CAS MA583 INTRODUCTION TO STOCHASTIC PROCESSES

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Contents

1 Lecture 1 – 1/21

Stocastic := random, process := anything that evolves through time.
Examples:

- Gambling
- Stocks
- Biology
- Geology
- Weather

The math used includes calculus (especially infinite sums), linear algebra (solving equations, eigenvalue diffeqs), and probability (know most common distributions including binomial, Poisson, exponential, normal).

1.1 Chapter 2

Conditional probability and conditional expectation.

Definition 1.1. Given two events A and B with $P(B) > 0$, the conditional probability of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Usually we know the conditional probabilities and use them to solve more difficult questions.

Example 1.2. Roll a six-sided die, and call the result X. Next we flip X coins. Let Y be the total number of coins that land on heads. What is the probability of $\mathbb{P}(Y = 4)$?
We know some probabilities, like $\mathbb{P}(X = 1) = \frac{1}{6}$, and $\mathbb{P}(X = 2) = \frac{1}{6}$. We also know conditional probabilities, like $\mathbb{P}(Y = 1|X = 1) = \frac{1}{2}$, and $\mathbb{P}(Y = 0|X = 1) = \frac{1}{2}$. Y is conditionally binomial if $n \le 4$: $\mathbb{P}(Y = n|X = k) = \binom{k}{n}\left(\frac{1}{2}\right)^k$.
So to answer the question,

$$\begin{aligned}
\mathbb{P}(Y = 4) &= \mathbb{P}(Y = 4 \text{ and } X = 4) + \mathbb{P}(Y = 4 \text{ and } X = 5) + \mathbb{P}(Y = 4 \text{ and } X = 6) \\
&= \mathbb{P}(Y = 4|X = 4)\mathbb{P}(X = 4) + \mathbb{P}(Y = 4|X = 5)\mathbb{P}(X = 5) + \mathbb{P}(Y = 4|X = 6)\mathbb{P}(X = 6) \\
&= \binom{4}{4}\left(\frac{1}{2}\right)^4 \cdot \frac{1}{6} + \binom{5}{4}\left(\frac{1}{2}\right)^5 \cdot \frac{1}{6} + \binom{6}{4}\left(\frac{1}{2}\right)^6 \cdot \frac{1}{6}.
\end{aligned}$$

Definition 1.3. The above example used what's called the law of total probability.

$$\mathbb{P}(Y = n) = \sum_x \mathbb{P}(Y = n|X = x)\mathbb{P}(X = x).$$

Example 1.4. What is $\mathbb{E}[Y]$ in the above example (or the expected number of heads)? The conditional expectation is clear:

$$\mathbb{E}[Y|X=4] = \frac{4}{2} = 2 \implies \mathbb{E}[Y|X=k] = \frac{k}{2}.$$

Thus the law of total expectation gives us

$$\mathbb{E}[Y] = \sum_{k=1}^{6} \mathbb{E}[Y|X=k]\mathbb{P}(X=4) = \sum_{k=1}^{6} \frac{k}{2} \cdot \frac{1}{6} = \frac{7}{4}.$$

Example 1.5. Based on the casino game craps: roll two standard six-sided dice over and over. If the sum is 7 then I lose. If the sum is 4, then I win. If the sum is anything else then I roll again. What is the probability that I win?

Well, in one roll of two dice, the probability of a 4 is $\frac{3}{36}$, the probability of a 7 is $\frac{6}{36}$, and the probability of neither is $\frac{27}{36}$.

Solving directly without conditionaly, the probability of winning is

$$\mathbb{P} = \frac{3}{36} + \frac{27}{36}\frac{3}{36} + \left(\frac{27}{36}\right)^2 \frac{3}{36} + \cdots = \sum_{n=0}^{\infty} \left(\frac{27}{36}\right)^n \frac{3}{36}.$$

Using an alternate approach, we could also use the law of total probability. Let W be the event that I win. Then

$$\begin{aligned} \mathbb{P}(W) &= \mathbb{P}(W|\text{roll } 4)\mathbb{P}(\text{roll } 4) + \mathbb{P}(W|\text{roll } 7)\mathbb{P}(\text{roll } 7) + \mathbb{P}(W|\text{roll other})\mathbb{P}(\text{roll other}) \\ &= 1 \cdot \frac{3}{36} + 0 \cdot \frac{6}{36} + \mathbb{P}(W) \cdot \frac{27}{36}. \end{aligned}$$

Solving for $\mathbb{P}(W)$ gives

$$\mathbb{P}(W) = \frac{3/36}{1 - 27/36} = \frac{1}{9}.$$