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## **CAS MA411 ADVANCED CALCULUS**

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**Fall 2025**

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# 1 Lecture 1 – 9/2

## 1.1 LinAlg Review

Essentially geometry.

### 1.1.1 $\mathbb{R}^2$

A vector that points from  $(0, 0)$  to  $(x, y)$  has a length  $\|\vec{v}\| = \sqrt{x^2 + y^2}$ . This utilizes the Pythagorean Theorem, which can be proved using a geometric proof.

Now, if we have a vector that points from  $(0, 0)$  to  $(a, b)$ , and another that points from  $(0, 0)$  to  $(-b, a)$ , how do we know that the angle between these two vectors is  $90^\circ$ ? We can use the dot product, where if  $\vec{v} = (a, b)$  and  $\vec{w} = (c, d)$ , then  $\vec{v} \cdot \vec{w} = ac + bd$ . Knowing this, the dot product between the two aforementioned vectors is  $-ba + ab$ , or 0. This means the angle between them is  $90^\circ$ , or there is no projection of one vector onto the other, or they are perpendicular to each other.

### 1.1.2 $\mathbb{R}^3$

Given  $\vec{x} = (x_1, x_2, x_3)$  and  $\vec{y} = (y_1, y_2, y_3)$ , the dot product between these is  $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3$ .

### 1.1.3 $\mathbb{R}^n$

Given  $\vec{x} = (x_1, x_2, \dots, x_n)$  and  $\vec{y} = (y_1, y_2, \dots, y_n)$  and that  $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ ,  $\vec{x} \perp \vec{y} \iff \|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$ . (To visualize, if  $n = 2$ , then the length of the hypotenuse above the vectors is  $\|\vec{y} - \vec{x}\|$ .)

Extending this dialogue on dot products, we can further identify it by using the law of cosines:  $\|\vec{x}\| \cdot \|\vec{y}\| \cos\theta = \vec{x} \cdot \vec{y}$ . (To visualize, draw  $\vec{v}$  and  $\vec{y}$  with an angle  $\theta$  between them.) Prove this for homework.

# 2 Lecture 2 – 9/4

## 2.1 Basics Review

**Definition 2.1.** A **function**  $(f, D, X)$  is defined as:  $f$  is a rule that maps set  $D$  onto  $X$ , or  $D \xrightarrow{f} X$ .

**Example 2.2.** Given  $f(x) = x^2$ :

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

is a surjective function (compared to an injective function, where everything is mapped one-to-one).

**Definition 2.3.** A **transformation**  $T$  is a special geometric function.

A transformation  $T$  that is  $(x, y) \mapsto x^2 + y^2$  is a  $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ , or  $T(x, y) = x^2 + y^2$ .

## 2.2 Linear Functions (Linear Mappings/Linear Transformations)

**Definition 2.4.** We say that a function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear if and only if for every  $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$  and every  $c_1, c_2 \in \mathbb{R}$  we have  $T(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1T(\vec{x}_1) + c_2T(\vec{x}_2)$

To describe a linear function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we need only to know  $T(\vec{e}_i)$ , where  $i = 1, \dots, n$ .

**Example 2.5.** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is linear, and  $T(\vec{e}_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ,  $T(\vec{e}_1) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, T(\vec{e}_1) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \text{ what is } T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}?$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3\vec{e}_1 + 2\vec{e}_2 + 1\vec{e}_3$$

$$T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3T(\vec{e}_1) + 2T(\vec{e}_2) + 1T(\vec{e}_3) = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 5 \\ -1 \end{pmatrix} =$$

$$\begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

Note: We can rewrite a lot of this using a matrix  $A = \begin{pmatrix} 2 & -1 & 5 \\ 3 & 0 & -1 \end{pmatrix}$ , which is the matrix associated with  $T$ .

Given a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the  $Ker(T) = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}\}$ , which says that the kernel of transformation  $T$ , or the null space of transformation  $T$ , is the set of vectors mapped to the zero vector.

**Example 2.6.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation with matrix  $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 5 \end{pmatrix}$ .

We know that  $T(\vec{x}) = A\vec{x}$  and that  $\vec{x} \in \mathbb{R}^3$ . What is the image of  $T$ ?

The image (or the range, or the column space) is the span (or the set of all possible linear combinations) of its column vectors.

### 3 Lecture 3 – 9/9

Given that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , it could also be described as  $f : D \rightarrow \mathbb{R}^n$  where  $D \subseteq \mathbb{R}^n$  is a region (an open domain, which doesn't include the boundaries) in  $\mathbb{R}^m$ .

**Definition 3.1.** A subset of  $\mathbb{R}^n$  is called a **region** if for every  $x \in D \exists$  a neighborhood of  $x$  that is contained in  $D$ .

**Example 3.2.**  $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$  is the region inside a circle, inclusive. By a **neighborhood** in the previous definition, we mean that a neighborhood of  $\vec{x}_0$  is an open circular disc centered at  $\vec{x}$ , or that  $B_\epsilon(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x} - \vec{x}_0| \leq \epsilon\}$ . This means it's not a region/open domain. But, if we changed it to  $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}$ , thus eliminating the boundary, it does become an region (every point in the region can have a neighborhood, while points on the boundary cannot have a neighborhood. This makes the former closed, while the latter is open).

**Definition 3.3.** A region  $X$  is said to be **connected** if and only if for any two points  $\vec{x}_1, \vec{x}_2 \in X$ , there is a broken line connecting  $\vec{x}_1$  to  $\vec{x}_2$ . (A broken line is just a line that can be drawn with multiple connected line segments.)

**Definition 3.4.** A function  $f : X \rightarrow \mathbb{R}^m$  where  $X$  is an open region is said to be **continuous** if  $\forall \vec{x}_0 \in X, f$  is continuous at  $\vec{x}_0$ .

If  $\forall \epsilon > 0$ , there is a  $\delta > 0$  such that  $f(B_\delta(\vec{x}_0)) \subseteq B_\epsilon(f(\vec{x}_0))$ . This is the delta-epsilon definition of continuity.

We say  $f$  is continuous at  $\vec{x}_0 \iff \lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$ . We say  $\lim_{x \rightarrow x_0} = L \in \mathbb{R}$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that  $f(B_\delta(\vec{x}_0)) \subseteq B_\epsilon(L)$ .

$$\frac{\partial f}{\partial x}(\vec{x}_0) = \lim_{\Delta \rightarrow 0} \frac{f(\vec{x}_0 + \Delta x) - f(\vec{x}_0)}{\Delta x}$$

### 4 Lecture 4 – 9/11

Use  $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$  as a concept and develop the Jacobian Matrix.

Given  $y = f(x_1, x_2, \dots, x_n)$ , then  $dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \dots + \frac{\partial f}{\partial x_n}dx_n$ .

Now suppose we have a collection of functions, like  $y_1 = f_1(x_1, \dots, x_n), y_2 = f_2(x_1, \dots, x_n), \dots, y_m = f_m(x_1, \dots, x_n)$ .

If we differentiate each of those functions like we did two sentences ago, we end up with  $d\vec{y} = \begin{pmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_m \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$ , or the Jacobian Matrix applied to the  $d\vec{x}$  vector.

**Example 4.1.** Given

$$\begin{aligned}y_1 &= x_1^2 + x_2^2 - x_3^2 \\y_2 &= x_1^2 - x_2^2 + x_3^2 \\y_3 &= -x_1^2 + x_2^2 + x_3^2\end{aligned}$$

Then the Jacobian matrix is

$$J = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & -2x_3 \\ 2x_1 & -2x_2 & 2x_3 \\ -2x_1 & 2x_2 & 2x_3 \end{pmatrix}.$$

So  $d\vec{y} = J \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$ , at which it becomes a matter of algebra.

## 5 Lecture 5 – 9/16

**Example 5.1.** Given  $z = f(x, y)$ ,  $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ . Now suppose  $x$  and  $y$  are parameterized:  $x = x(t)$ ,  $y = y(t)$ . Then  $z = z(t) = f(x(t), y(t))$ , thus creating  $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ . This is the **chain rule**.

The principle of the chain rule can always extend itself, like if  $x = x(u_1, u_2)$  and  $y = y(u_3, u_4)$  and  $z = z(x, y)$ , then  $\frac{\partial z}{\partial u_1} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u_1} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u_1}$  aka  $\frac{\partial z}{\partial u_1} = \left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) \begin{pmatrix} \frac{\partial x}{\partial u_1} \\ \frac{\partial y}{\partial u_1} \end{pmatrix}$  and on.

## 6 Lecture 6 – 9/18

Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  as a differentiable function, we obtain two functions:  $f(\vec{x}_0) = \begin{pmatrix} y_1(\vec{x}_0) \\ \vdots \\ y_m(\vec{x}_0) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ , which is just the application of the function onto a vector, as well as  $f(\vec{x}_0 + d\vec{x}) \sim f(\vec{x}_0) + D_f(x_0)d\vec{x}$ , which uses the Jacobian.

**Example 6.1.** Given  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$  where

$$\begin{aligned} y_1 &= x_1^2 + x_2^2 - x_3^2 \\ y_2 &= x_1^2 - x_2^2 + x_3^2 \\ y_3 &= -x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

Then,  $D_f$  is the Jacobian matrix, which is  $\mathbb{R}^3 \rightarrow Matrix_{3 \times 3}(\mathbb{R})$ , and the result is  $D_f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 & -2x_3 \\ 2x_1 & -2x_2 & 2x_3 \\ -2x_1 & 2x_2 & 2x_3 \end{pmatrix}$ .

With just function application, let's use  $\vec{x}_0 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ . We get  $f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$ .

Applying the Jacobian gives  $D_f(\vec{x}_0) = D_f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ 4 & -2 & 2 \\ -4 & 2 & 2 \end{pmatrix}$ . Plugging this into the equation for differentiation, we get  $f \left( \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + d\vec{x} \right) \sim f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + D_f \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} d\vec{x}$ , or  $f \left( \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + d\vec{x} \right) \sim$

$\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 & 2 & -2 \\ 4 & -2 & 2 \\ -4 & 2 & 2 \end{pmatrix} d\vec{x}$ . We can now use this to calculate any small change away from  $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ .

Using the above concept, imagine a cube centered at  $\vec{x}_0$ . Applying a Jacobian  $D_f$  onto the cube maps it into a region that seems like a parallelepiped that contains  $f(\vec{x}_0)$ . The volume of this new parallelepiped can be calculated using  $|\det(\vec{v})|$ , where  $\vec{v}$  are the transformed vectors on the edges from one corner of the original cube (aka the vectors that span the parallelepiped).

## 7 Lecture 7 – 9/23

Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the graph of  $f$  is  $\{(\vec{x}, f(\vec{x})) \mid \vec{x} \in \mathbb{R}^n\}$ . With this,  $\vec{x}$  could be a vector of functions,

such that the resulting vector  $\vec{y} = f(\vec{x}) = f \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1(\vec{x}) \\ \vdots \\ y_m(\vec{x}) \end{pmatrix}$ , where  $D_f(\vec{x}) = \vec{y}_{\vec{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$ .

This is significant because of  $f(\vec{x}) + df(\vec{x}, d\vec{x}) = f(\vec{x}) + \vec{y}_{\vec{x}}d\vec{x}$ . This allows us to come to the key point that  $(\vec{x}, f(\vec{x}) + df(\vec{x}, d\vec{x})) = (\vec{x}, f(\vec{x}) + \vec{y}_{\vec{x}}d\vec{x})$ .

**Example 7.1.** Given

$$\begin{aligned} y_1 &= u_1 u_2 - u_1 u_3 \\ y_2 &= u_1 u_3 + u_2^2 \\ u_1 &= x_1 \cos x_2 + (x_1 - x_2)^2 \\ u_2 &= x_1 \sin x_2 + x_1 x_2 \\ u_3 &= x_1^2 - x_1 x_2 + x_2^2 \end{aligned}$$

and knowing that

$$\vec{y}_{\vec{u}} = \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \frac{\partial y_1}{\partial u_3} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \frac{\partial y_2}{\partial u_3} \\ \frac{\partial y_3}{\partial u_1} & \frac{\partial y_3}{\partial u_2} & \frac{\partial y_3}{\partial u_3} \end{pmatrix}$$

we can come to the matrix  $\vec{y}_{\vec{u}} = \begin{pmatrix} u_2 - u_3 & u_1 & -u_1 \\ u_3 & 2u_2 & u_1 \\ \cos x_2 + 2(x_1 - x_2) & -\sin x_2 - 2(x_1 - x_2) \\ \sin x_2 + x_2 & x_1 \cos x_2 + x_1 \\ 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix}$  with and with the same process for the rest of the functions, we can get the matrix  $\vec{u}_{\vec{x}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

Take  $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , then  $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ .

So, for  $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\vec{y}_{\vec{u}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ , and  $\vec{u}_{\vec{x}} = \begin{pmatrix} 3 & -2 \\ 0 & 2 \\ 2 & -1 \end{pmatrix}$ .

Thus, finally, that means  $\vec{y}_{\vec{x}} = \begin{pmatrix} -7 & 8 \\ 7 & -4 \end{pmatrix}$ , the product of  $\vec{y}_{\vec{u}}$  and  $\vec{u}_{\vec{x}}$ .

**Definition 7.2.** Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , function  $f$  is **affine** if  $f(\vec{x}) = A\vec{x} + \vec{b}$ , where  $\vec{b} \in \mathbb{R}^m$ .

## 8 Lecture 8 – 9/25

Given  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \begin{cases} 2x^2 + y^2 - z^2 = 3 \\ xyz + 2x^2z + 3xz^2 = 1 \end{cases}$ , this says  $z(x, y)$  is a function where  $2x + y^2 - z(x, y)^2 = 3$  and  $xy \cdot z(x, y) + 2x^2 \cdot z(x, y) + 3x(z(x, y)^2) = 1$ .

## 9 Lecture 9 – 9/30

Suppose we have two functions  $2x + y - 3z - 2u = 0$  and  $x + 2y + z + u = 0$ . If we're asked to solve for  $x, u$  in terms of  $y, z$ , that's  $(\frac{\partial x}{\partial y})_z$  and  $(\frac{\partial u}{\partial z})_y$ . Take the differential of the first equation to get  $2dx + dy - 3dz - 2du = 0$ . Do the same for the second equation. Then rewrite both in the terms of the question:

$$\begin{aligned} 2dx - 2du &= -dy + 3dz \\ dx + du &= -2dy - dz. \end{aligned}$$

This can be rewritten as a matrix equation such that  $\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} dx \\ du \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} dy \\ dz \end{pmatrix}$ . To solve for the question,  $\begin{pmatrix} dx \\ du \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} dy \\ dz \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} dy \\ dz \end{pmatrix}$ . This gives us the final equations

$$\begin{aligned} dx &= -\frac{5}{4}dy + \frac{1}{4}dz \\ du &= -\frac{3}{4}dy - \frac{5}{4}dz \end{aligned}$$

**Example 9.1.** Given  $zx^2 + y^2 + z^3 - 1 = 0$  we can solve this equations for  $z$  in terms of  $x + y$ .

Let  $z = z(x, y)$ . Then  $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ .

$2xz \cdot dx + x^2 \cdot dz + 2y \cdot dy + 3z^2 \cdot dz = 0$ . We can solve for  $dz = -\frac{2xz}{x^2+3z^2}dx - \frac{2y}{x^2+3z^2}dy$ . That means  $\frac{\partial z}{\partial x} = -\frac{2xz}{x^2+3z^2}$  and  $\frac{\partial z}{\partial y} = -\frac{2y}{x^2+3z^2}$ .

This is different from the above example, because the last one had the partials as literal numbers, while this has functions.

## 10 Discussion 10/1

**Theorem 10.1.** Consider the simultaneous equations  $F(\vec{x}, \vec{u}) \in \vec{0}, \vec{x} \in \mathbb{R}^n, \vec{u} \in \mathbb{R}^m, F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ . Let  $(\vec{x}_0, \vec{u}_0)$  be a point at which  $F(\vec{x}_0, \vec{u}_0) = \vec{0}, \det(F_{\vec{x}}(\vec{x}_0, \vec{u}_0)) \neq 0$ . Suppose  $F$  has continuous partial derivatives in a neighborhood of  $(\vec{x}_0, \vec{u}_0)$ . Then there exists a neighborhood  $D$  of  $\vec{u}_0$  and a unique continuously differentiable function  $f : D \rightarrow \mathbb{R}^n$  such that  $\vec{x} = f(\vec{u})$  (or  $F(f(\vec{u}), \vec{u}) = 0$ ).

## 11 Lecture 10 – 10/16

Given a space  $X$ , where  $X \subseteq \mathbb{R}^n$ , a path in  $X$  is a function  $\pi : [a, b] \rightarrow X \xrightarrow{x_i} \mathbb{R}$ . For  $t \in [a, b], \pi(t) \in X$ , or  $\pi(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in X$ . Thus  $\pi'(t) = (x'_1(t), x'_2(t), \dots, x'_n(t)) \in X$ .

Let gradient  $\nabla F|_{(x_0, y_0)} = (\frac{\partial F}{\partial x}|_{(x_0, y_0)}, \frac{\partial F}{\partial y}|_{(x_0, y_0)})$ .

Given function  $F(x, y)$  and space above  $(x(t), y(t), F(x(t), y(t)))$ , with  $x_0 = x(0), y_0 = y(0), \frac{d}{dt}F(x(t), y(t))|_{(x_0, y_0)} = \frac{\partial F}{\partial x}(x_0, y_0) \cdot x'(0) + \frac{\partial F}{\partial y}(x_0, y_0) \cdot y'(0) = \nabla f|_{(x_0, y_0)} \cdot (x'(0), y'(0))$ .

**Definition 11.1.** The directional derivative of  $F$  in the direction  $\frac{x'(0)}{\|x'(0)\|}$  is  $\nabla_{\vec{x}(0)}F = \nabla F \cdot \frac{x'(0)}{\|x'(0)\|}$ .

**Example 11.2.** Let  $F(x, y, z) = 2x^2 - y^2 + z^2$  and  $(x_0, y_0, z_0) = (1, 2, 3)$  and  $\vec{u} = (2, 3, -3)$ .

Find the directional derivative of  $F$  in the direction of  $\vec{u}$  starting from the point  $(x_0, y_0, z_0)$ .

Then  $\nabla_{\vec{u}}F|_{(1,2,3)} = (\frac{\partial F}{\partial x}|_{(1,2,3)}, \frac{\partial F}{\partial y}|_{(1,2,3)}, \frac{\partial F}{\partial z}|_{(1,2,3)})(\frac{1}{\sqrt{22}} \cdot (2, 3, -3))$ .

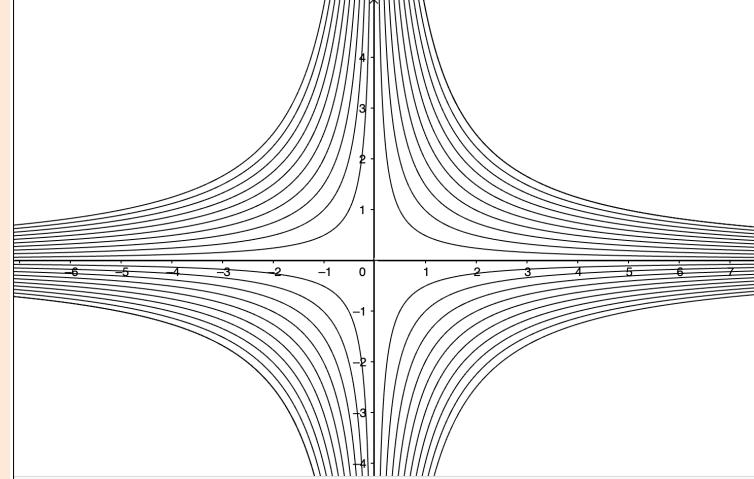
Solving this out gives  $\nabla F = (4x, -2y, 2z)|_{(1,2,3)} = (4, -4, -6)$ . Plug this into our expanded equation above gives  $(4, -4, -6) \cdot (\frac{1}{\sqrt{22}} \cdot (2, 3, -3)) = \frac{-22}{\sqrt{22}}$ .

## 12 Lecture 11 – 10/21

**Definition 12.1.** A scalar field is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , or a function that results in a single output value.

We can make **level curves** to identify how the result changes as a function of the inputs.

**Example 12.2.** Given  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  where  $f(x, y) = xy$ , the level curve is



Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . We can form the **gradient field** associated to  $f$  (assuming the function is a smooth, continuous function). We write it using  $\nabla f$ , which (for this  $f$ ) is calculated using  $\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$ . Notably, this means we took  $f$ , a *scalar field*, and turned it into a **vector field**.

**Definition 12.3. Vector fields** have multiple output values, such as  $\vec{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . In this instance,  $\vec{v}(x, y) = v_x(x, y)\vec{i} + v_y(x, y)\vec{j}$ , where the subscript denotes the component in that dimension and NOT the derivative.

Gradient vectors will always point perpendicular to any level set!

Because vector fields are a function that map multi-dimensions into multi-dimensions, we can also identify the Jacobian of a vector field:  $J(\vec{v}) = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}$ .

**Definition 12.4.** The **divergence** of a vector field is  $\text{Div}(\vec{v}) := \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$ . This is found from the *trace* of the Jacobian matrix of the vector field (found above). The trace is the diagonal of the matrix.

Physics relation: let the density of a fluid at  $(x, y)$  at time  $t$  be equal to  $\rho(t) = (x, y)$ . Then  $\text{Div}(\vec{v})|_{(x,y)} = -\rho'(t)$ .

**Definition 12.5.** The **curl** of a vector field  $\vec{v}$  in  $\mathbb{R}^3$  is  $\text{curl}(\vec{v}) = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\vec{i} - (\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z})\vec{j} +$

$(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\vec{k}$ . This is the expanded form of  $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$ .

## 13 Lecture 12 – 10/23

Some observations:

1. The curl of a gradient of a function,  $\text{curl}(\nabla f) = 0$ .
2.  $\text{div} \cdot \text{curl}(\vec{v}) = 0$

### 13.1 Integration

Given a function  $f$  defined on region  $[a, b]$  and that maps to  $\mathbb{R}$ , the integration over this region is  $\int_a^b f(x) dx$ .

Now, let's split the region into partitions, such that  $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ . The region from  $a = x_0$  to  $x_1$  is labeled as  $x_1^*$ , and we can do the same for each partition. If we define  $\Delta x_i = x_i - x_{i-1}$  for  $i = 1, \dots, n$ , then  $\int_a^b f(x) dx \sim \sum_{i=0}^n f(x_i^*) \Delta x_i$ .

**Definition 13.1.** The **mesh** is defined as  $h = \max(\Delta x_i)$ , for  $i = 1, \dots, n$ .

Given  $I \in \mathbb{R}$ , we say  $\int_a^b f(x) dx = I \iff \forall \epsilon > 0, \exists \delta > 0$  such that for any partition  $a = x_0, x_1, \dots, x_n = b$ , if the mesh  $h < \delta$ , then we have  $|\sum_{i=1}^n f(x_i^*) \Delta x_i - I| < \epsilon$ .

### 13.2 Double Integration

$$\iint_R f(x, y) dx dy, R \subseteq \mathbb{R}, f : \mathbb{R}^2 \rightarrow \mathbb{R}.$$

The mesh in two dimensions extends into... well, two dimensions, forming a grid over  $R$ . In two dimensions,  $h = \text{minimum side length of the squares in the mesh}$ .  $\iint_R f dx dy = \lim_{h \rightarrow 0} \sum_{i=1}^N f(x_i^*) \Delta x_i$ .

### 13.3 Average value of $f$ on $R$

**Definition 13.2.** Defined as

$$\frac{1}{\text{Area}(R)} \iint_R f(x, y) dx dy.$$

### 13.4 Mean Value Theorem

**Definition 13.3.** If  $f : R \rightarrow \mathbb{R}$  is continuous, then  $\exists (x_0, y_0) \in R$  such that  $f(x_0, y_0) = \frac{1}{\text{Area}(R)} \iint_R f dx dy$ .

## 14 Lecture 13 – 10/28

### 14.1 Triple Integral

$$\iiint_R f(x, y, z) dx dy dz.$$

**Example 14.1.** Say we're given a region  $R : \{0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq x + y\}$ . Given the function  $f(x, y, z) = 2x - y - z$ , we can find the volume in this region of this function with

$$\begin{aligned}
&= \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z) dz dy dx \\
&= \int_0^1 \int_0^{x^2} (2xz - yz - \frac{z^2}{2}) \Big|_0^{x+y} dy dx \\
&= \int_0^1 \int_0^{x^2} (2x(x+y) - y(x+y) - \frac{(x+y)^2}{2}) dy dx \\
&= \int_0^1 \int_0^{x^2} (\frac{3}{2}x^2 - \frac{3}{2}y^2) dy dx \\
&= \frac{3}{2} \int_0^1 (x^2y - \frac{y^3}{3}) \Big|_0^{x^2} dx \\
&= \frac{3}{2} \int_0^1 (x^4 - \frac{x^6}{3}) dx \\
&= \frac{3}{2} (\frac{x^5}{5} - \frac{x^7}{21}) \Big|_0^1 \\
&= \frac{8}{35}.
\end{aligned}$$

## 14.2 Change of Variables

**Theorem 14.2.** The **Fundamental Theorem of Calculus** says that if  $F$  is an antiderivative of  $f$ , i.e.  $F'(x) = f(x)$ , then  $\int_{x_1}^{x_2} f(x) dx = F(x) \Big|_{x_1}^{x_2} = F(x_2) - F(x_1)$ .

**Theorem 14.3.** The **Chain Rule** tells us that given  $F(x(u))$ , we can find  $\frac{d}{du}(F(x(u))) = f(x(u)) \cdot x'(u)$ , where  $F' = f$ .

### 14.2.1 One variable case

$\int_{x_1}^{x_2} f(x) dx$  can have its variables changed into  $\int_{u_1}^{u_2} f(x(u)) \frac{dx}{du} du$ .

### 14.2.2 Two variable case

$$\iint_{R_{xy}} f(x, y) dx dy = \iint_{R_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

**Example 14.4.** Given a parallelogram in  $R_{xy}$  with corners  $(0, 1), (2, 2), (3, 1), (1, 0)$ , and functions  $u = x + y$  and  $v = x - 2y$ , we find that this parallelogram gets mapped into corners  $(1, -2), (4, -2), (4, 1), (1, 1)$  in  $R_{uv}$ .

We do this because if we wanted the volume of the function  $(x + y)^3$  in the region of  $R_{xy}$ , setting up the bounds of integration would be tedious in this strange, slanted parallelogram. So changing it into  $\int_{R_{uv}} u^3 \cdot |\frac{\partial(x,y)}{\partial(u,v)}| du dv$  is a lot easier, since  $R_{uv}$  is a square.

Continuing this example, use the fact that the determinant  $\frac{\partial(u,v)}{\partial(x,y)}$  is equal to  $\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$ . So we take the derivatives of  $u$  and  $v$  for a determinant of  $-3$ . Take the inverse for  $\frac{-1}{3}$ . Thus,

$$\begin{aligned} &= \int_{R_{xy}} (x + y)^3 dx dy \\ &= \int_{R_{uv}} u^3 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv \\ &= \frac{-1}{3} \int_{-2}^1 \int_1^4 u^3 du dv \\ &= \frac{-1}{3} \int_{-2}^1 \frac{u^4}{4} \Big|_1^4 dv \\ &= \frac{-1}{3} \int_{-2}^1 \frac{255}{4} dv \\ &= \frac{-255}{4}. \end{aligned}$$

We got a negative answer because the original parallelogram was in the counter-clockwise direction, while the new mapping into the  $u, v$  plane made it clockwise.

## 15 Lecture 14 – 10/30

### 15.1 Paths

Suppose path  $\pi(t) = (x(t), y(t))$ . The length of  $\pi := \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b \sqrt{(dx)^2 + (dy)^2}$ .

**Example 15.1.** Find the circumference of a circle of radius  $\alpha > 0$ . Let  $\Pi : [0, 2\pi] \rightarrow \mathbb{R}^2$ ,  $\Pi(t) = \alpha \cdot \cos(t), \alpha \cdot \sin(t)$ . The circumference can then be calculated as the length of the path,  $l = \int_0^{2\pi} \sqrt{(\alpha^2 \sin^2(t) + \alpha^2 \cos^2(t))} dt = \int_0^{2\pi} \alpha \sqrt{\sin^2(t) + \cos^2(t)} dt = \alpha \int_0^{2\pi} 1 dt = 2\pi\alpha$ .

This works in more than 2 dimensions as well. In  $\mathbb{R}^3$ , the length of  $\pi$  is  $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function written as  $y = f(x)$ . This means that the path is the graph of  $f$ . Then the length of the graph  $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b \sqrt{1 + f'(t)^2} dt$ , since  $x(t) = x$ .

## 15.2 Surface Area

Find the surface area of the graph of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , where  $z = f(x, y)$  is defined on a region  $R \subseteq \mathbb{R}^2$ . The surface area is  $\iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dx dy$ .

**Example 15.2.** Find the surface area of a sphere of radius  $\alpha$ . The upper hemisphere is the graph of the function  $z = f(x, y) = \sqrt{\alpha^2 - x^2 - y^2}$  on the disk  $D := \{(x, y) \mid x^2 + y^2 \leq \alpha^2\}$ .

We find the surface area with the equation

$$S = 2 \iint_D \sqrt{1 + (-x(\alpha^2 - x^2 - y^2)^{-\frac{1}{2}})^2 + (-y(\alpha^2 - x^2 - y^2)^{-\frac{1}{2}})^2} dx dy.$$

We have a coefficient of 2 because  $z$  only gives us the top hemisphere, while we want the whole sphere's. We simplify this into

$$S = 2 \iint_D \sqrt{\frac{\alpha^2}{\alpha^2 - x^2 - y^2}} dx dy.$$

In order to identify the region, we need to parameterize the disk  $D$  into polar coordinates. Thus  $D = R_{r\theta} = \{(r \cos \theta, r \sin \theta) \mid 0 \leq r \leq \alpha, 0 \leq \theta \leq 2\pi\}$ . Doing so transforms our equation into

$$S = 2 \iint_{R_{r\theta}} \sqrt{\frac{\alpha^2}{\alpha^2 - (r \cos \theta)^2 - (r \sin \theta)^2}} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta.$$

Identify our region and simplify:

$$\begin{aligned} S &= 2 \int_0^{2\pi} \int_0^\alpha \sqrt{\frac{\alpha^2}{\alpha^2 - r^2}} r dr d\theta \\ &= 2\alpha \int_0^{2\pi} \int_0^\alpha \sqrt{\frac{1}{\alpha^2 - r^2}} r dr d\theta \\ &= 2\alpha \int_0^{2\pi} -\sqrt{\alpha^2 - r^2} \Big|_0^\alpha d\theta \\ &= 2\alpha \int_0^{2\pi} \alpha d\theta \\ &= 2\alpha \cdot 2\pi \cdot \alpha \\ &= 4\pi\alpha^2. \end{aligned}$$

## 16 Lecture 15 – 11/6

A path could also be the sum of two functions  $P(x, y)$  and  $Q(x, y)$ . Then, it would be  $\pi(t) = P(x, y) dx + Q(x, y) dy$ . If  $P(x, y) = y^2$  and  $Q(x, y) = x^2$ , we can find the line integral of a differential form using  $\int_C y^2 dx + x^2 dy$ . If  $C$  were over the points  $(0, -1)$  to  $(0, 1)$ , with parametrized  $(x(t), y(t)) = (\cos t, \sin t)$  on the range  $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , then we'd get  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t \sin t dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \cos t dt$ . This simplifies into  $\frac{4}{3}$ .

Different paths between the same endpoints could yield different results. For example, if I kept the same endpoints from above, but had  $(x(t), y(t)) = (0, t)$ , it would be a straight line from  $(0, -1)$  to  $(0, 1)$ , with a line integral of 0.

**Example 16.1.** Given the function  $y = (x - 1)^2$ , find  $\int_C (x^2 - y^2) dx$  from  $(1, 0)$  to  $(2, 1)$ .

First, we find the parametrized equations. We would do on  $t \in [0, 1]$  for  $(x(t), y(t)) = (1 + t, t^2)$ . This yields

$$\int_C (x^2 - y^2) dx = \int_0^1 ((1+t)^2 - t^4) x'(t) dt = \frac{32}{15}.$$

You may run into questions that require both  $dx$  and  $dy$ . This will be represented as  $\int_C f(x, y) ds := \int f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt$ .

### 16.1 Closed curves

Closed curves with a distinction in the direction of the curve is notated as  $\oint_C P dx + Q dy$  for a positive direction and  $\oint_C P dx + Q dy$  for negative.

**Example 16.2.** For  $\oint_C y^2 dx + y dy$ , with endpoints  $(1, 1), (-1, 1), (-1, -1), (1, -1)$ , we can add up each individual line integral to get a total of 0.

## 17 Lecture 16 – 11/11

**Theorem 17.1. Green's Theorem:** If we have a region in  $\mathbb{R}^2$  and if  $P(x, y), Q(x, y)$  are of type  $C^1$  (continuous and at least once differentiable), then

$$\oint_C P(x, y) dx + Q(x, y) dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

**Example 17.2.** Let  $C$  be the curve  $x^2 + y^2 = 1$  over the region  $R := \{(x, y) \mid x^2 + y^2 \leq 1\}$ . Use Green's/Stokes's Theorem to evaluate the line integral in the vector field:  $\oint_C (4xy^3 dx + 6x^2y dy)$ .

We can parameterize this region with  $(x(t), y(t)) = (\cos t, \sin t)$ ,  $t \in [0, 2\pi]$ . This gives us a solvable equation of

$$\begin{aligned} &= \int_0^{2\pi} (4 \cos t \sin^3 t (-\sin t) dt + 6 \cos^2 t \sin t \cos t dt) \\ &= \int_0^{2\pi} (-4 \cos t \sin^4 t + 6 \cos^3 t \sin t) dt. \end{aligned}$$

But if we instead used Green's Theorem to find the solution, then we know that  $\frac{\partial P}{\partial x} = 12xy^2$  and  $\frac{\partial Q}{\partial y} = 12xy$ , so the integrand is  $12xy(1-y)$ , which is odd in  $x$  over the symmetric region  $R$ . Therefore, the integral (and thus the line integral) is 0.

**Example 17.3.** Given curve  $C : x^2 + 4y^2 = 4$ , find  $\oint_C (2x - y) dx + (x + 3y) dy$ .

Using Green's,

$$\begin{aligned} &= \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R (1 - (-1)) dx dy \\ &= 2 \cdot \text{Area}(R) \end{aligned}$$

Since  $C$  creates an  $R$  that's an ellipse, we know that the area is  $2ab$  where  $a$  and  $b$  are the lengths of the axes, giving us a final answer of  $= 2 \cdot 2\pi = 4\pi$ .

**Example 17.4.** Given  $C : x^2 + y^2 = 1$ , find  $\oint_C \frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ . If we tried using Green's, it would suck. That's because it's not  $C^1$ . Therefore, Green's would not work.

**Example 17.5.** Given  $C$  as the boundary of the square with points  $(1, 1), (-1, 1), (-1, -1), (1, -1)$ , find  $\oint_C (x^2 + 2y^2) dx$ .

Using Green's:

$$\begin{aligned} &= \iint_R -4y dx dy \\ &= -4 \iint_R y dx dy \\ &= -4y \text{ (there's supposed to be a bar over the y, representing the "moment")} \\ &= 0. \end{aligned}$$

## 18 Lecture 17 – 11/13

Given vector field  $\vec{F} = P\vec{i} + Q\vec{j}$  in  $\mathbb{R}^2$ , we can define the **circulation** of  $\vec{F}$  along a closed curve  $C$  as  $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F}) dA$ , where  $\text{curl}(\vec{F}) := \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ . This is Green's Theorem.  
Then we did a proof of Green's but like I'm not writing that xd.

## 19 Lecture 18 – 12/2

### 19.1 Fourier Series

Given a range  $[-\pi, \pi]$ , we can define a function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  as  $C([-\pi, \pi]) = \{f : [-\pi, \pi] \rightarrow \mathbb{R} \mid f \text{ is piecewise "smooth"}\}$ .

In finite dimensional spaces, we find the inner product of two vectors using  $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^n u_i v_i$ .

In infinite dimensional spaces, we can define the inner product of two functions  $f, g \in C([-\pi, \pi])$  as  $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$ .

Thus, we can find  $\|f\|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$ .

**Definition 19.1.** An orthonormal basis for  $C([-\pi, \pi])$  with  $e_1, e_2, \dots, e_n, \dots$ ,

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So with functions  $\cos nx$  and  $\sin nx$ , and  $n = 1, 2, \dots$ , we can find the orthonormal basis for  $C([-\pi, \pi])$  as

$$\langle \cos nx, \cos mx \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\langle \sin nx, \sin mx \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\langle \cos mx, \sin nx \rangle = 0$$

Thus, for any  $f \in C([-\pi, \pi])$ , we consider the infinite Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \langle f(x), \cos nx \rangle$  and  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \langle f(x), \sin nx \rangle$ . This infinite series is called the **Fourier Series** of  $f$ .

**Example 19.2.** Let  $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \pi \\ -1 & \text{if } -\pi \leq x < 0 \end{cases}$ . Find the Fourier expansion of  $f$ .

Since the Fourier series just results in finding the Fourier coefficients, we just need to find  $a_n = \langle f, \cos nx \rangle$  and  $b_n = \langle f, \sin nx \rangle$  where  $n \geq 0$ .

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \left( \int_0^{\pi} 1 \cdot \cos nx dx + \int_{-\pi}^0 -1 \cdot \cos nx dx \right) \\ &= \frac{1}{\pi} \left( \frac{\sin nx}{n} \Big|_0^{\pi} - \frac{\sin nx}{n} \Big|_{-\pi}^0 \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \left( \int_0^{\pi} 1 \cdot \sin nx dx + \int_{-\pi}^0 -1 \cdot \sin nx dx \right) \\ &= \frac{1}{\pi} \left( -\frac{\cos nx}{n} \Big|_0^{\pi} + \frac{\cos nx}{n} \Big|_{-\pi}^0 \right) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Taking these coefficients, we find that the Fourier series of  $f$  is

$$\begin{aligned} f(x) &= \frac{0}{2} + \sum_{n=1}^{\infty} (0 \cdot \cos nx + b_n \sin nx) \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}. \end{aligned}$$

## 20 Lecture 19 – 12/4

**Example 20.1.** Find the Fourier series of  $f(x) = \sin x \cos x$ .

First, recognize that  $2 \cdot f(x) = 2 \sin x \cos x = \sin 2x$ . Thus,  $f(x) = \frac{1}{2} \sin 2x$ . From this, we can see that  $a_n = 0$  for all  $n \geq 0$  and  $b_n = \frac{1}{2}$  if  $n = 2$  and 0 otherwise. Therefore, the Fourier series of  $f$  is

$$\begin{aligned}f(x) &= \frac{0}{2} + \sum_{n=1}^{\infty} (0 \cdot \cos nx + b_n \sin nx) \\&= \frac{1}{2} \sin 2x.\end{aligned}$$

## 21 Lecture 20 – 12/9

**Example 21.1.** Find the Fourier Series of  $f(x) = \begin{cases} -x & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x \leq \pi \end{cases}$ .

Since  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ , we need to find  $a_0, a_n, b_n$ .

First, we find  $a_n$ :

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^\pi x \cos nx dx \\ &= \frac{2}{\pi n^2} \int_0^\pi ((nx) \cos nx) d(nx) \\ &= \frac{2}{\pi n^2} ((nx) \sin nx \Big|_0^\pi - \int_0^\pi \sin nx d(nx)) \\ &= \frac{2}{\pi n^2} (\pi n \sin n\pi + \cos nx \Big|_0^\pi) \\ &= \frac{2}{\pi n^2} (\pi n \cdot 0 + (-1)^n - 1) \\ &= \frac{2}{\pi n^2} ((-1)^n - 1). \end{aligned}$$

Use this to find  $a_0$ :

$$\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

Next, we find  $b_n$ :

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx dx \\ &= 0. \end{aligned}$$

Thus, the Fourier series of  $f$  is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$