AMTH 605 / ENAS 503 / STAT 667
Probabilistic Networks, Algorithms, and Applications

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Sept. 8, 2016 Problem Set #1

Problem Set #1 is due: **Thursday, September 22**

Problem 1: For each of the following statements, either give a proof of its correctness, or a counterexample to show incorrectness:

(a) If $X_1 \perp X_2$ then $X_1 \perp X_2 | X_3$.

(b) If $X_1 \perp X_2 | X_3$ for some X_3 then $X_1 \perp X_2$

(c) If $X_1 \perp X_2 | X_4$ and $X_1 \perp X_3 | X_4$ then $X_1 \perp (X_2, X_3) | X_4$.

(d) If $X_1 \perp (X_2, X_3) | X_4$ then $X_1 \perp X_2 | X_4$.

Problem 2: Problem 3.3 in Barber.

Problem 3: Problem 3.4 in Barber.

Problem 4: Problem 4.1 in Barber.

Problem 5: Problem 4.2 in Barber.

Problem 6: Consider an undirected tree with graph $G = (V, E)$. Let $p(x_V)$ be a distribution that is globally Markov with respect to G. We know that it is not possible in general to parameterize a distribution on a tree using marginal probabilities as the potentials. It is, however, possible to parameterize such a distribution using ratios of marginal probabilities. In particular, let:

$$\phi_i(x_i) = p(x_i) \ \forall i \in V \quad \text{and} \quad \phi_{ij}(x_i, x_j) = \frac{p(x_i, x_j)}{p(x_i)p(x_j)} \ \forall \{i, j\} \in E.$$

Show that this setting of potentials yields a parametrization of the joint probability distribution $p(x_V)$ on the tree. What is Z under this parametrization?

Problem 7: Problem 4.6 in Barber.

Problem 8: Find an optimal elimination ordering for the figure below. Let node $\star$ be the last node to be eliminated.

$$\begin{array}{ccccccc} \star & - & \bullet & - & \bullet & - & \bullet \\ | & & | & & | & & \\ \bullet & - & \bullet & - & \bullet & & \\ & & | & & | & & \\ & & \bullet & - & \bullet & & \end{array}$$

Problem 9: Problem 5.2 in Barber.

Problem 10: Give a proof by induction that the sum-product algorithm correctly computes all the marginal probabilities of the single nodes in an undirected tree.

Problem 11: Assume that we have a factor graph where each factor has at most 2 variables connected to it. Show that the sum-product message-passing rules for this factor graph simplify to the sum-product message-passing rule for the associated undirected graph.

Problem 12: Problem 6.5 in Barber.

Problem 13:(Optional) Let $\Omega = \{x_V \ : \ p(x_V) > 0\}$ represent the support of $p(x_V)$. Similarly for each $i \in V$ let $\Omega_{V \setminus \{i\}}$ represent the support of the marginal $p(x_{V \setminus \{i\}})$. We say a probability distribution $p(x_V)$ satisfies the *weak positivity condition* if there exists a realization $\overline{x}_V \in \Omega$ such that for every $i \in V$ we have

$$\{\overline{x}_i\} \times \Omega_{V \setminus \{i\}} \subseteq \Omega.$$

(a) Give an example of a distribution that is weak positive but not positive.

(b) Prove that the Hammersley-Clifford theorem continues to hold under the weak positivity condition.