Connecting distant points via random blobs

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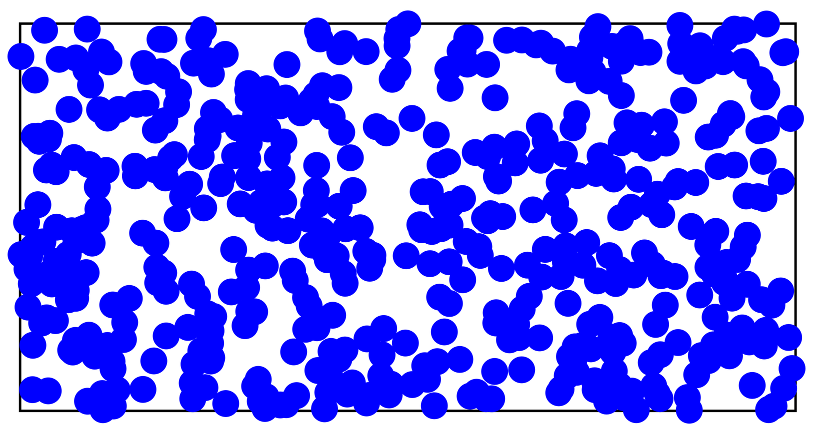
Based on joint work with Mathew Penrose

The "random blob" or Boolean model



The *Boolean model* or *random blob model* is the union of balls of radius r centred at the points of a point process $\mathcal{P}$ in a set A. It is closely related to the *random geometric graph*, in which the points of $\mathcal{P}$ are vertices and edges join the pairs of points which are within distance $2r$. Suppose $A \subseteq \mathbb{R}^d$ is bounded, and $\mathcal{P}$ is either n iid points with uniform distribution on A, or a homogeneous Poisson point process of intensity $n/|A|$. Denote the Boolean model by $Z(n, r)$. We often choose $r = r_n$ by setting the parameter $\Lambda_n = nr_n^d$, as it is Λ_n that determines many of the properties of $Z(n, r_n)$. If $x \in A^\circ$ is at least distance r_n from the boundary, then the expected number of balls which contain x is exactly proportional to Λ_n, and the probability $x \in Z(n, r_n)$ depends on Λ_n, for all n. For a given sequence of Λ_ns, this makes the question of *limiting behaviour* as $n \to \infty$ very interesting.

Continuum percolation

Suppose $A = \mathbb{R}^d$ or $\mathbb{H} := (0, \infty) \times \mathbb{R}^{d-1}$, fix $r = 1$ and let $\mathcal{P}$ be a homogeneous Poisson process of intensity $\lambda > 0$. In this case, the Boolean model is normally referred to as *continuum percolation*. There are also many variants on continuum percolation, particularly when the balls have random radii: Meester and Roy's book Continuum Percolation is a very good introduction.

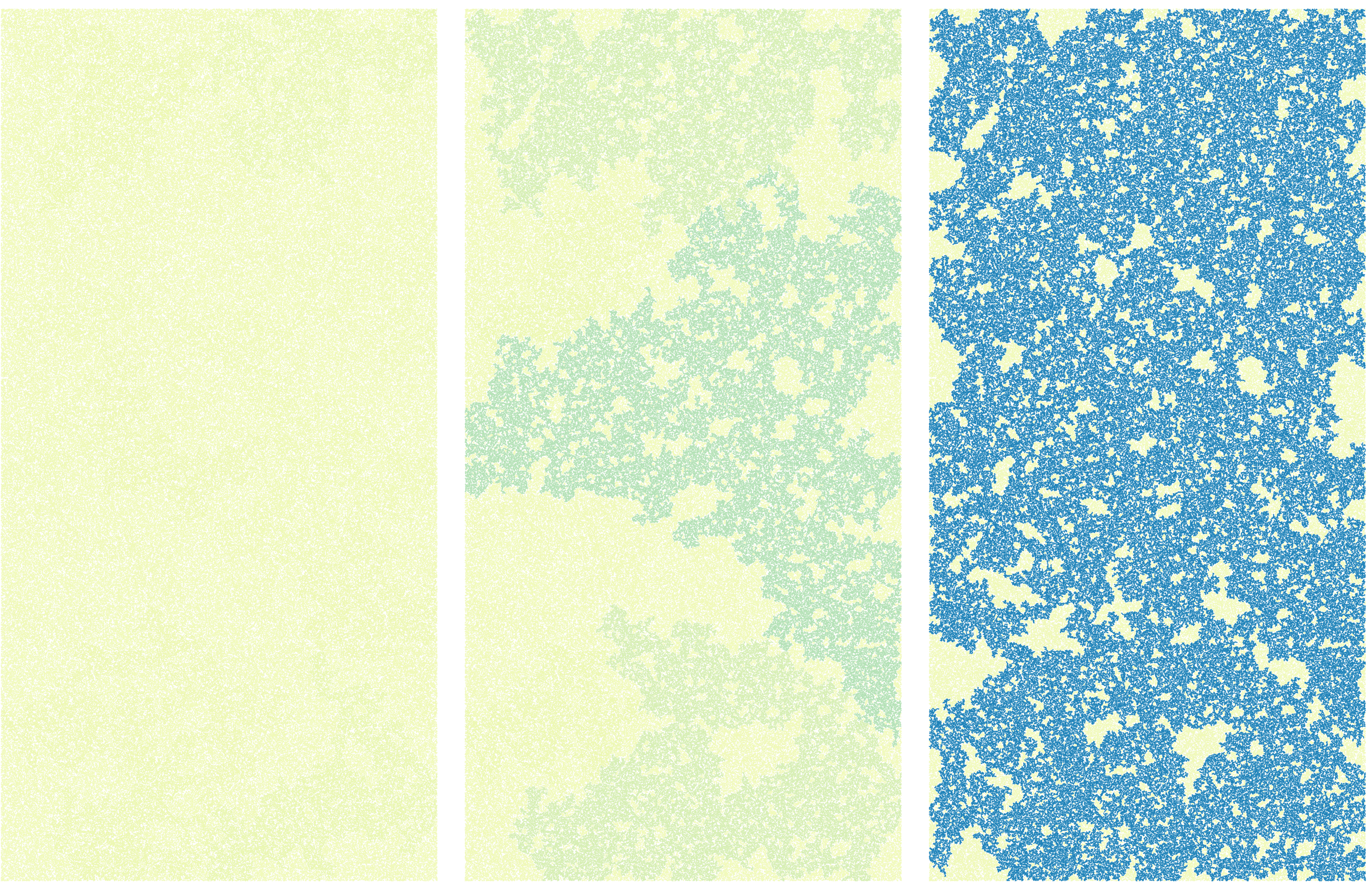
As the name suggests, continuum percolation has many behaviours similar to discrete percolation. For example, we can define the *percolation probability*

$$\theta_A(\lambda) := \mathbb{P}_\lambda[\,0 \text{ is in an unbounded component of } Z(n, r)\,]$$

and there is a non-trivial *critical intensity*

$$\lambda_c^A := \inf\{\lambda > 0 : \theta_A(\lambda) > 0\}.$$

It is also known that $\lambda_c^{\mathbb{H}} = \lambda_c^{\mathbb{R}^d}$, so we will can just write λ_c to denote the critical intensity.



Three realisations of the Boolean model in $\mathbb{R}^2$ corresponding to continuum percolation with $\lambda = 0.35$, $\lambda = 0.36$ and $\lambda = 0.37$ respectively. Each component is coloured according to its size (darker components are larger). Numerical studies show $\lambda_c \approx 0.36$.

Connection events

Suppose A has a complicated geometry. As long as the boundary is smooth, the Boolean model near ∂A should "look like" a rescaled verison of continuum percolation in $\mathbb{H}$. Can we relate events inside A to continuum percolation events in the canonical sets $\mathbb{R}^d$ and $\mathbb{H}$? For example, suppose $x, y \in \overline{A}$. Let $\{x \leftrightarrow y \text{ in } Z(n, r)\}$ denote the event that x and y are in the same connected component of the Boolean model.

Theorem (2024+, H. and Penrose)**.** *Suppose $A \subseteq \mathbb{R}^d$ is open, connected and has a C^2 boundary. Let x, y be two distinct points on ∂A and let $\lambda \in (0, \infty) \setminus \{\lambda_c\}$. If $\Lambda_n \to \lambda$ as $n \to \infty$, then*

$$\lim_{n \to \infty} \mathbb{P}[x \leftrightarrow y \text{ in } Z(n, r_n)] = \theta_{\mathbb{H}}(\lambda)^2.$$

Boundary effects

A similar result with x, y chosen *uniformly* in $[0, 1]^2$ (so almost surely in the interior of the square) was proven in 2022 by Penrose.

If A is any polytope, an analogous result to our recent theorem for points *on the boundary* should also hold, but comparing to events from continuum percolation in *cones*.

To work near the boundary, we fit the part of ∂A near x between the two tangent sphere meeting the boundary at x. Then we can relate events on one side of the tangent plane to continuum percolation in $\mathbb{H}$.

What about λ_c?

We required that $\lambda \neq \lambda_c$ for our result. Why?

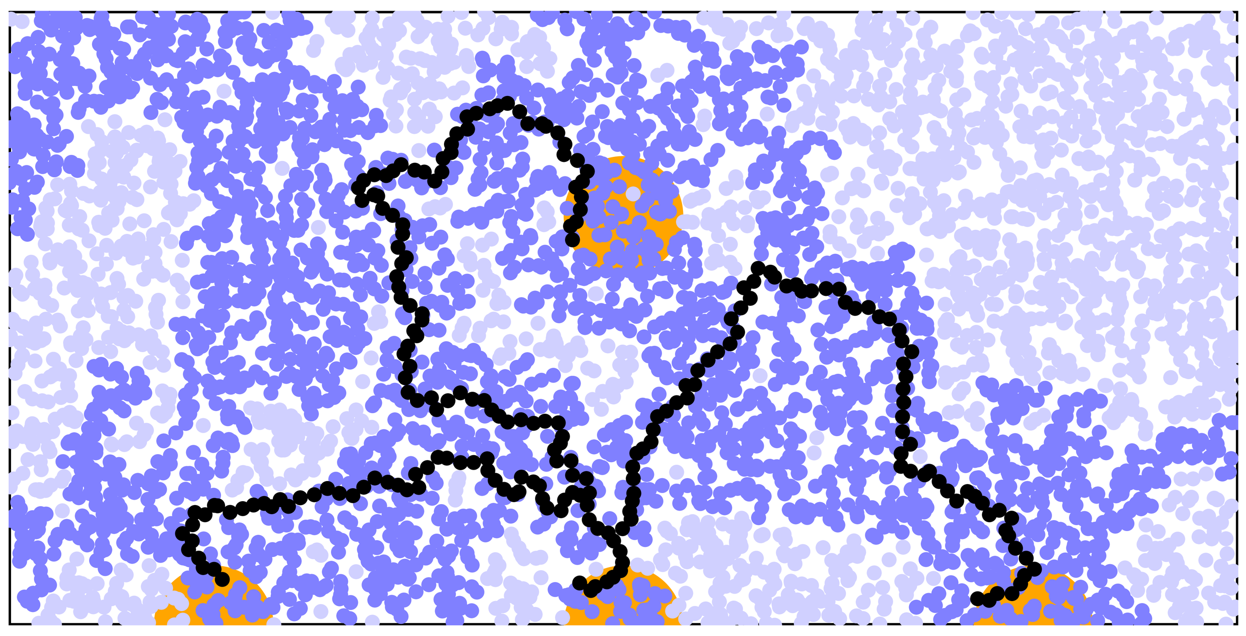
It is unknown whether $\theta_{\mathbb{R}^d}(\lambda_c) = 0$ except when $d = 2$ and $d \geq 11$. If $\theta(\lambda_c) > 0$, this would have a number of implications for percolation. In particular, points which *are* in the same component may not be connected inside quite a large box. So even if x and y are in huge components, they may not meet inside A.

Renormalisation

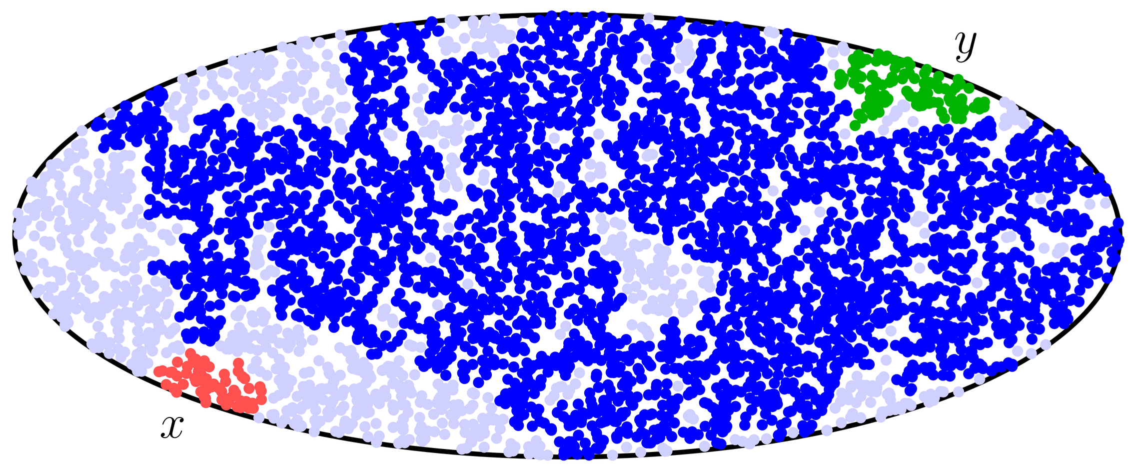
To construct long-range connections in the Boolean model, we define a finite grid of balls with spacing Mr_n (for some large constant M) and construct a *site percolation* model on this grid using the Boolean model.

A "good" vertex is the centre of a ball of radius Kr_n which intersects a unique large component of $Z(n, r_n)$ *inside a box of side lengths* $2dM$ which also intersects all neighbouring balls in the grid.

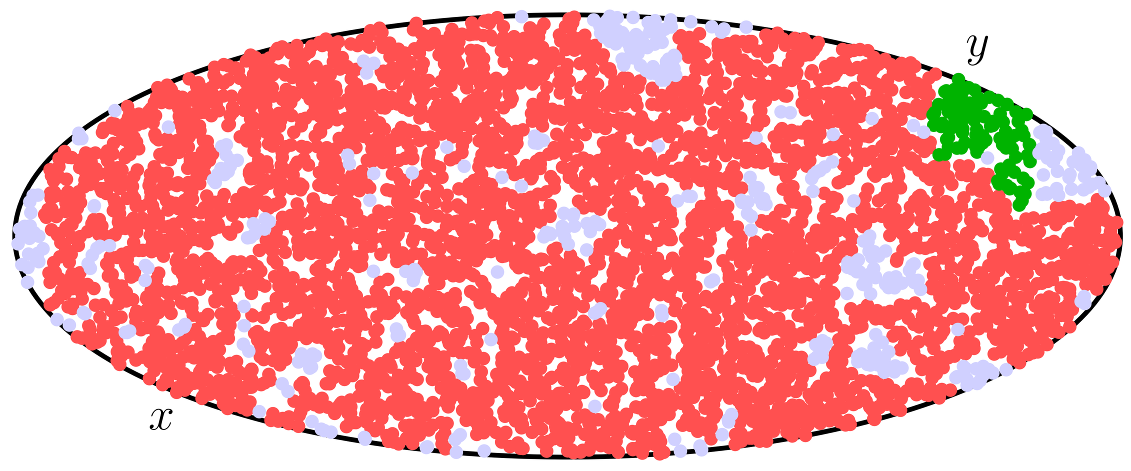
By choosing K, M and other parameters appropriately, we can ensure most balls in the grid are good. Since all these events occur *inside a box*, the dependence structure of the "good events" is simple.



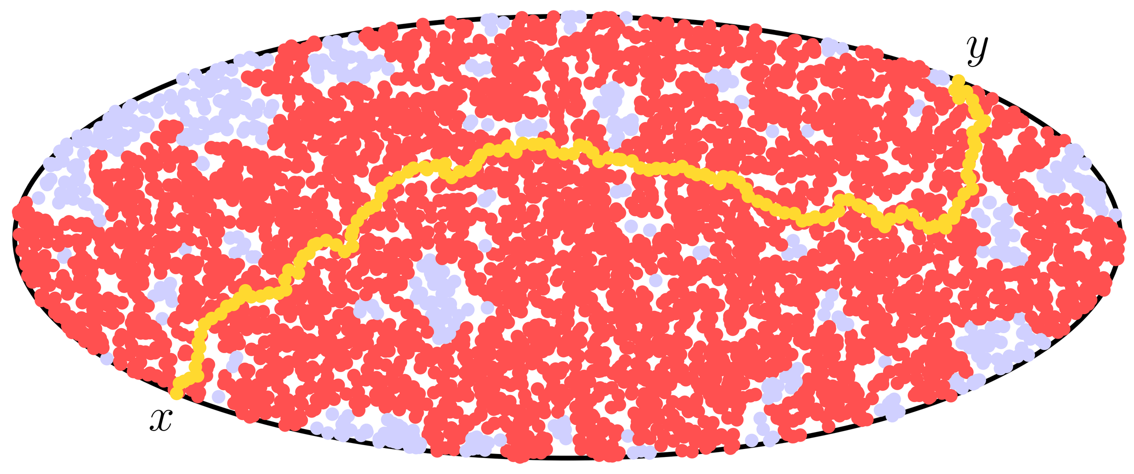
If you can prove that appropriate choices of K and M can be made under the assumption that $\theta_{\mathbb{H}}(\lambda) > 0$ (rather than $\lambda > \lambda_c$), then this implies $\theta_{\mathbb{R}^d}(\lambda_c) = 0$, which is a major open problem in percolation. Left here as an exercise for the reader...



When $\lambda > \lambda_c$, there will be a giant component. Above, neither x nor y is contained in the giant component.



x is in the giant component, but y is not.



x and y are connected via the giant component.

Extensions

There are some very easy extensions which would follow directly from our methods. For example, the probability that a collection of k distinct points are all in a single component should have a similar limit.

A more interesting extension is the question of whether x and y are in the same component of $\overline{A} \setminus Z(n, r)$. This *vacancy percolation* is very interesting, and quite challenging. For example, there is no lower bound on the size of a component, and the component can have much more complicated shapes than those of $Z(n, r)$.

What else can Λ_n do?

If $\Lambda_n = \Theta(\log n)$, other events like coverage $\{A \subseteq Z(n, r_n)\}$, connectivity of $Z(n, r_n)$ and interesting properties of the homology of $Z(n, r_n)$ become non-trivial.