Saddle-Point Approach to Large-Time Vol Smile

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By a saddle-point approximation to Lewis equation, we derive analytic form of large-time vol smile implied from model. In this note, we outline, justify and exemplify the approach in the case of Heston/VG/CGMY/BG. From here, we inspire some parametrizations of vol smile.

The approach was developed during my summer internship at Morgan Stanley, with some inputs from my supervisor King Wang.

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1 Lewis Equation

Define log-spot $X_T = \log(S_T/S_0)$ and log-strike $k = \log(K/S_0)$.

Lewis equation states that under characteristic function $\phi_T(u) = Ee^{iuX_T}$, call price

$$C(S,K) = S - \frac{\sqrt{SK}}{2\pi}\int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}\phi_T\left(u-\frac{i}{2}\right)$$

Under BS, $\phi_T^{BS}(u) = e^{-\frac{1}{2}u(u+i)\sigma^2 T}$. For each strike K, we quote $C(K)$ in $C_{BS}(K)$, thus

$$\int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}\phi_T\left(u-\frac{i}{2}\right) = \int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\sigma(k)^2 T}$$

where $\sigma(k)$ is implied vol (of our interest!).

2 Saddle-Point Equation

We assume characteristic function $\phi_T(u - i/2) \sim e^{-\psi(u)T}$ for large T.

Physically this says, at large time, log-spot X_T evolves like a Levy process, so that time T in characteristic exponent factors out. By definition, this is satisfied by all Levy processes, and some path-dependent processes that forget about its initial states over time e.g. Heston/SVJ.

Define time-scaled log-strike $x = k/T$, abbreviated strike below. Substituting the large-time ϕ_T into LHS we get

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-(iux+\psi(u))T}$$

Now, Taylor-expand the exponent around saddle-point $\tilde{u}$:

$$ix\tilde{u} + ix(u - \tilde{u}) + \psi(\tilde{u}) + \psi'(\tilde{u})(u - \tilde{u}) + \frac{\psi''(\tilde{u})}{2}(u - \tilde{u})^2 + O(u - \tilde{u})^3$$

Require $\psi'(\tilde{u}) = -ix$ (now keep in mind $\tilde{u} = \tilde{u}(x)$) to kill linear term, then LHS simplifies to

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-(ix\tilde{u}+\psi(\tilde{u}))T - \frac{\psi''(\tilde{u})T}{2}(u-\tilde{u})^2 - T\cdot O(u-\tilde{u})^3} \approx \frac{e^{-(ix\tilde{u}+\psi(\tilde{u}))T}}{\tilde{u}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''(\tilde{u})T}}$$

The approximation $e^{-(ix\tilde{u}+\psi(\tilde{u}))T} \sim$ Gaussian is valid when (1) $ix\tilde{u} + \psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$ (3) T is large, because

- for u far away from $\tilde{u}$ and T large, $e^{-T\cdot O(u-\tilde{u})^3} \to 0$ - tails flatten to zero
- for u close to $\tilde{u}$, constant/quadratic term in $e^{-(ix\tilde{u}+\psi(\tilde{u}))T - \frac{\psi''(\tilde{u})}{2}(u-\tilde{u})^2 - T\cdot O(u-\tilde{u})^3}$ dominate
- thus exponential approximates a Gaussian

For BS, $\psi_{BS}(u) = \frac{1}{2}\left(u^2 + \frac{1}{4}\right)v$ where $v = \sigma^2$ and noting $\psi'_{BS}(u) = uv$ and $\psi''_{BS}(u) = v$, by solving $\psi'_{BS}(\tilde{u}) = -ix$ we get

$$\tilde{u}_{BS} = -\frac{ix}{v}$$

Thus our saddle-point condition:

$$\begin{aligned}
\frac{e^{-(ix\tilde{u}+\psi(\tilde{u}))T}}{\tilde{u}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''(\tilde{u})T}} &\sim \frac{e^{-(ix\tilde{u}_{BS}+\psi_{BS}(\tilde{u}_{BS}))T}}{\tilde{u}_{BS}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''_{BS}(\tilde{u}_{BS})T}} \\
&= \frac{e^{-\left(\frac{x^2}{v} + \frac{v}{2}\left(\frac{1}{4} - \left(\frac{x}{v}\right)^2\right)\right)T}}{\frac{1}{4} - \left(\frac{x}{v}\right)^2} \sqrt{\frac{2\pi}{vT}} \\
&\approx 4\exp\left(-\left(\frac{v}{8} + \frac{x^2}{2v}\right)T\right)\sqrt{\frac{2\pi}{vT}}
\end{aligned}$$

Const. terms are of similar orders (dominated by $e^{-(\ldots)T}$) and we make exponent equal:

$$\omega(x) \equiv ix\tilde{u} + \psi(\tilde{u}) \sim \frac{v}{8} + \frac{x^2}{2v}$$

a quadratic equation, with solution

$$v(x) \sim 4\left(\omega(x) \pm \sqrt{\omega(x)^2 - \frac{x^2}{4}}\right)$$

Denote $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$, vanishing at $x_\pm$ which solve $\omega(x) = \pm x/2$, chosen to fulfill $\bar{\omega}((x_-, x_+)) < 0$ and $\bar{\omega}(\mathbb{R}\backslash(x_-, x_+)) > 0$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

We may express large-time asymptotic characteristic function in form $e^{-\psi(u)T}$ to reach a very wide class of model-inspired parametrizations.

3 In a Nutshell

Denote time-scaled log-strike $x = k/T$ and variance $v(x)$, implied from characteristic function $\phi_T(u) = Ee^{iuX_T}$ where $X_T = \log(S_T/S_0)$. Our saddle-point procedure reads:

1. evaluate characteristic function $\phi_T(u - i/2) \equiv e^{-\psi(u)T}$ to get $\psi(u)$
2. compute saddle-point $\tilde{u}$ which fulfills $\psi'(\tilde{u}) = -ix$
3. evaluate $\psi(\tilde{u})$ thus $\omega(x) \equiv i\tilde{u} \cdot x + \psi(\tilde{u})$ and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$
4. full smile given by $v(x) = 4\left(\omega(x) - \bar{\omega}(x)\right)$

Readers may first head to section Large-Time Heston Smile (and the like) to understand practical use and how $\omega, \bar{\omega}, v$ etc. actually look like before getting swamped by mathematical proofs below.

4 Gaussian Approximation

Our large-time Gaussian approximation in Lewis equation is valid when (1) $ix\tilde{u} + \psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$. We now show that these are always true from cumulant property.

We first show that $i\tilde{u}$ and $\psi(\tilde{u})$ are always real.

In large time, recall our characteristic function

$$\phi_T\left(u - \frac{i}{2}\right) = Ee^{\left(iu+\frac{1}{2}\right)X_T} = e^{-\psi(u)T}$$

thus $\psi(u)$ is a cumulant:

$$\psi(u) = -\frac{1}{T}\log Ee^{\left(iu+\frac{1}{2}\right)X_T}$$

Differentiating,

$$\psi'(u) = -\frac{1}{T}\frac{EiX_T e^{\left(iu+\frac{1}{2}\right)X_T}}{Ee^{\left(iu+\frac{1}{2}\right)X_T}}$$

$\tilde{u}$ satisfies

$$\psi'(\tilde{u}) = -\frac{1}{T}\frac{EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} = -ix$$

so

$$EX_T e^{\left(\hat{u}+\frac{1}{2}\right)X_T} = kEe^{\left(\hat{u}+\frac{1}{2}\right)X_T}$$

where we define $\hat{u} \equiv i\tilde{u}$. This is an equation in $\hat{u}$ - does a real $\hat{u}$ exist and is it unique?

A more insightful form: define Esscher measure $\mathbb{U}$ s.t.

$$\frac{d\mathbb{U}}{d\mathbb{P}} = \frac{e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}}$$

where $\mathbb{P}$ is our pricing measure (we have always been working in). Then

$$E^{\mathbb{U}}X_T = k$$

thus $\hat{u}(k)$ defines a measure under which expectation of log-spot is exactly log-strike. For $\mathbb{U}$ to be a properly defined measure, $\hat{u}$ has to be real - does it exist and is it unique?

Intuitively, $\hat{u}$ translates log-spot density - imagine Gaussian $f_{X_T}(x) \overset{\mathbb{P}}{\sim} e^{-x^2/2}$, then under $\mathbb{U}$, $f_{X_T}(x) \overset{\mathbb{U}}{\sim} e^{-(x-\hat{u})^2/2}$ and tanslation of density corresponds to translation of mean, so we can always suitably choose (solve for) $\hat{u}$ s.t. mean exactly matches log-strike i.e unique real $\hat{u}$ exists.

Now I make rigorous why $\hat{u}$, equivalently $\bar{u} = \hat{u} + 1/2$, always exists and is unique. Rewrite the expectation equation as

$$k = E^{\mathbb{U}}X_T = \frac{EX_T e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} = \frac{\partial}{\partial \bar{u}} \log Ee^{\bar{u}X_T}$$

This is the first derivative of cumulant (under pricing measure, not Esscher), which spans the support of $X_T \in \mathbb{R}$ (thus existence of at least a real root), and monotonically (thus one unique root $\hat{u}$), as its second denvative is variance - always positive.

With real $\hat{u}$, $\mathbb{U}$ is a property defined measure equivalent to $\mathbb{P}$.

Now,

$$\psi(\tilde{u}) = -\frac{1}{T}\log Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T} \in \mathbb{R}$$

For our saddle-point approximation to be valid, we need $\psi''(\tilde{u}) > 0$:

$$
\begin{aligned}
\psi''(\tilde{u}) &= -\frac{1}{T}\left(-\frac{EX_T^2 e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} - \frac{EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{\left(Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}\right)^2} EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}\right) \\
&= \frac{1}{T}\left(\frac{EX_T^2 e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} - \left(\frac{EX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}\right)^2\right) \\
&= \frac{1}{T}\left(\frac{EX_T^2 e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}} - \left(\frac{EX_T e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}}\right)^2\right) \\
&= \frac{1}{T}\left(E^{\mathbb{U}}X_T^2 - \left(E^{\mathbb{U}}X_T\right)^2\right) \\
&= \frac{V^{\mathbb{U}}X_T}{T}
\end{aligned}
$$

i.e. $\psi''(\tilde{u})$ is log-spot variance under $\mathbb{U}$. As expected, because differentiated cumulant gives central moments.

Thus as long as we can write down characteristic function, for large T, our saddle-point condition is always valid.

We can computatonally check how large is large, say approx closed-form formula vs. FFT.

5 Moment Expansion

To make things simplier, define

$$
\bar{u} = \hat{u} + \frac{1}{2}
$$

Then we rewrite our cumulant

$$
\psi(\tilde{u}) = -\frac{1}{T}\log Ee^{\bar{u}X_T}
$$

and saddle-point equation

$$
k = E\left[X_T\frac{e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}}\right]
$$

We now show that derivatives of $\psi(\tilde{u})$ and $\bar{u}$ are connected to central moments under Esscher measure.

First $\psi(\tilde{u})$.

$$
\begin{aligned}
\partial_x\psi(\tilde{u}) &= -\frac{1}{T}\frac{EX_T e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}}\bar{u}' = -x\bar{u}' \\
\Rightarrow \partial_x^{n+1}\psi(\tilde{u}) &= -\partial_x^n(x\bar{u}') = -n\bar{u}^{(n)} - x\bar{u}^{(n+1)}
\end{aligned}
$$

As long as we know derivatives $\partial_x^n \bar{u}$, we can compute all derivatives of $\psi(\tilde{u})$.

Equivalently we compute $\partial_k^n \bar{u}$, with strike dervatives related by $\partial_x = T\partial_k$.

Differentiate saddle-point equation to get

$$\partial_k^n k = E\left[X_T \partial_k^n \frac{e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}}\right]$$

where k-dependence is stressed.

We study $n = 1, 2$ doing explicit calculations. Note when $n = 0$, we have $k = E^{\mathbb{U}} X_T$.

For $n = 1$,

$$\begin{aligned}
1 &= E\left[X_T \partial_k \frac{e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}}\right] \\
&= E\left[X_T \left(\frac{X_T e^{\bar{u}X_T} \bar{u}'}{Ee^{\bar{u}X_T}} - \frac{e^{\bar{u}X_T}}{(Ee^{\bar{u}X_T})^2} EX_T e^{\bar{u}X_T} \bar{u}'\right)\right] \\
&= \bar{u}' \left(E^{\mathbb{U}} X_T^2 - (E^{\mathbb{U}} X_T)^2\right) \\
&= \bar{u}' E^{\mathbb{U}} (X_T - k)^2
\end{aligned}$$

so

$$\bar{u}'(k) = \frac{1}{E^{\mathbb{U}}(X_T - k)^2}$$

For $n = 2$,

$$\begin{aligned}
0 &= E\left[X_T \partial_k^2 \frac{e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}}\right] \\
&= E\left[X_T \left(\frac{\partial_k^2 e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} + 2\partial_k e^{\bar{u}X_T} \partial_k \frac{1}{Ee^{\bar{u}X_T}} + e^{\bar{u}X_T} \partial_k^2 \frac{1}{Ee^{\bar{u}X_T}}\right)\right]
\end{aligned}$$

which we have

$$\begin{aligned}
\partial_k e^{\bar{u}X_T} &= X_T e^{\bar{u}X_T} \bar{u}' \\
\partial_k^2 e^{\bar{u}X_T} &= X_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + X_T e^{\bar{u}X_T} \bar{u}'' \\
\partial_k \frac{1}{Ee^{\bar{u}X_T}} &= -\frac{EX_T e^{\bar{u}X_T}}{(Ee^{\bar{u}X_T})^2} \bar{u}' \\
\partial_k^2 \frac{1}{Ee^{\bar{u}X_T}} &= \frac{2(EX_T e^{\bar{u}X_T})^2}{(Ee^{\bar{u}X_T})^3} (\bar{u}')^2 - \frac{1}{(Ee^{\bar{u}X_T})^2} \left[EX_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + EX_T e^{\bar{u}X_T} \bar{u}''\right]
\end{aligned}$$

Combining,

$$
\begin{aligned}
0 &= E\left[X_T\left(\frac{\partial_k^2 e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} + 2\partial_k e^{\bar{u}X_T}\partial_k \frac{1}{Ee^{\bar{u}X_T}} + e^{\bar{u}X_T}\partial_k^2 \frac{1}{Ee^{\bar{u}X_T}}\right)\right] \\
&= E\left[X_T\left\{\frac{X_T^2 e^{\bar{u}X_T}(\bar{u}')^2 + X_T e^{\bar{u}X_T}\bar{u}''}{Ee^{\bar{u}X_T}} + 2\left(X_T e^{\bar{u}X_T}\bar{u}'\right)\left(-\frac{EX_T e^{\bar{u}X_T}}{(Ee^{\bar{u}X_T})^2}\bar{u}'\right) + \right.\right. \\
&\quad \left.\left. e^{\bar{u}X_T}\left(\frac{2(EX_T e^{\bar{u}X_T})^2}{(Ee^{\bar{u}X_T})^3}(\bar{u}')^2 - \frac{EX_T^2 e^{\bar{u}X_T}(\bar{u}')^2 + EX_T e^{\bar{u}X_T}\bar{u}''}{(Ee^{\bar{u}X_T})^2}\right)\right\}\right] \\
&= \left(E^{\mathbb{U}}X_T^2 - (E^{\mathbb{U}}X_T)^2\right)\bar{u}'' + \left(E^{\mathbb{U}}X_T^3 - 3E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T + 2(E^{\mathbb{U}}X_T)^3\right)(\bar{u}')^2 \\
&= E^{\mathbb{U}}(X_T - k)^2\bar{u}'' + E^{\mathbb{U}}(X_T - k)^3(\bar{u}')^2
\end{aligned}
$$

Solving,

$$
\bar{u}''(k) = -\frac{E^{\mathbb{U}}(X_T - k)^3}{(E^{\mathbb{U}}(X_T - k)^2)^3}
$$

Calculations for $n = 3$ are a pain ty type here, thus I state the result:

$$
0 = E^{\mathbb{U}}(X_T - k)^2\bar{u}''' + 3E^{\mathbb{U}}(X_T - k)^3\bar{u}'\bar{u}'' + \left(E^{\mathbb{U}}(X_T - k)^4 - 3(E^{\mathbb{U}}(X_T - k)^2)^2\right)(\bar{u}')^3
$$

solved to give

$$
\bar{u}'''(k) = \frac{3(E^{\mathbb{U}}(X_T - k)^2)^3 + 3(E^{\mathbb{U}}(X_T - k)^3)^2 - E^{\mathbb{U}}(X_T - k)^2 E^{\mathbb{U}}(X_T - k)^4}{(E^{\mathbb{U}}(X_T - k)^2)^5}
$$

Casting back to x, we have

$$
\begin{aligned}
\bar{u}'(x) &= T\frac{1}{E^{\mathbb{U}}(X_T - k)^2} \\
\bar{u}''(x) &= -T^2\frac{E^{\mathbb{U}}(X_T - k)^3}{(E^{\mathbb{U}}(X_T - k)^2)^3} \\
\bar{u}'''(x) &= T^3\frac{3(E^{\mathbb{U}}(X_T - k)^2)^3 + 3(E^{\mathbb{U}}(X_T - k)^3)^2 - E^{\mathbb{U}}(X_T - k)^2 E^{\mathbb{U}}(X_T - k)^4}{(E^{\mathbb{U}}(X_T - k)^2)^5}
\end{aligned}
$$

ATM $k = 0$, choose $\bar{u} = \bar{u}_0$ s.t. $E^{\mathbb{U}}X_T = 0$ - now $\mathbb{U}$ is the ATM Esscher measure.

Thus a moment expansion of $\bar{u}$ to third order

$$
\bar{u}(k) = \bar{u}_0 + \frac{1}{E^{\mathbb{U}}X_T^2}k - \frac{1}{2}\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3}k^2 + \frac{1}{6}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}k^3 ...
$$

Differentiation of ψ gives

$$
\begin{aligned}
\partial_x \psi(\tilde{u}_0) &= -x\bar{u}' = 0 \quad \text{saddle-point condition} \\
\partial_x^2 \psi(\tilde{u}_0) &= -\bar{u}' - x\bar{u}'' = -T\frac{1}{E^{\mathbb{U}}X_T^2} \\
\partial_x^3 \psi(\tilde{u}_0) &= -2\bar{u}'' - x\bar{u}''' = 2T^2\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3} \\
\partial_x^4 \psi(\tilde{u}_0) &= -3\bar{u}''' - x\bar{u}'''' = -3T^3\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}
\end{aligned}
$$

ATM, we have

$$
\psi(\tilde{u}_0) \equiv \frac{\psi_0}{T} = -\frac{1}{T}\log E e^{\bar{u}_0 X_T}
$$

Thus a moment expansion of ψ to forth order

$$
\psi(k) = \frac{\psi_0}{T} - \frac{1}{2}\frac{1}{E^{\mathbb{U}}X_T^2}\frac{k^2}{T} + \frac{1}{3}\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3}\frac{k^3}{T} - \frac{1}{8}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}\frac{k^4}{T}...
$$

5.1 Moment Expansion of ω

Var-like quantity ω in our smile expands as

$$
\begin{aligned}
&\omega(k) \equiv \hat{u}(k)\frac{k}{T} + \psi(k) \\
&\quad = \frac{k}{T}\left(\bar{u}_0 - \frac{1}{2} + \frac{1}{E^{\mathbb{U}}X_T^2}k - \frac{1}{2}\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3}k^2 + \frac{1}{6}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}k^3...\right) + \\
&\quad\quad \frac{\psi_0}{T} - \frac{1}{2}\frac{1}{E^{\mathbb{U}}X_T^2}\frac{k^2}{T} + \frac{1}{3}\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3}\frac{k^3}{T} - \frac{1}{8}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}\frac{k^4}{T}...
\end{aligned}
$$

so

$$
\omega(k)T = \psi_0 + \left(\bar{u}_0 - \frac{1}{2}\right)k + \frac{1}{2}\frac{1}{E^{\mathbb{U}}X_T^2}k^2 - \frac{1}{6}\frac{E^{\mathbb{U}}X_T^3}{(E^{\mathbb{U}}X_T^2)^3}k^3 + \frac{1}{24}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(E^{\mathbb{U}}X_T^2)^5}k^4...
$$

Denote $\sigma_0^2 T \equiv E^{\mathbb{U}}X_T^2$, and we rewrite the smile in terms of normalized strike

$$
\begin{aligned}
\omega(k)T = \psi_0 &+ \left(\bar{u}_0 - \frac{1}{2}\right)\sigma_0\sqrt{T}\left(\frac{k}{\sigma_0\sqrt{T}}\right) + \frac{1}{2}\left(\frac{k}{\sigma_0\sqrt{T}}\right)^2 - \frac{1}{6}\frac{E^{\mathbb{U}}X_T^3}{(\sigma_0\sqrt{T})^3}\left(\frac{k}{\sigma_0\sqrt{T}}\right)^3 + \\
&\frac{1}{24}\frac{3(E^{\mathbb{U}}X_T^2)^3 + 3(E^{\mathbb{U}}X_T^3)^2 - E^{\mathbb{U}}X_T^2 E^{\mathbb{U}}X_T^4}{(\sigma_0\sqrt{T})^6}\left(\frac{k}{\sigma_0\sqrt{T}}\right)^4...
\end{aligned}
$$

with dimensionless coefficients.

5.2 Derivatives of $\bar{u}$

For $n = 1, 2$, respectively we have

$$
\begin{aligned}
1 &= E^{\mathbb{U}}(X_T - k)^2 \bar{u}' \\
0 &= E^{\mathbb{U}}(X_T - k)^2 \bar{u}'' + E^{\mathbb{U}}(X_T - k)^3 (\bar{u}')^2
\end{aligned}
$$

Can we generalize this line? Note each term carries dimension $1/k^{n-1}$.

Let's back out a bit and consider our saddle-point equation in an alternative form

$$\partial_k \log E e^{\bar{u} X_T} = k \bar{u}'$$

so its nth derivative

$$\partial_k^{n+1} \log E e^{\bar{u} X_T} = \partial_k^n (k \bar{u}') = n \bar{u}^{(n)} + k \bar{u}^{(n+1)} \quad n \geq 0$$

from which we recursively solve $\bar{u}^{(n)}$ in terms of $\bar{u}^{(j)}$, $j \leq n-1$, and collect its coefficients into central moments. Derivatives of cumulant of becomputed from Faà di Bruno's formula for derivatives of function composition (of $\log \cdot$ and $E e^{u(\cdot) X_T}$), which has very unintuitive coefficients - I will omit.

5.3 Moment Expansion of Smile v

We obtain a moment expansion of $\bar{\omega}(k)T$ hence smile v.

We defined $\bar{\omega}(x) \equiv \sqrt{\omega(x)^2 - x^2/4}$, shown in next section Existence of Variance Solution to be always real, and variance smile is given by

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

We express total variance in terms of log-strike

$$w(k) \equiv v(k)T \sim 4(\omega(k)T - \bar{\omega}(k)T)$$

with moment expansion

$$\omega(k)T = \psi_0 - \left(\frac{1}{2} - \bar{u}_0\right) k + \frac{1}{2} \frac{1}{E^{\mathbb{U}} X_T^2} k^2 - \frac{1}{6} \frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} k^3 + \frac{1}{24} \frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} k^4 \ldots$$

$|\bar{\omega}(k)T|$ expands ATM as

$$
\begin{aligned}
|\bar{\omega}(k)T| = -\bar{\omega}(k)T = \psi_0 - \left(\frac{1}{2} - \bar{u}_0\right) k + \frac{1}{2}\left(\frac{1}{E^{\mathbb{U}} X_T^2} - \frac{1}{4\psi_0}\right) k^2 - \frac{1}{6}\left(\frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} - \frac{6\bar{u}_0 - 3}{8\psi_0^2}\right) k^3 + \\
\frac{1}{24}\left(\frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} + \frac{3}{2\psi_0^2 E^{\mathbb{U}} X_T^2} - \frac{3(16\bar{u}_0^2 - 16\bar{u}_0 + 5)}{16\psi_0^3}\right) k^4 \ldots
\end{aligned}
$$

Therefore, we obtain full variance smile moment expansion

$$
\begin{aligned}
w(k) = 8\Bigg\{ & \psi_0 - \left(\frac{1}{2} - \bar{u}_0\right) k + \frac{1}{2}\left(\frac{1}{E^{\mathbb{U}} X_T^2} - \frac{1}{8\psi_0}\right) k^2 - \frac{1}{6}\left(\frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} - \frac{6\bar{u}_0 - 3}{16\psi_0^2}\right) k^3 + \\
& \frac{1}{24}\left(\frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} + \frac{3}{4\psi_0^2 E^{\mathbb{U}} X_T^2} - \frac{3(16\bar{u}_0^2 - 16\bar{u}_0 + 5)}{32\psi_0^3}\right) k^4 \ldots \Bigg\}
\end{aligned}
$$

Note that we make no approx here and this formula is exact.

Notably, var skew is difference of ATM Esscher shift between model's $\bar{u}_0$ and BS's $1/2$; var curvature is inverse difference between ATM Esscher variance and ATM var. They are measures of deviation from BS diffusion progressively to higher orders.

$$
\begin{aligned}
\frac{\partial w}{\partial k} &= -8\left(\frac{1}{2} - \bar{u}_0\right) \\
\frac{\partial^2 w}{\partial k^2} &= 8\left(\frac{1}{E^{\mathbb{U}} X_T^2} - \frac{1}{w(0)}\right)
\end{aligned}
$$

Under BS flat var w_0 and Esscher shift $\bar{u}_0 = 1/2$, we have exactly $w(k) = w_0$.

A practical use is, we fit a polynomial, say quartic, to ATM and from coefficients we imply the moments. But the moments are under ATM Esscher measure, which makes them less useful. If log-spot density is close enough to Gaussian, moments (above quadratic) under pncing/Esscher measure are exactly identical, as effect of Esscher is merely a shift (translation of mean). But at least Esscher moments give an order-of-magnitude estimates of true implied moments.

6 Existence of Variance Solution

Our variance $v(x)$ exists if $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$ is well-defined.

My proposition is: $\omega(x)$ tangentially touches $|x/2|$ at points $x_\pm$ (small and off by a sign) but is everywhere else bounded below.

Equivalently, this may be stated as: $\omega(x) \mp x/2$ and its derivative vanish at some $x_\pm$ and second derivative is positive everywhere. We write

$$
\omega(x) \equiv x\hat{u} + \psi(\tilde{u}(x))
$$

thus derivatives

$$
\begin{aligned}
\omega'(x) &= \hat{u}(x) + x\hat{u}'(x) + \psi'(\tilde{u}(x)) = \hat{u}(x) \\
\omega''(x) &= \hat{u}'(x) = \frac{T}{E^{\mathbb{U}}(X_T - k)^2} > 0
\end{aligned}
$$

so second derivative is everywhere positive, and

$$\partial_x\left(\omega(x) \mp x/2\right) = \hat{u}(x) \mp \frac{1}{2}$$

which vanishes at some $x_\pm$,

$$\hat{u}(x_\pm) = \pm\frac{1}{2}$$

Does such $x_\pm$ always exist? Yes. Recall that $\hat{u}$ is roughly the log-spot density shift (via Esscher measure $\mathbb{U}$) s.t. expectation of log-spot $E^{\mathbb{U}}X_T$, exactly matches log-strike k, equivalently x, thus $\hat{u}$ is monotonic increasing in x, spanning $\mathbb{R}$. We can always find such an $x_\pm$ s.t. $\hat{u}(x_\pm)$ exactly matches $\pm1/2$ - smallness/sign of $\pm1/2$ leads to smaliness/sign of $x_\pm$.

By our moment expansion for $\bar{u}$, for small x

$$\bar{u}(x) \approx \bar{u}_0 + \frac{T}{E^{\mathbb{U}}X_T^2}x \equiv \frac{1}{2} \pm \frac{1}{2}$$

which approximates

$$x_\pm \approx \frac{E^{\mathbb{U}}X_T^2}{T}\left(\frac{1}{2} \pm \frac{1}{2} - \bar{u}_0\right) \approx \pm\frac{E^{\mathbb{U}}X_T^2}{2T} \sim \pm\frac{\sigma^2}{2}$$

as $\bar{u}_0 \approx 1/2$, from Ito correction, and σ is some characteristic vol in model. In Heston case (see section Large-Time Heston Smile), $\sigma^2 \sim \bar{v}$ so $x_\pm \sim \pm\bar{v}/2 = \pm0.02$ - consistent with our plot!

To reason about this, think about BS diffusion - log-spot $X_T = -\sigma^2T/2 + \sigma Z_T$ and denote total var $w = \sigma^2 T$ so density $f_{X_T}(x) \overset{\mathbb{P}}{\sim} e^{-(x+w/2)^2/2w}$. Under ATM Esscher measure defined by $\bar{u}_0$, $f_{X_T}(x) \overset{\mathbb{U}}{\sim} e^{-(x+w/2-w\bar{u}_0)^2/2w}$. For $E^{\mathbb{U}}X_T = 0$, we demand $\bar{u}_0 = 1/2$ - this is exact. At-the-money, other processes may be expressed as perturbation around BS diffusion so densities near $x = 0$ behave roughly identical, so approximation $\bar{u}_0 \approx 1/2$ is not unreasonable.

Typically, if model produces heavy left-tail, mode of log-spot density is biased to right, so we require a smaller Esscher shift to match match expectation of log-spot to zero - thus $\bar{u}_0 < 1/2$, a negatively skewed $\omega(k)T$.

Lastly, $\omega(x) \mp x/2$ vanishes at $x_\pm$ as

$$\omega(x_\pm) \mp \frac{x_\pm}{2} = x_\pm\hat{u}(x_\pm) - \frac{1}{T}\log Ee^{\left(\hat{u}(x_\pm)+\frac{1}{2}\right)X_T} \mp \frac{x_\pm}{2} = \pm\frac{x_\pm}{2} \mp \frac{x_\pm}{2} - \frac{1}{T}\log Ee^{\left(\pm\frac{1}{2}+\frac{1}{2}\right)X_T} = 0$$

as $Ee^{X_T} = 1$ by martingale condition.

So, $\omega(x) \geq |x/2|$ for all strike x and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - x^2/4} \leq \omega(x)$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

is always well-defined (positive).

7 Large-Time Heston Smile

We derive the large-time Heston full smile, unapproximated thus valid for all strike $x \in \mathbb{R}$.

8 Large-Time VG Smile

9 Large-Time BG Smile

10 Large-Time CGMY Smile

11 Comments on Saddle-Point Trick

Our starting point is Lewis equation expanded around saddle-point $\tilde{u}$ with integral approximated as a Gaussian in large time - for a full discussion, see section Saddle-Point Equation.

I thought about the analog formula for small time, but no - this approach relies on large time and in small time non-linear terms come in, so we can no longer make Gaussian approx.

We reach following smile equation for implied variance, assuming characteristic function in large time scales as $\phi_T(u - i/2) \sim e^{-\psi(u)T}$:

$$\frac{v(x)}{8} + \frac{x^2}{2v(x)} \sim i\tilde{u}(x)x + \psi(\tilde{u}(x)) \equiv \omega(x)$$

We proved that $\omega(x) \geq |x/2|$, so quadratic determinant is always positive. Variance $v(x)$ always exists and is positive.

Condition $\phi_T(u-i/2) \sim e^{-\psi(u)T}$ says: in large time log-spot evolves like a Levi process (of independent and stationary increments), and large-time smile converges to a stationary state. Independence leads to scaling in T in exponent; staionarity leads to time-independent factor $\psi(u)$.

Our saddle-point trick works for Levy processes (by definition), and some path-dependent processes that forget about its inital states over time e.g. Heston/SVJ due to mean-reversion of vol.

Smile properties depend entirely on factor $\omega(x)$. If we want a complex smile of rich curvatures (say crazy W-shape), we have to design a $\psi(u)$ complex enough, that fulfills martingale condition $\psi(-i/2) = 0$ - is $\psi(u)$, rather than log-spot process X_T, the night thing to start with? By Levy-Khintchine representation it seems there exists some Levy measure integrated to match ψ, at least in theory, though we risk a totally incomprehensible Levy measure. Another queston is, does arbitrary $\psi(u)$ (that fulfills margingale conditon) contain arbitrage?

To make this precise, say we want a pure-jump process (ignore drift/diffusion for now) obeying some Levy measure $\mu(\xi)$ that fulfills $\psi(u)$. By Levy-Khintchine, characteristic function reads

$$\phi_T(u) \equiv Ee^{iuX_T} = \exp\left(T\int(e^{iu\xi} - 1)\mu(\xi)d\xi\right)$$

$$\phi_T\left(u - \frac{i}{2}\right) = \exp\left(T\int\left(e^{\left(iu+\frac{1}{2}\right)\xi} - 1\right)\mu(\xi)d\xi\right) \equiv e^{-\psi(u)T}$$

so

$$\psi(u) = -\int \left(e^{\left(iu+\frac{1}{2}\right)\xi} - 1\right)\mu(\xi)d\xi$$

which fulfills martingale condition

$$\psi(-i/2) = 0$$

Levy measure μ can be obtained via inverse-Fourier:

$$\psi\left(u+\frac{i}{2}\right) = -\int e^{iu\xi}\mu(\xi)d\xi + \int \mu(\xi)d\xi$$
$$\mu(\xi) = -\mathcal{F}^{-1}\psi\left(u+\frac{i}{2}\right) + \left(\int \mu(\xi)d\xi\right)\delta(\xi)$$

Let's ignore the delta function explosion because Levy measure cannot count zero jump anyway - they are indistinguishable from diffusion! Either it is part of μ or will cancel with some infinity in inverse-Fourier term.

It seems from ψ we can back out its underlying Levy measure μ, which corresponds to a Levy process. What extra conditions do we need on ψ s.t. the Levy process is well-defined?

11.1 Some Opinions

I think Merton-jump model with following extensions can potentially fit a smile of rich curvatures. In Merton we assume Gaussian log-jumps obeying Poisson arrival - this has a (finite-activity) Gaussian Levy measure. With sufficiently many Gaussians (with different arrival rates λ and spreads σ^2) this seems to match arbitrary Levy measure. More precisely, any Levy measure may be represented in superposition of compound Poisson processes

$$\mu(\xi) = \int d\alpha\lambda(\alpha)\frac{e^{-(\xi-\alpha)^2/2\sigma^2(\alpha)}}{\sqrt{2\pi}}$$

Activity is controlled by λ - say for infinite activity, λ integrates to infinity. This probably has too many degrees of freedom $\{\lambda(\alpha), \sigma^2(\alpha)\}$ - extra constraints are needed. This essentially says, any function may be expanded in Gaussian basis - I think this is very plausible. Now that we have a Levy measure flexible enough, this fits complex smile.

The first thing to do is, work out large-time Merton smile and study its curvature, then progressively add Poisson jumps and observe how this changes curvature.

12 References

- Gatheral, The Volatility Surface: A Practitioner's Guide
- Gatheral/Jacquier, Convergence of Heston to SVI
- Madan/Wang, Additive Processes with Bilateral Gamma Marginals