

Saddle-Point Approach to Large-Time Vol Smile

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By a saddle-point approximation to Lewis equation, we derive analytic form of large-time vol smile implied from model. In this note, we outline, justify and exemplify the approach in the case of Heston/VG/CGMY/BG. From here, we inspire some parametrizations of vol smile.

The approach was developed during my summer internship at Morgan Stanley, with some inputs from my supervisor King Wang.

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1 Lewis Equation

Define log-spot $X_T = \log(S_T/S_0)$ and log-strike $k = \log(K/S_0)$.

Lewis equation states that under characteristic function $\phi_T(u) = Ee^{iuX_T}$, call price

$$C(S, K) = S - \frac{\sqrt{SK}}{2\pi} \int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-iuk} \phi_T\left(u - \frac{i}{2}\right)$$

Under BS, $\phi_T^{BS}(u) = e^{-\frac{1}{2}u(u+i)\sigma^2 T}$. For each strike K , we quote $C(K)$ in $C_{BS}(K)$, thus

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-iuk} \phi_T\left(u - \frac{i}{2}\right) = \int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-iuk} e^{-\frac{1}{2}(u^2 + \frac{1}{4})\sigma(k)^2 T}$$

where $\sigma(k)$ is implied vol (of our interest!).

2 Saddle-Point Equation

We assume characteristic function $\phi_T(u - i/2) \sim e^{-\psi(u)T}$ for large T .

Physically this says, at large time, log-spot X_T evolves like a Levy process, so that time T in characteristic exponent factors out. By definition, this is satisfied by all Levy processes, and some path-dependent processes that forget about its initial states over time e.g. Heston/SVJ.

Define time-scaled log-strike $x = k/T$, abbreviated strike below. Substituting the large-time ϕ_T into LHS we get

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-(iux + \psi(u))T}$$

Now, Taylor-expand the exponent around saddle-point \tilde{u} :

$$ix\tilde{u} + ix(u - \tilde{u}) + \psi(\tilde{u}) + \psi'(\tilde{u})(u - \tilde{u}) + \frac{\psi''(\tilde{u})}{2}(u - \tilde{u})^2 + O(u - \tilde{u})^3$$

Require $\psi'(\tilde{u}) = -ix$ (now keep in mind $\tilde{u} = \tilde{u}(x)$) to kill linear term, then LHS simplifies to

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}} e^{-(ix\tilde{u} + \psi(\tilde{u}))T - \frac{\psi''(\tilde{u})}{2}(u - \tilde{u})^2 - T \cdot O(u - \tilde{u})^3} \approx \frac{e^{-(ix\tilde{u} + \psi(\tilde{u}))T}}{\tilde{u}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''(\tilde{u})T}}$$

The approximation $e^{-(ix\tilde{u} + \psi(\tilde{u}))T} \sim \text{Gaussian}$ is valid when (1) $ix\tilde{u} + \psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$ (3) T is large, because

- for u far away from \tilde{u} and T large, $e^{-T \cdot O(u - \tilde{u})^3} \rightarrow 0$ – tails flatten to zero
- for u close to \tilde{u} , constant/quadratic term in $e^{-(ix\tilde{u} + \psi(\tilde{u}))T - \frac{\psi''(\tilde{u})}{2}(u - \tilde{u})^2 - T \cdot O(u - \tilde{u})^3}$ dominate
- thus exponential approximates a Gaussian

For BS, $\psi_{BS}(u) = \frac{1}{2} \left(u^2 + \frac{1}{4} \right) v$ where $v = \sigma^2$ and noting $\psi'_{BS}(u) = uv$ and $\psi''_{BS}(u) = v$, by solving $\psi'_{BS}(\tilde{u}) = -ix$ we get

$$\tilde{u}_{BS} = -\frac{ix}{v}$$

Thus our saddle-point condition:

$$\begin{aligned} \frac{e^{-(ix\tilde{u} + \psi(\tilde{u}))T}}{\tilde{u}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''(\tilde{u})T}} &\sim \frac{e^{-(ix\tilde{u}_{BS} + \psi_{BS}(\tilde{u}_{BS}))T}}{\tilde{u}_{BS}^2 + \frac{1}{4}} \sqrt{\frac{2\pi}{\psi''_{BS}(\tilde{u}_{BS})T}} \\ &= \frac{e^{-\left(\frac{x^2}{v} + \frac{v}{2} \left(\frac{1}{4} - \left(\frac{x}{v}\right)^2\right)\right)T}}{\frac{1}{4} - \left(\frac{x}{v}\right)^2} \sqrt{\frac{2\pi}{vT}} \\ &\approx 4 \exp\left(-\left(\frac{v}{8} + \frac{x^2}{2v}\right)T\right) \sqrt{\frac{2\pi}{vT}} \end{aligned}$$

Const. terms are of similar orders (dominated by $e^{-(\dots)T}$) and we make exponent equal:

$$\omega(x) \equiv ix\tilde{u} + \psi(\tilde{u}) \sim \frac{v}{8} + \frac{x^2}{2v}$$

a quadratic equation, with solution

$$v(x) \sim 4 \left(\omega(x) \pm \sqrt{\omega(x)^2 - \frac{x^2}{4}} \right)$$

Denote $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$, vanishing at x_{\pm} which solve $\omega(x) = \pm x/2$, chosen to fulfill $\bar{\omega}((x_-, x_+)) < 0$ and $\bar{\omega}(\mathbb{R} \setminus (x_-, x_+)) > 0$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

We may express large-time asymptotic characteristic function in form $e^{-\psi(u)T}$ to reach a very wide class of model-inspired parametrizations.

3 In a Nutshell

Denote time-scaled log-strike $x = k/T$ and variance $v(x)$, implied from characteristic function $\phi_T(u) = Ee^{iuX_T}$ where $X_T = \log(S_T/S_0)$. Our saddle-point procedure reads:

1. evaluate characteristic function $\phi_T(u - i/2) \equiv e^{-\psi(u)T}$ to get $\psi(u)$
2. compute saddle-point \tilde{u} which fulfills $\psi'(\tilde{u}) = -ix$
3. evaluate $\psi(\tilde{u})$ thus $\omega(x) \equiv i\tilde{u} \cdot x + \psi(\tilde{u})$ and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$
4. full smile given by $v(x) = 4(\omega(x) - \bar{\omega}(x))$

Readers may first head to section **Large-Time Heston Smile** (and the like) to understand practical use and how $\omega, \bar{\omega}, v$ etc. actually look like before getting swamped by mathematical proofs below.

4 Gaussian Approximation

Our large-time Gaussian approximation in Lewis equation is valid when (1) $ix\tilde{u} + \psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$. We now show that these are always true from cumulant property.

We first show that $i\tilde{u}$ and $\psi(\tilde{u})$ are always real.

In large time, recall our characteristic function

$$\phi_T \left(u - \frac{i}{2} \right) = Ee^{(iu + \frac{1}{2})X_T} = e^{-\psi(u)T}$$

thus $\psi(u)$ is a cumulant:

$$\psi(u) = -\frac{1}{T} \log Ee^{(iu + \frac{1}{2})X_T}$$

Differentiating,

$$\psi'(u) = -\frac{1}{T} \frac{E i X_T e^{(iu + \frac{1}{2})X_T}}{E e^{(iu + \frac{1}{2})X_T}}$$

\tilde{u} satisfies

$$\psi'(\tilde{u}) = -\frac{1}{T} \frac{E i X_T e^{(i\tilde{u} + \frac{1}{2})X_T}}{E e^{(i\tilde{u} + \frac{1}{2})X_T}} = -ix$$

so

$$E X_T e^{(\hat{u} + \frac{1}{2})X_T} = k E e^{(\hat{u} + \frac{1}{2})X_T}$$

where we define $\hat{u} \equiv i\tilde{u}$. This is an equation in \hat{u} – does a real \hat{u} exist and is it unique?

A more insightful form: define Esscher measure \mathbb{U} s.t.

$$\frac{d\mathbb{U}}{d\mathbb{P}} = \frac{e^{(\hat{u} + \frac{1}{2})X_T}}{E e^{(\hat{u} + \frac{1}{2})X_T}}$$

where \mathbb{P} is our pricing measure (we have always been working in). Then

$$E^{\mathbb{U}} X_T = k$$

thus $\hat{u}(k)$ defines a measure under which expectation of log-spot is exactly log-strike. For \mathbb{U} to be a properly defined measure, \hat{u} has to be real – does it exist and is it unique?

Intuitively, \hat{u} translates log-spot density – imagine Gaussian $f_{X_T}(x) \stackrel{\mathbb{P}}{\sim} e^{-x^2/2}$, then under \mathbb{U} , $f_{X_T}(x) \stackrel{\mathbb{U}}{\sim} e^{-(x-\hat{u})^2/2}$ and translation of density corresponds to translation of mean, so we can always suitably choose (solve for) \hat{u} s.t. mean exactly matches log-strike i.e unique real \hat{u} exists.

Now I make rigorous why \hat{u} , equivalently $\bar{u} = \hat{u} + 1/2$, always exists and is unique. Rewrite the expectation equation as

$$k = E^{\mathbb{U}} X_T = \frac{E X_T e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} = \frac{\partial}{\partial \bar{u}} \log E e^{\bar{u}X_T}$$

This is the first derivative of cumulant (under pricing measure, not Esscher), which spans the support of $X_T \in \mathbb{R}$ (thus existence of at least a real root), and monotonically (thus one unique root \hat{u}), as its second denvative is variance – always positive.

With real \hat{u} , \mathbb{U} is a property defined measure equivalent to \mathbb{P} .

Now,

$$\psi(\tilde{u}) = -\frac{1}{T} \log E e^{(\hat{u} + \frac{1}{2})X_T} \in \mathbb{R}$$

For our saddle-point approximation to be valid, we need $\psi''(\tilde{u}) > 0$:

$$\begin{aligned}
\psi''(\tilde{u}) &= -\frac{1}{T} \left(-\frac{EX_T^2 e^{(i\tilde{u}+\frac{1}{2})X_T}}{Ee^{(i\tilde{u}+\frac{1}{2})X_T}} - \frac{EiX_T e^{(i\tilde{u}+\frac{1}{2})X_T}}{\left(Ee^{(i\tilde{u}+\frac{1}{2})X_T}\right)^2} EiX_T e^{(i\tilde{u}+\frac{1}{2})X_T} \right) \\
&= \frac{1}{T} \left(\frac{EX_T^2 e^{(i\tilde{u}+\frac{1}{2})X_T}}{Ee^{(i\tilde{u}+\frac{1}{2})X_T}} - \left(\frac{EX_T e^{(i\tilde{u}+\frac{1}{2})X_T}}{Ee^{(i\tilde{u}+\frac{1}{2})X_T}} \right)^2 \right) \\
&= \frac{1}{T} \left(\frac{EX_T^2 e^{(\hat{u}+\frac{1}{2})X_T}}{Ee^{(\hat{u}+\frac{1}{2})X_T}} - \left(\frac{EX_T e^{(\hat{u}+\frac{1}{2})X_T}}{Ee^{(\hat{u}+\frac{1}{2})X_T}} \right)^2 \right) \\
&= \frac{1}{T} \left(E^{\mathbb{U}} X_T^2 - \left(E^{\mathbb{U}} X_T \right)^2 \right) \\
&= \frac{V^{\mathbb{U}} X_T}{T}
\end{aligned}$$

i.e. $\psi''(\tilde{u})$ is log-spot variance under \mathbb{U} . As expected, because differentiated cumulant gives central moments.

Thus as long as we can write down characteristic function, for large T , our saddle-point condition is always valid.

We can computationally check how large is large, say approx closed-form formula vs. FFT.

5 Moment Expansion

To make things simpler, define

$$\bar{u} = \hat{u} + \frac{1}{2}$$

Then we rewrite our cumulant

$$\psi(\tilde{u}) = -\frac{1}{T} \log Ee^{\bar{u}X_T}$$

and saddle-point equation

$$k = E \left[X_T \frac{e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} \right]$$

We now show that derivatives of $\psi(\tilde{u})$ and \bar{u} are connected to central moments under Esscher measure.

First $\psi(\tilde{u})$.

$$\begin{aligned}
\partial_x \psi(\tilde{u}) &= -\frac{1}{T} \frac{EX_T e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} \bar{u}' = -x\bar{u}' \\
\Rightarrow \partial_x^{n+1} \psi(\tilde{u}) &= -\partial_x^n (x\bar{u}') = -n\bar{u}^{(n)} - x\bar{u}^{(n+1)}
\end{aligned}$$

As long as we know derivatives $\partial_x^n \bar{u}$, we can compute all derivatives of $\psi(\bar{u})$.

Equivalently we compute $\partial_k^n \bar{u}$, with strike derivatives related by $\partial_x = T \partial_k$.

Differentiate saddle-point equation to get

$$\partial_k^n k = E \left[X_T \partial_k^n \frac{e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} \right]$$

where k -dependence is stressed.

We study $n = 1, 2$ doing explicit calculations. Note when $n = 0$, we have $k = E^\mathbb{U} X_T$.

For $n = 1$,

$$\begin{aligned} 1 &= E \left[X_T \partial_k \frac{e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} \right] \\ &= E \left[X_T \left(\frac{X_T e^{\bar{u}X_T} \bar{u}'}{E e^{\bar{u}X_T}} - \frac{e^{\bar{u}X_T}}{(E e^{\bar{u}X_T})^2} E X_T e^{\bar{u}X_T} \bar{u}' \right) \right] \\ &= \bar{u}' \left(E^\mathbb{U} X_T^2 - (E^\mathbb{U} X_T)^2 \right) \\ &= \bar{u}' E^\mathbb{U} (X_T - k)^2 \end{aligned}$$

so

$$\bar{u}'(k) = \frac{1}{E^\mathbb{U} (X_T - k)^2}$$

For $n = 2$,

$$\begin{aligned} 0 &= E \left[X_T \partial_k^2 \frac{e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} \right] \\ &= E \left[X_T \left(\frac{\partial_k^2 e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} + 2 \partial_k e^{\bar{u}X_T} \partial_k \frac{1}{E e^{\bar{u}X_T}} + e^{\bar{u}X_T} \partial_k^2 \frac{1}{E e^{\bar{u}X_T}} \right) \right] \end{aligned}$$

which we have

$$\begin{aligned} \partial_k e^{\bar{u}X_T} &= X_T e^{\bar{u}X_T} \bar{u}' \\ \partial_k^2 e^{\bar{u}X_T} &= X_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + X_T e^{\bar{u}X_T} \bar{u}'' \\ \partial_k \frac{1}{E e^{\bar{u}X_T}} &= - \frac{E X_T e^{\bar{u}X_T}}{(E e^{\bar{u}X_T})^2} \bar{u}' \\ \partial_k^2 \frac{1}{E e^{\bar{u}X_T}} &= \frac{2(E X_T e^{\bar{u}X_T})^2}{(E e^{\bar{u}X_T})^3} (\bar{u}')^2 - \frac{1}{(E e^{\bar{u}X_T})^2} [E X_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + E X_T e^{\bar{u}X_T} \bar{u}''] \end{aligned}$$

Combining,

$$\begin{aligned}
0 &= E \left[X_T \left(\frac{\partial_k^2 e^{\bar{u}X_T}}{E e^{\bar{u}X_T}} + 2\partial_k e^{\bar{u}X_T} \partial_k \frac{1}{E e^{\bar{u}X_T}} + e^{\bar{u}X_T} \partial_k^2 \frac{1}{E e^{\bar{u}X_T}} \right) \right] \\
&= E \left[X_T \left\{ \frac{X_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + X_T e^{\bar{u}X_T} \bar{u}''}{E e^{\bar{u}X_T}} + 2 (X_T e^{\bar{u}X_T} \bar{u}') \left(-\frac{E X_T e^{\bar{u}X_T}}{(E e^{\bar{u}X_T})^2} \bar{u}' \right) + \right. \right. \\
&\quad \left. \left. e^{\bar{u}X_T} \left(\frac{2(E X_T e^{\bar{u}X_T})^2}{(E e^{\bar{u}X_T})^3} (\bar{u}')^2 - \frac{E X_T^2 e^{\bar{u}X_T} (\bar{u}')^2 + E X_T e^{\bar{u}X_T} \bar{u}''}{(E e^{\bar{u}X_T})^2} \right) \right\} \right] \\
&= \left(E^{\mathbb{U}} X_T^2 - (E^{\mathbb{U}} X_T)^2 \right) \bar{u}'' + \left(E^{\mathbb{U}} X_T^3 - 3E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T + 2(E^{\mathbb{U}} X_T)^3 \right) (\bar{u}')^2 \\
&= E^{\mathbb{U}} (X_T - k)^2 \bar{u}'' + E^{\mathbb{U}} (X_T - k)^3 (\bar{u}')^2
\end{aligned}$$

Solving,

$$\bar{u}''(k) = -\frac{E^{\mathbb{U}} (X_T - k)^3}{(E^{\mathbb{U}} (X_T - k)^2)^3}$$

Calculations for $n = 3$ are a pain ty type here, thus I state the result:

$$0 = E^{\mathbb{U}} (X_T - k)^2 \bar{u}''' + 3E^{\mathbb{U}} (X_T - k)^3 \bar{u}' \bar{u}'' + \left(E^{\mathbb{U}} (X_T - k)^4 - 3(E^{\mathbb{U}} (X_T - k)^2)^2 \right) (\bar{u}')^3$$

solved to give

$$\bar{u}'''(k) = \frac{3(E^{\mathbb{U}} (X_T - k)^2)^3 + 3(E^{\mathbb{U}} (X_T - k)^3)^2 - E^{\mathbb{U}} (X_T - k)^2 E^{\mathbb{U}} (X_T - k)^4}{(E^{\mathbb{U}} (X_T - k)^2)^5}$$

Casting back to x , we have

$$\begin{aligned}
\bar{u}'(x) &= T \frac{1}{E^{\mathbb{U}} (X_T - k)^2} \\
\bar{u}''(x) &= -T^2 \frac{E^{\mathbb{U}} (X_T - k)^3}{(E^{\mathbb{U}} (X_T - k)^2)^3} \\
\bar{u}'''(x) &= T^3 \frac{3(E^{\mathbb{U}} (X_T - k)^2)^3 + 3(E^{\mathbb{U}} (X_T - k)^3)^2 - E^{\mathbb{U}} (X_T - k)^2 E^{\mathbb{U}} (X_T - k)^4}{(E^{\mathbb{U}} (X_T - k)^2)^5}
\end{aligned}$$

ATM $k = 0$, choose $\bar{u} = \bar{u}_0$ s.t. $E^{\mathbb{U}} X_T = 0$ – now \mathbb{U} is the ATM Esscher measure.

Thus a moment expansion of \bar{u} to third order

$$\bar{u}(k) = \bar{u}_0 + \frac{1}{E^{\mathbb{U}} X_T^2} k - \frac{1}{2} \frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} k^2 + \frac{1}{6} \frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} k^3 \dots$$

Differentiation of ψ gives

$$\begin{aligned}
\partial_x \psi(\tilde{u}_0) &= -x\tilde{u}' = 0 \quad \text{saddle-point condition} \\
\partial_x^2 \psi(\tilde{u}_0) &= -\tilde{u}' - x\tilde{u}'' = -T \frac{1}{E^\mathbb{U} X_T^2} \\
\partial_x^3 \psi(\tilde{u}_0) &= -2\tilde{u}'' - x\tilde{u}''' = 2T^2 \frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} \\
\partial_x^4 \psi(\tilde{u}_0) &= -3\tilde{u}''' - x\tilde{u}'''' = -3T^3 \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5}
\end{aligned}$$

ATM, we have

$$\psi(\tilde{u}_0) \equiv \frac{\psi_0}{T} = -\frac{1}{T} \log E e^{\tilde{u}_0 X_T}$$

Thus a moment expansion of ψ to forth order

$$\psi(k) = \frac{\psi_0}{T} - \frac{1}{2} \frac{1}{E^\mathbb{U} X_T^2} \frac{k^2}{T} + \frac{1}{3} \frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} \frac{k^3}{T} - \frac{1}{8} \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5} \frac{k^4}{T} \dots$$

5.1 Moment Expansion of ω

Var-like quantity ω in our smile expands as

$$\begin{aligned}
\omega(k) &\equiv \hat{u}(k) \frac{k}{T} + \psi(k) \\
&= \frac{k}{T} \left(\bar{u}_0 - \frac{1}{2} + \frac{1}{E^\mathbb{U} X_T^2} k - \frac{1}{2} \frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} k^2 + \frac{1}{6} \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5} k^3 \dots \right) + \\
&\quad \frac{\psi_0}{T} - \frac{1}{2} \frac{1}{E^\mathbb{U} X_T^2} \frac{k^2}{T} + \frac{1}{3} \frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} \frac{k^3}{T} - \frac{1}{8} \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5} \frac{k^4}{T} \dots
\end{aligned}$$

so

$$\omega(k)T = \psi_0 + \left(\bar{u}_0 - \frac{1}{2} \right) k + \frac{1}{2} \frac{1}{E^\mathbb{U} X_T^2} k^2 - \frac{1}{6} \frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} k^3 + \frac{1}{24} \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5} k^4 \dots$$

Denote $\sigma_0^2 T \equiv E^\mathbb{U} X_T^2$, and we rewrite the smile in terms of normalized strike

$$\begin{aligned}
\omega(k)T &= \psi_0 + \left(\bar{u}_0 - \frac{1}{2} \right) \sigma_0 \sqrt{T} \left(\frac{k}{\sigma_0 \sqrt{T}} \right) + \frac{1}{2} \left(\frac{k}{\sigma_0 \sqrt{T}} \right)^2 - \frac{1}{6} \frac{E^\mathbb{U} X_T^3}{(\sigma_0 \sqrt{T})^3} \left(\frac{k}{\sigma_0 \sqrt{T}} \right)^3 + \\
&\quad \frac{1}{24} \frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(\sigma_0 \sqrt{T})^6} \left(\frac{k}{\sigma_0 \sqrt{T}} \right)^4 \dots
\end{aligned}$$

with dimensionless coefficients.

5.2 Derivatives of \bar{u}

For $n = 1, 2$, respectively we have

$$\begin{aligned} 1 &= E^{\mathbb{U}}(X_T - k)^2 \bar{u}' \\ 0 &= E^{\mathbb{U}}(X_T - k)^2 \bar{u}'' + E^{\mathbb{U}}(X_T - k)^3 (\bar{u}')^2 \end{aligned}$$

Can we generalize this line? Note each term carries dimension $1/k^{n-1}$.

Let's back out a bit and consider our saddle-point equation in an alternative form

$$\partial_k \log E e^{\bar{u} X_T} = k \bar{u}'$$

so its n th derivative

$$\partial_k^{n+1} \log E e^{\bar{u} X_T} = \partial_k^n (k \bar{u}') = n \bar{u}^{(n)} + k \bar{u}^{(n+1)} \quad n \geq 0$$

from which we recursively solve $\bar{u}^{(n)}$ in terms of $\bar{u}^{(j)}$, $j \leq n-1$, and collect its coefficients into central moments. Derivatives of cumulant of becomputed from Faà di Bruno's formula for derivatives of function composition (of $\log \cdot$ and $E e^{u(\cdot) X_T}$), which has very unintuitive coefficients – I will omit.

5.3 Moment Expansion of Smile v

We obtain a moment expansion of $\bar{\omega}(k)T$ hence smile v .

We defined $\bar{\omega}(x) \equiv \sqrt{\omega(x)^2 - x^2/4}$, shown in next section [Existence of Variance Solution](#) to be always real, and variance smile is given by

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

We express total variance in terms of log-strike

$$w(k) \equiv v(k)T \sim 4(\omega(k)T - \bar{\omega}(k)T)$$

with moment expansion

$$\omega(k)T = \psi_0 - \left(\frac{1}{2} - \bar{u}_0\right)k + \frac{1}{2} \frac{1}{E^{\mathbb{U}} X_T^2} k^2 - \frac{1}{6} \frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} k^3 + \frac{1}{24} \frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} k^4 \dots$$

$|\bar{\omega}(k)T|$ expands ATM as

$$\begin{aligned} |\bar{\omega}(k)T| = -\bar{\omega}(k)T &= \psi_0 - \left(\frac{1}{2} - \bar{u}_0\right)k + \frac{1}{2} \left(\frac{1}{E^{\mathbb{U}} X_T^2} - \frac{1}{4\psi_0} \right) k^2 - \frac{1}{6} \left(\frac{E^{\mathbb{U}} X_T^3}{(E^{\mathbb{U}} X_T^2)^3} - \frac{6\bar{u}_0 - 3}{8\psi_0^2} \right) k^3 + \\ &\quad \frac{1}{24} \left(\frac{3(E^{\mathbb{U}} X_T^2)^3 + 3(E^{\mathbb{U}} X_T^3)^2 - E^{\mathbb{U}} X_T^2 E^{\mathbb{U}} X_T^4}{(E^{\mathbb{U}} X_T^2)^5} + \frac{3}{2\psi_0^2 E^{\mathbb{U}} X_T^2} - \frac{3(16\bar{u}_0^2 - 16\bar{u}_0 + 5)}{16\psi_0^3} \right) k^4 \dots \end{aligned}$$

Therefore, we obtain full variance smile moment expansion

$$w(k) = 8 \left\{ \psi_0 - \left(\frac{1}{2} - \bar{u}_0 \right) k + \frac{1}{2} \left(\frac{1}{E^\mathbb{U} X_T^2} - \frac{1}{8\psi_0} \right) k^2 - \frac{1}{6} \left(\frac{E^\mathbb{U} X_T^3}{(E^\mathbb{U} X_T^2)^3} - \frac{6\bar{u}_0 - 3}{16\psi_0^2} \right) k^3 + \right. \\ \left. \frac{1}{24} \left(\frac{3(E^\mathbb{U} X_T^2)^3 + 3(E^\mathbb{U} X_T^3)^2 - E^\mathbb{U} X_T^2 E^\mathbb{U} X_T^4}{(E^\mathbb{U} X_T^2)^5} + \frac{3}{4\psi_0^2 E^\mathbb{U} X_T^2} - \frac{3(16\bar{u}_0^2 - 16\bar{u}_0 + 5)}{32\psi_0^3} \right) k^4 \dots \right\}$$

Note that we make no approx here and this formula is exact.

Notably, var skew is difference of ATM Esscher shift between model's \bar{u}_0 and BS's $1/2$; var curvature is inverse difference between ATM Esscher variance and ATM var. They are measures of deviation from BS diffusion progressively to higher orders.

$$\frac{\partial w}{\partial k} = -8 \left(\frac{1}{2} - \bar{u}_0 \right) \\ \frac{\partial^2 w}{\partial k^2} = 8 \left(\frac{1}{E^\mathbb{U} X_T^2} - \frac{1}{w(0)} \right)$$

Under BS flat var w_0 and Esscher shift $\bar{u}_0 = 1/2$, we have exactly $w(k) = w_0$.

A practical use is, we fit a polynomial, say quartic, to ATM and from coefficients we imply the moments. But the moments are under ATM Esscher measure, which makes them less useful. If log-spot density is close enough to Gaussian, moments (above quadratic) under pricing/Esscher measure are exactly identical, as effect of Esscher is merely a shift (translation of mean). But at least Esscher moments give an order-of-magnitude estimates of true implied moments.

6 Existence of Variance Solution

Our variance $v(x)$ exists if $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$ is well-defined.

My proposition is: $\omega(x)$ tangentially touches $|x/2|$ at points x_\pm (small and off by a sign) but is everywhere else bounded below.

Equivalently, this may be stated as: $\omega(x) \mp x/2$ and its derivative vanish at some x_\pm and second derivative is positive everywhere. We write

$$\omega(x) \equiv x\hat{u} + \psi(\tilde{u}(x))$$

thus derivatives

$$\omega'(x) = \hat{u}(x) + x\hat{u}'(x) + \psi'(\tilde{u}(x)) = \hat{u}(x) \\ \omega''(x) = \hat{u}'(x) = \frac{T}{E^\mathbb{U}(X_T - k)^2} > 0$$

so second derivative is everywhere positive, and

$$\partial_x (\omega(x) \mp x/2) = \hat{u}(x) \mp \frac{1}{2}$$

which vanishes at some x_{\pm} ,

$$\hat{u}(x_{\pm}) = \pm \frac{1}{2}$$

Does such x_{\pm} always exist? Yes. Recall that \hat{u} is roughly the log-spot density shift (via Esscher measure \mathbb{U}) s.t. expectation of log-spot $E^{\mathbb{U}} X_T$, exactly matches log-strike k , equivalently x , thus \hat{u} is monotonic increasing in x , spanning \mathbb{R} . We can always find such an x_{\pm} s.t. $\hat{u}(x_{\pm})$ exactly matches $\pm 1/2$ – smallness/sign of $\pm 1/2$ leads to smallness/sign of x_{\pm} .

By our moment expansion for \bar{u} , for small x

$$\bar{u}(x) \approx \bar{u}_0 + \frac{T}{E^{\mathbb{U}} X_T^2} x \equiv \frac{1}{2} \pm \frac{1}{2}$$

which approximates

$$x_{\pm} \approx \frac{E^{\mathbb{U}} X_T^2}{T} \left(\frac{1}{2} \pm \frac{1}{2} - \bar{u}_0 \right) \approx \pm \frac{E^{\mathbb{U}} X_T^2}{2T} \sim \pm \frac{\sigma^2}{2}$$

as $\bar{u}_0 \approx 1/2$, from Ito correction, and σ is some characteristic vol in model. In Heston case (see section **Large-Time Heston Smile**), $\sigma^2 \sim \bar{v}$ so $x_{\pm} \sim \pm \bar{v}/2 = \pm 0.02$ – consistent with our plot!

To reason about this, think about BS diffusion – log-spot $X_T = -\sigma^2 T/2 + \sigma Z_T$ and denote total var $w = \sigma^2 T$ so density $f_{X_T}(x) \stackrel{\mathbb{P}}{\sim} e^{-(x+w/2)^2/2w}$. Under ATM Esscher measure defined by \bar{u}_0 , $f_{X_T}(x) \stackrel{\mathbb{U}}{\sim} e^{-(x+w/2-w\bar{u}_0)^2/2w}$. For $E^{\mathbb{U}} X_T = 0$, we demand $\bar{u}_0 = 1/2$ – this is exact. At-the-money, other processes may be expressed as perturbation around BS diffusion so densities near $x = 0$ behave roughly identical, so approximation $\bar{u}_0 \approx 1/2$ is not unreasonable.

Typically, if model produces heavy left-tail, mode of log-spot density is biased to right, so we require a smaller Esscher shift to match match expectation of log-spot to zero – thus $\bar{u}_0 < 1/2$, a negatively skewed $\omega(k)T$.

Lastly, $\omega(x) \mp x/2$ vanishes at x_{\pm} as

$$\omega(x_{\pm}) \mp \frac{x_{\pm}}{2} = x_{\pm} \hat{u}(x_{\pm}) - \frac{1}{T} \log E e^{(\hat{u}(x_{\pm}) + \frac{1}{2}) X_T} \mp \frac{x_{\pm}}{2} = \pm \frac{x_{\pm}}{2} \mp \frac{x_{\pm}}{2} - \frac{1}{T} \log E e^{(\pm \frac{1}{2} + \frac{1}{2}) X_T} = 0$$

as $E e^{X_T} = 1$ by martingale condition.

So, $\omega(x) \geq |x/2|$ for all strike x and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - x^2/4} \leq \omega(x)$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

is always well-defined (positive).

7 Large-Time Heston Smile

We derive the large-time Heston full smile, unapproximated thus valid for all strike $x \in \mathbb{R}$.

It remains to evaluate \tilde{u} and $\psi(\tilde{u})$ under Heston. We shall see for large T , characteristic function ϕ_T factorizes into $e^{-\psi(u)T}$.

Recall Heston ϕ_T is given by $e^{\mathcal{C}_T(u)\bar{v} + \mathcal{D}_T(u)v}$ where

$$\begin{aligned}\mathcal{C}_T(u) &= \lambda \left(r_- T - \frac{2}{\eta^2} \log \left(\frac{1 - ge^{-dT}}{1 - g} \right) \right) \\ \mathcal{D}_T(u) &= r_- \frac{1 - e^{-dT}}{1 - ge^{-dT}}\end{aligned}$$

with

$$\begin{aligned}\alpha &= -\frac{u^2}{2} - \frac{iu}{2} \\ \beta &= \lambda - \rho\eta iu \\ \gamma &= \frac{\eta^2}{2} \\ d &= \sqrt{\beta^2 - 4\alpha\gamma} \\ r_{\pm} &= \frac{\beta \pm d}{2\gamma} \\ g &= \frac{r_+}{r_-}\end{aligned}$$

following usual definitions in Heston, e.g. correlation ρ , mean-reversion λ and vol-of-vol η .

In large-time limit,

$$\phi_T \left(u - \frac{i}{2} \right) \sim e^{\lambda\bar{v}r_-(u-i/2)T} \equiv e^{-\psi(u)T}$$

thus

$$\psi(u) = -\lambda\bar{v}r_- = \frac{\lambda\bar{v}}{\eta^2} \left(\sqrt{\beta^2 - 2\eta^2\alpha} - \beta \right)$$

Note we are evaluating at $u - i/2$, so

$$\begin{aligned}\alpha &= -\frac{1}{2} \left(u^2 + \frac{1}{4} \right) \\ \beta &= \left(\lambda - \frac{\rho\eta}{2} \right) - \rho\eta iu\end{aligned}$$

Denote $\xi = \eta^2/\lambda\bar{v}$, and differentiating,

$$\psi'(u) = \frac{1}{\xi} \left(\frac{\beta \frac{\partial \beta}{\partial u} - \eta^2 \frac{\partial \alpha}{\partial u}}{\sqrt{\beta^2 - 2\eta^2 \alpha}} - \frac{\partial \beta}{\partial u} \right) = \frac{1}{\xi} \left(\frac{-i\rho\eta\beta + \eta^2 u}{\sqrt{\beta^2 - 2\eta^2 \alpha}} + i\rho\eta \right) = -ix$$

giving

$$-(\xi x + \rho\eta)^2(\beta^2 - 2\eta^2 \alpha) = \left(i\rho\eta \left(\lambda - \frac{\rho\eta}{2} - \rho\eta iu \right) - \eta^2 u \right)^2 = \left(i\rho\eta \left(\lambda - \frac{\rho\eta}{2} \right) - \eta^2(1 - \rho^2)u \right)^2$$

On LHS we expand $\beta^2 - 2\eta^2 \alpha$ to get

$$\left(\lambda - \frac{\rho\eta}{2} \right)^2 + \frac{\eta^2}{4} - 2\rho\eta \left(\lambda - \frac{\rho\eta}{2} \right) iu + \eta^2(1 - \rho^2)u^2$$

Now, define characteristic quantities to simplify things:

$$\begin{aligned} A^2 &= \eta^2(1 - \rho^2) \\ B &= \rho\eta \left(\lambda - \frac{\rho\eta}{2} \right) \\ C^2 &= \left(\lambda - \frac{\rho\eta}{2} \right)^2 + \frac{\eta^2}{4} \\ m &= -\frac{\rho\eta}{\xi} \\ a &= \frac{\rho\eta}{\lambda} \end{aligned}$$

Plugging back in,

$$-\xi^2(x - m)^2 (C^2 - 2Biu + A^2 u^2) = (iB - A^2 u)^2,$$

a quadratic equation in u :

$$(C^2 \xi^2 (x - m)^2 - B^2) - 2Biu (A^2 + \xi^2 (x - m)^2) + A^2 (A^2 + \xi^2 (x - m)^2) u^2 = 0$$

where we define quantities involving strike x :

$$\begin{aligned} \Theta^2 &= B^2 - C^2 \xi^2 (x - m)^2 \\ \Sigma^2 &= A^2 + \xi^2 (x - m)^2 \end{aligned}$$

Rewriting quadratic equation,

$$\begin{aligned} \Theta^2 &= \Sigma^2 \left(A^2 u^2 - 2Biu + \left(i\frac{B}{A} \right)^2 - \left(i\frac{B}{A} \right)^2 \right) \\ \left(\frac{\Theta}{\Sigma} \right)^2 &= \left(Au - i\frac{B}{A} \right)^2 + \left(\frac{B}{A} \right)^2 \end{aligned}$$

thus we have

$$\begin{aligned}\tilde{u} &= \frac{1}{A} \left(i \frac{B}{A} \pm \sqrt{\left(\frac{\Theta}{\Sigma}\right)^2 - \left(\frac{B}{A}\right)^2} \right) \\ \hat{u} \equiv i\tilde{u} &= \frac{1}{A} \left(-\frac{B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \left(\frac{\Theta}{\Sigma}\right)^2} \right) \in \mathbb{R}\end{aligned}$$

where plus corresponds to call domain and minus corresponds to put domain, matching at strike $x_0 = m$ which solves

$$\left(\frac{B}{A}\right)^2 = \left(\frac{\Theta}{\Sigma}\right)^2 = \frac{B^2 - C^2 \xi^2 (x - m)^2}{A^2 + \xi^2 (x - m)^2}$$

Finally we evaluate $\psi(\tilde{u})$:

$$\begin{aligned}\psi(\tilde{u}) &= \frac{1}{\xi} \left(\sqrt{\left(\lambda - \frac{\rho\eta}{2} - \rho\eta i\tilde{u}\right)^2 + \eta^2 \left(\tilde{u}^2 + \frac{1}{4}\right)} - \left(\lambda - \frac{\rho\eta}{2} - \rho\eta i\tilde{u}\right) \right) \\ &= \frac{1}{\xi} \left(\sqrt{C^2 + \left(\frac{\Theta}{\Sigma}\right)^2} - \lambda \left(1 - \frac{a}{2} - a \cdot \hat{u}\right) \right) \in \mathbb{R}\end{aligned}$$

\hat{u} and $\psi(\tilde{u})$ are combined to yield variance quantity $\omega(x) \equiv \hat{u}(x) \cdot x + \psi(\tilde{u}(x))$.

With simplification, we shall see that $\omega(x)$ hence smile $v(x)$ are nothing other than SVI.

7.1 Towards SVI

Consider scaled version of our strike quantities:

$$\begin{aligned}\bar{\Theta}^2 &= 1 - \xi^2 \left(\frac{C}{B}\right)^2 (x - m)^2 \\ \bar{\Sigma}^2 &= 1 + \xi^2 \left(\frac{1}{A}\right)^2 (x - m)^2\end{aligned}$$

Denote

$$D^2 = \left(\frac{1}{A}\right)^2 + \left(\frac{C}{B}\right)^2$$

thus

$$\bar{\Sigma}^2 - \bar{\Theta}^2 = \xi^2 D^2 (x - m)^2$$

Assume negative correlation regime $-\rho < 0$ hence $B < 0$, typical of equity market. This assumption is not necessary but we can remove absolute signs for clarity.

We simplify

$$\hat{u} = \frac{B}{A^2} \left(-1 \mp \sqrt{1 - \left(\frac{\bar{\Theta}}{\bar{\Sigma}} \right)^2} \right) = \frac{B}{A^2} \left(-\frac{\xi D(x-m)}{\bar{\Sigma}} - 1 \right)$$

Now,

$$\psi(\tilde{u}) = \frac{1}{\xi} \left(\sqrt{C^2 + \left(\frac{B}{A} \right)^2 \left(\frac{\bar{\Theta}}{\bar{\Sigma}} \right)^2} - \lambda \left(1 - \frac{a}{2} - a \cdot \hat{u} \right) \right)$$

Write out the square-root term:

$$\sqrt{C^2 + \left(\frac{B}{A} \right)^2 \left(\frac{\bar{\Theta}}{\bar{\Sigma}} \right)^2} = \frac{1}{\bar{\Sigma}} \sqrt{C^2 \left(1 + \xi^2 \left(\frac{1}{A} \right)^2 (x-m)^2 \right) + \left(\frac{B}{A} \right)^2 \left(1 - \xi^2 \left(\frac{C}{B} \right)^2 (x-m)^2 \right)} = -\frac{BD}{\bar{\Sigma}}$$

hence

$$\psi(\tilde{u}) = \frac{1}{\xi} \left(-\frac{BD}{\bar{\Sigma}} - \lambda \left(1 - \frac{a}{2} \right) + \lambda a \hat{u} \right)$$

Final piece

$$\begin{aligned} \omega(x) &= \hat{u} \cdot x + \psi(\tilde{u}) \\ &= \hat{u} \cdot (x-m) + \hat{u} \cdot m + \frac{1}{\xi} \left(-\frac{BD}{\bar{\Sigma}} - \lambda \left(1 - \frac{a}{2} \right) \right) + \hat{u} \cdot \frac{\lambda a}{\xi} \\ &= \frac{B}{A^2} \left(-\frac{\xi D(x-m)}{\bar{\Sigma}} - 1 \right) (x-m) + \frac{1}{\xi} \left(-\frac{BD}{\bar{\Sigma}} - \lambda \left(1 - \frac{a}{2} \right) \right) \\ &= -\frac{\lambda}{\xi} \left(1 - \frac{a}{2} \right) - \frac{B}{A^2} (x-m) - \frac{1}{\bar{\Sigma}} \frac{BD}{\xi} \left(1 + \xi^2 \left(\frac{1}{A} \right)^2 (x-m)^2 \right) \\ &= -\frac{\lambda}{\xi} \left(1 - \frac{a}{2} \right) - \frac{B}{A^2} (x-m) - \frac{BD}{\xi} \sqrt{1 + \xi^2 \left(\frac{1}{A} \right)^2 (x-m)^2} \end{aligned}$$

which is SVI-like.

To reach full smile $v(x)$, we require $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - x^2/4}$. We conjecture that $\bar{\omega}(x)$ is also SVI-like, of form

$$-K_0 - K_1(x-m) - K_2 \sqrt{1 + \xi^2 \left(\frac{1}{A} \right)^2 (x-m)^2}$$

This turns out correct – we now show this.

Expand $\omega(x)^2 - x^2/4$:

$$\begin{aligned}
& \left(\frac{\lambda}{\xi}\right)^2 \left(1 - \frac{a}{2}\right)^2 + \left(\frac{B}{A^2}\right)^2 (x-m)^2 + \left(\frac{BD}{\xi}\right)^2 \left(1 + \xi^2 \left(\frac{1}{A}\right)^2 (x-m)^2\right) + \\
& 2\frac{\lambda}{\xi} \frac{B}{A^2} \left(1 - \frac{a}{2}\right) (x-m) + 2\frac{\lambda}{\xi} \frac{BD}{\xi} \left(1 - \frac{a}{2}\right) \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x-m)^2} + \\
& 2\frac{B}{A^2} \frac{BD}{\xi} (x-m) \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x-m)^2} \\
& - \frac{1}{4} ((x-m)^2 + 2m(x-m) + m^2) \\
& = \left(\left(\frac{\lambda}{\xi}\right)^2 \left(1 - \frac{a}{2}\right)^2 + \left(\frac{BD}{\xi}\right)^2 - \frac{m^2}{4}\right) + \left(2\frac{\lambda}{\xi} \frac{B}{A^2} \left(1 - \frac{a}{2}\right) - \frac{m}{2}\right) (x-m) + \\
& \left(\left(\frac{B}{A^2}\right)^2 + \xi^2 \left(\frac{BD}{\xi}\right)^2 \left(\frac{1}{A}\right)^2 - \frac{1}{4}\right) (x-m)^2 + \\
& 2\frac{\lambda}{\xi} \frac{BD}{\xi} \left(1 - \frac{a}{2}\right) \sqrt{\dots} + 2\frac{B}{A^2} \frac{BD}{\xi} (x-m) \sqrt{\dots} \\
& \equiv \left(-K_0 - K_1(x-m) - K_2 \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x-m)^2}\right)^2
\end{aligned}$$

thus

$$\begin{aligned}
K_0^2 + K_2^2 &= \left(\frac{\lambda}{\xi}\right)^2 \left(1 - \frac{a}{2}\right)^2 + \left(\frac{BD}{\xi}\right)^2 - \frac{m^2}{4} \\
K_1^2 + K_2^2 \xi^2 \left(\frac{1}{A}\right)^2 &= \left(\frac{B}{A^2}\right)^2 + \xi^2 \left(\frac{BD}{\xi}\right)^2 \left(\frac{1}{A}\right)^2 - \frac{1}{4} \\
K_0 K_1 &= \frac{\lambda}{\xi} \frac{B}{A^2} \left(1 - \frac{a}{2}\right) - \frac{m}{4} = \frac{\lambda}{\xi} \frac{B}{A^2} \left(1 - \frac{a}{2} + \frac{aA^2}{4B}\right) = \frac{\lambda}{\xi} \frac{B}{A^2} \left(1 - \frac{a}{2}\right) \left(1 + \frac{aA^2/4B}{1 - a/2}\right) \\
K_1 K_2 &= \frac{B}{A^2} \frac{BD}{\xi} \\
K_0 K_2 &= \frac{\lambda}{\xi} \frac{BD}{\xi} \left(1 - \frac{a}{2}\right)
\end{aligned}$$

Five equations for three unknowns – overconstraint! Solving from last three:

$$\begin{aligned}
K_0 &= \frac{\lambda}{\xi} \left(1 - \frac{a}{2}\right) K \\
K_1 &= \frac{B}{A^2} K \\
K_2 &= \frac{BD}{\xi} \frac{1}{K}
\end{aligned}$$

with

$$K = \sqrt{1 + \frac{aA^2/4B}{1 - a/2}}$$

$K_{0,1,2}$ have to automatically satisfy first two equations, if our guess $\omega^2 - x^2/4 = (\dots)^2$ is correct. We provide a numerical check here, given our typical Heston params in next section:

Eq. (1): LHS = 50.76705882352938 RHS = 50.76705882352938

Eq. (2): LHS = 613.898500576701 RHS = 613.8985005767013

Now, our smile is given by difference of two SVIs:

$$\omega(x) = -\frac{\lambda}{\xi} \left(1 - \frac{a}{2}\right) - \frac{B}{A^2}(x - m) - \frac{BD}{\xi} \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x - m)^2}$$

and

$$\bar{\omega}(x) = -K \frac{\lambda}{\xi} \left(1 - \frac{a}{2}\right) - K \frac{B}{A^2}(x - m) - \frac{1}{K} \frac{BD}{\xi} \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x - m)^2}$$

Therefore,

$$\begin{aligned} v(x) &= 4 \left((K - 1) \frac{\lambda}{\xi} \left(1 - \frac{a}{2}\right) + (K - 1) \frac{B}{A^2}(x - m) - \left(1 - \frac{1}{K}\right) \frac{BD}{\xi} \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x - m)^2} \right) \\ &= 4(K - 1) \left(\frac{\lambda}{\xi} \left(1 - \frac{a}{2}\right) + \frac{B}{A^2}(x - m) - \frac{1}{K} \frac{BD}{\xi} \sqrt{1 + \xi^2 \left(\frac{1}{A}\right)^2 (x - m)^2} \right) \end{aligned}$$

This echoes with Gatheral's proof that **large-time Heston implied vol smile is exactly SVI**. Our saddle-point trick turns out valid.

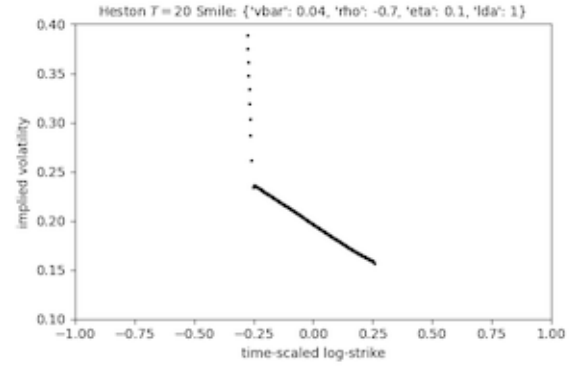
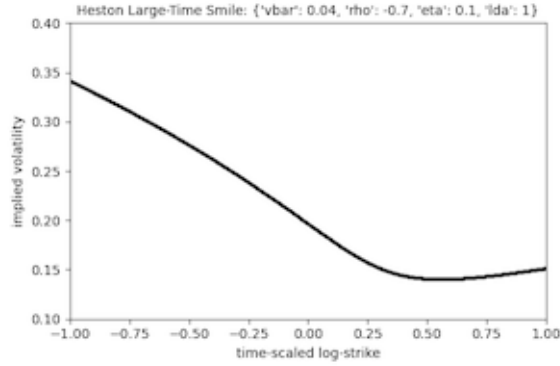
7.2 Numerical Experiment

We assume typical Heston params and contrast our closed-form smile formula with FFT calculation, which in large time only works for small strikes though I am already using very fine grids. But over this computable domain, they exactly match.

$$\bar{v} = 0.04 \quad \rho = -0.7 \quad \eta = 0.1 \quad \lambda = 1$$

Full-Smile Formula

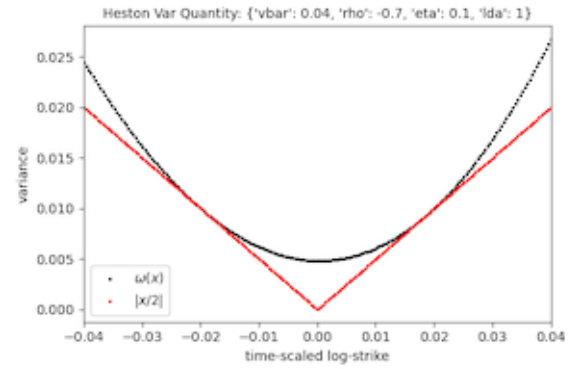
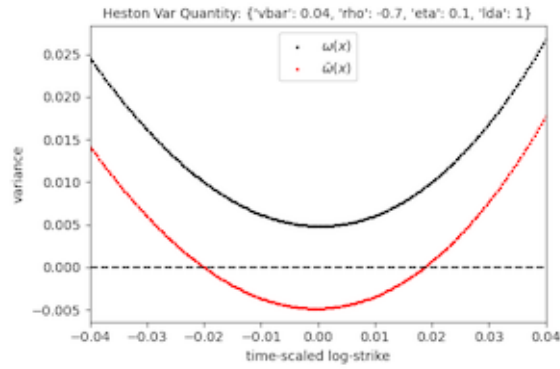
FFT Calculation



We visualize $\omega(x)$ and $\bar{\omega}(x)$. Recall that $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - x^2/4}$, where we choose $\bar{\omega}((x_-, x_+)) < 0$ and $\bar{\omega}(\mathbb{R} \setminus (x_-, x_+)) > 0$, and x_{\pm} are solution to $\omega(x) = \pm x/2$.

$\omega(x)$ vs. $\bar{\omega}(x)$

$\omega(x)$ vs. $|x/2|$



Readers may make use of following code snippet to reproduce full-smile plot.

```
[1]: import numpy as np
import matplotlib.pyplot as plt

params = {'vbar': 0.04, 'rho': -0.7, 'eta': 0.1, 'lda': 1}
vbar = params['vbar']
rho = params['rho']
eta = params['eta']
lda = params['lda']

x = np.arange(-2, 2, 0.002)

xi = eta**2/(lda*vbar)
a = rho*eta/lda
m = -rho*eta/xi
```

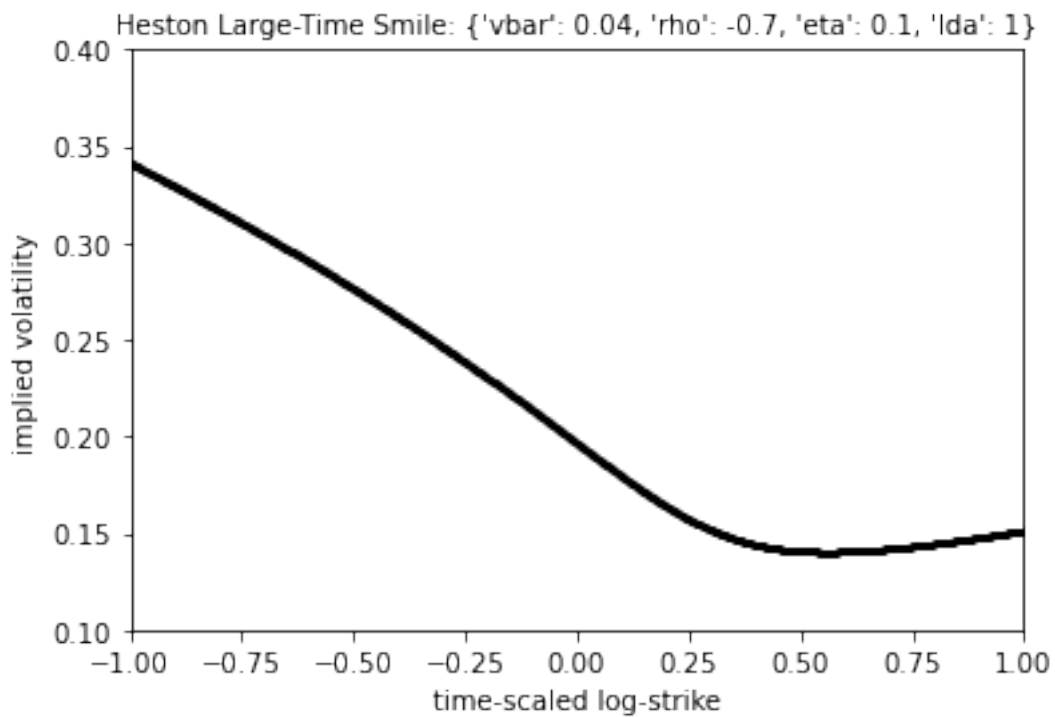
```

A = eta*np.sqrt(1-rho**2)
B = rho*eta*(lda-rho*eta/2)
C = np.sqrt(eta**2/4+(lda-rho*eta/2)**2)
D = np.sqrt((1/A)**2+(C/B)**2)
K = np.sqrt(1+(a*A**2/(4*B))/(1-a/2))

v0 = -lda/xi*(1-a/2)-B/A**2*(x-m)-B*D/xi*np.sqrt(1+(xi/A)**2*(x-m)**2)
v1 = -lda/xi*(1-a/2)*K-B/A**2*K*(x-m)-B*D/xi/K*np.sqrt(1+(xi/A)**2*(x-m)**2)
v = 4*(v0-v1)
sig = np.sqrt(v)

plt.scatter(x,sig,c='black',s=2)
plt.xlabel('time-scaled log-strike')
plt.ylabel('implied volatility')
plt.title(f'Heston Large-Time Smile: {params}',fontsize=10)
plt.xlim([-1,1])
plt.ylim([0.1,0.4])
plt.show()

```



8 Large-Time VG Smile

9 Large-Time BG Smile

10 Large-Time CGMY Smile

11 Comments on Saddle-Point Trick

Our starting point is Lewis equation expanded around saddle-point \tilde{u} with integral approximated as a Gaussian in large time – for a full discussion, see section [Saddle-Point Equation](#).

I thought about the analog formula for small time, but no – this approach relies on large time and in small time non-linear terms come in, so we can no longer make Gaussian approx.

We reach following smile equation for implied variance, assuming characteristic function in large time scales as $\phi_T(u - i/2) \sim e^{-\psi(u)T}$:

$$\frac{v(x)}{8} + \frac{x^2}{2v(x)} \sim i\tilde{u}(x)x + \psi(\tilde{u}(x)) \equiv \omega(x)$$

We proved that $\omega(x) \geq |x/2|$, so quadratic determinant is always positive. Variance $v(x)$ always exists and is positive.

Condition $\phi_T(u - i/2) \sim e^{-\psi(u)T}$ says: in large time log-spot evolves like a Levi process (of independent and stationary increments), and large-time smile converges to a stationary state. Independence leads to scaling in T in exponent; stationarity leads to time-independent factor $\psi(u)$.

Our saddle-point trick works for Levy processes (by definition), and some path-dependent processes that forget about its initial states over time e.g. Heston/SVJ due to mean-reversion of vol.

Smile properties depend entirely on factor $\omega(x)$. If we want a complex smile of rich curvatures (say crazy W-shape), we have to design a $\psi(u)$ complex enough, that fulfills martingale condition $\psi(-i/2) = 0$ – is $\psi(u)$, rather than log-spot process X_T , the right thing to start with? By Levy-Khintchine representation it seems there exists some Levy measure integrated to match ψ , at least in theory, though we risk a totally incomprehensible Levy measure. Another question is, does arbitrary $\psi(u)$ (that fulfills martingale condition) contain arbitrage?

To make this precise, say we want a pure-jump process (ignore drift/diffusion for now) obeying some Levy measure $\mu(\xi)$ that fulfills $\psi(u)$. By Levy-Khintchine, characteristic function reads

$$\begin{aligned}\phi_T(u) &\equiv Ee^{iuX_T} = \exp\left(T \int (e^{iu\xi} - 1)\mu(\xi)d\xi\right) \\ \phi_T\left(u - \frac{i}{2}\right) &= \exp\left(T \int \left(e^{(iu+\frac{1}{2})\xi} - 1\right)\mu(\xi)d\xi\right) \equiv e^{-\psi(u)T}\end{aligned}$$

so

$$\psi(u) = - \int \left(e^{(iu+\frac{1}{2})\xi} - 1\right)\mu(\xi)d\xi$$

which fulfills martingale condition

$$\psi(-i/2) = 0$$

Levy measure μ can be obtained via inverse-Fourier:

$$\begin{aligned}\psi\left(u + \frac{i}{2}\right) &= - \int e^{iu\xi} \mu(\xi) d\xi + \int \mu(\xi) d\xi \\ \mu(\xi) &= -\mathcal{F}^{-1}\psi\left(u + \frac{i}{2}\right) + \left(\int \mu(\xi) d\xi\right) \delta(\xi)\end{aligned}$$

Let's ignore the delta function explosion because Levy measure cannot count zero jump anyway – they are indistinguishable from diffusion! Either it is part of μ or will cancel with some infinity in inverse-Fourier term.

It seems from ψ we can back out its underlying Levy measure μ , which corresponds to a Levy process. What extra conditions do we need on ψ s.t. the Levy process is well-defined?

11.1 Some Opinions

I think Merton-jump model with following extensions can potentially fit a smile of rich curvatures. In Merton we assume Gaussian log-jumps obeying Poisson arrival – this has a (finite-activity) Gaussian Levy measure. With sufficiently many Gaussians (with different arrival rates λ and spreads σ^2) this seems to match arbitrary Levy measure. More precisely, any Levy measure may be represented in superposition of compound Poisson processes

$$\mu(\xi) = \int d\alpha \lambda(\alpha) \frac{e^{-(\xi-\alpha)^2/2\sigma^2(\alpha)}}{\sqrt{2\pi}}$$

Activity is controlled by λ – say for infinite activity, λ integrates to infinity. This probably has too many degrees of freedom $\{\lambda(\alpha), \sigma^2(\alpha)\}$ – extra constraints are needed. This essentially says, any function may be expanded in Gaussian basis – I think this is very plausible. Now that we have a Levy measure flexible enough, this fits complex smile.

The first thing to do is, work out large-time Merton smile and study its curvature, then progressively add Poisson jumps and observe how this changes curvature.

12 References

- Gatheral, The Volatility Surface: A Practitioner's Guide
- Gatheral/Jacquier, Convergence of Heston to SVI
- Madan/Wang, Additive Processes with Bilateral Gamma Marginals