| | $C^* = 100$ | $\lvert C - C^* \rvert$ | $\rho$ | $\varepsilon$ |
|---|---|---|---|---|
| min | $C = 120$ | 20 | ? | ? |

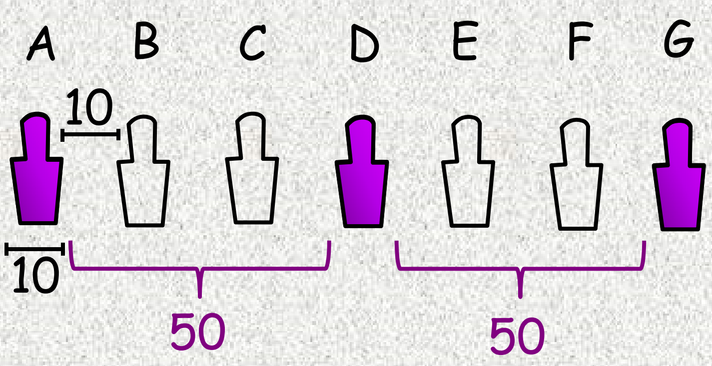
| | $C^* = 100$ | $\lvert C - C^* \rvert$ | $\rho$ | $\varepsilon$ |
|---|---|---|---|---|
| max | $C = 80$ | 20 | ? | ? |

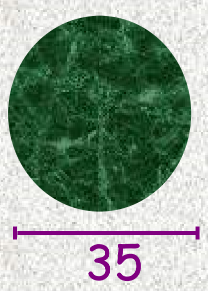
Note: $\rho \geq 1$ and $\varepsilon \geq 0$
(larger value means larger inaccuracy)



## Establish a lower bound on an optimal solution

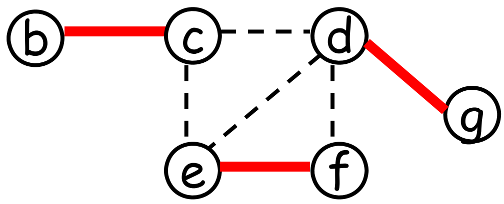
Idea: an independent set implies a lower bound




should be 21, I know

$G \supseteq A$: three disjoint edges



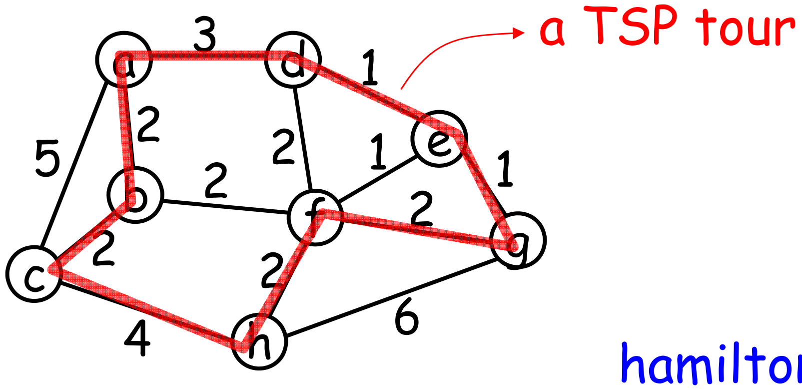

A needs at least $|A| = 3$ vertices

⇨ G needs at least $|A| = 3$ vertices

⇨ $|C^*| \geq |A|$ — ② (a lower bound on $C^*$)



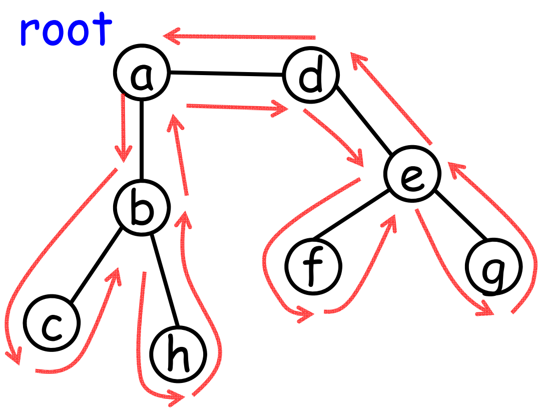
The TSP Problem




hamiltonian cycle

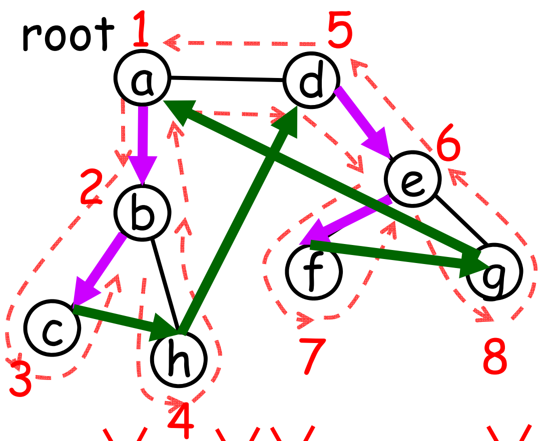
① visit each vertex exactly once

② minimum total length




full walk W=(a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)






full walk W=(a, b, c, ~~b~~, h, ~~b~~, ~~a~~, d, e, f, ~~e~~, g, ~~e~~, ~~d~~, a)

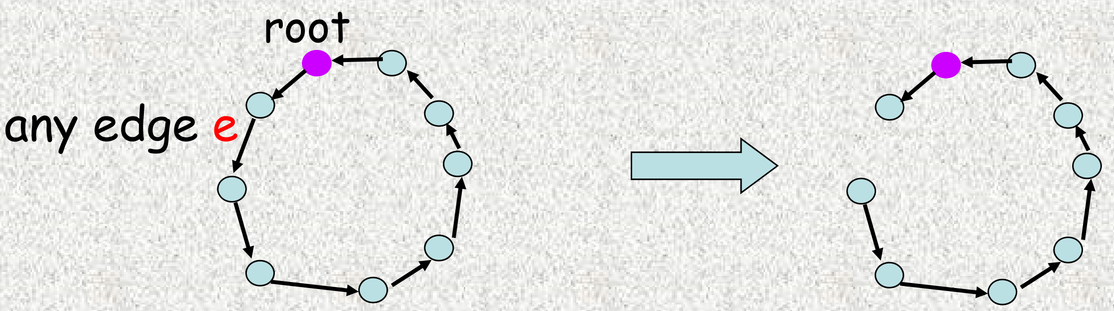
1 2 3 4 5 6 7 8 1

H=(a, b, c, h, d, e, f, g, a)

pre-order traversal on T

# Establish a lower bound on an optimal TSP tour $H^*$

Idea: an MST implies a lower bound




an optimal TSP tour $H^*$
(n edges)

a path $P = H^* - \{e\}$
(n-1 edges)

$|H^*| \geq |P|$

$|P| \geq |T|$ (since P is a tree and T is MST)

① $|H^*| \geq |T|$ (a lower bound on $H^*$)

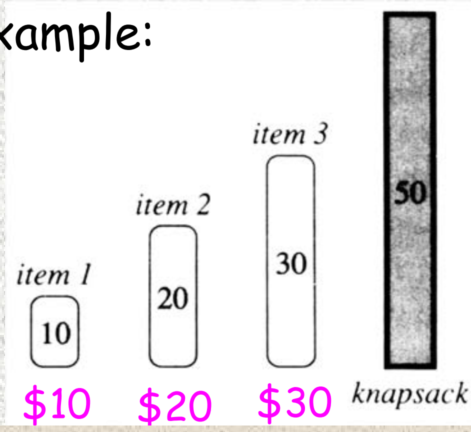


## 0-1 knapsack problem (integer):

Input: n items with weight $w_i$ and value $v_i$
capacity $C$

Output: a subset of items with
weight $\leq C$ and maximum value

Example:



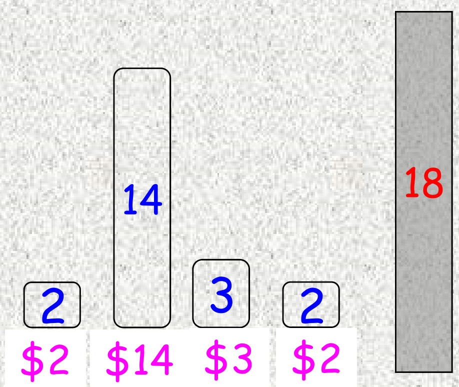

special case, in which $v_i = w_i$

## Subset-Sum problem (integer):

Input: a set of n integers $x_i$
target $t$

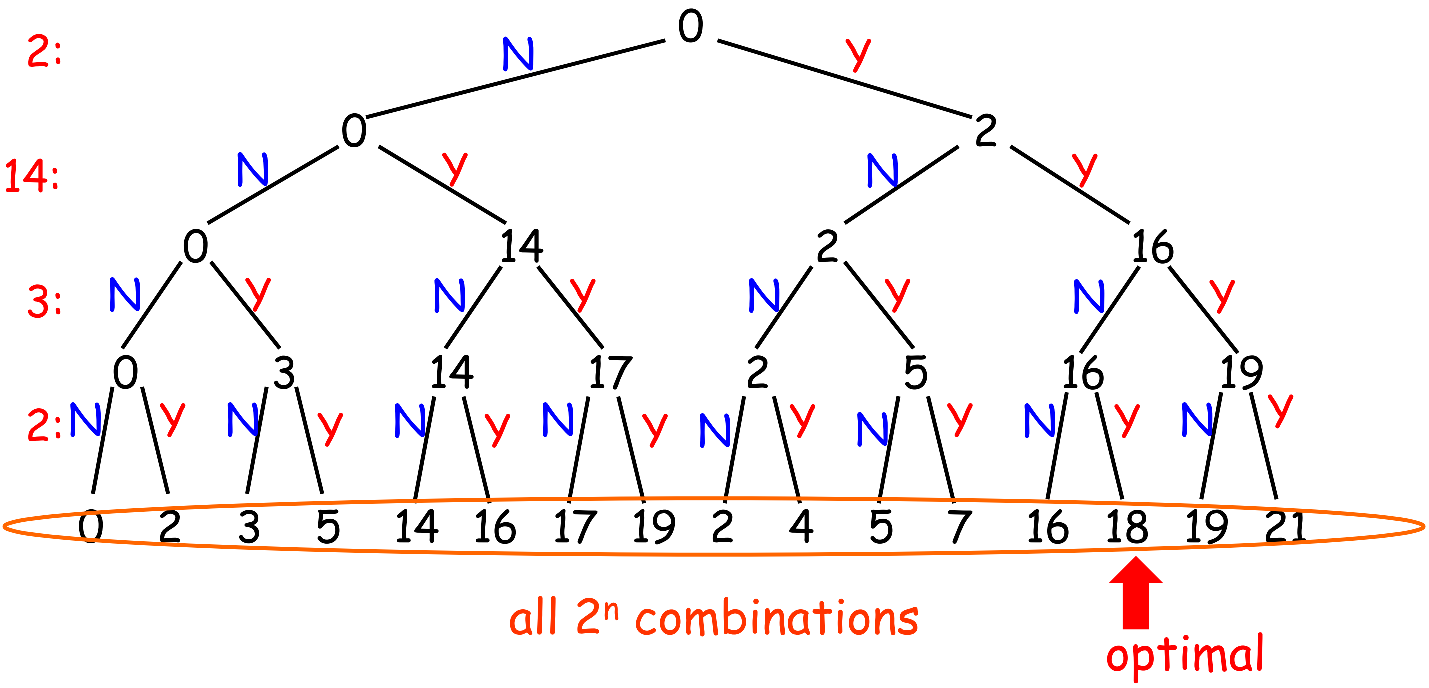
Output: a subset of integers whose
sum $\leq t$ and is maximum

Example: $S = \{2, 14, 3, 2\}$, $t = 18$



t = 18

S = (2, 14, 3, 2)

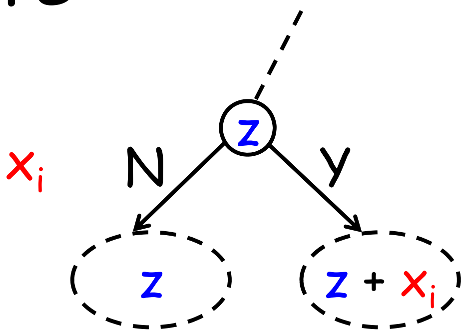




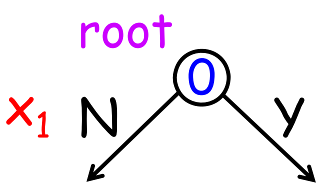

## Brute-Force-DFS

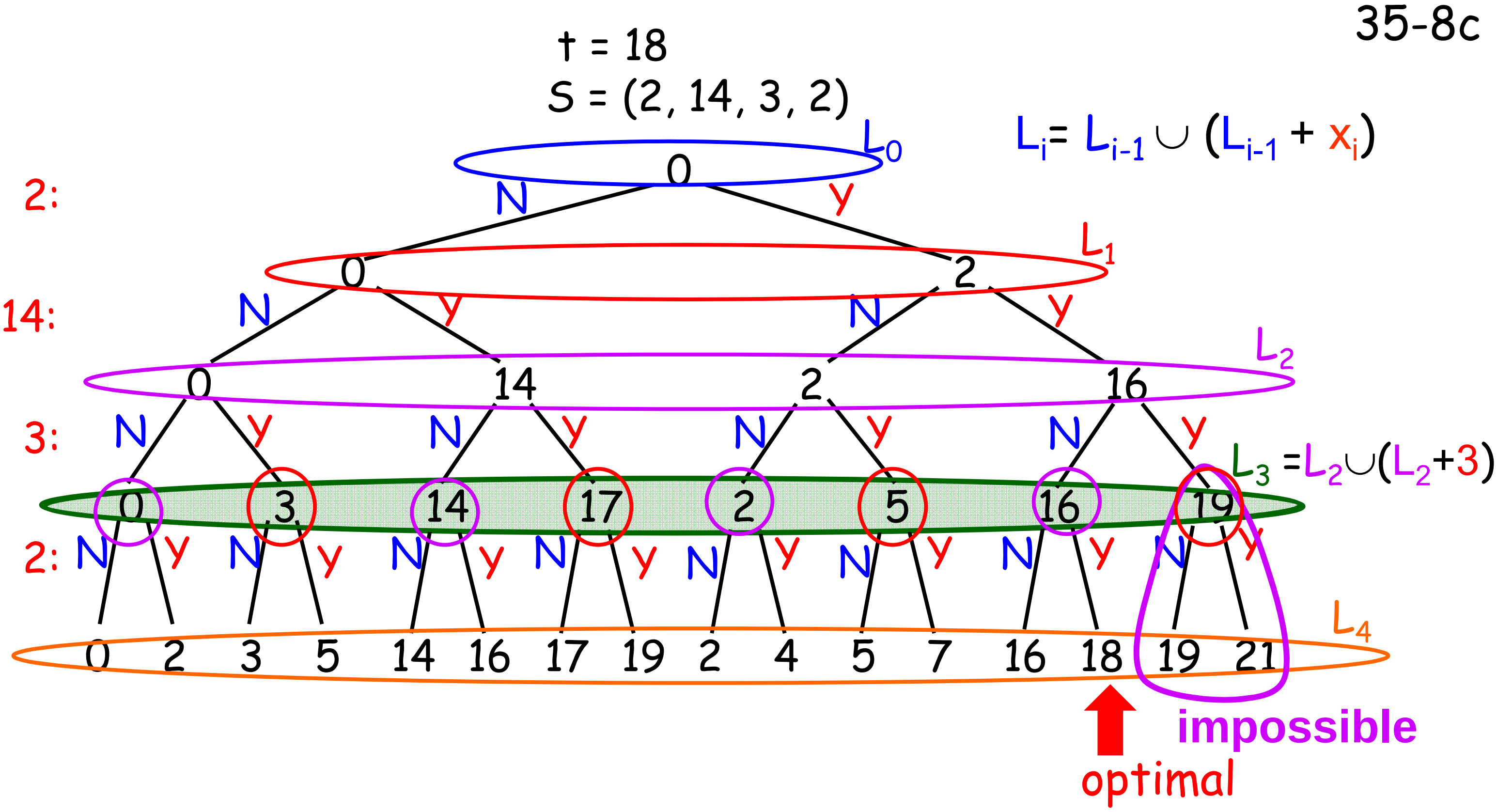
```
DFS(i, z)
begin
   if i ≤ n then
       DFS(i + 1, z)
       DFS(i + 1, z + x_i)
   else /* z is a leaf-candidate
       z* = better(z*, z)
end
```

also check valid or not

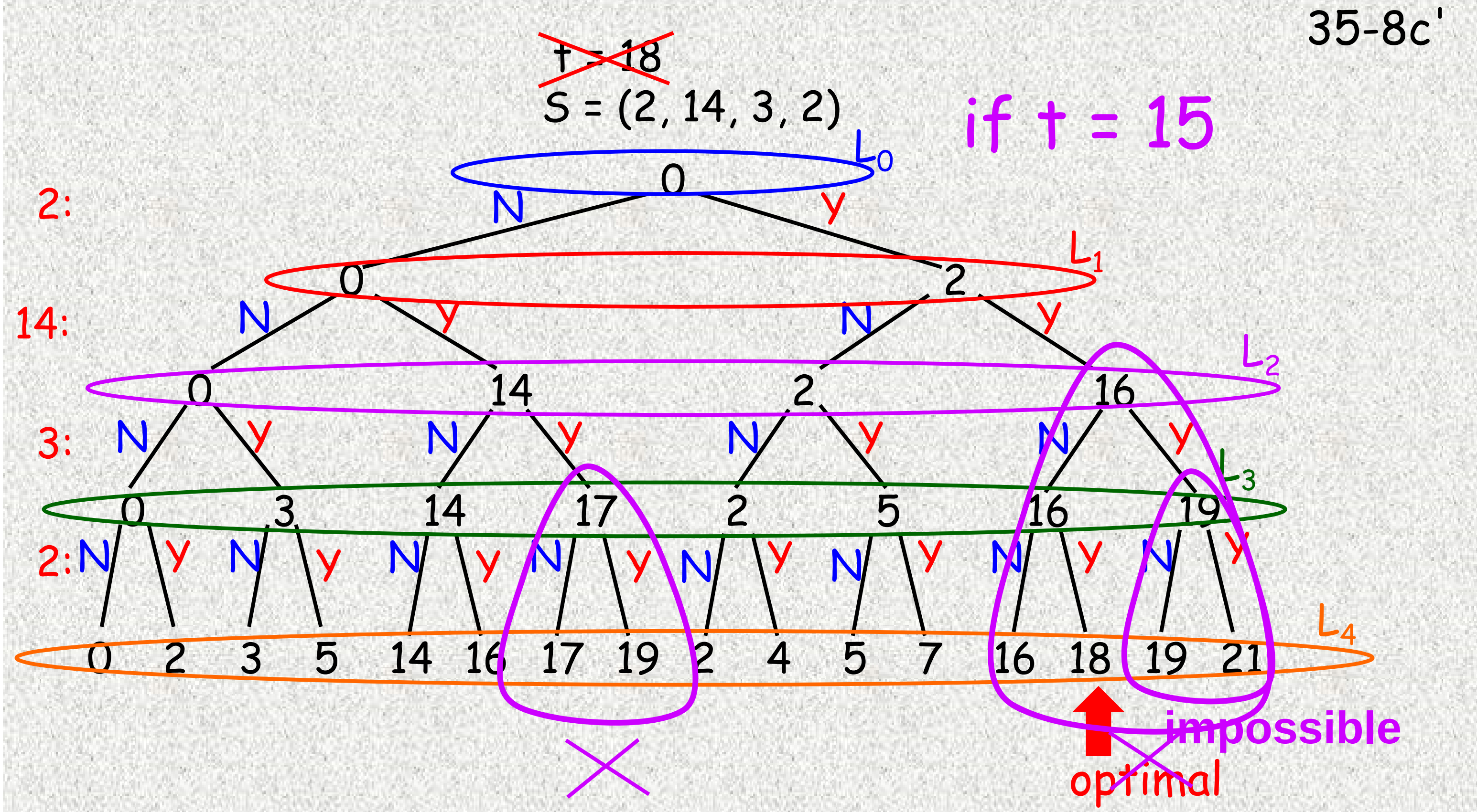



```
SubsetSum(S = (x_1, x_2, ..., x_n), t)
begin
   z* = 0
   DFS(1, 0)
   Output z*
end
```

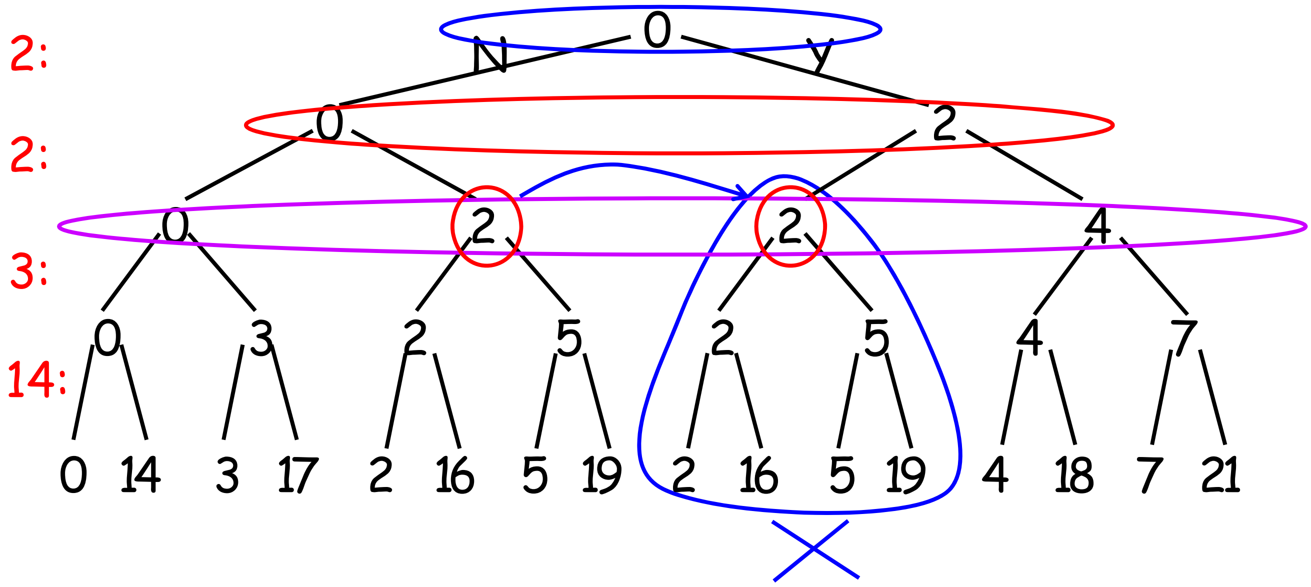



t = 18
S = (2, 14, 3, 2)
$L_i = L_{i-1} \cup (L_{i-1} + x_i)$
$L_0$
$L_1$
$L_2$
$L_3 = L_2 \cup (L_2 + 3)$
$L_4$
2:
14:
3:
2:
N
y
0
2
14
16
3
17
5
19
0 2 3 5 14 16 17 19 2 4 5 7 16 18 19 21
impossible
optimal




~~t = 18~~
S = (2, 14, 3, 2)
if t = 15
$L_0$
$L_1$
$L_2$
$L_3$
$L_4$
2:
14:
3:
2:
N
y
0 2 3 5 14 16 17 19 2 4 5 7 16 18 19 21
impossible
optimal

$S = (2, 2, 3, 14)$




---

$L_i$ = Merge-List$(L_{i-1}, x_i)$ Time: $O(2|L_{i-1}|)$

Cut I: 拿掉相同的

Example: $L_{i-1} = (0, 2, 3, 5, 14, 16, 17)$ , $x_i = 2$
(sorted)

$L_i = L_{i-1} \cup (L_{i-1} + 2)$

$(L_{i-1} + 2)$
$O(|L_{i-1}|)$

$= (0, 2, 3, 5, 14, 16, 17) \cup (2, 4, 5, 7, 16, 18, 19)$

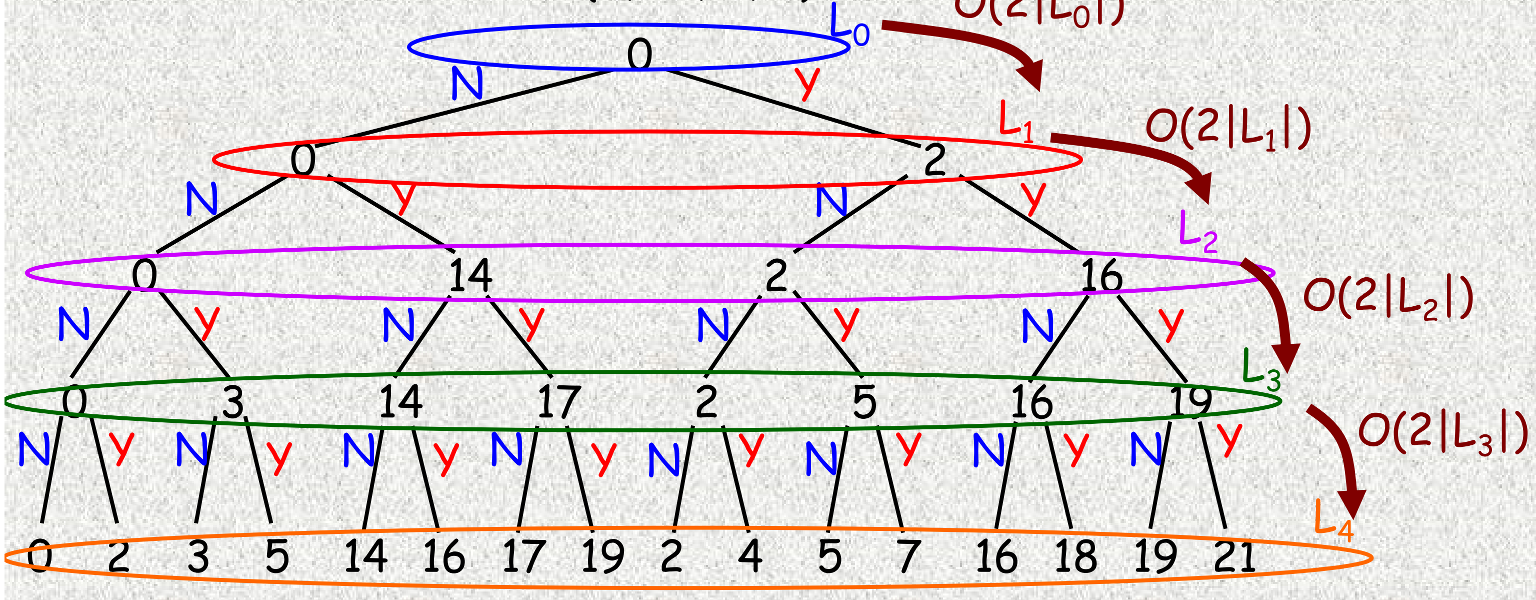
merge
$O(2|L_{i-1}|)$

$= (0, 2, \cancel{2}, 3, 4, 5, \cancel{5}, 7, 14, 16, \cancel{16}, 17, 18, 19)$

$= (0, 2, 3, 4, 5, 7, 14, 16, 17, 18, 19, 21)$
(sorted)

t = 18

S = (2, 14, 3, 2)




$$T(n) = 2|L_0| + 2|L_1| + 2|L_2| + 2|L_3| = \sum_{i=0}^{n-1} 2|L_i|$$

35-9x

35-9a

$$T(n) = \sum_{i=0}^{n-1} 2|L_i| \qquad L_i = (y_1, y_2, \ldots, y_k)$$

① $|L_i| \le 2^i$

$$T(n) = 2^0 + 2^1 + 2^2 + \ldots + 2^{n-1}$$
$$= O(2^n)$$

$L_i = (y_1 = 0, y_2 \geq 1, y_3, \; y_4, \; \ldots, y_k)$

**Effect of two cuts**: all $l_j$ are distinct integers and $t_k \leq t$

(從 0 開始，越來越大，但不得超過 t)

$\Rightarrow |L_i| \leq t + 1$

**Example**: $t = 1024 \Rightarrow |L_i| \leq 1024 + 1$

("=" when $L_i = (0, 1, 2, 3, \ldots, 1024)$)

$$\Rightarrow T(n) = \sum_{i=0}^{n-1} 2|L_i| = O(nt)$$

35-9w

35-9a

$$T(n) = \sum_{i=0}^{n-1} 2|L_i| \qquad L_i = (y_1, y_2, \ldots, \underset{\underset{100}{\wedge|}}{y_k})$$

all $y_j$ are distinct $\Rightarrow |L_i| \leq \underset{\underset{101}{\wedge|}}{y_k + 1}$ and $y_k \leq t = 100$

① $|L_i| \leq 2^i$

$T(n) = 2^0 + 2^1 + 2^2 + \ldots + 2^{n-1}$
$= O(2^n)$

② $y_k \leq t \Rightarrow |L_i| \leq t+1$

$T(n) = O(nt)$

③ $y_k \leq \overset{W}{\cancel{sum(S)}} \Rightarrow |L_i| \leq W+1$

$T(n) = O(nW)$

④ let $m = \max(S) \Rightarrow W \leq nm$

$\Rightarrow |L_i| \leq nm+1$

$T(n) = O(n \times nm) = O(n^2 m)$

T(n) is polynomial if one of t, W, m is polynomial !

$\Rightarrow$ T(n) is pseudo-polynomial! (t, W, m may be $\infty$)

## Pseudo-Polynomial:

If time is in the numeric value of an integer x, we consider x as a $\lg_2 x$-bit integer (input size)

e.g. x = 60000, s = lg x = 16 bits

Example
input: N
output: IsPrime(N)

input siz : $s = \lg_2 N$

Algorithm 1:
$O(N) = O(2^s)$
exponential in s
pseudo-polynomial

Algorithm 2:
$O(N^{1/2}) = O(2^{s/2})$
pseudo-polynomial

Example
input: a, X
output: $X^a$

input siz : $s = \lg_2 a$

Algorithm 1:
$O(a) = O(2^s)$
exponential in s
pseudo-polynomial

Algorithm 2:
$O(\lg a) = O(s)$
polynomial





## Pseudo-Polynomial:

polynomial in the numeric value of an integer
(exponential in the length (# of bits) of the integer)

The subset sum problem ($S = \{x_1, ..., x_n\}, t$)

* Consider s = lg t as the "input size" of t. (t is an s-bit integer)
  e.g. t = 60000, s = lg t = 16 bits
* $T(n) = O(nt) = O(n2^s)$ is exponential in s (pseudo-polynomial)
* $A(n) = O(n^2 \log t) = O(n^2 s)$ is polynomial (in n and s)

Examples:

pseudo-polynomial
Counting sort - O(n + k)
Knapsack - O(nC)
GCD- O(b)
$X^a$- O(a)

polynomial
GCD- O(lg b)
$X^a$- O(lg a)

$$T(n) = \sum_{i=0}^{n-1} 2|L_i|$$

① $T(n) = O(2^n)$  ③ $T(n) = O(nW)$

② $T(n) = O(nt)$  ④ $T(n) = O(n^2m)$

T(n) is polynomial if one of t, W, m is polynomial !

⇨ T(n) is pseudo-polynomial! (t, W, m may be ∞)

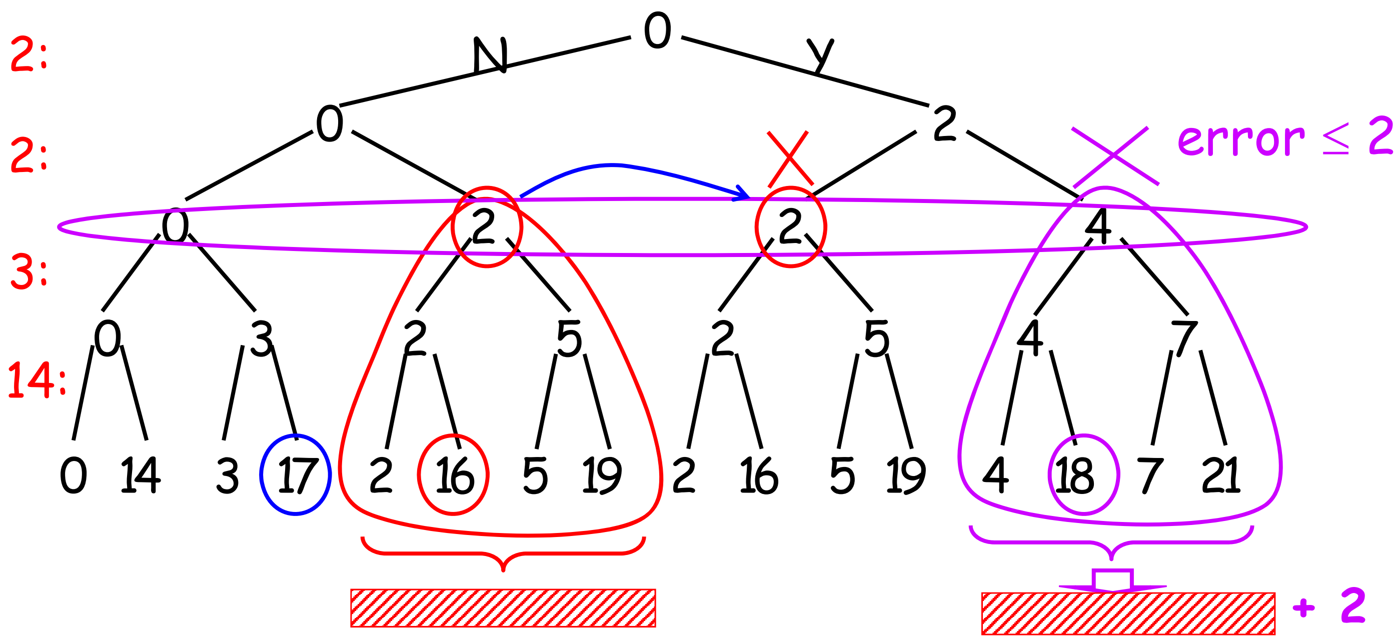
$$T(n) = O(\min\{2^n, nt, nW, n^2m\})$$

Note: $2^n$ may be the best.
(e.g., $n = 10$, $t = 10^{100}$, $W = 3 \times 10^{100}$, $m = 10^{99}$)

35-9z

35-9c

$S = (2, 2, 3, 14)$



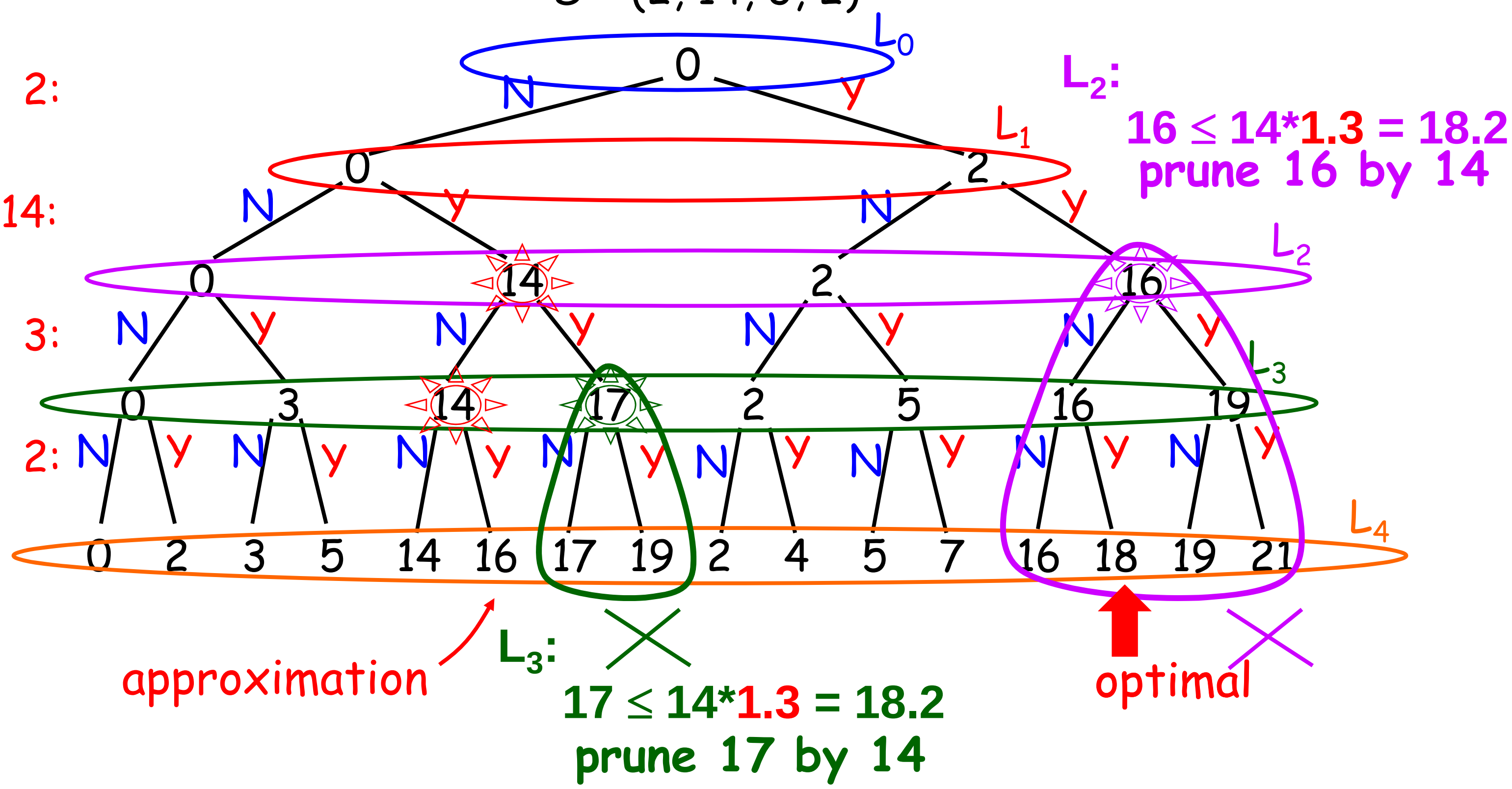

e.g., t = 18 ⇒ error ≤ 2  ⇒ error = 1

⇒ ~~error~~ = 2

$t = 18$

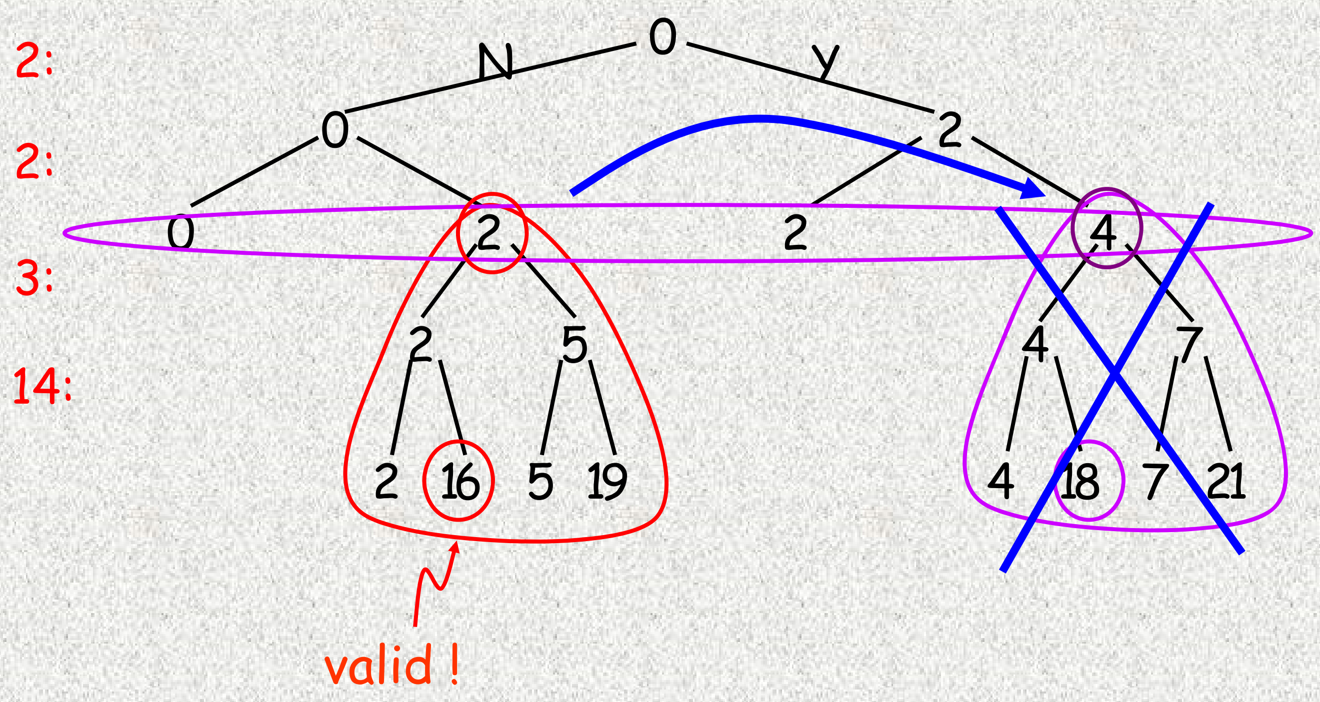
$S = (2, 14, 3, 2)$

$\delta = 0.3$




**Question: Can we prune 14 by 16 at $L_2$?**

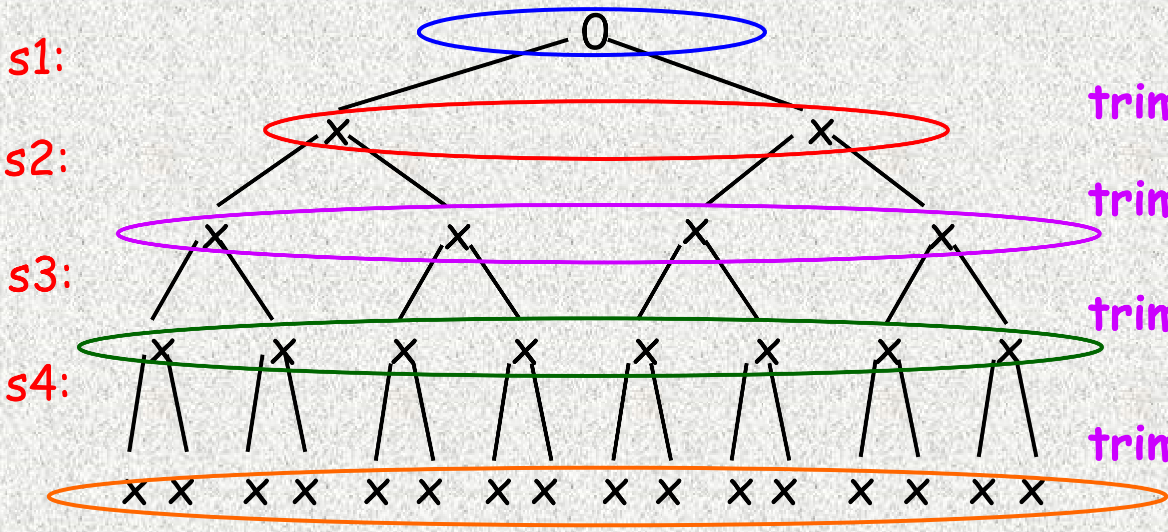
Assume $t = 18$ and "4" is cut.



# Why $\delta = \varepsilon/n$ ?

Example: take $\delta$ = 5% for n = 4 and $\varepsilon$ = 20%

S = (s1, s2, s3, s4)




trim: error ≤ 5%

trim: error ≤ 5%

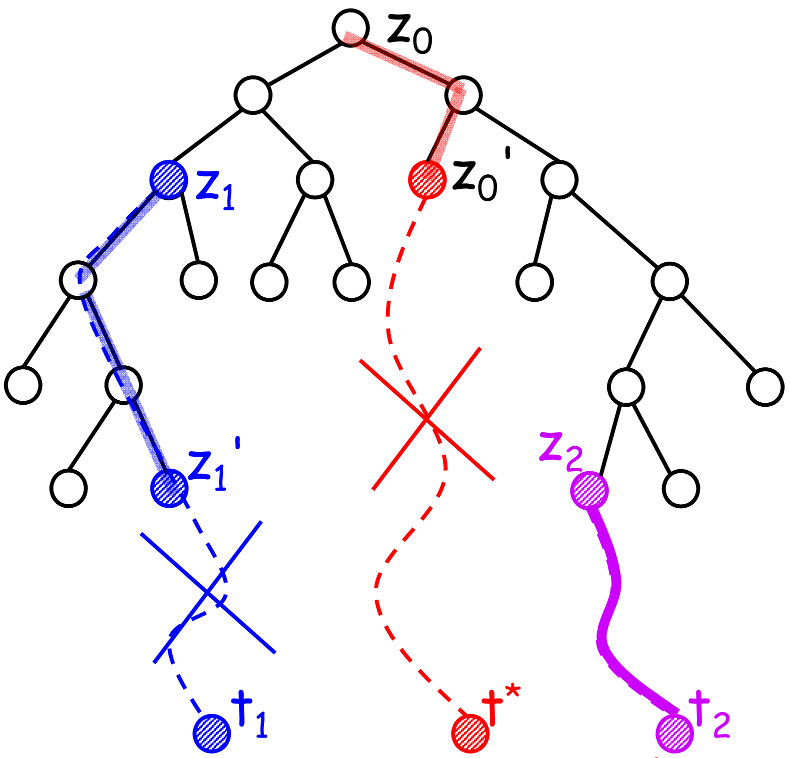
trim: error ≤ 5%

trim: error ≤ 5%

total error ≤ 20%








$z_0' \le z_1(1+\delta)$
error $\le \delta \times z_1$

$z_1' \le z_2(1+\delta)$
error $\le \delta \times z_2$

total error $\le \delta \times (z_1 + z_2)$

Note: $z_1$ and $z_2$ are valid ($\le t$)

$z_k \leftarrow \cdots \quad \leftarrow z_3 \leftarrow z_2 \leftarrow z_1 \leftarrow z_0$

Step i:

seed for $t^*$

$L_i = \{ \_, \_, \ldots, \square, \square, \ldots \}$

trim

$L_i = \{ \_, \_, \ldots, \square, \ldots \}$

error $\le \delta \times \square$

Step j:

$L_j = \{ \_, \_, \ldots, \square, \square, \ldots \}$

trim

$L_j = \{ \_, \_, \ldots, \square, \ldots \}$

error $\le \delta \times \square$

Note: all $z_i$ are valid ($\le t$)

total error

$\le \sum z'_{i-1} - z_i$

$\le \sum_{i=1}^{k} \delta \times z_i$

$\le \sum_{i=1}^{k} \delta \times t^*$ $\quad (z_i \le t^*)$

$\le k\delta t^*$

$\le n\delta t^*$ $\quad (k \le n)$

$\le t^* \varepsilon$ $\quad (\delta = \varepsilon / n)$

(reason for this setting)

---

$X = (x_1, \quad x_2, \quad x_3, \quad x_4, \quad \ldots, \quad x_k)$

1. 從 1 開始，越來越大，但不得超過 1024

$\Rightarrow$ $|X| \le$ ?

("=" when X= (1, 2, 3, ..., 1024))

2. 從 1 開始，每次至少成長 2 倍，但不得超過 1024

$\Rightarrow$ $|X| \le$ ?

("=" when $L_i$ = (1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024)

$= (2^0, 2^1, 2^2, 2^3, \quad \ldots, \quad , 2^{\log_2 1024}))$

3. 從 1 開始，每次至少成長 1.5 倍，但不得超過 1024

$\Rightarrow$ $|X| \le$ ?

("=" when $L_i = (1.5^0, 1.5^1, 1.5^2, 1.5^3, \quad \ldots, \quad , 1.5^{\log_{1.5} 1024}))$

$L_i = (y_1 = 0,\ y_2 \geq 1,\ y_3,\ \ y_4,\ \ \ldots\ ,\ \ \ y_k)$

**Effect of two cuts**: all $y_j$ are distinct integers and $y_k \leq t$

(從 0 開始, 越來越大, 但不得超過 $t$)

$\Rightarrow$ $|L_i| \leq t + 1$

**Example**: $t = 1024$

$\Rightarrow$ $|L_i| \leq 1024 + 1$

("=" when $L_i = (0, 1, 2, 3, \ldots, 1024)$)



$L_i = (y_1 = 0,\ y_2 \geq 1,\ y_3,\ \ y_4,\ \ \ldots\ ,\ \ \ y_k)$

**Effect of trimming**: $y_j$ is at least $(1 + \delta) \times y_{j-1}$ for $j \geq 3$

(每次至少成長 $(1 + \delta)$ 倍, 但不得超過 $t$)

$\Rightarrow$ $|L_i| = k \leq \lg_{(1 + \delta)} t + 2$

Example: $t = 1024$, $(1 + \delta) = 2$

$\Rightarrow$ $|L_i| \leq \lg_2 1024 + 2 = 10 + 2$

("=" when $L_i = (0, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024)$)

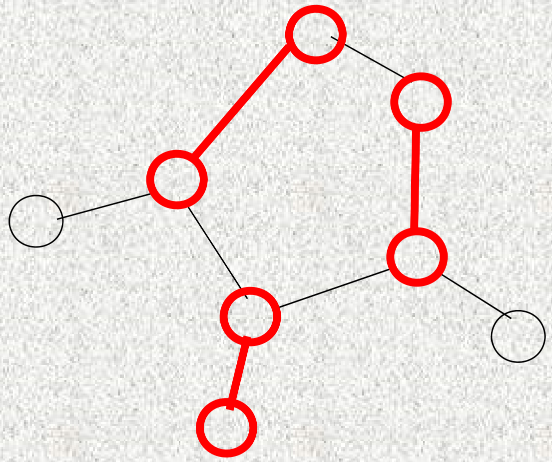
Example: $t = 1024$, $(1 + \delta) = 3$

$\Rightarrow$ $|L_i| \leq \lg_3 1024 + 2 = 6 + 2$

Example: $t = 1024$, $(1 + \delta) = 1.1$

$\Rightarrow$ $|L_i| \leq \lg_{1.1} 1024 + 2$

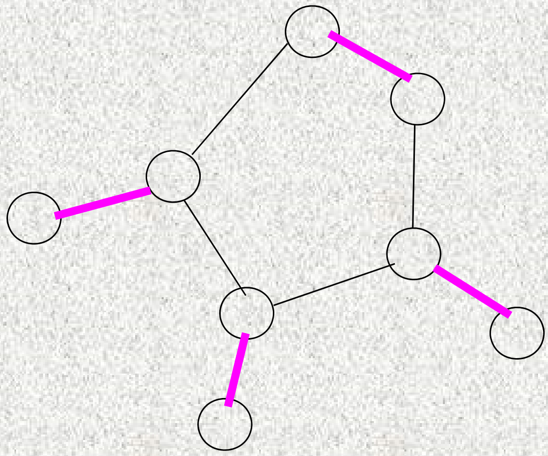
Q: $|C^*| = |A|$ or $|C^*| \geq |A|$



$|A| = 3$, $|C| = 6$

$|C^*| \cancel{=} 3$ ???

$|C^*| \geq 3$



$|C^*| \geq 4$



Q: Polynomial and Pseudo Polynomial

problem sizes: $n$, $s = \lg t$

Cut 1&2: $|L_i| = O(t)$ (pseudo)

Time: $O(n \times t) = O(n\ 2^s)$

Cut 1&2&3: $|L_i| \leq O(n \lg t)$ (real)

Time: $O(n \times n \lg t) = O(n^2 \lg t) = O(n^2\ s)$