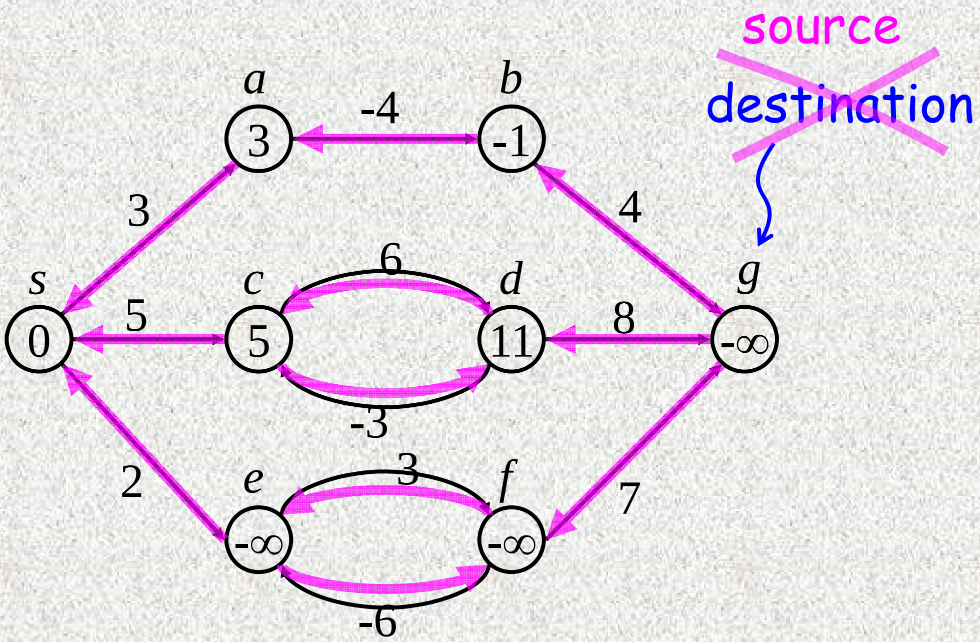
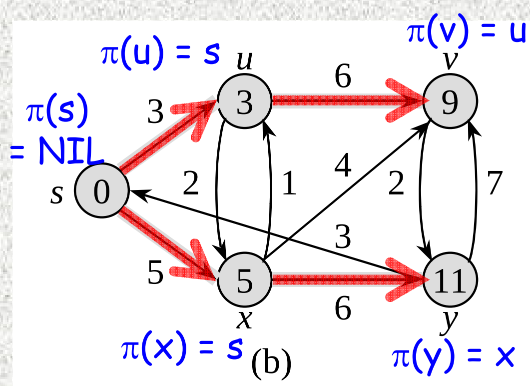
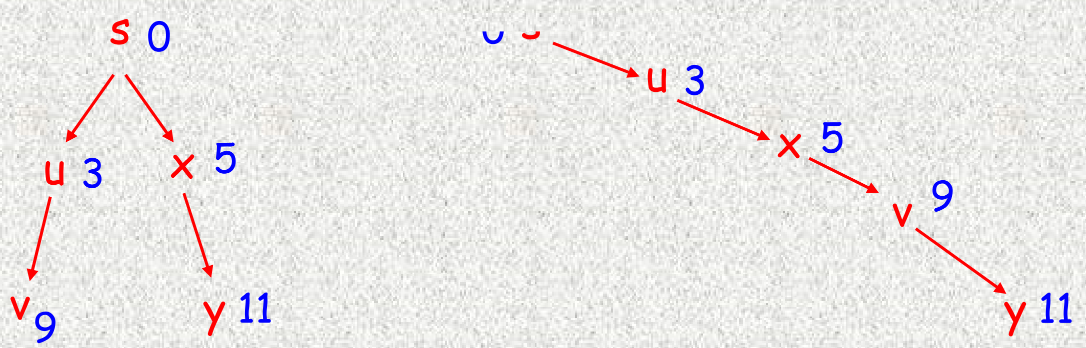


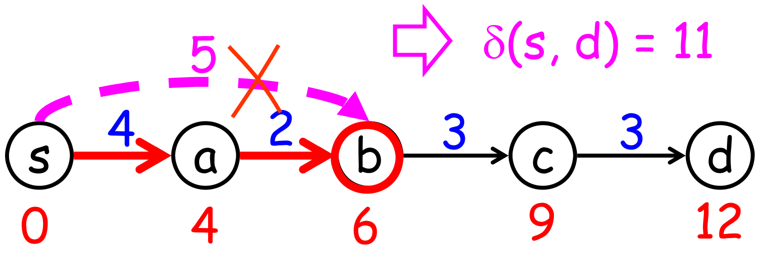
Shortest-paths tree (not unique)





Main Idea ----- 1

If



$\pi(d) = c$

is a shortest path from s to d

Then

(i) all subpaths are shortest **optimal substructure !**

(ii) After $\delta(s, \pi(v))$ is known,
we can get $\delta(s, v)$ by Relax($\pi(v)$, v, w)

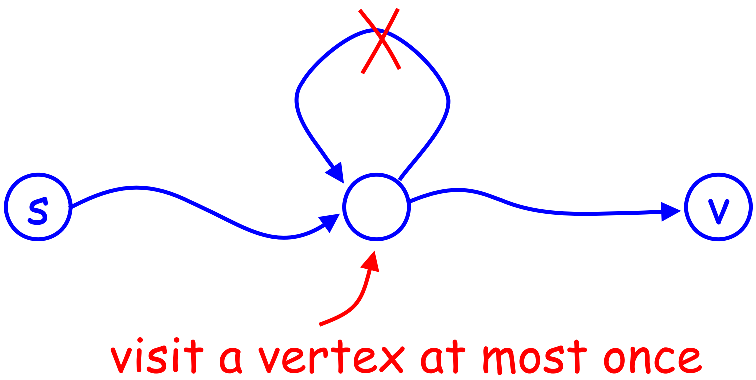
e.g. After $\delta(s, c) = 9$ is known,
we have $\delta(s, d) = 9 + w(c, d) = 12$ **Relax(c, d, w)**

Main Idea ----- 2

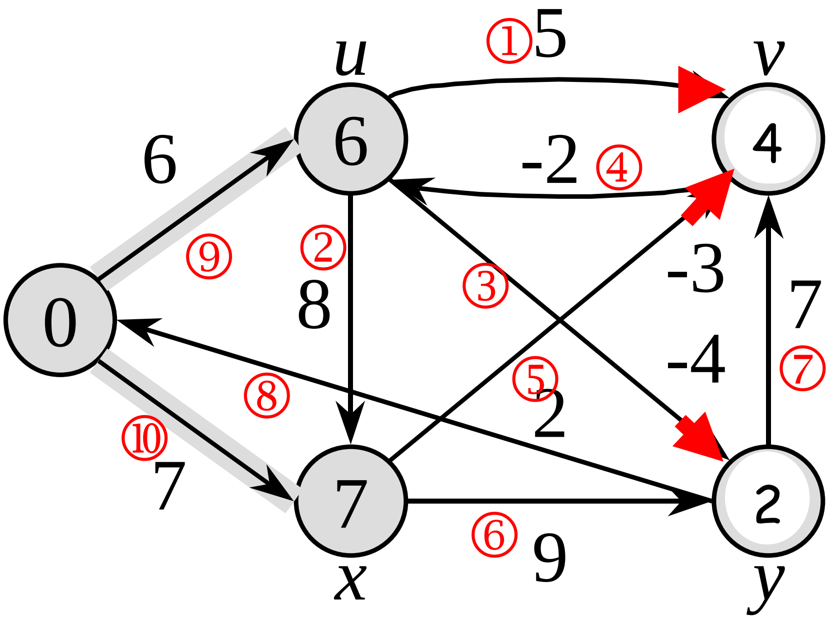
If G contains no negative cycles,

(i) every shortest path is a simple path

(ii) every shortest path has at most n – 1 edges



(For ease of discussion, assume that there are no 0-cycles)



(b)

Main Idea: Bellman-Ford
(no negative cycles)

shortest path tree

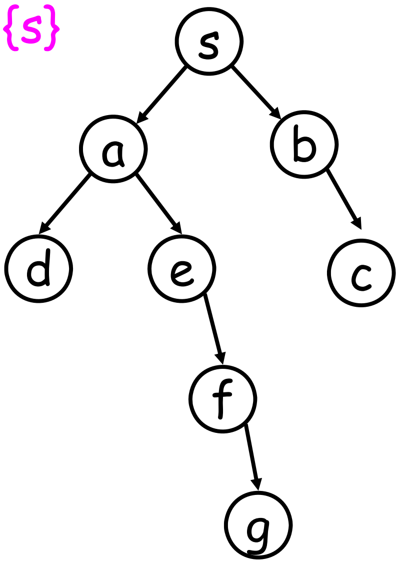
U_0 = {s}

U_1

U_2

U_3

U_4



* U_i: vertices whose shortest paths having i edges

* $U_0 \xrightarrow{\text{phase 1}} U_1 \xrightarrow{\text{phase 2}} U_2 \longrightarrow \cdots$

  ok ok ok

  main idea 1 - correctness

* A simple path has at most n – 1 edges

  ⇨ $U_n = U_{n+1} = U_{n+2} = \ldots = \varnothing$

  ⇨ n – 1 phases is sufficient!

  main idea 2 - time complexity

Authors: Bellman 1958, Ford 1956 (Moore 1957)

Simple Speedups:

(1) Phase 1: relax(s, •) only

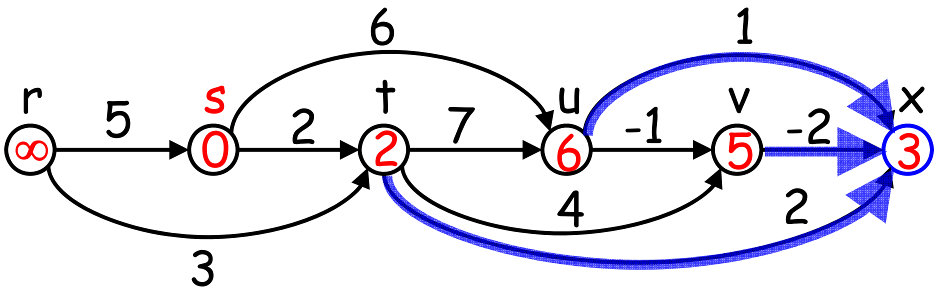
(2) Phase i: relax(v, •) only if d(v) changes

(3) stop once there are no changes

Remark: mentioned early in 1959

Remark: "discovered" by a Chinese in 1994 and named as SPFA

Traditional approach: DP (See 15-14a)



$$\begin{cases} d(s) = 0 \\ d(v) = \underset{(u,v) \in E}{\mathrm{MIN}} \{d(u) + w(u, v)\} \end{cases}$$

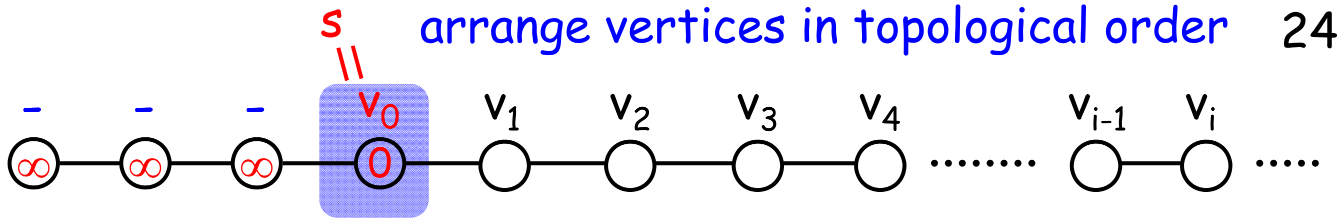
bottom-up computation (left-to-right)

$\pi(x) \in \{t, u, v\}$

$$d(x) = \begin{cases} d(t) + 2, \\ d(u) + 1, \\ d(v) + (-2) \end{cases}$$

DP: 有答案的存起來等別人問 (t, u, v 等 x 來問答案)

24.2: 有答案的主動去修正有需要的人 (t, u, v 主動用答案修正 x) arrange vertices in topological order



* all edges are from left to right →
* $\pi(v_i)$ is one of $v_0, v_1, v_2, \ldots, v_{i-1}$ (or NIL)
* Once $v_0, v_1, v_2, \ldots, v_{i-1}$ ok ⇨ v_i ok!
* Initially, $d(v_0)$ is correct

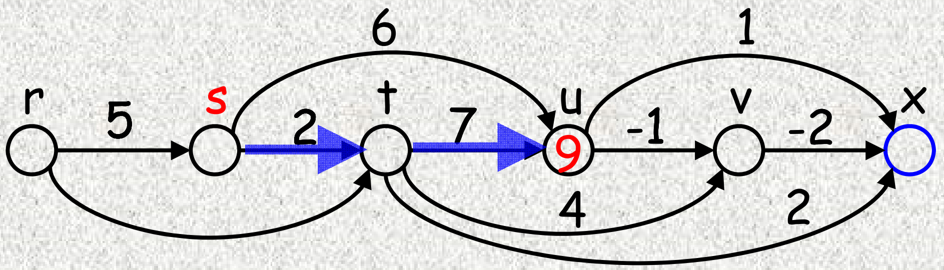
  v_0 does "relax" with correct $d(v_0)$ ⇨ $d(v_1)$ is correct

⇨ v_1 does "relax" with correct $d(v_1)$ ⇨ $d(v_2)$ is correct

⇨ v_2 does "relax" with correct $d(v_2)$ ⇨ $d(v_3)$ is correct

⇨ • • • all $d(v_i)$ are correct (by induction)

The longest path problem on a DAG



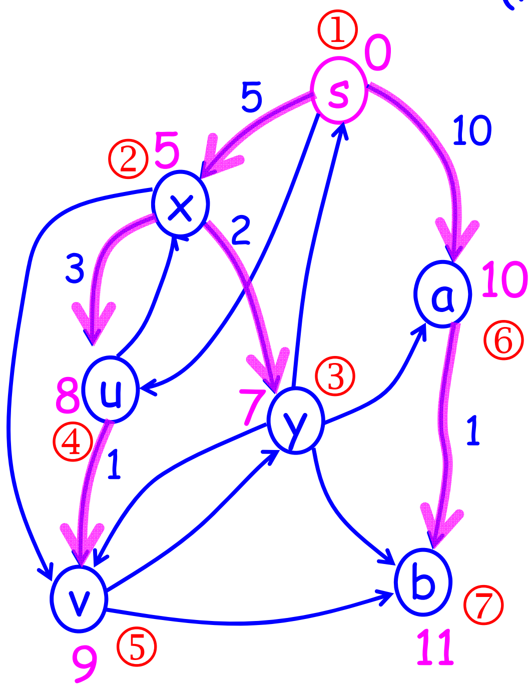
Negating the edge weights

* edge weights: 5, -2, 7, -1, ... ⇨ -5, +2, - 7, +1, ...
* path lengths: -3, 12, 73, 24, ... ⇨ +3, -12, -73, -24, ...

longest

shortest

Main Idea: Dijkstra
(no negative edge)



No negative edge → $\delta(v) > \delta(\pi(v))$

⇒ $\text{rank}(v) > \text{rank}(\pi(v))$

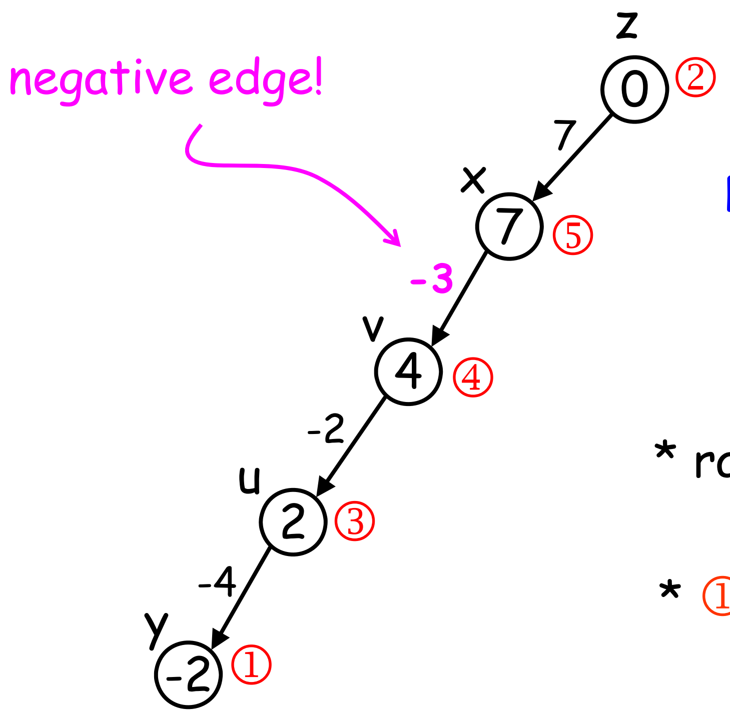
⇒ Once 1 2 3 • • • k ok, k+1 can be computed.

⇒ 1 → 2 → 3 → 4 → • • •

ok ok ok ok

必然是 s

Why all weights should be nonnegative?



(shortest path tree of 24-5 Fig.)

Dijkstra's idea :

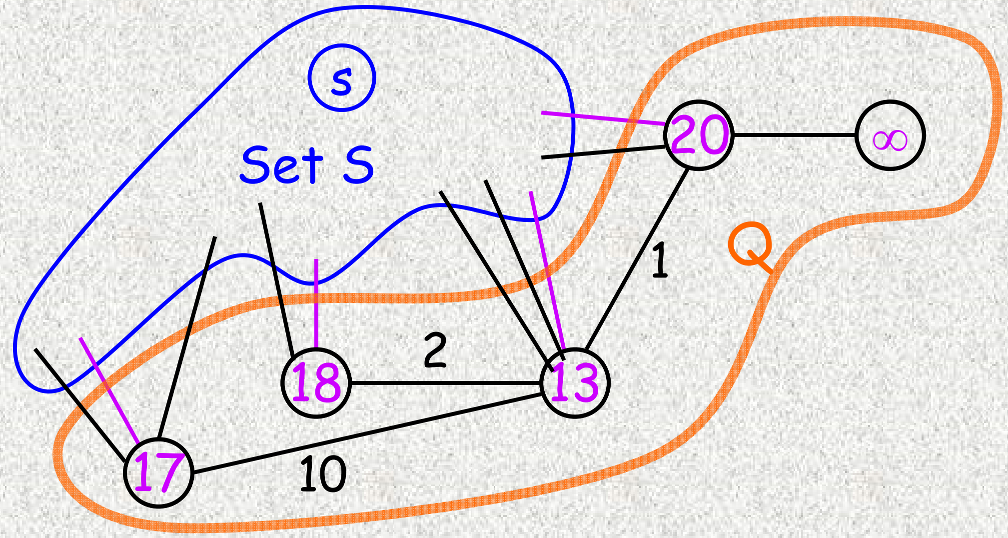
~~rank(v) > rank(π(v))~~

* $\underset{④}{\text{rank}(v)} < \underset{⑤}{\text{rank}(\pi(v))}$

* ① ② ③ ok ⇒ v not ok (④)

Dijkstra's shortest path algorithm

* d[u] 記住 u 和 s 之間目前已知的最短距離
(π[u] 記住目前的 predecessor)



Dijkstra's shortest path algorithm

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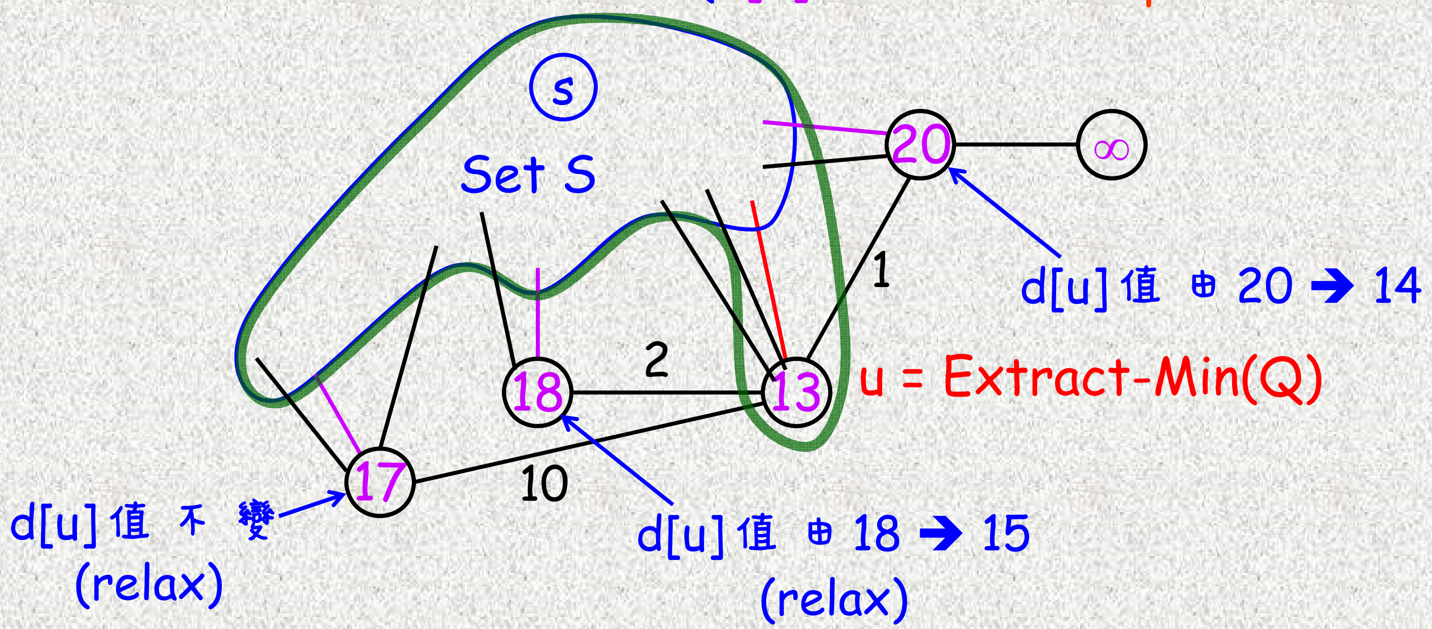
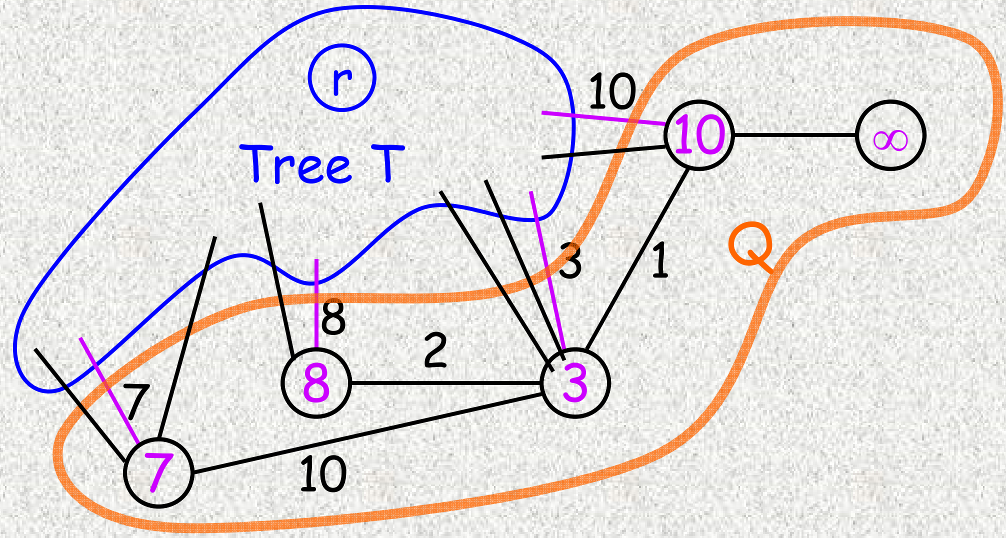


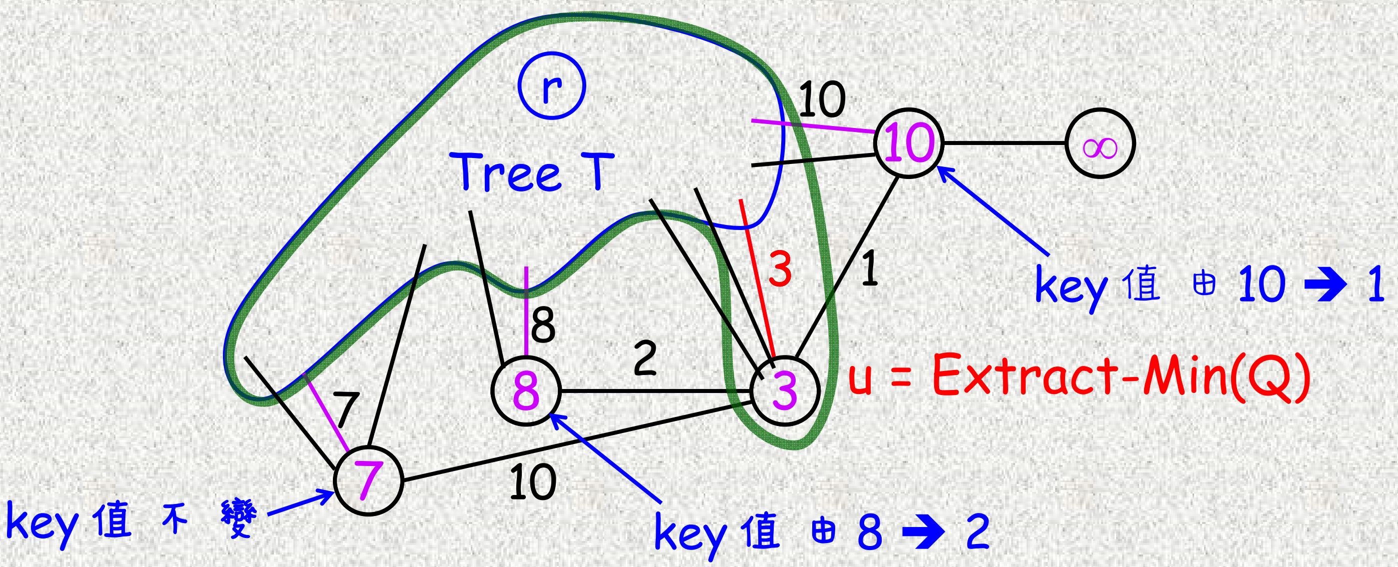
Prim's MST

* key[u] 記住 u 和 T 之間最短的一條 edge

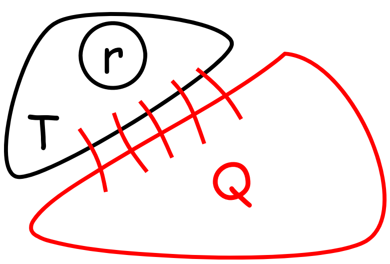


Prim's MST

* key[u] 記住 u 和 T 之間最短的一條 edge



Prim's MST



key[v] : shortest edge to T

π[v] : nearest vertex in T

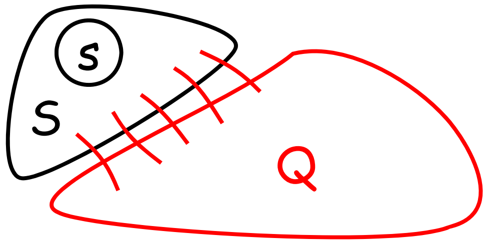
u ← ExtractMin(Q)

T ← T ∪ {u}

reduce key[·] of Adj(u)

(decrease-key)

Dijkstra's shortest path



d[v] : known shortest distance to s

π[v] : current predecessor

u ← ExtractMin(Q)

S ← S ∪ {u}

relax d[·] of Adj(u)

(decrease-key)

		array	b. heap	f. heap
Steps 1~3:	Build Q	O(V)	O(V)	O(V)
Step 5:	V times Extract-Min	O(V^2)	O(V lg V)	O(V lg V)
Steps 7~9:	E times Decrease-Key	O(E)	O(E lg V)	O(E)
		O(V^2+E)	O(E lg V)	O(E + Vlg V)

Procedure	Binary heap (worst-case)	Fibonacci heap (amortized)	array
MAKE-HEAP	$\Theta(1)$	$\Theta(1)$	O(1)
INSERT	$\Theta(\lg n)$	$\Theta(1)$	O(1)
MINIMUM	$\Theta(1)$	$\Theta(1)$	O(n)
EXTRACT-MIN	$\Theta(\lg n)$	$O(\lg n)$	O(n)
UNION	$\Theta(n)$	$\Theta(1)$	O(n)
DECREASE-KEY	$\Theta(\lg n)$	$\Theta(1)$	O(1)
DELETE	$\Theta(\lg n)$	$O(\lg n)$	O(1)
build	O(n)	O(n)	O(n)

(See 22-1)

Single-Source Shortest Paths Algorithms - Review

Main Ideas

Optimal substructure: $\underset{\text{ok}}{\pi(v)} \xrightarrow{\text{relax}} \underset{\text{ok}}{v}$

No negative cycles: simple path (at most n-1 edges)

Bellman-Ford (no negative cycles, can detect) O(VE)

$\underset{\text{ok}}{U_0^{=\{s\}}} \rightarrow \underset{\text{ok}}{U_1} \rightarrow \underset{\text{ok}}{U_2} \rightarrow \underset{\text{ok}}{U_3} \rightarrow \dots \rightarrow \underset{\text{ok}}{U_{n-1}}$

Dijkstra (no negative edges) O(Vlg V+E)

$\underset{\text{ok}}{\text{rank}(1)^{=\{s\}}} \rightarrow \underset{\text{ok}}{\text{rank}(2)} \rightarrow \underset{\text{ok}}{\text{rank}(3)} \rightarrow \dots \rightarrow \underset{\text{ok}}{\text{rank}(n)}$

Two important special cases

Single-Source on un-weighted graph O(V+E)

BFS

Single-Source on a DAG: shortest/longest O(V+E)

(1) Bellman-Ford: one phase – left to right

(2) classical: DP