CAS MA411 ADVANCED CALCULUS

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Contents

1 Lecture 1 – 9/2

1.1 LinAlg Review

Essentially geometry.

1.1.1 $\mathbb{R}^2$

A vector that points from $(0, 0)$ to (x, y) has a length $||\vec{v}|| = \sqrt{x^2 + y^2}$. This utilizes the Pythagorean Theorem, which can be proved using a geometric proof.
Now, if we have a vector that points from $(0, 0)$ to (a, b), and another that points from $(0, 0)$ to $(-b, a)$, how do we know that the angle between these two vectors is 90º? We can use the dot product, where if $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$, then $\vec{v} \cdot \vec{w} = ac + bd$. Knowing this, the dot product between the two aforementioned vectors is $-ba + ab$, or 0. This means the angle between them is 90, or there is no projection of one vector onto the other, or they are perpendicular to each other.

1.1.2 $\mathbb{R}^3$

Given $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$, the dot product between these is $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$.

1.1.3 $\mathbb{R}^n$

Given $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{y} = (y_1, y_2, ..., y_n)$ and that $||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, $\vec{x} \perp \vec{y} \iff ||x||^2 + ||y||^2 = ||y - x||^2$. (To visualize, if $n = 2$, then the length of the hypotenuse above the vectors is $||y - x||$.)
Extending this dialogue on dot products, we can further identify it by using the law of cosines: $||\vec{x}|| \cdot ||\vec{y}|| cos\theta = \vec{x} \cdot \vec{y}$. (To visualize, draw $\vec{v}$ and $\vec{y}$ with an angle θ between them.) Prove this for homework.

2 Lecture 2 – 9/4

2.1 Basics Review

Definition 2.1. A **function** (f, D, X) is defined as: f is a rule that maps set D onto X, or $D \xrightarrow{f} X$.

Example 2.2. Given $f(x) = x^2$:

$$f : \mathbb{R} \to \mathbb{R}$$

is a surjective function (compared to an injective function, where everything is mapped one-to-one).

Definition 2.3. A **transformation** T is a special geometric function.

A transformation T that is $(x, y) \mapsto x^2 + y^2$ is a $T : \mathbb{R}^2 \to \mathbb{R}$, or $T(x, y) = x^2 + y^2$.

2.2 Linear Functions (Linear Mappings/Linear Transformations)

Definition 2.4. We say that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if for every $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ and every $c_1, c_2 \in \mathbb{R}$ we have $T(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1T(\vec{x}_1) + c_2T(\vec{x}_2)$

To describe a linear function $T : \mathbb{R}^n \to \mathbb{R}^m$, we need only to know $T(\vec{e}_i)$, where $r = 1, \dots, n$.

Example 2.5. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, and $T(\vec{e}_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$, $T(\vec{e}_1) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$, $T(\vec{e}_1) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$, what is $T\begin{pmatrix}3\\2\\1\end{pmatrix}$?

$$\begin{pmatrix}3\\2\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix} = 3\vec{e}_1 + 2\vec{e}_2 + 1\vec{e}_3$$

$$T\begin{pmatrix}3\\2\\1\end{pmatrix} = 3T(\vec{e}_1) + 2T(\vec{e}_2) + 1T(\vec{e}_3) = 3\begin{pmatrix}2\\3\end{pmatrix} + 2\begin{pmatrix}-1\\0\end{pmatrix} + 1\begin{pmatrix}5\\-1\end{pmatrix} =$$

$$\begin{pmatrix}6\\9\end{pmatrix} + \begin{pmatrix}-2\\0\end{pmatrix} + \begin{pmatrix}5\\-1\end{pmatrix} = \begin{pmatrix}9\\8\end{pmatrix}$$

Note: We can rewrite a lot of this using a matrix $A = \begin{pmatrix}2 & -1 & 5\\3 & 0 & -1\end{pmatrix}$, which is the matrix associated with T.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the $Ker(T) = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x} = \vec{0})\}$, which says that the kernel of transformation T, or the null space of transformation T, is the set of vectors mapped to the zero vector.

Example 2.6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix $A = \begin{pmatrix}2 & 3 & 4\\-1 & 2 & 5\end{pmatrix}$.
We know that $T(\vec{x}) = A\vec{x}$ and that $\vec{x} \in \mathbb{R}^3$. What is the image of T?
The image (or the range, or the column space) is the span (or the set of all possible linear combinations) of its column vectors.

3 Lecture 3 – 9/9

Given that a function $f : \mathbb{R}^n \to \mathbb{R}^m$, it could also be described as $f : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is a region (an open domain, which doesn't include the boundaries) in $\mathbb{R}^m$.

Definition 3.1. A subset of $\mathbb{R}^n$ is called a **region** if for every $x \in D$ $\exists$ a neighborhood of x that is contained in D.

Example 3.2. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$ is the region inside a circle, inclusive. By a **neighborhood** in the previous definition, we mean that a neighborhood of $\vec{x}_0$ is an open circular disc centered at $\vec{x}$, or that $B_\epsilon(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x} - \vec{x}_0| \leq \epsilon\}$. This means it's not a region/open domain. But, if we changed it to $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}$, thus eliminating the boundary, it does become an region (every point in the region can have a neighborhood, while points on the boundary cannot have a neighborhood. This makes the former closed, while the latter is open).

Definition 3.3. A region X is said to be **connected** if and only if for any two points $\vec{x}_1, \vec{x}_2 \in X$, there is a broken line connecting $\vec{x}_1$ to $\vec{x}_2$. (A broken line is just a line that can be drawn with multiple connected line segments.)

Definition 3.4. A function $f : X \to \mathbb{R}^m$ where X is an open region is said to be **continuous** if $\forall$ $\vec{x}_0 \in X$, f is continuous at $\vec{x}_0$.
If $\forall$ $\epsilon > 0$,there is a $\delta > 0$ such that $f(B_\delta(\vec{x}_0)) \subseteq B_\epsilon(f(\vec{x}_0))$. This is the delta-epsilon definition of continuity.
We say f is continuous at $\vec{x}_0 \iff \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$. We say $\lim_{x\to x_\delta} = L \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $f(B_\delta(\vec{x}_0)) \subseteq B_\epsilon(L)$.

$$\frac{\partial f}{\partial x}(\vec{x}_0) = \lim_{\Delta\to 0} \frac{f(\vec{x}_0 + \Delta x) - f(\vec{x}_0)}{\Delta x}$$

4 Lecture 4 – 9/11

Use $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ as a concept and develop the Jacobian Matrix.
Given $y = f(x_1, x_2, ..., x_n)$, then $dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n$.
Now suppose we have a collection of functions, like $y_1 = f_1(x_1, ..., x_n), y_2 = f_2(x_1, ..., x_n), ..., y_m = f_m(x_1, ..., x_n)$.
If we differentiate each of those functions like we did two sentences ago, we end up with $d\vec{y} = \begin{pmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_m \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$, or the Jacobian Matrix applied to the $d\vec{x}$ vector.

Example 4.1. Given

$$\begin{aligned} y_1 &= x_1^2 + x_2^2 - x_3^2 \\ y_2 &= x_1^2 - x_2^2 + x_3^2 \\ y_3 &= -x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

Then the Jacobian matrix is

$$J = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & -2x_3 \\ 2x_1 & -2x_2 & 2x_3 \\ -2x_1 & 2x_2 & 2x_3 \end{pmatrix}.$$

So $d\vec{y} = J \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$, at which it becomes a matter of algebra.

5 Lecture 5 – 9/16

Example 5.1. Given $z = f(x, y)$, $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$. Now suppose x and y are parameterized: $x = x(t), y = y(t)$. Then $z = z(t) = f(x(t), y(t))$, thus creating $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. This is the **chain rule**.

The principle of the chain rule can always extend itself, like if $x = x(u_1, u_2)$ and $y = y(u_3, u_4)$ and $z = z(x, y)$, then $\frac{\partial z}{\partial u_1} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u_1} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u_1}$ aka $\frac{\partial z}{\partial u_1} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u_1} \\ \frac{\partial y}{\partial u_1} \end{pmatrix}$ and on.

6 Lecture 6 – 9/18

Given $f : \mathbb{R}^n \to \mathbb{R}^m$ as a differentiable function, we obtain two functions: $f(\vec{x}_0) = \begin{pmatrix} y_1(\vec{x}_0) \\ \vdots \\ y_m(\vec{x}_0) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, which is just the application of the function onto a vector, as well as $f(\vec{x}_0 + d\vec{x}) \sim f(\vec{x}_0) + D_f(x_0)d\vec{x}$, which uses the Jacobian.

Example 6.1. Given $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ where

$$\begin{aligned} y_1 &= x_1^2 + x_2^2 - x_3^2 \\ y_2 &= x_1^2 - x_2^2 + x_3^2 \\ y_3 &= -x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

.

Then, D_f is the Jacobian matrix, which is $\mathbb{R}^3 \to Matrix_{3x3}(\mathbb{R})$, and the result is $D_f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & -2x_3 \\ 2x_1 & -2x_2 & 2x_3 \\ -2x_1 & 2x_2 & 2x_3 \end{pmatrix}$.

With just function application, let's use $\vec{x}_0 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. We get $f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$.

Applying the Jacobian gives $D_f(\vec{x}_0) = D_f \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ 4 & -2 & 2 \\ -4 & 2 & 2 \end{pmatrix}$. Plugging this into the equation for differentiation, we get $f(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + d\vec{x}) \sim f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + D_f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} d\vec{x}$, or $f(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + d\vec{x}) \sim \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 & 2 & -2 \\ 4 & -2 & 2 \\ -4 & 2 & 2 \end{pmatrix} d\vec{x}$. We can now use this to calculate any small change away from $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Using the above concept, imagine a cube centered at $\vec{x}_0$. Applying a Jacobian D_f onto the cube maps it into a region that seems like a parallelepiped that contains $f(\vec{x}_0)$. The volume of this new parallelepiped can be calculated using $|\det(\vec{v})|$, where $\vec{v}$ are the transformed vectors on the edges from one corner of the original cube (aka the vectors that span the parallelepiped).

7 Lecture 7 – 9/23

Given $f : \mathbb{R}^n \to \mathbb{R}^m$, the graph of f is $\{(\vec{x}, f(\vec{x})) \mid \vec{x} \in \mathbb{R}^n\}$. With this, $\vec{x}$ could be a vector of functions, such that the resulting vector $\vec{y} = f(\vec{x}) = f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1(\vec{x}) \\ \vdots \\ y_m(\vec{x}) \end{pmatrix}$, where $D_f(\vec{x}) = \vec{y}_{\vec{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$.
This is significant because of $f(\vec{x}) + df(\vec{x}, d\vec{x}) = f(\vec{x}) + \vec{y}_{\vec{x}} d\vec{x}$. This allows us to come to the key point that $(\vec{x}, f(\vec{x}) + df(\vec{x}, d\vec{x})) = (\vec{x}, f(\vec{x}) + \vec{y}_{\vec{x}} d\vec{x})$.

Example 7.1. Given

$$\begin{aligned} y_1 &= u_1u_2 - u_1u_3 \\ y_2 &= u_1u_3 + u_2^2 \\ u_1 &= x_1 \cos x_2 + (x_1 - x_2)^2 \\ u_2 &= x_1 \sin x_2 + x_1x_2 \\ u_3 &= x_1^2 - x_1x_2 + x_2^2 \end{aligned}$$

and knowing that

$$\vec{y}_{\vec{u}} = \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \frac{\partial y_1}{\partial u_3} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \frac{\partial y_2}{\partial u_3} \end{pmatrix}$$

we can come to the matrix $\vec{y}_{\vec{u}} = \begin{pmatrix} u_2 - u_3 & u_1 & -u_1 \\ u_3 & 2u_2 & u_1 \end{pmatrix}$ with and with the same process for the rest of the functions, we can get the matrix $\vec{u}_{\vec{x}} = \begin{pmatrix} \cos x_2 + 2(x_1 - x_2) & -\sin x_2 - 2(x_1 - x_2) \\ \sin x_2 + x_2 & x_1 \cos x_2 + x_1 \\ 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix}$.

Take $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

So, for $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{y}_{\vec{u}} = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 0 & 2 \end{pmatrix}$, and $\vec{u}_{\vec{x}} = \begin{pmatrix} 3 & -2 \\ 0 & 2 \\ 2 & -1 \end{pmatrix}$.

Thus, finally, that means $\vec{y}_{\vec{x}} = \begin{pmatrix} -7 & 8 \\ 7 & -4 \end{pmatrix}$, the product of $\vec{y}_{\vec{u}}$ and $\vec{u}_{\vec{x}}$.

Definition 7.2. Given $f : \mathbb{R}^n \to \mathbb{R}^m$, function f is **affine** if $f(\vec{x}) = A\vec{x} + \vec{b}$, where $\vec{b} \in \mathbb{R}^m$.

8 Lecture 8 – 9/25

Given $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \begin{cases} 2x^2 + y^2 - z^2 = 3 \\ xyz + 2x^2z + 3xz^2 = 1 \end{cases}\}$, this says $z(x, y)$ is a function where $2x + y^2 - z(x, y)^2 = 3$ and $xy \cdot z(x, y) + 2x^2 \cdot z(x, y) + 3x(z(x, y)^2) = 1$.

9 Lecture 9 – 9/30

Suppose we have to functions $2x + y - 3z - 2u = 0$ and $x + 2y + z + u = 0$. If we're asked to solve for x, u in terms of y, z, that's $(\frac{\partial x}{\partial y})_z$ and $(\frac{\partial u}{\partial z})_y$. Take the differential of the first equation to get $2dx + dy - 3dz - 2du = 0$ Do the same for the second equation. Then rewrite both in the terms of the question:

$$2dx - 2du = -dy + 3dz$$
$$dx + du = -2dy - dz.$$

This can be rewritten as a matrix equation such that $\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} dx \\ du \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} dy \\ dz \end{pmatrix}$. To solve for the question, $\begin{pmatrix} dx \\ du \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} dy \\ dz \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} dy \\ dz \end{pmatrix}$. This gives us the final equations

$$dx = -\frac{5}{4}dy + \frac{1}{4}dz$$
$$du = -\frac{3}{4}dy - \frac{5}{4}dz$$

Example 9.1. Given $zx^2 + y^2 + z^3 - 1 = 0$ we can solve this equations for z in terms of $x + y$. Let $z = z(x, y)$. Then $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.
$2xz \cdot dx + x^2 \cdot dz + 2y \cdot dy + 3z^2 \cdot dz = 0$. We can solve for $dz = -\frac{2xz}{x^2+3z^2}dx - \frac{2y}{x^2+3z^2}dy$. That means $\frac{\partial z}{\partial x} = -\frac{2xz}{x^2+3z^2}$ and $\frac{\partial z}{\partial y} = -\frac{2y}{x^2+3z^2}$.
This is different from the above example, because the last one had the partials as literal numbers, while this has functions.

10 Discussion 10/1

Theorem 10.1. Consider the simultaneous equations $F(\vec{x}, \vec{u}) \in \vec{0}, \vec{x} = \mathbb{R}^n, \vec{u} \in \mathbb{R}^m, F : \mathbb{R}^{n+m} \to \mathbb{R}^n$. Let $(\vec{x}_0, \vec{u}_0)$ be a point at which $F(\vec{x}_0, \vec{u}_0) = \vec{0}, det(F_{\vec{x}}(\vec{x}_0, \vec{u}_0)) \neq 0$. Suppose F has continuous partial derivatives in a neighborhood of $(\vec{x}_0, \vec{u}_0)$. Then there exists a neighborhood D of $\vec{u}_0$ and a unique continuously differentiable function $f : D \to \mathbb{R}^n$ such that $\vec{x} = f(\vec{u})$ (or $F(f(\vec{u}, \vec{u}) = 0)$).

11 Lecture 10 – 10/16

Given a space X, where $X \subseteq \mathbb{R}^n$, a path in X is a function $\pi : [a, b] \to X \xrightarrow{x_i} \mathbb{R}$. For $t \in [a, b], \pi(t) \in X$, or $\pi(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in X$. Thus $\pi'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)) \in X$.
Let gradient $\nabla F |_{(x_0,y_0)} = (\frac{\partial F}{\partial x} |_{(x_0,y_0)}, \frac{\partial F}{\partial y} |_{(x_0,y_0)})$.
Given function $F(x, y)$ and space above $(x(t), y(t), F(x(t), y(t)))$, with $x_0 = x(0), y_0 = y(0), \frac{d}{dt}F(x(t), y(t)) |_{(x_0,y_0)} = \frac{\partial F}{\partial x}(x_0, y_0) \cdot x'(0) + \frac{\partial F}{\partial y} \cdot y'(0) = \nabla f |_{(x_0,y_0)} \cdot (x'(0), y'(0))$.

Definition 11.1. The directional derivative of F in the direction $\frac{x'(0)}{||x'(0)||}$ is $\nabla_{\vec{x}'(0)} F = \nabla F \cdot \frac{\vec{x}'(0)}{||\vec{x}'(0)||}$.

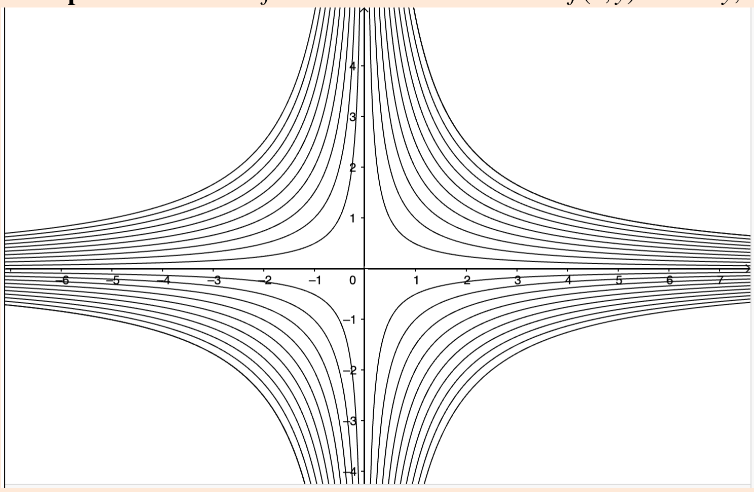
Example 11.2. Let $F(x, y, z) = 2x^2 - y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 3)$ and $\vec{u} = (2, 3, -3)$. Find the directional derivative of F in the direction of $\vec{u}$ starting from the point (x_0, y_0, z_0).
Then $\nabla_{\vec{u}} F |_{(1,2,3)} = (\frac{\partial F}{\partial x} |_{(1,2,3)}, \frac{\partial F}{\partial y} |_{(1,2,3)}, \frac{\partial F}{\partial z} |_{(1,2,3)})(\frac{1}{\sqrt{22}} \cdot (2, 3, -3))$.
Solving this out gives $\nabla F = (4x, -2y, 2z) |_{(1,2,3)} = (4, -4, -6)$. Plug this into our expanded equation above gives $(4, -4, 6) \cdot (\frac{1}{\sqrt{22}} \cdot (2, 3, -3)) = \frac{-22}{\sqrt{22}}$.

12 Lecture 11 – 10/21

Definition 12.1. A **scalar field** is a function $f : \mathbb{R}^n \to \mathbb{R}$, or a function that results in a single output value.

We can make **level curves** to identify how the result changes as a function of the inputs.

Example 12.2. Given $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = xy$, the level curve is



Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. We can form the **gradient field** associated to f (assuming the function is a smooth, continuous function). We write it using ∇f, which (for this f) is calculated using $\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$. Notably, this means we took f, a *scalar field*, and turned it into a **vector field**.

Definition 12.3. Vector fields have multiple output values, such as $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$. In this instance, $\vec{v}(x,y) = v_x(x,y)\vec{i} + v_y(x,y)$, where the subscript denotes the component in that dimension and NOT the derivative.

Gradient vectors will always point perpendicular to any level set!

Because vector fields are a function that map multi-dimensions into multi-dimensions, we can also identify the Jacobian of a vector field: $J(\vec{v}) = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}$.

Definition 12.4. The **divergence** of a vector field is $Div(\vec{v}) := \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$. This is found from the *trace* of the Jacobian matrix of the vector field (found above). The trace is the diagonal of the matrix.

Physics relation: let the density of a fluid at (x,y) at time t be equal to $\rho(t) = (x,y)$. Then $Div(\vec{v}) \mid_{(x,y)} = -\rho'(t)$.

Definition 12.5. The **curl** of a vector field $\vec{v}$ in $\mathbb{R}^3$ is $curl(\vec{v}) = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\vec{i} - (\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z})\vec{j} + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\vec{k}$. This is the expanded form of $curl(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$.

13 Lecture 12 – 10/23

Some observations:

1. The curl of a gradient of a function, $curl(\nabla f) = 0$.
2. $div \cdot curl(\vec{v}) = 0$

13.1 Integration

Given a function f defined on region $[a, b]$ and that maps to $\mathbb{R}$, the integration over this region is $\int_a^b f(x)\,dx$.
Now, let's split the region into partitions, such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. The region from $a = x_0$ to x_1 is labeled as x_1^*, and we can do the same for each partition. If we define $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$, then $\int_a^b f(x)\,dx \sim \sum_{i=0}^n f(x_i^*)\Delta x_i$.

> **Definition 13.1.** The **mesh** is defined as $h = max(\Delta x_i)$, for $i = 1, \ldots, n$.

Given $I \in \mathbb{R}$, we say $\int_a^b f(x)\,dx = I \iff \forall\ \epsilon > 0, \exists\ \delta > 0$ such that for any partition $a = x_0, x_1, \ldots, x_n = b$, if the mesh $h < \delta$, then we have $|\sum_{i=1}^n f(x_i^*)\Delta x_i - I| < \epsilon$.

13.2 Double Integration

$\iint_R f(x, y)\,dx\,dy$, $R \subseteq \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$.
The mesh in two dimensions extends into... well, two dimensions, forming a grid over R. In two dimensions, h = minimum side length of the squares in the mesh. $\iint_R f\,dx\,dy = \lim_{h\to 0}\sum_{i=1}^N f(x_i^*)\Delta x_i$.

13.3 Average value of f on R

> **Definition 13.2.** Defined as
> $$\frac{1}{Area(R)}\iint_R f(x, y)\,dx\,dy.$$

13.4 Mean Value Theorem

> **Definition 13.3.** If $f : R \to \mathbb{R}$ is continuous, then $\exists\ (x_0, y_0) \in R$ such that $f(x_0, y_0) = \frac{1}{Area(R)}\iint_R f\,dx\,dy$.

14 Lecture 13 – 10/28

14.1 Triple Integral

$\iiint_R f(x, y, z)\,dx\,dy\,dz$.

Example 14.1. Say we're given a region $R : \{0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le x+y\}$. Given the function $f(x,y,z) = 2x - y - z$, we can find the volume in this region of this function with

$$
\begin{aligned}
&= \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z)\, dz\, dy\, dx \\
&= \int_0^1 \int_0^{x^2} (2xz - yz - \frac{z^2}{2}) \mid_0^{x+y} \; dy\, dx \\
&= \int_0^1 \int_0^{x^2} (2x(x+y) - y(x+y) - \frac{(x+y)^2}{2})\, dy\, dx \\
&= \int_0^1 \int_0^{x^2} (\frac{3}{2}x^2 - \frac{3}{2}y^2)\, dy\, dx \\
&= \frac{3}{2} \int_0^1 (x^2 y - \frac{y^3}{3}) \mid_y^{x^2} \; dx \\
&= \frac{3}{2} \int_0^1 (x^4 - \frac{x^6}{3})\, dx \\
&= \frac{3}{2}(\frac{x^5}{5} - \frac{x^7}{21}) \mid_0^1 \\
&= \frac{8}{35}.
\end{aligned}
$$

14.2 Change of Variables

Theorem 14.2. The **Fundamental Theorem of Calculus** says that if F is an antiderivative of f, i.e. $F'(x) = f(x)$, then $\int_{x_1}^{x_2} f(x)\, dx = F(x) \mid_{x_1}^{x_2} = F(x_2) - F(x_1)$.

Theorem 14.3. The **Chain Rule** tells us that given $F(x(u))$, we can find $\frac{d}{du}(F(x(u))) = f(x(u)) \cdot x'(u)$, where $F' = f$.

14.2.1 One variable case

$\int_{x_1}^{x_2} f(x)\, dx$ can have it's variables changed into $\int_{u_1}^{u_2} f(x(u))\, \frac{dx}{du} du$.

14.2.2 Two variable case

$$\iint_{R_{xy}} f(x,y)\, dx\, dy = \iint_{R_{uv}} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

Example 14.4. Given a parallelogram in R_{xy} with corners $(0,1),(2,2),(3,1),(1,0)$, and functions $u = x + y$ and $v = x - 2y$, we find that this parallelogram gets mapped into corners $(1,-2),(4,-2),(4,1),(1,1)$ in R_{uv}.
We do this because if we wanted the volume of the function $(x+y)^3$ in the region of R_{xy}, setting up the bounds of integration would be tedious in this strange, slanted parallelogram. So changing it into $\int_{R_{uv}} u^3 \cdot |\frac{\partial(x,y)}{\partial(u,v)}|\,du\,dv$ is a lot easier, since R_{uv} is a square.
Continuing this example, use the fact that the determinant $\frac{\partial(u,v)}{\partial(x,y)}$ is equal to $\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$. So we take the derivatives of u and v for a determinant of -3. Take the inverse for $\frac{-1}{3}$. Thus,

$$\begin{aligned}
&= \int_{R_{xy}} (x+y)^3\,dx\,dy \\
&= \int_{R_{uv}} u^3 |\frac{\partial(x,y)}{\partial(u,v)}|\,du\,dv \\
&= \frac{-1}{3}\int_{-2}^{1}\int_{1}^{4} u^3\,du\,dv \\
&= \frac{-1}{3}\int_{-2}^{1} \frac{u^4}{4}|_1^4\,dv \\
&= \frac{-1}{3}\int_{-2}^{1} \frac{255}{4}\,dv \\
&= \frac{-255}{4}.
\end{aligned}$$

We got a negative answer because the original parallelogram was in the counter-clockwise direction, while the new mapping into the u,v plane made it clockwise.

15 Lecture 14 – 10/30

15.1 Paths

Suppose path $\pi(t) = (x(t), y(t))$. The length of $\pi := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_a^b \sqrt{(dx)^2 + (dy)^2}$.

Example 15.1. Find the circumference of a circle of radius $\alpha > 0$. Let $\Pi : [0, 2\pi] \to \mathbb{R}^2, \Pi(t) = \alpha \cdot \cos(t), \alpha \cdot \sin(t)$. The circumference can then be calculated as the length of the path, $l = \int_0^{2\pi} \sqrt{(\alpha^2 \sin^2(t) + \alpha^2 \cos^2(t))}\,dt = \int_0^{2\pi} \alpha \sqrt{\sin^2(t) + \cos^2(t)}\,dt = \alpha \int_0^{2\pi} 1\,dt = 2\pi\alpha$.

This works in more than 2 dimensions as well. In $\mathbb{R}^3$, the length of π is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$.
Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function written as $y = f(x)$. This means that the path *is* the graph of f. Then the length of the graph $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \sqrt{1 + f'(t)^2}\,dt$, since $x(t) = x$.

15.2 Surface Area

Find the surface area of the graph of $f : \mathbb{R}^2 \to \mathbb{R}$, where $z = f(x, y)$ is defined on a region $R \subseteq \mathbb{R}^2$. The surface area is $\iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\, dx\, dy$.

Example 15.2. Find the surface area of a sphere of radius α. The upper hemisphere is the graph of the function $z = f(x, y) = \sqrt{\alpha^2 - x^2 - y^2}$ on the disk $D := \{(x, y) \mid x^2 + y^2 \leq \alpha^2\}$. We find the surface area with the equation

$$S = 2 \iint_D \sqrt{1 + (-x(\alpha^2 - x^2 - y^2)^{-\frac{1}{2}})^2 + (-y(\alpha^2 - x^2 - y^2)^{-\frac{1}{2}})^2}\, dx\, dy.$$

We have a coefficient of 2 because z only gives us the top hemisphere, while we want the whole sphere's. We simplify this into

$$S = 2 \iint_D \sqrt{\frac{\alpha^2}{\alpha^2 - x^2 - y^2}}\, dx\, dy.$$

In order to identify the region, we need to parameterize the disk D into polar coordinated. Thus $D = R_{r\theta} = \{(r\cos\theta, r\sin\theta) \mid 0 \leq r \leq \alpha, 0 \leq \theta \leq 2\pi\}$. Doing so transforms our equation into

$$S = 2 \iint_{R_{r\theta}} \sqrt{\frac{\alpha^2}{\alpha^2 - (r\cos\theta)^2 - (r\sin\theta)^2}} \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\, d\theta.$$

Identify our region and simplify:

$$\begin{aligned} S &= 2 \int_0^{2\pi} \int_0^{\alpha} \sqrt{\frac{\alpha^2}{\alpha^2 - r^2}}\, r\, dr\, d\theta \\ &= 2\alpha \int_0^{2\pi} \int_0^{\alpha} \sqrt{\frac{1}{\alpha^2 - r^2}}\, r\, dr\, d\theta \\ &= 2\alpha \int_0^{2\pi} -\sqrt{\alpha^2 - r^2}\, \Big|_0^{\alpha}\, d\theta \\ &= 2\alpha \int_0^{2\pi} \alpha\, d\theta \\ &= 2\alpha \cdot 2\pi \cdot \alpha \\ &= 4\pi\alpha^2. \end{aligned}$$

16 Lecture 15 – 11/6

A path could also be the sum of two functions $P(x, y)$ and $Q(x, y)$. Then, it would be $\pi(t) = P(x, y)\,dx + Q(x, y)\,dy$. If $P(x, y) = y^2$ and $Q(x, y) = x^2$, we can find the line integral of a differential form using $\int_C y^2\,dx + x^2\,dy$. If C were over the points $(0, -1)$ to $(0, 1)$, with parametrized $(x(t), y(t)) = (\cos t, \sin t)$ on the range $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then we'd get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t \sin t\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \cos t\,dt$. This simplifies into $\frac{4}{3}$.

Different paths between the same endpoints could yield different results. For example, if I kept the same endpoints from above, but had $(x(t), y(t)) = (0, t)$, it would be a straight line from $(0, -1)$ to $(0, 1)$, with a line integral of 0.

Example 16.1. Given the function $y = (x-1)^2$, find $\int_C (x^2 - y^2)\,dx$ from $(1, 0)$ to $(2, 1)$.
First, we find the parametrized equations. We would do on $t \in [0, 1]$ for $(x(t), y(t)) = (1 + t, t^2)$. This yields

$$\int_C (x^2 - y^2)\,dx = \int_0^1 ((1+t)^2 - t^4)x'(t)\,dt = \frac{32}{15}.$$

You may run into questions that require both dx and dy. This will be represented as $\int_C f(x, y)\,ds := \int f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.

16.1 Closed curves

Closed curves with a distinction in the direction of the curve is notated as $\oint_C P\,dx + Q\,dy$ for a positive direction and $\oint_C P\,dx + Q\,dy$ for negative.

Example 16.2. For $\oint_C y^2\,dx + y\,dy$, with endpoints $(1, 1), (-1, 1), (-1, -1), (1, -1)$, we can add up each individual line integral to get a total of 0.

17 Lecture 16 – 11/11

Theorem 17.1. Green's Theorem: If we have a region in $\mathbb{R}^2$ and if $P(x, y), Q(x, y)$ are of type C^1 (continuous and at least once differentiable), then

$$\oint_C P(x, y)\,dx + Q(x, y)\,dx = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Example 17.2. Let C be the curve $x^2 + y^2 = 1$ over the region $R := \{(x, y) \mid x^2 + y^2 \leq 1\}$. Use Green's/Stokes's Theorem to evaluate the line integral in the vector field: $\oint_C (4xy^3\,dx + 6x^2y\,dy)$.
We can parameterize this region with $(x(t), y(t)) = (\cos t, \sin t)), t \in [0, 2\pi]$. This gives us a solvable equation of

$$\begin{aligned} &= \int_0^{2\pi} (4\cos t \sin^3 t(-\sin t)\,dt + 6\cos^2 t \sin t \cos t\,dt) \\ &= \int_0^{2\pi} (-4\cos t \sin^4 t + 6\cos^3 t \sin t)\,dt. \end{aligned}$$

But if we instead used Green's Theorem to find the solution, then we know that $\frac{\partial P}{\partial x} = 12xy^2$ and $\frac{\partial Q}{\partial y} = 12xy$, so the integrand is $12xy(1 - y)$, which is odd in x over the symmetric region R. Therefore, the integral (and thus the line integral) is 0.

Example 17.3. Given curve $C : x^2 + 4y^2 = 4$, find $\oint_C (2x - y)\,dx + (x + 3y)\,dy$.
Using Green's,

$$\begin{aligned} &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \\ &= \iint_R (1 - (-1))\,dx\,dy \\ &= 2 \cdot \text{Area(R)} \end{aligned}$$

Since C creates an R that's an ellipse, we know that the area is $2ab$ where a and b are the lengths of the axes, giving us a final answer of $= 2 \cdot 2\pi = 4\pi$.

Example 17.4. Given $C : x^2 + y^2 = 1$, find $\oint_C \frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$. If we tried using Green's, it would suck. That's because it's not C^1. Therefore, Green's would not work.

Example 17.5. Given C as the boundary of the square with points $(1, 1), (-1, 1), (-1, -1), (1, -1)$, find $\oint_C (x^2 + 2y^2)\,dx$.
Using Green's:

$$\begin{aligned} &= \iint_R -4y\,dx\,dy \\ &= -4 \iint_R y\,dx\,dy \\ &= -4y \text{ (there's supposed to be a bar over the y, representing the "moment")} \\ &= 0. \end{aligned}$$

18 Lecture 17 – 11/13

Given vector field $\vec{F} = P\vec{i} + Q\vec{j}$ in $\mathbb{R}^2$, we can define the **circulation** of $\vec{F}$ along a closed curve C as $\oint_C \vec{F} \cdot d\vec{r} = \iint_R curl(\vec{F})\, dA$, where $curl(\vec{F}) := \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. This is Green's Theorem.
Then we did a proof of Green's but like I'm not writing that xd.

19 Lecture 18 – 12/2

19.1 Fourier Series

Given a range $[-\pi, \pi]$, we can define a function $f : [-\pi, \pi] \to \mathbb{R}$ as $C([-\pi, \pi]) = \{f : [-\pi, \pi] \to \mathbb{R} \mid f \text{ is piecewise "smooth"}\}$.
In finite dimensional spaces, we find the inner product of two vectors using $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i$.
In infinite dimensional spaces, we can define the inner product of two functions $f, g \in C([-\pi, \pi])$ as $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$.
Thus, we can find $||f||^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx$.

Definition 19.1. An orthornmal basis for $C([-\pi, \pi])$ with $e_1, e_2, \ldots, e_n, \ldots,$

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So with functions $\cos nx$ and $\sin nx$, and $n = 1, 2, \ldots,$ we can find the orthornmal basis for $C([-\pi, \pi])$ as

$$\langle \cos nx, \cos mx \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\langle \sin nx, \sin mx \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\langle \cos mx, \sin nx \rangle = 0$$

Thus, for any $f \in C([-\pi, \pi])$, we consider the infinite Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \langle f(x), \cos nx \rangle$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \langle f(x), \sin nx \rangle$. This infinite series is called the **Fourier Series** of f.

Example 19.2. Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \pi \\ -1 & \text{if } -\pi \leq x < 0 \end{cases}$. Find the Fourier expansion of f.

Since the Fourier series just results in finding the Fourier coefficients, we just need to find $a_n = \langle f, \cos nx \rangle$ and $b_n = \langle f, \sin nx \rangle$ where $n \geq 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \left(\int_0^{\pi} 1 \cdot \cos nx \, dx + \int_{-\pi}^{0} -1 \cdot \cos nx \, dx \right) \\ &= \frac{1}{\pi} \left(\frac{\sin nx}{n} \Big|_0^{\pi} - \frac{\sin nx}{n} \Big|_{-\pi}^{0} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \left(\int_0^{\pi} 1 \cdot \sin nx \, dx + \int_{-\pi}^{0} -1 \cdot \sin nx \, dx \right) \\ &= \frac{1}{\pi} \left(-\frac{\cos nx}{n} \Big|_0^{\pi} + \frac{\cos nx}{n} \Big|_{-\pi}^{0} \right) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Taking these coefficients, we find that the Fourier series of f is

$$\begin{aligned} f(x) &= \frac{0}{2} + \sum_{n=1}^{\infty} (0 \cdot \cos nx + b_n \sin nx) \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}. \end{aligned}$$

20 Lecture 19 – 12/4

Example 20.1. Find the Fourier series of $f(x) = \sin x \cos x$.
First, recognize that $2 \cdot f(x) = 2\sin x \cos x = \sin 2x$. Thus, $f(x) = \frac{1}{2}\sin 2x$. From this, we can see that $a_n = 0$ for all $n \geq 0$ and $b_n = \frac{1}{2}$ if $n = 2$ and 0 otherwise. Therefore, the Fourier series of f is

$$\begin{aligned} f(x) &= \frac{0}{2} + \sum_{n=1}^{\infty}(0 \cdot \cos nx + b_n \sin nx) \\ &= \frac{1}{2}\sin 2x. \end{aligned}$$

21 Lecture 20 – 12/9

Example 21.1. Find the Fourier Series of $f(x) = \begin{cases} -x & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x \leq \pi \end{cases}$.

Since $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, we need to find a_0, a_n, b_n.
First, we find a_n:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi n^2}\int_0^{\pi}((nx)\cos nx)\,d(nx) \\ &= \frac{2}{\pi n^2}((nx)\sin nx\,|_0^{\pi} - \int_0^{\pi}\sin nx\,d(nx)) \\ &= \frac{2}{\pi n^2}(\pi n\sin n\pi + \cos nx\,|_0^{\pi}) \\ &= \frac{2}{\pi n^2}(\pi n\cdot 0 + (-1)^n - 1) \\ &= \frac{2}{\pi n^2}((-1)^n - 1). \end{aligned}$$

Use this to find a_0:

$$\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

Next, we find b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= 0. \end{aligned}$$

Thus, the Fourier series of f is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$