Saddle-Point Approach to Large-Time Vol Smile

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August 20, 2022

By a saddle-point approximation to Lewis equation, we derive analytic form of large-time vol smile implied from model. In this note, we outline, justify and exemplify the approach in the case of Heston/VG/CGMY/BG. From here, we inspire some parametrizations of vol smile.

The approach was developed during my summer internship at Morgan Stanley.

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1 Lewis Equation

Define log-spot $X_T = \log(S_T/S_0)$ and log-strike $k = \log(K/S_0)$.

Lewis equation states that under characteristic function $\phi_T(u) = Ee^{iuX_T}$, call price

$$C(S,K) = S - \frac{\sqrt{SK}}{2\pi}\int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}\phi_T\left(u-\frac{i}{2}\right)$$

Under BS, $\phi_T^{BS}(u) = e^{-\frac{1}{2}u(u+i)\sigma^2 T}$. For each strike K, we quote $C(K)$ in $C_{BS}(K)$, thus

$$\int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}\phi_T\left(u-\frac{i}{2}\right) = \int_{-\infty}^{\infty}\frac{du}{u^2+\frac{1}{4}}e^{-iuk}e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\sigma(k)^2 T}$$

where $\sigma(k)$ is implied vol (of our interest!).

2 Saddle-Point Equation

We assume characteristic function $\phi_T(u-i/2) \sim e^{-\psi(u)T}$ for large T.

Physically this says, at large time, log-spot X_T evolves like a Levy process, so that time T in characteristic exponent factors out. By definition, this is satisfied by all Levy processes, and some path-dependent processes that forget about its initial state over time e.g. Heston/SVJ.

Define time-scaled log-strike $x = k/T$, abbreviated strike below. Substituting the large-time ϕ_T into LHS we get

$$\int_{-\infty}^{\infty} \frac{du}{u^2+\frac{1}{4}} e^{-(iux+\psi(u))T}$$

Now, Taylor-expand the exponent around saddle-point $\tilde{u}$:

$$ix\tilde{u} + ix(u-\tilde{u}) + \psi(\tilde{u}) + \psi'(\tilde{u})(u-\tilde{u}) + \frac{\psi''(\tilde{u})}{2}(u-\tilde{u})^2 + O(u-\tilde{u})^3$$

Require $\psi'(\tilde{u}) = -ix$ (now keep in mind $\tilde{u} = \tilde{u}(x)$) to kill linear term, then LHS simplifies to

$$\int_{-\infty}^{\infty} \frac{du}{u^2+\frac{1}{4}} e^{-(ix\tilde{u}+\psi(\tilde{u}))T - \frac{\psi''(\tilde{u})T}{2}(u-\tilde{u})^2 - T\cdot O(u-\tilde{u})^3} \approx \frac{e^{-(ix\tilde{u}+\psi(\tilde{u}))T}}{\tilde{u}^2+\frac{1}{4}}\sqrt{\frac{2\pi}{\psi''(\tilde{u})T}}$$

The approximation $e^{-(ix\tilde{u}+\psi(\tilde{u}))T} \sim$ Gaussian is valid when (1) $ix\tilde{u}+\psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$ (3) T is large, because

- for u far away from $\tilde{u}$ and T large, $e^{-T\cdot O(u-\tilde{u})^3} \to 0$ - tails flatten to zero
- for u close to $\tilde{u}$, constant/quadratic term in $e^{-(ix\tilde{u}+\psi(\tilde{u}))T - \frac{\psi''(\tilde{u})}{2}(u-\tilde{u})^2 - T\cdot O(u-\tilde{u})^3}$ dominate
- thus exponential approximates a Gaussian

For BS, $\psi_{BS}(u) = \frac{1}{2}\left(u^2+\frac{1}{4}\right)v$ where $v=\sigma^2$ and noting $\psi'_{BS}(u) = uv$ and $\psi''_{BS}(u) = v$, by solving $\psi'_{BS}(\tilde{u}) = -ix$ we get

$$\tilde{u}_{BS} = -\frac{ix}{v}$$

Thus our saddle-point condition:

$$\begin{aligned}
\frac{e^{-(ix\tilde{u}+\psi(\tilde{u}))T}}{\tilde{u}^2+\frac{1}{4}}\sqrt{\frac{2\pi}{\psi''(\tilde{u})T}} &\sim \frac{e^{-(ix\tilde{u}_{BS}+\psi_{BS}(\tilde{u}_{BS}))T}}{\tilde{u}_{BS}^2+\frac{1}{4}}\sqrt{\frac{2\pi}{\psi''_{BS}(\tilde{u}_{BS})T}} \\
&= \frac{e^{-\left(\frac{x^2}{v}+\frac{v}{2}\left(\frac{1}{4}-\left(\frac{x}{v}\right)^2\right)\right)T}}{\frac{1}{4}-\left(\frac{x}{v}\right)^2}\sqrt{\frac{2\pi}{vT}} \\
&\approx 4\exp\left(-\left(\frac{v}{8}+\frac{x^2}{2v}\right)T\right)\sqrt{\frac{2\pi}{vT}}
\end{aligned}$$

Const. terms are of similar orders (dominated by $e^{-(\dots)T}$) and we make exponent equal:

$$\omega(x) \equiv ix\tilde{u} + \psi(\tilde{u}) \sim \frac{v}{8} + \frac{x^2}{2v}$$

a quadratic equation, with solution

$$v(x) \sim 4\left(\omega(x) \pm \sqrt{\omega(x)^2 - \frac{x^2}{4}}\right)$$

Denote $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$, vanishing at $x_\pm$ which solve $\omega(x) = \pm x/2$, chosen to fulfill $\bar{\omega}((x_-, x_+)) < 0$ and $\bar{\omega}(\mathbb{R}\backslash(x_-, x_+)) > 0$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

We may express large-time asymptotic characteristic function in form $e^{-\psi(u)T}$ to reach a very wide class of model-inspired parametrizations.

3 In a Nutshell

Denote time-scaled log-strike $x = k/T$ and variance $v(x)$, implied from characteristic function $\phi_T(u) = Ee^{iuX_T}$ where $X_T = \log(S_T/S_0)$. Our saddle-point procedure reads:

1. evaluate characteristic function $\phi_T(u - i/2) \equiv e^{-\psi(u)T}$ to get $\psi(u)$
2. compute saddle-point $\tilde{u}$ which fulfills $\psi'(\tilde{u}) = -ix$
3. evaluate $\psi(\tilde{u})$ thus $\omega(x) \equiv i\tilde{u} \cdot x + \psi(\tilde{u})$ and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$
4. full smile given by $v(x) = 4\left(\omega(x) - \bar{\omega}(x)\right)$

Readers may first head to section Large-Time Heston Smile (and the like) to understand practical use and how $\omega, \bar{\omega}, v$ etc. actually look like before getting swamped by mathematical proofs below.

4 Gaussian Approximation

Our large-time Gaussian approximation in Lewis equation is valid when (1) $ix\tilde{u} + \psi(\tilde{u})$ is real (2) $\psi''(\tilde{u}) > 0$. We now show that these are always true from cumulant property.

We first show that $i\tilde{u}$ and $\psi(\tilde{u})$ are always real.

In large time, recall our characteristic function

$$\phi_T\left(u - \frac{i}{2}\right) = Ee^{\left(iu+\frac{1}{2}\right)X_T} = e^{-\psi(u)T}$$

thus $\psi(u)$ is a cumulant:

$$\psi(u) = -\frac{1}{T}\log Ee^{\left(iu+\frac{1}{2}\right)X_T}$$

Differentiating,

$$\psi'(u) = -\frac{1}{T}\frac{EiX_T e^{\left(iu+\frac{1}{2}\right)X_T}}{Ee^{\left(iu+\frac{1}{2}\right)X_T}}$$

$\tilde{u}$ satisfies

$$\psi'(\tilde{u}) = -\frac{1}{T}\frac{EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} = -ix$$

so

$$EX_T e^{\left(\hat{u}+\frac{1}{2}\right)X_T} = kEe^{\left(\hat{u}+\frac{1}{2}\right)X_T}$$

where we define $\hat{u} \equiv i\tilde{u}$. This is an equation in $\hat{u}$ - does a real $\hat{u}$ exist and is it unique?

A more insightful form: define Esscher measure $\mathbb{U}$ s.t.

$$\frac{d\mathbb{U}}{d\mathbb{P}} = \frac{e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}}$$

where $\mathbb{P}$ is our pricing measure (we have always been working in). Then

$$E^{\mathbb{U}}X_T = k$$

thus $\hat{u}(k)$ defines a measure under which expectation of log-spot is exactly log-strike. For $\mathbb{U}$ to be a properly defined measure, $\hat{u}$ has to be real - does it exist and is it unique?

Intuitively, $\hat{u}$ translates log-spot density - imagine Gaussian $f_{X_T}(x) \overset{\mathbb{P}}{\sim} e^{-x^2/2}$, then under $\mathbb{U}$, $f_{X_T}(x) \overset{\mathbb{U}}{\sim} e^{-(x-\hat{u})^2/2}$ and tanslation of density corresponds to translation of mean, so we can always suitably choose (solve for) $\hat{u}$ s.t. mean exactly matches log-strike i.e unique real $\hat{u}$ exists.

Now I make rigorous why $\hat{u}$, equivalently $\bar{u} = \hat{u} + 1/2$, always exists and is unique. Rewrite the expectation equation as

$$k = E^{\mathbb{U}}X_T = \frac{EX_T e^{\bar{u}X_T}}{Ee^{\bar{u}X_T}} = \frac{\partial}{\partial\bar{u}}\log Ee^{\bar{u}X_T}$$

This is the first derivative of cumulant (under pricing measure, not Esscher), which spans the support of $X_T \in \mathbb{R}$ (thus existence of at least a real root), and monotonically (thus one unique root $\hat{u}$), as its second denvative is variance - always positive.

With real $\hat{u}$, $\mathbb{U}$ is a property defined measure equivalent to $\mathbb{P}$.

Now,

$$\psi(\tilde{u}) = -\frac{1}{T}\log Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T} \in \mathbb{R}$$

For our saddle-point approximation to be valid, we need $\psi''(\tilde{u}) > 0$:

$$
\begin{aligned}
\psi''(\tilde{u}) &= -\frac{1}{T}\left(-\frac{EX_T^2 e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} - \frac{EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{\left(Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}\right)^2} EiX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}\right) \\
&= \frac{1}{T}\left(\frac{EX_T^2 e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}} - \left(\frac{EX_T e^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(i\tilde{u}+\frac{1}{2}\right)X_T}}\right)^2\right) \\
&= \frac{1}{T}\left(\frac{EX_T^2 e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}} - \left(\frac{EX_T e^{\left(\hat{u}+\frac{1}{2}\right)X_T}}{Ee^{\left(\hat{u}+\frac{1}{2}\right)X_T}}\right)^2\right) \\
&= \frac{1}{T}\left(E^{\mathbb{U}}X_T^2 - \left(E^{\mathbb{U}}X_T\right)^2\right) \\
&= \frac{V^{\mathbb{U}}X_T}{T}
\end{aligned}
$$

i.e. $\psi''(\tilde{u})$ is log-spot variance under $\mathbb{U}$. As expected, because differentiated cumulant gives central moments.

Thus as long as we can write down characteristic function, for large T, our saddle-point condition is always valid.

We can computatonally check how large is large, say approx closed-form formula vs. FFT.

5 Moment Expansion

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5.1 Moment Expansion of ω

5.2 Moment Expansion of Smile v

6 Existence of Variance Solution

Our variance $v(x)$ exists if $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - \frac{x^2}{4}}$ is well-defined.

My proposition is: $\omega(x)$ tangentially touches $|x/2|$ at points $x_\pm$ (small and off by a sign) but is everywhere else bounded below.

Equivalently, this may be stated as: $\omega(x) \mp x/2$ and its derivative vanish at some $x_\pm$ and second derivative is positive everywhere. We write

$$
\omega(x) \equiv x\hat{u} + \psi(\tilde{u}(x))
$$

thus derivatives

$$
\begin{aligned}
\omega'(x) &= \hat{u}(x) + x\hat{u}'(x) + \psi'(\tilde{u}(x)) = \hat{u}(x) \\
\omega''(x) &= \hat{u}'(x) = \frac{T}{E^{\mathbb{U}}(X_T - k)^2} > 0
\end{aligned}
$$

so second derivative is everywhere positive, and

$$\partial_x \left(\omega(x) \mp x/2\right) = \hat{u}(x) \mp \frac{1}{2}$$

which vanishes at some $x_\pm$,

$$\hat{u}(x_\pm) = \pm\frac{1}{2}$$

Does such $x_\pm$ always exist? Yes. Recall that $\hat{u}$ is roughly the log-spot density shift (via Esscher measure $\mathbb{U}$) s.t. expectation of log-spot $E^{\mathbb{U}}X_T$, exactly matches log-strike k, equivalently x, thus $\hat{u}$ is monotonic increasing in x, spanning $\mathbb{R}$. We can always find such an $x_\pm$ s.t. $\hat{u}(x_\pm)$ exactly matches $\pm 1/2$ - smallness/sign of $\pm 1/2$ leads to smaliness/sign of $x_\pm$.

By our moment expansion for $\bar{u}$, for small x

$$\bar{u}(x) \approx \bar{u}_0 + \frac{T}{E^{\mathbb{U}}X_T^2}x \equiv \frac{1}{2} \pm \frac{1}{2}$$

which approximates

$$x_\pm \approx \frac{E^{\mathbb{U}}X_T^2}{T}\left(\frac{1}{2} \pm \frac{1}{2} - \bar{u}_0\right) \approx \pm\frac{E^{\mathbb{U}}X_T^2}{2T} \sim \pm\frac{\sigma^2}{2}$$

as $\bar{u}_0 \approx 1/2$, from Ito correction, and σ is some characteristic vol in model.

To reason about this, think about BS diffusion - log-spot $X_T = -\sigma^2 T/2 + \sigma Z_T$ and denote total var $w = \sigma^2 T$ so density $f_{X_T}(x) \overset{\mathbb{P}}{\sim} e^{-(x+w/2)^2/2w}$. Under ATM Esscher measure defined by $\bar{u}_0$, $f_{X_T}(x) \overset{\mathbb{U}}{\sim} e^{-(x+w/2-w\bar{u}_0)^2/2w}$. For $E^{\mathbb{U}}X_T = 0$, we demand $\bar{u}_0 = 1/2$ - this is exact. At-the-money, other processes may be expressed as perturbation around BS diffusion so densities near $x = 0$ behave roughly identical, so approximation $\bar{u}_0 \approx 1/2$ is not unreasonable.

Typically, if model produces heavy left-tail, mode of log-spot density is biased to right, so we require a smaller Esscher shift to match match expectation of log-spot to zero - thus $\bar{u}_0 < 1/2$, a negatively skewed $\omega(k)T$.

Lastly, $\omega(x) \mp x/2$ vanishes at $x_\pm$ as

$$\omega(x_\pm) \mp \frac{x_\pm}{2} = x_\pm\hat{u}(x_\pm) - \frac{1}{T}\log Ee^{\left(\hat{u}(x_\pm)+\frac{1}{2}\right)X_T} \mp \frac{x_\pm}{2} = \pm\frac{x_\pm}{2} \mp \frac{x_\pm}{2} - \frac{1}{T}\log Ee^{\left(\pm\frac{1}{2}+\frac{1}{2}\right)X_T} = 0$$

as $Ee^{X_T} = 1$ by martingale condition.

So, $\omega(x) \geq |x/2|$ for all strike x and $|\bar{\omega}(x)| \equiv \sqrt{\omega(x)^2 - x^2/4} \leq \omega(x)$. Our variance smile

$$v(x) \sim 4(\omega(x) - \bar{\omega}(x))$$

is always well-defined (positive).

7 Large-Time Heston Smile

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8 Large-Time VG Smile

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9 Large-Time BG Smile

10 Large-Time CGMY Smile

11 Comments on Saddle-Point Trick

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