CAS MA583 INTRODUCTION TO STOCHASTIC PROCESSES

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Contents

1 Lecture 1 – 1/21

Stocastic := random, process := anything that evolves through time.
Examples:

- Gambling
- Stocks
- Biology
- Geology
- Weather

The math used includes calculus (especially infinite sums), linear algebra (solving equations, eigenvalue diffeqs), and probability (know most common distributions including binomial, Poisson, exponential, normal).

1.1 Chapter 2

Conditional probability and conditional expectation.

Definition 1.1. Given two events A and B with $P(B) > 0$, the conditional probability of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Usually we know the conditional probabilities and use them to solve more difficult questions.

Example 1.2. Roll a six-sided die, and call the result X. Next we flip X coins. Let Y be the total number of coins that land on heads. What is the probability of $\mathbb{P}(Y = 4)$?
We know some probabilities, like $\mathbb{P}(X = 1) = \frac{1}{6}$, and $\mathbb{P}(X = 2) = \frac{1}{6}$. We also know conditional probabilities, like $\mathbb{P}(Y = 1|X = 1) = \frac{1}{2}$, and $\mathbb{P}(Y = 0|X = 1) = \frac{1}{2}$. Y is conditionally binomial if $n \leq 4$: $\mathbb{P}(Y = n|X = k) = \binom{k}{n}\left(\frac{1}{2}\right)^k$.
So to answer the question,

$$\begin{aligned}
\mathbb{P}(Y = 4) &= \mathbb{P}(Y = 4 \text{ and } X = 4) + \mathbb{P}(Y = 4 \text{ and } X = 5) + \mathbb{P}(Y = 4 \text{ and } X = 6) \\
&= \mathbb{P}(Y = 4|X = 4)\mathbb{P}(X = 4) + \mathbb{P}(Y = 4|X = 5)\mathbb{P}(X = 5) + \mathbb{P}(Y = 4|X = 6)\mathbb{P}(X = 6) \\
&= \binom{4}{4}\left(\frac{1}{2}\right)^4 \cdot \frac{1}{6} + \binom{5}{4}\left(\frac{1}{2}\right)^5 \cdot \frac{1}{6} + \binom{6}{4}\left(\frac{1}{2}\right)^6 \cdot \frac{1}{6}.
\end{aligned}$$

Definition 1.3. The above example used what's called the law of total probability.

$$\mathbb{P}(Y = n) = \sum_x \mathbb{P}(Y = n|X = x)\mathbb{P}(X = x).$$

Example 1.4. What is $\mathbb{E}[Y]$ in the above example (or the expected number of heads)? The conditional expectation is clear:

$$\mathbb{E}[Y|X=4] = \frac{4}{2} = 2 \implies \mathbb{E}[Y|X=k] = \frac{k}{2}.$$

Thus the law of total expectation gives us

$$\mathbb{E}[Y] = \sum_{k=1}^{6} \mathbb{E}[Y|X=k]\mathbb{P}(X=4) = \sum_{k=1}^{6} \frac{k}{2} \cdot \frac{1}{6} = \frac{7}{4}.$$

Example 1.5. Based on the casino game craps: roll two standard six-sided dice over and over. If the sum is 7 then I lose. If the sum is 4, then I win. If the sum is anything else then I roll again. What is the probability that I win?
Well, in one roll of two dice, the probability of a 4 is $\frac{3}{36}$, the probability of a 7 is $\frac{6}{36}$, and the probability of neither is $\frac{27}{36}$.
Solving directly without conditionaly, the probability of winning is

$$\mathbb{P} = \frac{3}{36} + \frac{27}{36}\frac{3}{36} + \left(\frac{27}{36}\right)^2 \frac{3}{36} + \cdots = \sum_{n=0}^{\infty} \left(\frac{27}{36}\right)^n \frac{3}{36}.$$

Using an alternate approach, we could also use the law of total probability. Let W be the event that I win. Then

$$\begin{aligned}\mathbb{P}(W) &= \mathbb{P}(W|\text{roll } 4)\mathbb{P}(\text{roll } 4) + \mathbb{P}(W|\text{roll } 7)\mathbb{P}(\text{roll } 7) + \mathbb{P}(W|\text{roll other})\mathbb{P}(\text{roll other}) \\ &= 1 \cdot \frac{3}{36} + 0 \cdot \frac{6}{36} + \mathbb{P}(W) \cdot \frac{27}{36}.\end{aligned}$$

Solving for $\mathbb{P}(W)$ gives

$$\mathbb{P}(W) = \frac{3/36}{1 - 27/36} = \frac{1}{9}.$$

2 Lecture 2 – 1/23

2.1 Random sums of random variables

Setting: N is a variable taking values $\{0, 1, 2, \ldots\}$, and $X_1, X_2, \ldots$ are iid random variables independent of N. We study $S = \sum_{i=1}^{N} X_i = X_1 + X_2 + \cdots + X_N$.

Example 2.1. Car insurance: N total number of car accidents. X_i is the cost of the i-th accident. S is the total cost of all accidents. $\sum_{i=1}^{N}$ is total costs for insurer.

Mean: $\mathbb{E}(\sum_{i=1}^{n} X_i) = \mathbb{E}(N)\mathbb{P}(X_i)$. Also, by the independence of X_n and N (where independence means conditional probability and nonconditional probabilities match), $\mathbb{E}[\sum_{k=1}^{N} X_k|N=n] = \mathbb{E}[\sum_{k=1}^{n} X_n]$.

Definition 2.2. The **law of total probability** looks like

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{N} X_k\right] &= \sum_{n=0}^{\infty} \mathbb{E}[\sum_{k=1}^{n} X_k | N = n]\mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}[\sum_{k=1}^{n} X_k]\mathbb{P}(N = n)\text{because we assumed } X_k \text{ have id. dists.} \\
&= \sum_{n=0}^{\infty} n\mathbb{E}[X_i]\mathbb{P}(N = n) \\
&= \mathbb{E}(X_1)\sum_{n=0}^{\infty} n\mathbb{P}(N = n) \\
&= \mathbb{E}(X_1)\mathbb{E}(N).
\end{aligned}$$

What about the variance($\sum_{i=1}^{N} X_i$)? Let's consider two cases:

- Case 1: $N = n$ is fixed with $n = 7$. Then $Var(\sum_{i=1}^{7} X_i) = 7Var(X_i)$ since the X_i are independent.
- Case 2: N is random. Using the law of total variance,

$$Var(\sum_{i=1}^{N} X_i) = \mathbb{E}[(\sum_{i=1}^{N} X_i)^2] - (\mathbb{E}[\sum_{i=1}^{N} X_i])^2 = (\mathbb{E}[X_1])^2 Var(N) + Var(X_1)\mathbb{E}(N)$$

$$\mathbb{E}[(\sum_{i=1}^{N} X_i)^2] = \mathbb{E}[N]Var(X_1) + (\mathbb{E}[X_1])^2\mathbb{E}[N^2]$$

3 Lecture 3 – 1/28

3.1 Chapter 3: Markov Chains

Abstract setting: Finite number of states, like $\{0, 1, 2, 3, 4\}$ or rainy, snowy, sunny, sleet or cold, medium, hot. Some system can jump randomly between these states. We can say X_0 is the (random) starting point, X_1 is the state at time 1, X_15 at time 15, etc. We can ask questions like, "What is the probability of going from 0 to state 2 in exactly two steps?"
To answer that question, we could use the law of total probability:

$$\mathbb{P}(X_2 = 2|X_0 = 0) = \sum_k \mathbb{P}(X_2 = 2|X_1 = k, X_0 = 0)\mathbb{P}(X_1 = k|X_0 = 0).$$

We could put it into a matrix:

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

where rows need to add to 1 and each entry must be ≥ 0. The law of total probability and matrix multiplication are the same formula!

Example 3.1. Flip a fair coin 100 times and write down the result. What is the probability of getting a run of at least 7 heads in a row in 100 flips?
Build a useful Markov chain: X_n keeps track of number of heads in a row on flip n. If X_n is ever 7 then it stays at 7 forever. Then $\mathbb{P}(\text{at least 7 heads in a row}) = \mathbb{P}(X_{100} = 7|X_0 = 0)$. The transition matrix is

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this 100 times and looking at the first row, last column gives $\mathbb{P}(X_{100} = 7|X_0 = 0) \approx 0.3175$.

4 Lecture 4 – 1/30