSS21], Γ and $\tilde{\Gamma}$ have the same partial automorphism monoid, hence the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in $\tilde{\Gamma}$. If $r = |V\Gamma| - 1$, then w is an isolated vertex in $\tilde{\Gamma}$, and so the vertices in A cannot be pseudo-similar in $\tilde{\Gamma}$. This is a contradiction.
- In the case $0 < r < |V\Gamma| - 1$, there must exist $u, v \in A$ such that $u \notin N_\Gamma(w)$ but $v \in N_\Gamma(w)$. We show by a degree argument that $\Gamma - u \ncong \Gamma - v$. First consider $\Gamma - u$. Since $w \notin N_\Gamma(u)$, the degree of w in $\Gamma - u$ is r. Moreover, for any $w' \in N_\Gamma(u)$, the vertex w' has degree $k - 1$ in $\Gamma - u$. It follows that the degree sequence of $\Gamma - u$ is
$$(r, \underbrace{k-1, \ldots, k-1}_{k}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-2}).$$
Now consider $\Gamma - v$. Since $w \in N_\Gamma(v)$, the degree of w in $\Gamma - v$ is $r - 1$. Given any $w' \in N_\Gamma(v) \setminus \{w\}$, the degree of w' in $\Gamma - u$ is $k - 1$. Therefore $\Gamma - v$ has degree sequence
$$(r-1, \underbrace{k-1, \ldots, k-1}_{k-1}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-1}).$$
Consider in particular the number of vertices of degree k. In the case $r = k + 1$, the graph $\Gamma - u$ has $|V\Gamma| - k - 2$ vertices of degree k, and $\Gamma - v$ has $|V\Gamma| - k$ vertices of degree k. Therefore, in this case, $\Gamma - u \ncong \Gamma - v$. Moreover, if $r \neq k + 1$, then $\Gamma - u$ has $|V\Gamma| - k - 2$ vertices of degree k, and $\Gamma - v$ has $|V\Gamma| - k - 1$ vertices of degree k, so in this case also $\Gamma - u \ncong \Gamma - v$. This is a contradiction.

$\square$

It is therefore sensible to ask if this method can be extended to show that a graph Γ cannot contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$.

Theorem 2.12. *Let Γ be a graph. Then Γ does not contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$.*

The proof of Theorem 2.12 is by contradiction. The argument is not simple, and is therefore structured into four main claims about Γ from which the desired contradiction is obtained. These are Propositions (a), (b), (c) and (d).

Suppose that Γ contains a set $A \subseteq V\Gamma$ of mutually pseudo-similar vertices, and that $|A| = |V\Gamma| - 2$. Assume $|A| \geq 2$. Let $w, w' \in V\Gamma \setminus A$ be the two remaining vertices. Then, there exists an integer k such that $\deg_\Gamma(u) = k$ for all $u \in A$. Notice that $k \geq 1$, otherwise the vertices in A would all be isolated vertices and hence would be similar. Let $\deg_\Gamma(w) = r$ and $\deg_\Gamma(w') = s$.

Our first claim about Γ is the following.

Proposition (a). For each $u \in A$, either $u \in N_\Gamma(w)$ or $N_\Gamma(w')$, but not both.

Proof. We prove this by showing that there cannot exist $u \in A$ such that both $u \notin N_\Gamma(w)$ and $u \notin N_\Gamma(w')$. The case where both $u \in N_\Gamma(w)$ and $u \in N_\Gamma(w')$ follows from the symmetric argument, or by considering the complement $\tilde{\Gamma}$ of Γ.

Suppose that there exists $u \in A$ is such that $u \notin N_\Gamma(w)$ and $N_\Gamma(w')$. If $v \notin N_\Gamma(w)$ and $N_\Gamma(w')$ for all $v \in A$, then Γ is not connected, and A is a connected component of Γ. It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma[A]$. Therefore, by Theorem 2.10, the vertices in A are not mutually pseudo-similar in Γ.

It follows that there exists $v \in A$ such that at least one of $v \in N_\Gamma(w)$ or $v \in N_\Gamma(w')$ is true. Suppose, without loss of generality, that $v \in N_\Gamma(w)$. We show, by a degree argument, that $v \notin N_\Gamma(w')$. Suppose that $v \in N_\Gamma(w')$. Since $u \notin N_\Gamma(w)$, the degree of w in $\Gamma - u$ is r. Similarly, the degree of w' in $\Gamma - u$ is s. Since $w, w' \notin N_\Gamma(u)$, any $u' \in N_\Gamma(u)$ has degree $k - 1$ in the graph $\Gamma - u$. Finally, any $u' \notin \{w, w'\} \cup N_\Gamma(u)$ has degree k in the graph $\Gamma - u$. Therefore, the degree sequence of $\Gamma - u$ is

$$(r, s, \underbrace{k-1, \ldots, k-1}_{k}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-3}).$$

By a similar argument, taking into account the fact that $w, w' \in N_\Gamma(v)$, the degree sequence of $\Gamma - v$ is

$$(r-1, s-1, \underbrace{k-1, \ldots, k-1}_{k-2}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-1}).$$

Since $\Gamma - u \cong \Gamma - v$, these degree sequences must be equal. Hence, we must have $r = s = k$, so Γ is regular. This contradicts Lemma 2.9. Therefore, it follows that $v \notin N_\Gamma(w')$.

Now, by a similar degree argument, we claim that $r = k$. By the previous argument, the degree sequence of $\Gamma - u$ is

$$(r, s, \underbrace{k-1, \ldots, k-1}_{k}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-3}).$$

Moreover, taking into account the fact that $v \in N_\Gamma(w)$ and $v \notin N_\Gamma(w')$, it follows that the degree sequence of $\Gamma - v$ is

$$(r-1, s, \underbrace{k-1, \ldots, k-1}_{k-1}, \underbrace{k, \ldots, k}_{|V\Gamma|-k-2}).$$

Since $\Gamma - u \cong \Gamma - v$, these degree sequences must be equal. This is only possible if $r = k$, proving our claim.

Now suppose that there exists $v' \in A$ such that $v' \notin N_\Gamma(w)$ and $v' \in N_\Gamma(w')$. By the symmetric argument, $s = k$, which implies that Γ is regular. This contradicts Lemma 2.9. Moreover, since there cannot exist $v' \in A$ such that $v' \in N_\Gamma(w)$ and $v' \in N_\Gamma(w')$, we must conclude that

$$N_\Gamma(w') \setminus \{w\} = \emptyset.$$

There are now two cases to consider.

- In the case $N_\Gamma(w') = \emptyset$, the vertex w' is an isolated vertex in Γ. It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma - w'$. By Theorem 2.11, the vertices in A are not mutually pseudo-similar in $\Gamma - w'$, therefore they are not mutually pseudo-similar in Γ. This is a contradiction.
- In the case $N_\Gamma(w') = \{w\}$, the graph Γ is of the form in Figure 4. There are three sub-cases to be considered, depending on the value of k.
  1. In the case $k = 1$, each $u \in A$ is adjacent to only w. Therefore, Γ is a star graph with center w. In this case, all the vertices in A are similar, which is a contradiction.
  2. In the case $k = 2$, each vertex $u \in A$ is adjacent to w and exactly one $v \in A$. Since the same is true of the vertex v, it follows that there is an automorphism of Γ exchanging u and v. Therefore, $u \sim_S v$, which is a contradiction. This is illustrated in Figure 5.
  3. In the case $k > 2$, suppose that $u, v \in A$ and

     $$\varphi : \Gamma - u \to \Gamma - v$$

     is a partial automorphism. In the graph $\Gamma - u$, the vertex w' has degree 1, w has degree $|A| + 1 > 1$, each $u' \in N_\Gamma(u)$ has degree $k - 1 > 1$, and the remaining vertices have degree $k > 2$. A similar observation is true for the graph $\Gamma - v$. It follows that φ fixes w', and since

     $$N_{\Gamma-u}(w') = N_{\Gamma-v}(w') = \{w\},$$

     φ fixes w. Consider the image of $N_\Gamma(u)$ under φ. By a degree argument, φ induces the partial automorphism

     $$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$

     Since φ fixes w, it then follows that φ induces the partial automorphism

     $$\psi' : N_\Gamma(u) \to N_\Gamma(v).$$

     Therefore, φ can be extended to an automorphism of Γ mapping u to v. This is a contradiction.

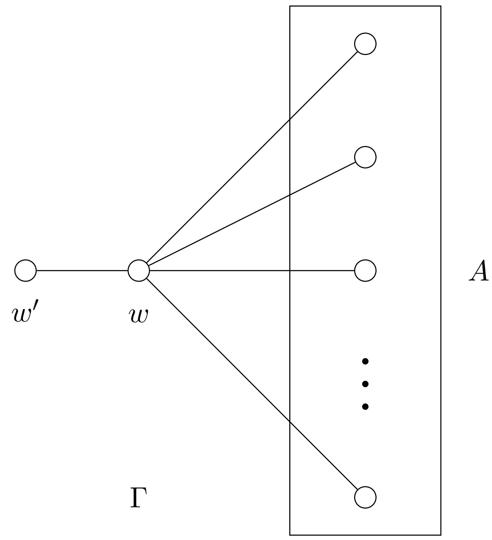


Figure 4: The graph Γ in the case $N_\Gamma(w') = \{w\}$.

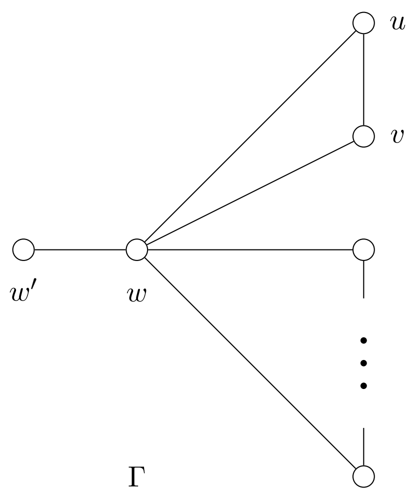


Figure 5: The graph Γ in the case $N_\Gamma(w') = \{w\}$ and $k = 2$.

Therefore, we have demonstrated that, in all cases, $N_\Gamma(w') \setminus \{w\} = \emptyset$ leads to a contradiction. The only conclusion left is that our initial assumption that there exists $u \in A$ such that $u \notin N_\Gamma(w)$ and $u \notin N_\Gamma(w')$ is false. Then, the fact that there cannot exist $u \in A$ such that $u \in N_\Gamma(w)$ and $u \in N_\Gamma(w')$ can then be deduced from considering the complement $\tilde{\Gamma}$ of Γ, since Γ and $\tilde{\Gamma}$ have the same partial automorphism monoid (Corollary 6.5 in [JJSS21]).

Therefore, for every $u \in A$, either $u \in N_\Gamma(w)$ or $u \in N_\Gamma(w')$, but not both. This proves Proposition (a).

$\square$

Proposition (b). $r = s = k + 1$.

Proof. Let $u, v \in A$ be such that $u \in N_\Gamma(w)$ and $v \in N_\Gamma(w')$. Since, by Proposition (a), $u \notin N_\Gamma(w')$

and $v \notin N_\Gamma(w)$, it follows that the degree sequence of $\Gamma - u$ is

$$(r-1, s, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}),$$

and the degree sequence in $\Gamma - v$ is

$$(r, s-1, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Since $\Gamma - u \cong \Gamma - v$, if follows that these degree sequences must be equal. Therefore, we must have $r = s$.

We now move on to showing that $r = k+1$. The proof is by a degree argument, however the special case where $r = k-1$ must be treated separately. Assume, for a contradiction, that $r \neq k+1$.

Let $u, v \in A$ be such that $u, v \in N_\Gamma(w)$. Let

$$\varphi : \Gamma - u \to \Gamma - v$$

be a partial automorphism.

In the graph $\Gamma - u$, each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k-1$. Since $w \in N_\Gamma(u)$, the vertex w has degree $r-1$ in the graph $\Gamma - u$, and since $w' \notin N_\Gamma(u)$, the vertex w' has degree r in the graph $\Gamma - u$. The remaining vertices all have degree k in $\Gamma - u$. A similar observation is true for the graph $\Gamma - v$.

Suppose first that $r \neq k-1$. Since $r \neq k+1$, the vertex w is the unique vertex of degree $r-1$ in the graph $\Gamma - u$ and in the graph $\Gamma - v$. It follows that $w\varphi = w$. Moreover, since $r \neq k-1$, the vertices in $N_\Gamma(u) \setminus \{w\}$ are only vertices in the graph $\Gamma - u$ of degree $k-1$. A similar observation is true of $N_\Gamma(v) \setminus \{w\}$ in the graph $\Gamma - v$. Therefore, φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$

Thus, combining these two facts, φ induces a partial automorphism

$$\psi' : N_\Gamma(u) \to N_\Gamma(v),$$

and therefore can be extended to an automorphism of Γ mapping u to v. This is a contradiction.

We now consider the special case when $r = k-1$. First, in both $\Gamma - u$ and $\Gamma - v$, the vertex w is the unique vertex of degree $k-2$, and so $w\varphi = w$. It follows that φ maps the set $N_{\Gamma-u}(w)$ onto the set $N_{\Gamma-v}(w)$. That is, φ induces a partial automorphism

$$\theta : N_\Gamma(w) \setminus \{u\} \to N_\Gamma(w) \setminus \{v\}.$$

Moreover, each $u' \in N_\Gamma(w') \setminus \{w\}$ has degree k in the graph $\Gamma - u$. Since $N_\Gamma(w') \setminus \{w\}$ is disjoint to both the sets $N_\Gamma(w) \setminus \{u\}$ and $N_\Gamma(w) \setminus \{v\}$, and moreover since $w\varphi = w$ and $\deg_{\Gamma-u}(w') = \deg_{\Gamma-v}(w') = k-1$, it follows that $N_\Gamma(w') \setminus \{w\}$ is fixed setwise by φ.

Now consider the image of w' under φ. Since $r = k-1$, the set of vertices of degree $k-1$ in the graph $\Gamma - u$ is

$$\{w'\} \cup (N_\Gamma(u) \setminus \{w\}).$$

Similarly, the set of vertices of degree $k-1$ in the graph $\Gamma - v$ is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Therefore, by a degree argument, φ induces a partial automorphism

$$\psi : \{w'\} \cup (N_\Gamma(u) \setminus \{w\}) \to \{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Using the fact that $w\varphi = w$, it follows that φ induces a partial automorphism

$$\psi' : \{w'\} \cup N_\Gamma(u) \to \{w'\} \cup N_\Gamma(v).$$

Let $u' \in \{w'\} \cup N_\Gamma(u)$ be such that $u'\varphi = w'$. If $u' = w'$ then we are done. Otherwise, $w'\varphi \in N_\Gamma(v)$, and so the image of $N_\Gamma(u)$ under φ is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w'\varphi\}).$$

Consider the neighbours of w' in the graph $\Gamma - u$.

If w and w' are adjacent, then

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

It follows, since w is fixed by φ and $N_\Gamma(w') \setminus \{w\}$ is fixed setwise by φ, that

$$N_{\Gamma-v}(w'\varphi) = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Moreover,

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Similarly, in the case that w and w' are not adjacent,

$$N_{\Gamma-u}(w') = N_\Gamma(w') = N_\Gamma(w') \setminus \{w\}$$

which is fixed setwise by φ, and so

$$N_{\Gamma-v}(w'\varphi) = N_\Gamma(w') \setminus \{w\} = N_{\Gamma-v}(w').$$

Therefore, $\Gamma - v$ has an automorphism which exchanges w' and $w'\varphi$, and fixes every other vertex pointwise. This is an automorphism of Γ mapping

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w'\varphi\}) = (N_\Gamma(u))\varphi$$

to $N_\Gamma(v)$, and hence, by Theorem 2.8, u and v are similar in Γ. This is a contradiction.

Therefore, we must conclude that $r = s = k+1$. This proves Proposition (b).

□

Proposition (c). Each $u \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$.

Proof. Let $u \in A$. Since $N_\Gamma(w) \cap N_\Gamma(w') = \emptyset$ and $(N_\Gamma(w) \cup N_\Gamma(w')) \setminus \{w, w'\} = A$, we can partition the set $N_\Gamma(u) \setminus \{w\}$ as follows

$$N_\Gamma(u) \setminus \{w\} = N_\Gamma^w(u) \cup N_\Gamma^{w'}(u),$$

where $N_\Gamma^w(u)$ is the set of neighbours of u in A which are adjacent to w, and $N_\Gamma^{w'}(u)$ is the set of neighbours of u in A which are adjacent to w'.

Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and suppose that

$$\varphi : \Gamma - u \to \Gamma - v$$

is a partial automorphism. In the graph $\Gamma - u$, each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k-1$, whereas every vertex not in this set has degree at least k. A similar observation is true in the graph $\Gamma - v$. Therefore, by a degree argument, φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$

Consider $w'\varphi$. Since $w' \notin N_\Gamma(u)$, the vertex w' has degree $k+1$ in the graph $\Gamma - u$. Every other vertex in $\Gamma - u$ has degree k or $k-1$. A similar observation is true of w' in the graph $\Gamma - v$. Therefore, by a degree argument, $w'\varphi = w'$.

Since φ fixes w', it follows that φ maps the set $N_{\Gamma-u}(w')$ onto the set $N_{\Gamma-v}(w')$, and hence the set

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

is fixed setwise. Therefore, φ must induce partial automorphisms

$$\theta : N_\Gamma^w(u) \to N_\Gamma^w(v)$$
$$\theta' : N_\Gamma^{w'}(u) \to N_\Gamma^{w'}(v).$$

Hence, $N_\Gamma^w(u) \cong N_\Gamma^w(v)$ and $N_\Gamma^{w'}(u) \cong N_\Gamma^{w'}(v)$, and so

$$|N_\Gamma^w(u)| = |N_\Gamma^w(v)|$$
$$|N_\Gamma^{w'}(u)| = |N_\Gamma^{w'}(v)|.$$

Since u and v were arbitrary vertices in $N_\Gamma(w) \setminus \{w'\}$, it follows that there exist integers p and q such that, for all $u \in N_\Gamma(w) \setminus \{w'\}$,

$$|N_\Gamma^w(u)| = p \text{ and } |N_\Gamma^{w'}(u)| = q.$$

By the symmetric argument, with $u, v \in N_\Gamma(w') \setminus \{w\}$, it also follows that, for all $u \in N_\Gamma(w') \setminus \{w\}$,

$$|N_\Gamma^{w'}(u)| = p \text{ and } |N_\Gamma^w(u)| = q.$$

Now suppose that $q = 0$. We distinguish between the case when w, w' are adjacent and when w, w' are not adjacent.

- In the case where w, w' are not adjacent, then Γ is not connected. The two connected components of Γ are $N_\Gamma(w) \cup \{w\}$ and $N_\Gamma(w') \cup \{w'\}$. This is illustrated in Figure 6. Consider, in particular, the component $N_\Gamma(w) \cup \{w\}$. The vertices in $N_\Gamma(w)$ are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph

  $$\Gamma[N_\Gamma(w) \cup \{w\}].$$

  By Theorem 2.11, they cannot be mutually pseudo-similar in the graph $\Gamma[N_\Gamma(w) \cup \{w\}]$, hence they are not mutually pseudo-similar in Γ. This is a contradiction.

- In the case where w, w' are adjacent, first notice that we must have $|A| \geq 4$, so that $|N_\Gamma(w) \setminus \{w'\}| = |N_\Gamma(w') \setminus \{w\}| \geq 2$. This is illustrated in Figure 7. Otherwise, Γ is a path graph with an automorphism exchanging $N_\Gamma(w) \setminus \{w'\}$ and $N_\Gamma(w') \setminus \{w\}$, making the vertices in A similar, which is a contradiction.

  Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and let
  $$\varphi : \Gamma - u \to \Gamma - v$$
  be a partial automorphism. In the graph $\Gamma - u$, the vertex w' has degree $k+1$; each $u' \in N_\Gamma(w')$ has degree k; the vertex w has degree k; each $u' \in N_\Gamma(w) \setminus (N_\Gamma(u) \cup \{w'\})$ has degree k; and each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k-1$. A similar observation is true for the graph $\Gamma - v$. It follows that φ induces a partial automorphism
  $$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$
  Moreover, since w' is the only vertex of degree $k+1$ in both $\Gamma - u$ and $\Gamma - v$, it follows that $w'\varphi = w'$. Hence, φ fixes
  $$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$
  setwise. Consider $w\varphi$. Since $w \in N_\Gamma(w')$, it follows that
  $$w\varphi \in N_\Gamma(w').$$
  Since $N_\Gamma(w) \geq 2$, the vertex w has a neighbour $u' \in N_\Gamma(w) \setminus \{w'\}$ which has degree $k-1$ in the graph $\Gamma - u$. A similar observation is true for the graph $\Gamma - v$. In contrast, since $q = 0$, each $u' \in N_\Gamma(w') \setminus \{w\}$ has no neighbour in the set $N_\Gamma(w) \setminus \{w'\}$, therefore each neighbour of u' has degree k or $k+1$ in the graph $\Gamma - u$. Similarly in the graph $\Gamma - v$. It follows that, if φ is a partial automorphism, $w\varphi = w$, and therefore φ induces the partial automorphism
  $$\psi' : N_\Gamma(u) \to N_\Gamma(v).$$
  Hence, φ can be extended to an automorphism of Γ mapping u to v, making u and v similar. This is a contradiction.

Therefore, $q \geq 1$, so every $u \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$, as desired. This proves Proposition (c).

Proposition (d). Each $u \in N_\Gamma(w') \setminus \{w\}$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$.

Continuing from the proof of Proposition (c), suppose instead that $p = 0$. Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and suppose that
$$\varphi : \Gamma - u \to \Gamma - v$$
is a partial automorphism. Once again, by a degree argument φ fixes w', and hence fixes $N_\Gamma(w')$ setwise. Moreover, by a degree argument, φ induces a partial automorphism
$$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$

- In the case w, w' are not adjacent, since $N_\Gamma(w')$ is fixed setwise, it follows that $N_\Gamma(w) \setminus \{u\}$ is mapped onto $N_\Gamma(w) \setminus \{v\}$, and hence w is fixed. It follows that φ induces a partial automorphism
  $$\psi' : N_\Gamma(u) \to N_\Gamma(v),$$
  and so φ can be extended to an automorphism of Γ mapping u to v. This implies $u \sim_S v$ in Γ, which is a contradiction.

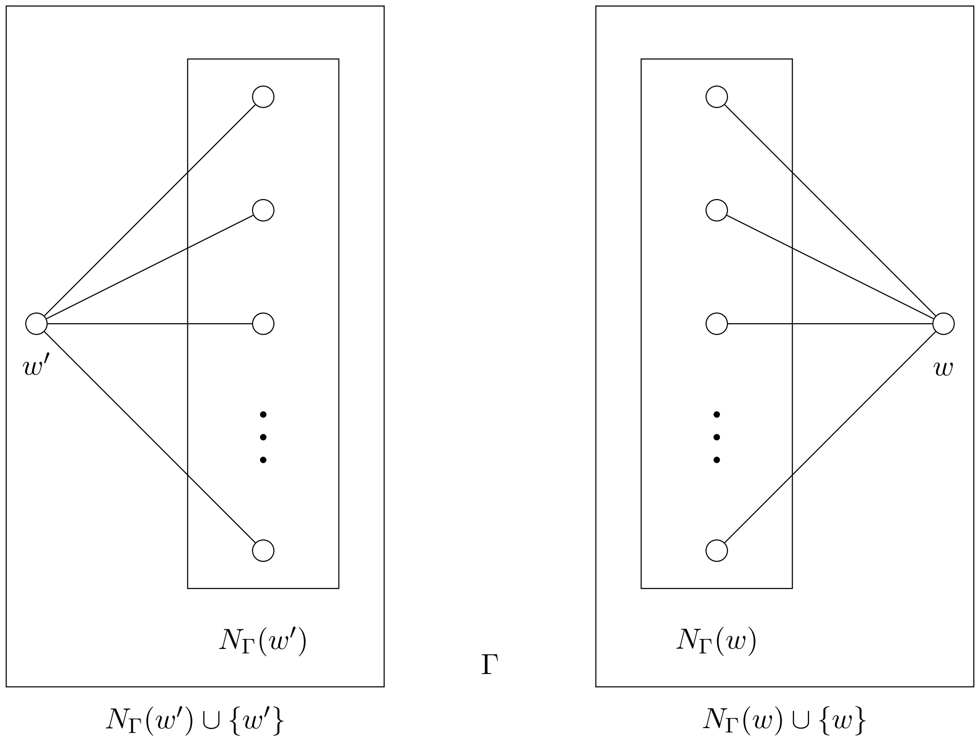


Figure 6: The graph Γ in the case that w, w' are not adjacent.

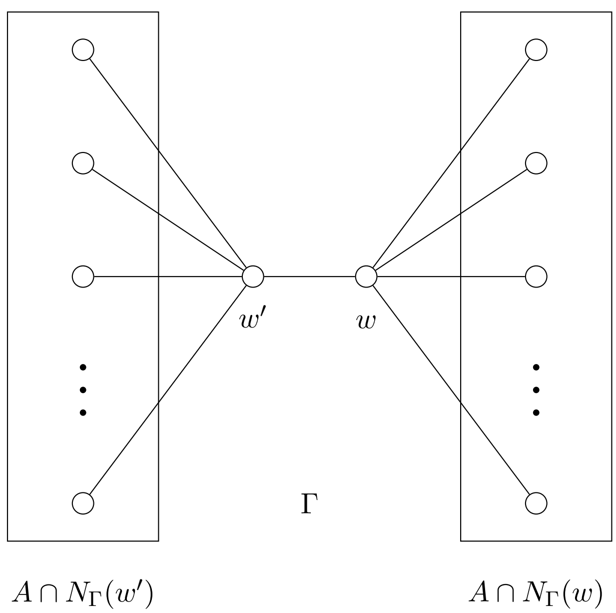


Figure 7: The graph Γ in the case that w, w' are adjacent.

- In the case w, w' are adjacent, since $N_\Gamma(w')$ is fixed setwise, it follows that $w\varphi \in N_\Gamma(w')$. Moreover, by a degree argument, $w\varphi \in \{w\} \cup (N_\Gamma(w') \setminus N_\Gamma(v))$ in the graph $\Gamma - v$. In the graph $\Gamma - v$, the vertex w is adjacent to w and all of the $k-1$ vertices in the set $N_\Gamma(w) \setminus \{v\}$. Moreover, in the graph $\Gamma - v$ every $v' \in N_\Gamma(w') \setminus N_\Gamma(v)$ is adjacent to w' and all of the $k-1$

vertices in the set $N_\Gamma(w) \setminus \{v, w'\}$. Hence,

$$N_{\Gamma - v}(w) = N_{\Gamma - v}(w\varphi),$$

and therefore $\Gamma - v$ has an automorphism which exchanges φ and $w\varphi$, and fixes all other vertices pointwise. This is an automorphism of $\Gamma - v$ mapping the set $(N_\Gamma(u))\varphi$ onto $N_\Gamma(v)$, and so by Theorem 2.8, $u \sim_S v$ in Γ. This is a contradiction.

It follows that $p \geq 1$, and so each $u \in N_\Gamma(w') \setminus \{w\}$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. This proves Proposition (d). □

Proof of Theorem 2.9. For the remainder of the argument, let $u, v \in A$ be such that $u, v \in N_\Gamma(w)$. Suppose that

$$\varphi : \Gamma - u \to \Gamma - v$$

is a partial automorphism. By Claim 2, the vertex w' has degree $k + 1$ in the graph $\Gamma - u$, and also has degree $k + 1$ in the graph $\Gamma - v$. Every vertex not equal to w' has degree at most k in $\Gamma - u$, and similarly in $\Gamma - v$. Thus, $w'\varphi = w'$. It follows that φ maps the set $N_{\Gamma - u}(w')$ onto the set $N_{\Gamma - v}(w')$, and therefore

$$N_\Gamma(w') = N_{\Gamma - u}(w') = N_{\Gamma - v}(w')$$

is fixed setwise.

Moreover, every $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k - 1$ in $\Gamma - u$. Similarly, every $v' \in N_\Gamma(v)$ has degree $k - 1$ in $\Gamma - v$. The remaining vertices all have degree at least k. It follows, by a degree argument, that φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \to N_\Gamma(v) \setminus \{w\}.$$

Now consider $w\varphi$. By Proposition (b), w has degree $k + 1$ in Γ, and so has degree k in $\Gamma - u$. Similarly, w has degree k in $\Gamma - v$. Therefore, by a degree argument, either

$$w\varphi \in N_\Gamma(w'),$$

or

$$w\varphi \in \{w\} \cup N_\Gamma(w) \setminus (\{w'\} \cup N_\Gamma(v)).$$

We again distinguish between the cases when w, w' are adjacent and w, w' are not adjacent.

- In the case w, w' are not adjacent, then $w\varphi \in \{w\} \cup N_\Gamma(w) \setminus N_\Gamma(v)$. Suppose that $w\varphi \neq w$. By Claim 3, every $u' \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. It follows that $w\varphi$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$. However, w has no neighbour in $N_\Gamma(w') \setminus \{w\}$, and since $N_\Gamma(w')$ is fixed setwise by φ, this is a contradiction.
- In the case w, w' are adjacent, then $w\varphi \in N_\Gamma(w)$. Suppose for a contradiction that $w\varphi \neq w$. In $\Gamma - u$, the vertex w has no neighbours in the set $N_\Gamma(w') \setminus \{w\}$. However, by Claim 3, if $w\varphi \in N_\Gamma(w') \setminus \{w\}$, then $w\varphi$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. Denote this neighbour u'. Since φ fixes $N_\Gamma(w')$ setwise, it follows that $u'\varphi^{-1} \in N_\Gamma(w')$. Moreover, since $w\varphi$ and u' are adjacent, it follows that $u' \neq w\varphi$, and hence $u'\varphi^{-1} \neq w$. Therefore, $u'\varphi^{-1} \in N_\Gamma(w') \setminus \{w\}$, so w has a neighbour in the set $N_\Gamma(w') \setminus \{w\}$. This contradicts Proposition (a).

Therefore, $w\varphi = w$. It follows that φ induces the partial automorphism

$$\psi' : N_\Gamma(u) \to N_\Gamma(v),$$

and hence by Lemma 2.7, $u \sim_S v$ in Γ. This contradicts their pseudo-similarity. It follows that there cannot exist a graph Γ containing a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$. This proves Theorem 2.12.

□

The method of reduction used to prove Theorem 2.12 is clearly limited. As we decrease $|A|$ relative to $|V\Gamma|$, the amount of case analysis required rapidly becomes untenable. In fact, the method will eventually fail entirely: there exists a graph Γ with a set of mutually pseudo-similar vertices of size $|V\Gamma| - 6$. For an example, see Figure 9.

One possible route for further progress may be to consider generalisations of Proposition (a). If such a generalisation could be used to show that, for each $u \in A$, a sufficiently large fraction of the vertices in $N_\Gamma(u)$ are not pseudo-similar to u in Γ, then this could be used to prove a new lower bound for Question 1.3.

2.5 Digraphs and constructing graphs with arbitrarily large sets of mutually pseudo-similar vertices

The case when Γ is digraph contrasts sharply with the case when Γ is a graph. For example, Theorem 2.10 is false for digraphs: there exist digraphs with every vertex mutually pseudo-similar to every other vertex. An important example is the *transitive tournament*, T_k, defined on the vertex set

$$VT_k = \{1, 2, \dots, k\}$$

with $u \to v$ if and only if $u < v$. For a picture, see Figure 8.

Proposition 2.13. *In T_k, every vertex is mutually pseudo-similar to every other vertex.*

Proof. First, note that T_k has no automorphism other than the identity, so no pair of distinct vertices $u, v \in VT_k$ is similar. By considering T_k as a linear order on the vertices $\{1, 2, \dots, k\}$, we can see that $T_k - u \cong T_{k-1}$ for every $u \in VT_k$. Therefore, for every pair $u, v \in T_k$, we have $T_k - u \cong T_k - v$. Thus, any pair $u, v \in T_k$ is pseudo-similar. □

Given a transitive tournament T_k, it is possible to replace the directed edges with undirected 'gadgets' while maintaining the mutual pseudo-similarity of the vertices. We describe on such construction, due to [KJSS81]. Consider the undirected gadget Γ in Figure 9. The vertices u and v are pseudo-similar in Γ, and, moreover, are cutvertices of Γ, that is, the graphs $\Gamma - u$ and $\Gamma - v$ are not connected. For each pair $u, v \in VT_k$, replace the directed edge $u \to v$ with the gadget Γ, including the extra vertices on each side. See Figure 10 for an example. This produces a sequence of graphs, Γ_k, containing a set of k mutually pseudo-similar vertices and

$$|V\Gamma_k| = k + (|V\Gamma| - 2)\binom{k}{2} = k + 6\binom{k}{2} = \mathcal{O}(k^2).$$

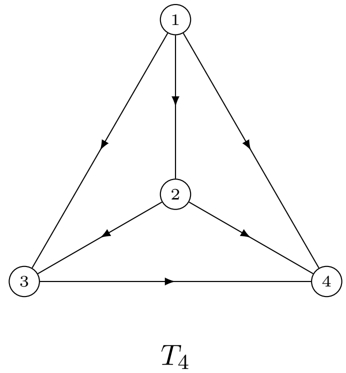


Figure 8: The transitive tournament T_4.

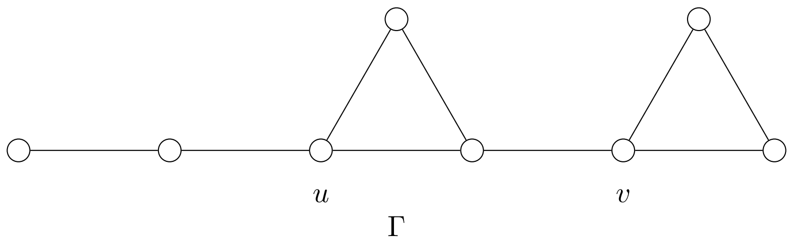


Figure 9: The gadget Γ, with pseudo-similar cutvertices u and v.

The problem with transforming T_k into an undirected graph in this manner is that, while every every vertex in T_k is mutually pseudo-similar to every other vertex, and hence contains a set of k mutually pseudo-similar vertices, T_k contains $\binom{k}{2} = \mathcal{O}(k^2)$ edges. When replacing a directed edge with a gadget, at least one more vertex is added, so the resulting undirected graph has $\mathcal{O}(k^2)$ vertices.

A natural question is therefore to ask, does there exist a sequence of digraphs with every vertex mutually pseudo-similar to every other vertex and with fewer than $\mathcal{O}(k^2)$ directed edges?

Problem 2.14. *Let Γ_k be a digraph with $|V\Gamma_k| = k$ such that every vertex is mutually pseudo-similar to every other vertex. Does this then imply $\Gamma_k \cong T_k$?*

This problem is deceptively difficult. One subtlety occurs when considering extensions of partial automorphisms of the form

$$\varphi : \Gamma_k - u \to \Gamma_k - v$$

to automorphisms of Γ_k. Unlike in graphs, it is not sufficient for φ to be able to be extended to an automorphism of Γ_k that φ maps the (undirected) neighbours of u to the (undirected) neighbours of v. It also required that φ respects the partition of $N_\Gamma(u)$ and $N_\Gamma(v)$ into the in-neighbours and out-neighbours. This subtlety causes degree arguments of the kind used in Lemma 2.9 to lose their power.

Consider, for instance, the following proposition about extending partial automorphisms of digraphs, which may be viewed as an attempt at an analogue of Lemma 2.7 for digraphs.

Proposition 2.15. *Let Γ be a digraph, and $u, v \in V\Gamma$ be removal similar. Suppose that $\Gamma - u - v$ has no automorphism mapping an in-neighbour of u to an out-neighbour of u, and vice-versa. Let*

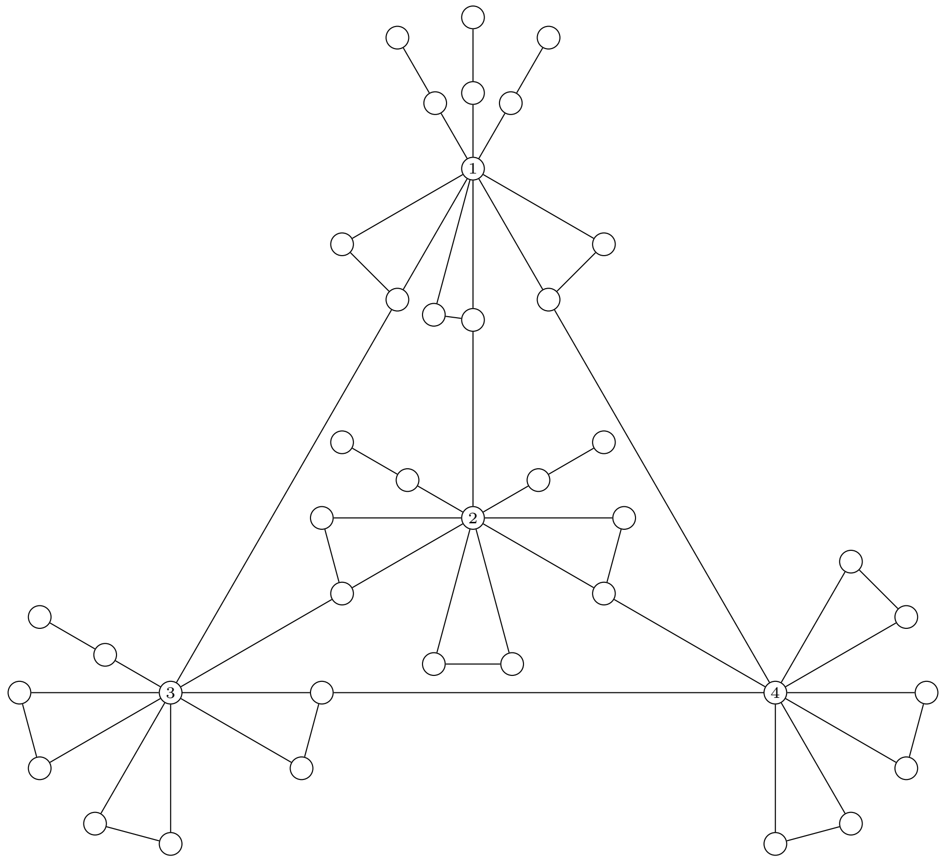


Figure 10: The transitive tournament T_4, with directed edges replaced with appropriate undirected gadgets.

$\varphi : \Gamma - u \to \Gamma - v$ be a partial automorphism, and suppose that restricting φ induces two partial automorphisms

$$\psi : N_\Gamma(u) \to N_\Gamma(v)$$
$$\theta : N'_\Gamma(v) \to N'_\Gamma(u)$$

such that $v\theta = u$. Then u and v are similar in Γ.

Proof. Let Γ, u and v be as given in the statement of the lemma, and let $\varphi : \Gamma - u \to \Gamma - v$ be a partial automorphism which induces two partial automorphisms

$$\psi : N_\Gamma(u) \to N_\Gamma(v)$$
$$\theta : N'_\Gamma(v) \to N'_\Gamma(u)$$

Notice that, since $\varphi : \Gamma - u \to \Gamma - v$, and $v\varphi = u$, it follows that φ^2 is an automorphism of $\Gamma - u - v$. Suppose, for a contradiction, that there is a $w \to u$ such that $v \to w\varphi$. Then, $w\varphi^2 = (w\varphi)\theta$, so if $v \to w\varphi$ then $v\theta \to (w\varphi)\theta$ and so $u \to w\varphi^2$. Therefore, φ^2 is an automorphism of $\Gamma - u - v$ which maps an in-neighbour of u to an out-neighbour of u, which is a contradiction. Since, in addition to this, φ induces the map

$$\psi : N_\Gamma(u) \to N_\Gamma(v)$$

it follows that φ must map in-neighbours of u to in-neighbours of v, and by the symmetric argument, map out-neighbours of u onto out-neighbours of v. It follows that φ can be extended to an automorphism of Γ mapping u to v, so $u \sim_S v$ in Γ. □

Proposition 2.5 is weak in the sense that the requirements placed on Γ and φ are very strong. If Question 2.14 is to be approached, then these conditions must be weakened significantly.

Another approach is to weaken the condition that $|V\Gamma_k| = k$. If instead, we only require that Γ_k contains a set of k mutually pseudo-similar vertices with $|V\Gamma_k| = \mathcal{O}(k)$, then a wider class of digraphs may be considered.

Problem 2.16. *Construct a sequence of digraphs Γ_k such that*

1. *Γ_k contains a set of k mutually pseudo-similar vertices;*
2. *$|V\Gamma_k| = \mathcal{O}(k)$; and*
3. *$|E\Gamma_k| = \mathcal{O}(k)$.*

If such a sequence Γ_k were found, then it could possibly be transformed using undirected gadgets to produce a sequence of graphs having a set of k mutually pseudo-similar vertices and $\mathcal{O}(k)$ vertices.

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