Fundamentals of Algorithms*

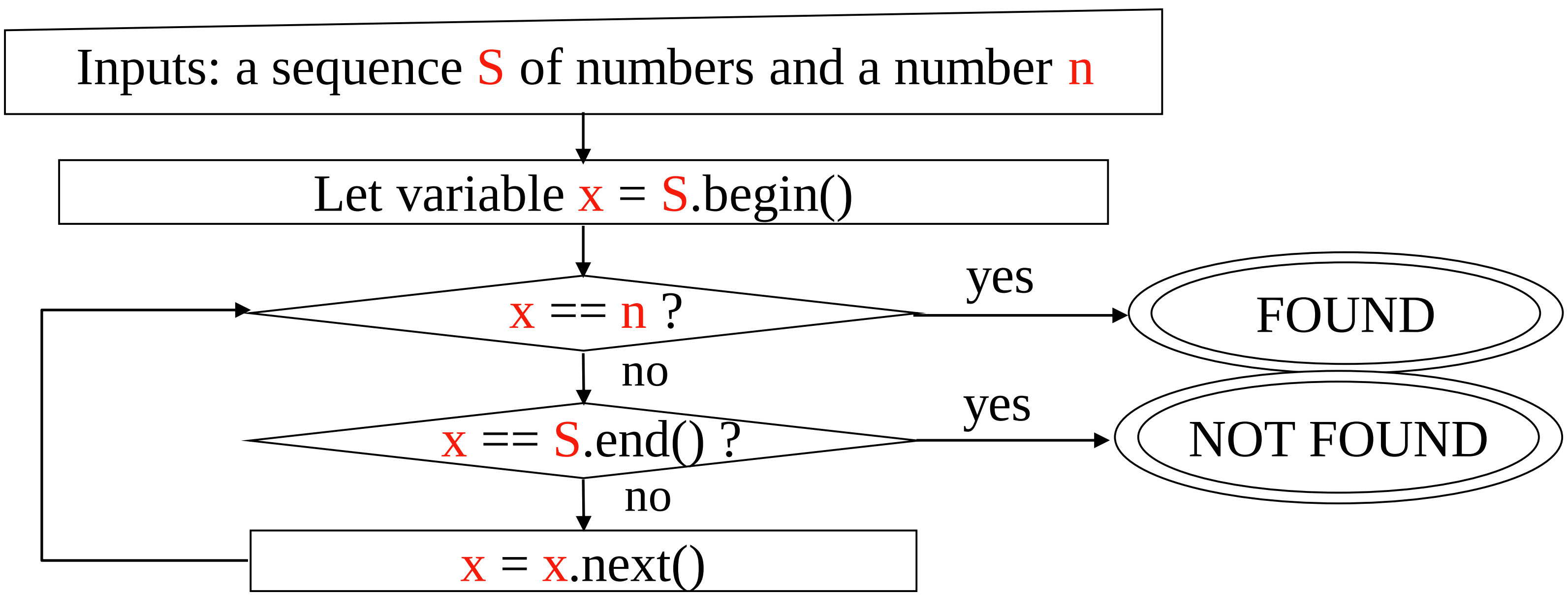
* Slides are based on Chapter 4 of the following book:
Electronic Design Automation: Synthesis, Verification, and Test, edited by L.-T. Wang, Y.-W. Chang, K.-T. Cheng, Morgan Kaufmann, 2009.

Algorithm

- What is an “algorithm”?
  - A sequence of well-defined instructions for completing a task or solving a problem
  - Can be described in a natural language, pseudo code, a flow chart, or even a programming language

Example of an Algorithm

- Problem
  - Whether a specific number is contained in a given sequence of numbers
- Algorithm: Linear Search



Evaluation of an Algorithm

- Is the algorithm in the previous slide efficient?
  - Can we make a better algorithm?
- Can the algorithm complete the task within an acceptable amount of time for a specific set of data derived from a practical application?
  - How to quantify the “efficiency” of an algorithm?

Some Well-Developed Algorithms

- Graph algorithms
  - Converting problems into graphs, and using existing graph algorithms to solve them
  - Breadth-First Search and Depth-First Search, Topological Sort, Shortest and Longest Path Algorithms, etc.
- Heuristic algorithms
  - Fact: Many (or most) EDA problems are intrinsically very difficult because finding an optimal solution within a reasonable runtime is not always possible

  ➔ Find an acceptable solution first. If time or computer resources permit, further improve the result incrementally
- Mathematical programming
  - Transforming problems into certain mathematical models, such as linear inequalities or non-linear equations
  - Well developed algorithms or even tools are available to solve these mathematical formulae very efficiently

Computational Complexity

- Computational Complexity
  - A measurement of efficiency of algorithms
  - Measured with *memory* and *time* used
    - Space complexity: used memory
    - Time complexity: used time, more important than space complexity
  - A mathematical function of the *input size*
    - Sorting n words: input size n
    - Graph: #vertices ($|V|$) and #edges ($|E|$)

Average- and Worst-Case Complexity

- Time complexity
  - Amount of required elementary computational steps
  - Differs from input to input because of the existence of conditional constructs
    - If-else statements
  - Average-case complexity is needed to leverage the differences among inputs
    - Hard to compute, however
    - Use *worst-case complexity* instead in practice

Asymptotic Notations

- Asymptotic functions
  - Expressions of complexity, or runtime
  - Only care about the rate of growth, or the order of growth
  - When expressed in functions of the input size n
    - Lower-order terms ignored, e.g., $n^2 + n \rightarrow n^2$
    - Coefficients ignored, e.g., $3n \rightarrow n$
  - Most used notations
    - O, Θ and Ω

O-notation

- O: upper bounds of complexity functions
- Definition:
  - $O(g(n)) = \{ f(n) : \ c, n_0 > 0$ exist such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0 \}$
  - Conventionally written as $f(n) = O(g(n))$, but it's NOT equivalence
- Examples
  - $n^3 + 1000n^2 + n = O(n^3)$
  - $n^2 = O(2^n)$

Common O-notations

- Polynomial-time complexity: $O(n^k)$, where n is the input size and k is a constant
  - $O(1)$: constant time
  - $O(\lg n)$: logarithmic time
  - $O(n)$: linear time
  - $O(n^2)$: quadratic time
  - $O(n^3)$: cubic time
- Non-polynomial time complexity:
  - $O(2^n)$, $O(3^n)$: exponential time
  - $O(n!)$: factorial time

Ω-notation

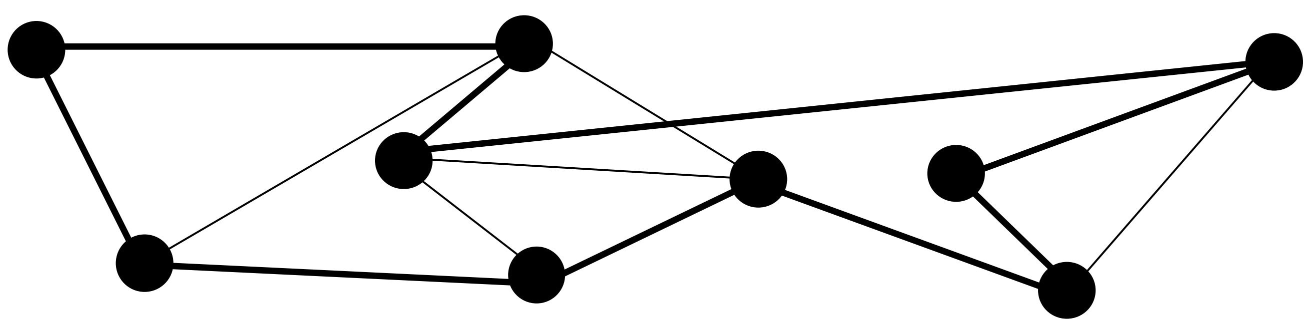
- Ω-notation
  - the inverse of *O*-notation
    - $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$
  - lower bounds of complexity functions
- Definition:
  - $\Omega(g(n)) = \{\ f(n) :\ c, n_0 > 0$ exist such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0\ \}$
- Much less common than *O*-notation

Θ-notation

- Θ-notation
  - tight bounds of complexity functions
    - $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- Definition:
  - $\Theta(g(n)) = \{ f(n) : c_1, c_2, n_0 > 0$ exist such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0 \}$
- Examples
  - $0.1n^2 + 1000n = \Theta(n^2)$
  - $10000n^2 + 2^n = \Theta(2^n)$

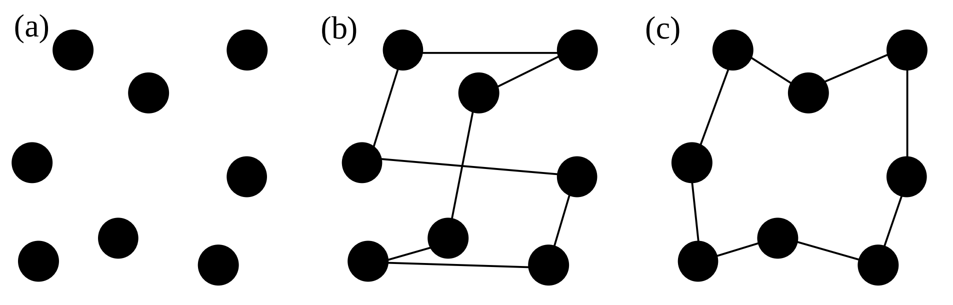
Decision Problems

- Decision problems
  - Can only be answered with “yes” or “no”
  - Names in capital letters
- HAMILTONIAN CYCLE
  - Given a graph, whether a loop that goes through all nodes exists



Optimization Problems

- An optimization problem
  - Looks for an optimum of the objective function
  - Can be decomposed into a series of decision problems by binary search
- Traveling Salesman Problem (TSP)
  - Finding a shortest route for a salesman who needs to visit a set of cities in a round tour



Complexity Classes

- Complexity classes
  - A set of problems with the same degree of complexity, where the complexity of a problem is the complexity of its most efficient possible algorithm
- 4 most frequently seen classes
  - P (Polynomial)
  - NP (Nondeterministic Polynomial)
  - NP-complete
  - NP-hard

Complexity Class P

- Complexity Class P
  - P stands for polynomial
  - Contains the problems that can be solved in polynomial time in terms of the input size on a deterministic computer
  - For a problem in P with input size n, an algorithm of complexity $O(n^k)$ exists, where k is a constant
  - Problems in P are considered *tractable*

Complexity Class NP

- Problems in class NP (Nondeterministic Polynomial)
  - Decision problems
  - Can be solved in polynomial time on a *nondeterministic* computer
  - The solutions can be verified for correctness in polynomial time on a *deterministic* computer
- P is contained in NP
- But the question of if P = NP remains open

Polynomial Transformation

- Polynomial transformation from problems P_a to P_b
  - Expresses an instance of P_a as an instance of P_b
  - The transformation is in polynomial time complexity
- P_a is polynomially reducible to P_b
  - A polynomial transformation (or reduction) from P_a to P_b exists

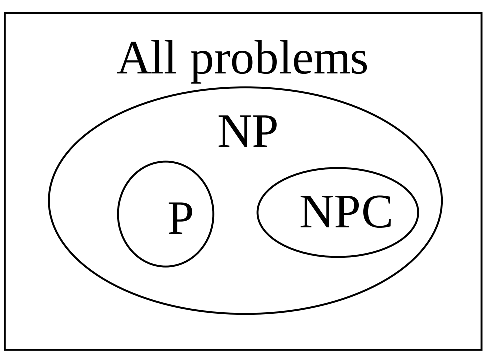
Complexity Class NP-complete (NPC)

- Informal definition
  - The most difficult problems in class NP
- Formal definition
  - The problem P_b is in NPC if
    - P_b is in NP
    - For any problem P_a in NP, a *polynomial transformation* from P_a to P_b always exists
- Problems in NPC are *polynomially reducible* to one another

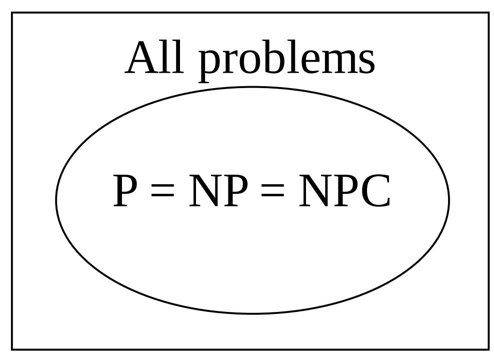
Complexity Class NPC

- Checking if a problem P_a is in NPC
  - P_a is in NP
    - Solution checking can be done in polynomial time
  - A known NPC problem is polynomially reducible to P_a
- P = NP iff P = NPC
  - Still an open question

(a) $P \neq NP$



(b) $P = NP$

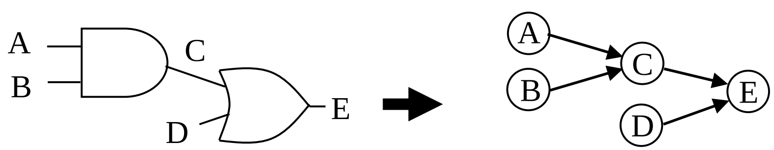


Complexity Class NP-hard

- NP-hard problems
  - At least as hard as NPC problems (most of the time harder)
  - Solution checking cannot be done in polynomial time
- In practice, most optimization versions of NPC problems are NP-hard
  - TSP as the optimization version of TRAVELING SALESMAN: if a round tour of length under a constant k exists

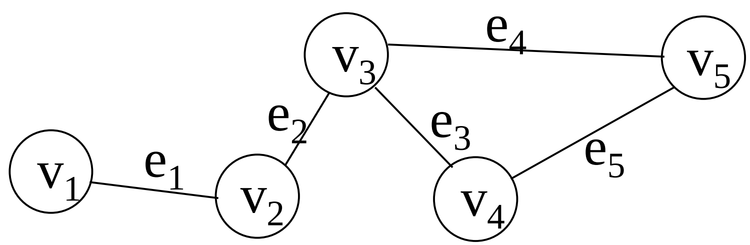
Graph

- Graph
  - Models pairwise relationships among items of certain form
  - We can model circuits with directed graphs



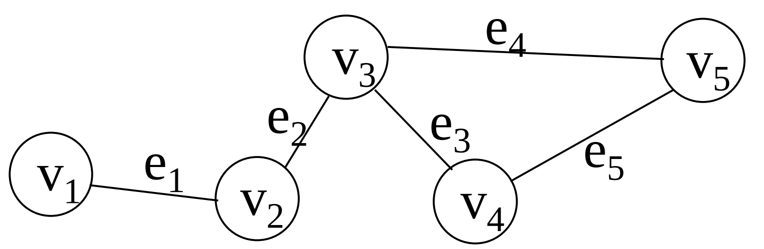
  - Defined by 2 sets: a vertex (node) set V and an edge set E
    - Edges can be either directed or undirected

Graph Terminology



- An undirected graph $G = (V, E)$
  - $V = \{v_1, v_2, v_3, v_4, v_5\}$, $|V| = 5$
  - $E = \{e_1, e_2, e_3, e_4, e_5\}$, $|E| = 5$
  - An edge has 2 endpoints, e.g., $e_1 = (v_1, v_2)$
    - v_1 and v_2 are adjacent
    - e_1 is incident with v_1 and v_2
  - Degree of a vertex: number of incident edges
    - degree(v_3) = 3, degree(v_5) = 2

Graph Terminology (Cont'd)



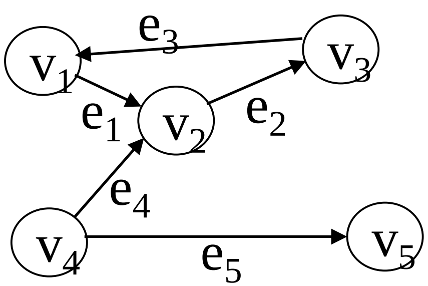
- Path: <v_5, e_4, v_3, e_3, v_4>, its length (number of edges in the path) is 2
- Cycle: <v_5, e_4, v_3, e_3, v_4, e_5, v_5> starts and ends at the same vertex
- Simple path: no cycles in the path

Graph Terminology (Cont'd)

- Loop
  - An edge starting and ending at the same vertex
- Parallel edges
  - Plural edges incident with the same two vertices
- Simple graph: no loops or parallel edges
- $|V|$ and $|E|$ can be simply V and E inside asymptotic notations, e.g. $O(V + E)$
- Weighted graphs
  - Various values (weights) are assigned to edges

Terminology for Directed Graphs

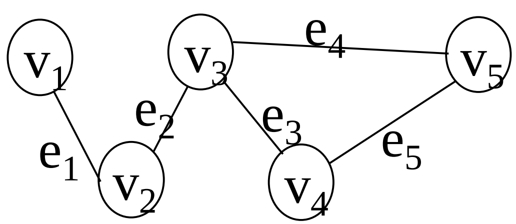
- Edge: $e_1 = (v_1, v_2) \neq (v_2, v_1)$
  - e_1 is incident from v_1 *to* v_2
  - v_2 is the head of e_1
  - v_1 is the tail of e_1



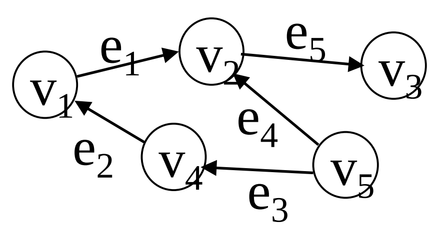
- Path: $<v_4, e_4, v_2, e_2, v_3>$
  - v_4 appears before v_3 in the path
  - v_4 is v_3's predecessor; v_3 is v_4's successor
- DAG: directed acyclic graphs

Representation: Adjacency Matrix

- A $|V|$ x $|V|$ matrix
  - $A_{mn} = 1$ if $(v_m, v_n) \in E$
  - $A_{mn} = 0$ if $(v_m, v_n) \notin E$
- Symmetric for undirected graphs
- $\Theta(V^2)$ space, efficient for dense graphs



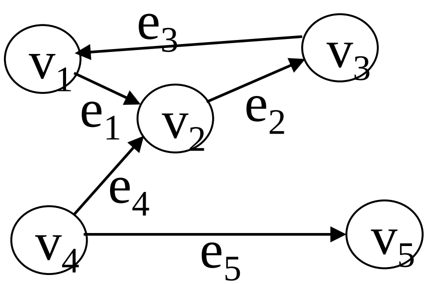
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

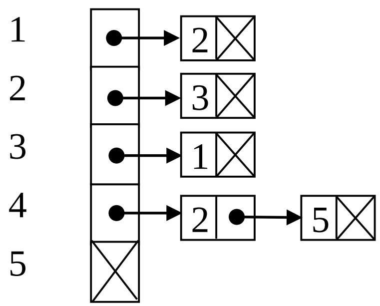


$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Representation: Adjacency List

- An array of $|V|$ linked lists each of which stores all the adjacent vertices to a respective vertex in V
- $\Theta(V + E)$ space, efficient for sparse graphs



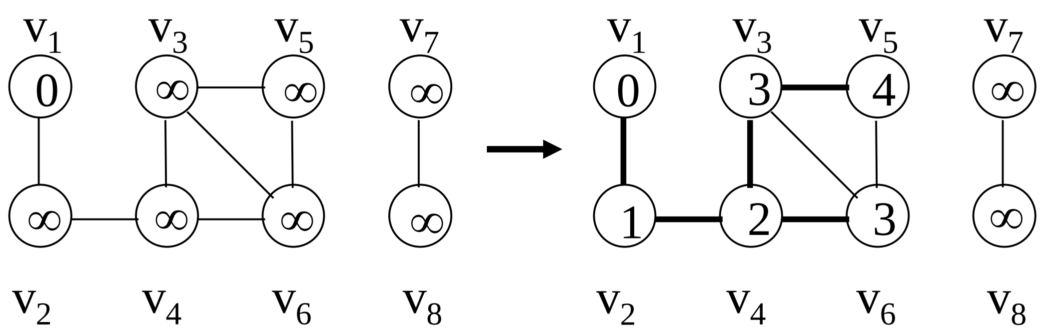


Breadth-First Search (BFS)

```
BFS(Graph G, Vertex s)
  FIFO_Queue Q = {s};
  for (each v ∈ V){
     v.visited = false;
     v.distance = ∞;
     v.predecessor = NIL;
  }
  s.visited = true;
  s.distance = 0;
  while (Q ≠ ∅){
     Vertex u = Dequeue(Q);
     for (each (u, w) ∈ E){
        if (!(w.visited)){
           w.visited = true;
           w.distance = u.distance + 1;
           w.predecessor = u;
           Enqueue(Q, w);
        }
     }
  }
```

- BFS: visit neighbors of the source first, then neighbors of neighbors, and so on
- *s:* source of the graph
- distance: length of the shortest path from source to the vertex

Breadth-First Search (BFS) (Cont'd)



- BFS
  - Builds a breadth-first tree (thick edges)
  - Finds connected components (nodes with finite distances)
  - Finds shortest paths to the source in unweighted graphs

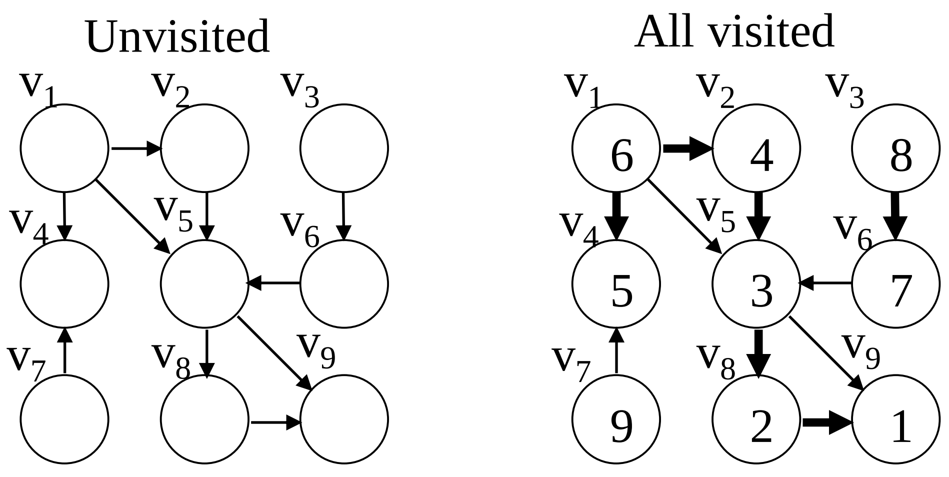
Depth-First Search (DFS)

```
DFS(Graph G)
for (each vertex v ∈ V){
   v.visited = false;
   v.predecessor = NIL;
}
time = 0;
for (each vertex v ∈ V)
   if (!(v.visited)) DFSVisit(v);

DFSVisit(Vertex v)
v.visited = true;
for (each (v, u) ∈ E){
   if (!(u.visited)){
      u.predecessor = v;
      DFSVisit(u);
   }
}
time = time + 1;
v.PostOrderTime = time;
```

- DFS: traverse as deeply as possible before backtracking
- PostOrderTime: the visiting sequence of each node in post order
- Time complexity:

  $O(V + E)$

Depth-First Search (DFS) (Cont’d)



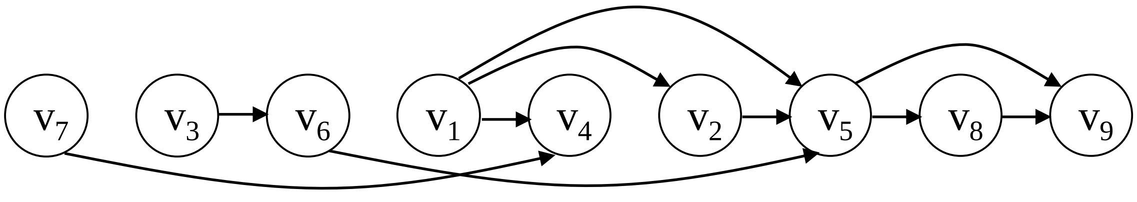
- Values in the nodes: PostOrderTime
- Guaranteed to visit all vertices
  – Forms a depth-first forest (instead of a tree)
- Important applications
  – E.g., topological sort

Topological Sort

- Applied on DAGs
- Gives a linear ordering of vertices, where no vertex appears before its predecessors
- $O(V + E)$ complexity (because of DFS)

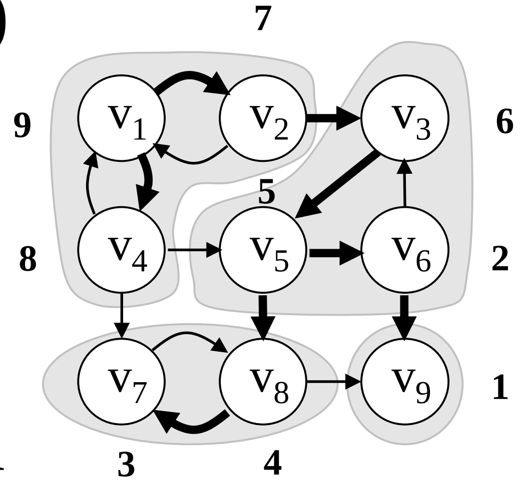
TopologicalSortByDFS(Graph G)
1 call DFS(G), and insert v onto the front of a linked list L as PostOrderTime of each vertex v is computed;
2 return L;

Topological sort of the graph on the previous page



Strongly Connected Components

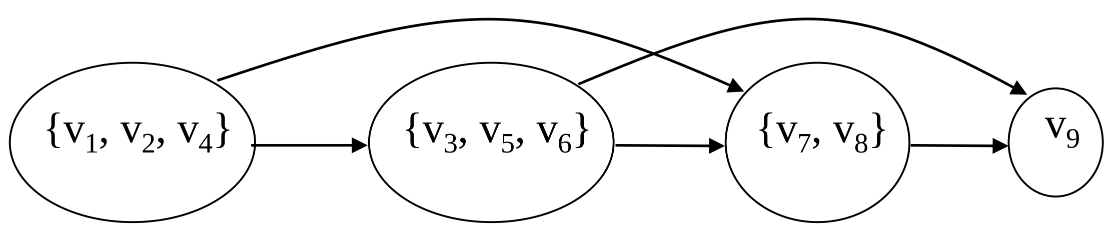
- Strongly connected graphs
  - Directed
  - All vertices can reach each other through directed edges
- Strongly connected components (SCC)
  - Partitioning of a directed graph
  - Each partition is
    - Strongly connected
    - Maximal, no vertex can be added



Strongly Connected Components

SCC(Graph G)
1 call DFS(G) for PostOrderTime;
2 G^T = transpose(G);
3 call DFS(G^T), replacing line 6 of DFS with a procedure examining vertices in order of decreasing PostOrderTime;
4 return different trees in depth-first forest built in DFS(G^T) as separate SCCs;

- Algorithm:
  – A DFS, a transpose, then another DFS
  – $O(V + E)$ time complexity (because of DFS)
- Resultant SCCs form a DAG



Shortest Path Algorithms

- Applied on directed, weighted graphs
- Different algorithms for different graphs
  - DAG shortest path algorithm: on DAGs
  - Dijkstra’s Algorithm: no negative weights
  - The Bellman-Ford Algorithm: most general
- All 3 algorithms share the same kernel
  - initialization and relaxation

Initialization and Relaxation

```
Initialize(graph G, Vertex s)
for (each vertex v ∈ V){
    pre(v) = NIL;
    est(v) = ∞;
}
est(s) = 0;
```

```
Relax(Vertex u, Vertex v)
if (est(v) > est(u) + w((u, v))){
    est(v) = est(u) + w((u, v));
    pre(v) = u;
}
```

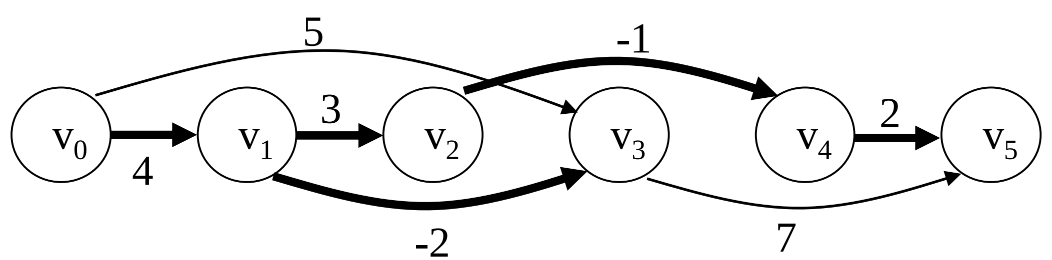
- source *s*: the origin of shortest paths
- pre(*v*): predecessor of node *v*
- est(*v*): shortest-path estimate of *v*, changes along the algorithm

- Relaxation:
  – checks if the shortest path from source *s* to *v* can be shortened (relaxed) by taking a path through *u*

DAG Shortest Path Algorithm

- Perform a topological sort first
- Visit each node in topological order and relax all edges incident from it
- $\Theta(V + E)$ runtime

```
DAGShortestPaths(Graph G, vertex s)
topologically sort the vertices of G;
Initialize(G, s);
for (each vertex u in topological sorted order)
   for (each vertex v such that (u, v) ∈ E)
      Relax(u, v);
```

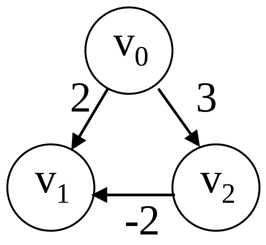


Dijkstra's Algorithm

- Works on graphs with non-negative weights
- Needs a priority queue *minQ* with est(*v*) as keys
- Extracts the node *u* with minimum est(*u*) in *minQ* and relaxes all edges incident from *u* to all nodes still in *minQ*

```
Dijkstra(Graph G, Vertex s)
Initialize(G, s);
Priority_Queue minQ = {all vertices in V};
while (minQ ≠ ∅){
   Vertex u = ExtractMin(minQ); // minimum est(u)
   for (each v ∈ minQ such that (u, v) ∈ E)
      Relax(u, v);
}
```

Dijkstra's Algorithm (Cont'd)



	Predecessors			Shortest-Path Estimates		
	v_0	v_1	v_2	v_0	v_1	v_2
Dijkstra's	NIL	v_0	v_0	0	2	3
Correct path	NIL	v_2	v_0	0	1	3

- Produces incorrect results if weights are negative
- Time complexity depends on the implementation of the priority queue *minQ*
  - A linear array: $O(V^2)$
  - A Fibonacci heap: $O(E + V \cdot \lg V)$

The Bellman-Ford Algorithm

- Relax every edge ($|V|$ - 1) times
  - Since negative cycles should not exist in a shortest-path problem
- The most general algorithm
  - Also the most time-consuming: $O(VE)$

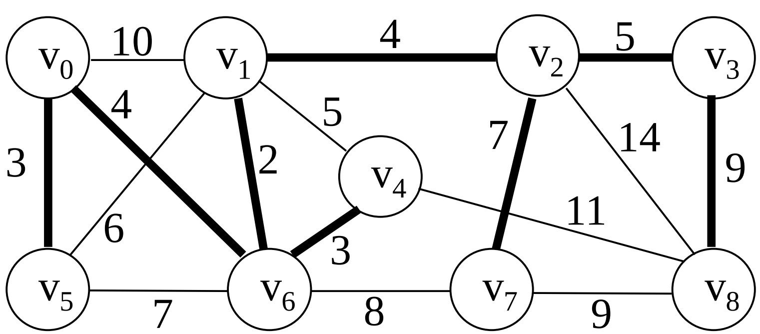
```
Bellman-Ford(Graph G, Vertex s)
Initialize(G, s);
for (counter = 1 to |V| - 1)
   for (each edge (u, v) ∈ E)
      Relax(u ,v);
for (each edge (u, v) ∈ E)
   if (est(v) > est(u) + w((u, v)))
      report "negative-weight cycles exist";
```

The Longest Path Problem

- Similar to the shortest-path problem if no positive-cycle exists
  - Changing the value of est(v) to $-\infty$ in initialization and “>” to “<” in relaxation finds longer paths
- If positive cycles exist and want to find the longest simple path
  - An NP-hard problem

Minimum Spanning Tree (MST)

- Spanning trees
  - Defined on undirected, weighted graphs
  - Connects all vertices without cycles
  - Tree weight: sum of all edge weights
- Minimum Spanning Tree
  - A spanning tree with minimum tree weight



Prim’s Algorithm

- Builds a set with a starting vertex
  - Incrementally adds the shortest edge to the set
  - The set ends with an MST

```
PrimMST(Graph G)
Priority_Queue minQ = {all vertices in V};
for (each vertex u ∈ minQ) u.key = ∞;
randomly select a vertex r in V as root;
r.key = 0;
r.predecessor = NIL;
while (minQ ≠ ∅){
   vertex u = ExtractMin(minQ);
   for (each vertex v such that (u, v) ∈ E)
      if (v ∈ minQ and w((u, v)) < v.key) {
          v.predecessor = u;
          v.key = w((u, v));
      }
}
```

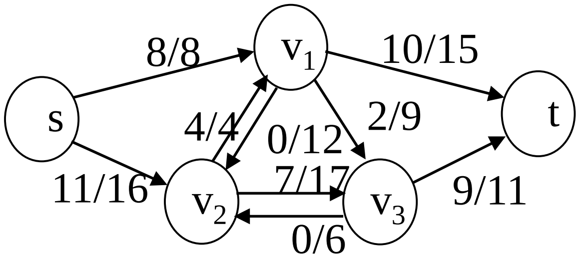
- Use a priority queue
  - key: edge weight
- Runtime depends on *minQ*
  - Like Dijkstra’s Algortihm
  - $O(V^2)$ with an array
  - $O(E + V\lg V)$ with a Fibonacci heap

Flow Network

- A variant of connected, directed graphs
- Two special nodes
  - Source *s*: no edge incident to *s*
  - Sink *t*: no edge incident from *t*
  - Every flow starts at *s* and ends at *t*
- Every edge (u, v) has two attributes
  - Capacity $c(u, v)$: the flow it can hold
  - Flow $f(u, v)$ satisfies 3 constraints
    - Capacity constraint: $f(u, v) \leq c(u, v)$
    - Skew symmetry: $f(u, v) = -f(v, u)$
    - Flow conservation (exceptions: *s* and *t*): $\sum_{v \in V} f(u, v) = 0$

Maximum-Flow Problem

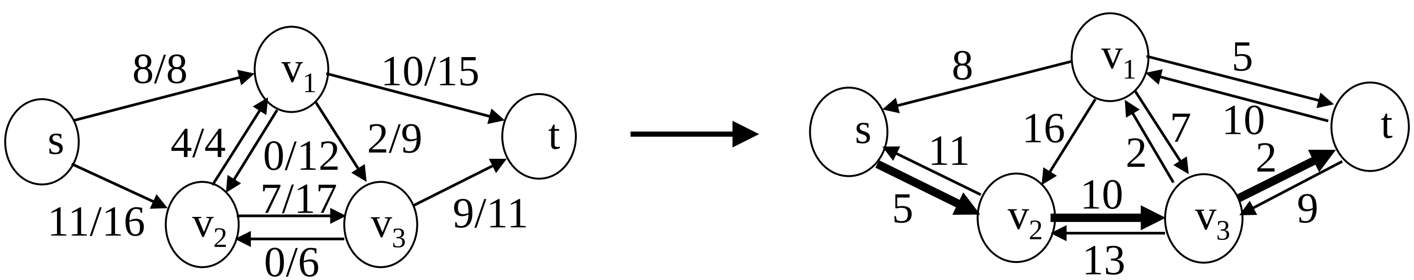
- The value of a flow: $|f| = \sum_{v \in V} f(s, v)$
- Maximum flow problem
  - Finds a flow with the maximum value in a flow network



- Numbers on edges: $f(u, v)/c(u, v)$
- $|f| = 19$, not maximum
  - More flow can be pushed into path $s \rightarrow v_2 \rightarrow v_3 \rightarrow t$
    - An augmenting path

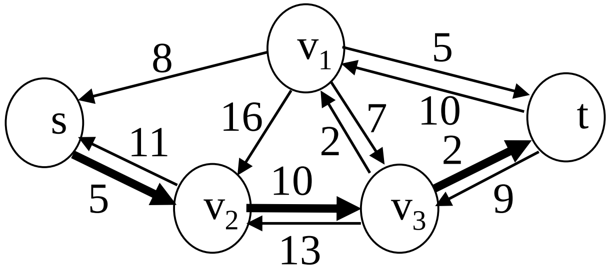
Residual Network

- Facilitates finding augmenting paths
- Residual capacity
  - Defined with respect to a flow *f*
  - $c_f(u, v) = c(u, v) - f(u, v)$
  - For both directions of every pairs of nodes
- $G_f = (V, E_f)$
  - E_f: edges with residual capacity as weights



Residual Network (Cont'd)

- Augmenting paths
  - Paths in the residual network from s to t
  - E.g., $p = s \rightarrow v_2 \rightarrow v_3 \rightarrow t$
- Residual capacity of a path
  - Minimum edge weight on the path
  - $c_f(p) = c_f(v_3, t) = 2$



- Intuitive algorithm for maximum flow
  - Finds augmenting paths in residual networks and push flows equal to their residual capacity
  - Updates residual networks according to the new flows until no augmenting paths can be found

The Ford-Fulkerson Method

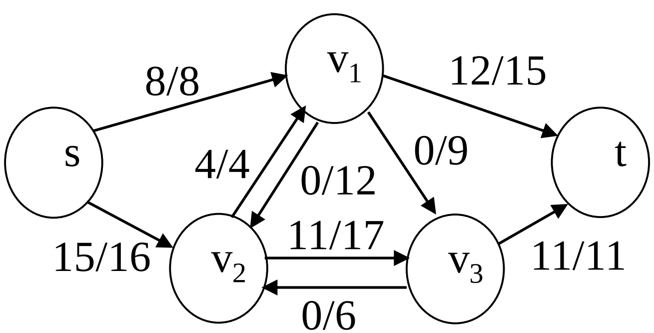
- An intuitive method that repeats the following steps
  - Finds an augmenting path *p* on the residual network
  - Push more flow according to $c_f(p)$
  - Update the residual network

```
Ford-Fulkerson(Graph G, Source s, Sink t)
for (each (u, v) ∈ E) f[u, v] = f[v, u] = 0;
Build a residual network G_f based on flow f;
while (there is an augmenting path p in G_f){
   c_f(p) = min(c_f(u, v) : (u, v) ∈ p);
   for (each edge (u, v) ∈ p){
      f[u, v] = f[u, v] + c_f(p);
      f[v, u] = -f[u ,v];
   }
   Rebuild G_f based on new flow f;
}
```

- Time complexity: $O(E \cdot |f^*|)$
  - f^*: the maximum flow
  - $|f^*|$ can be very large
  - Very inefficient if $|f^*|$ is large

The Edmonds-Karp Algorithm

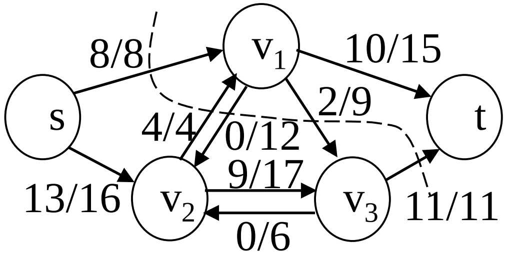
- In Ford-Fulkerson, how to find augmenting paths is unspecified
- Edmonds-Karp algorithm uses breadth-first search to find an augmenting path with a minimum number of edges
- Time complexity: $O(VE^2)$



- Resultant network
  - The maximum flow
  - $|f^*| = 23$

Cuts in Flow Networks

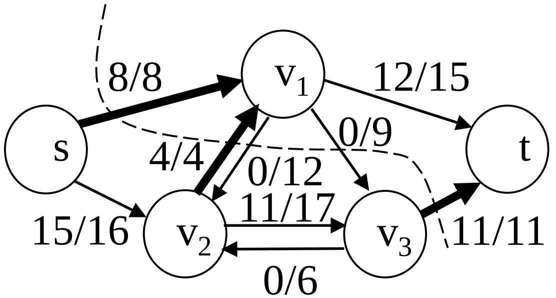
- A cut (S, T)
  - A partition of the node set V into S and $T = V - S$
  - Source $s \in S$ and sink $t \in T$
    - $S = \{s, v_2, v_3\}$, $T = \{t, v_1\}$
  - Net flow across the cut, $f(S, T)$: $f(S,T) = \sum_{u \in S, v \in T} f(u,v)$
    - $f(S, T) = 21$
  - Capacity of the cut, $c(S, T)$: $c(S,T) = \sum_{u \in S, v \in T} c(u,v)$
    - $c(S, T) = 23$
    - $f(S, T) \le c(S, T)$



The Max-Flow Min-Cut Theorem

- The following 3 things are equivalent
  - f is a maximum flow in G
  - The residual network G_f has no augmenting paths
  - $|f| = c(S, T)$ for some cut of G
- Finding maximum flow = finding minimum cut
  - $|f^*| = 23 = c(S, T) =$

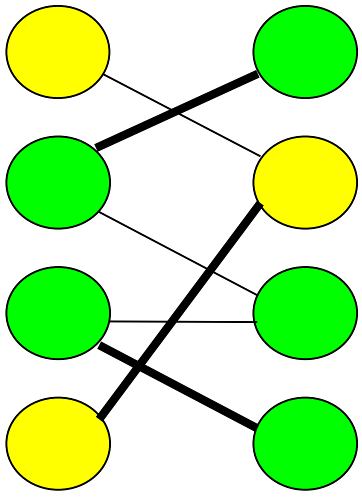
    $c(\{s, v_2, v_3\}, \{t, v_1\})$



Maximum Bipartite Matching

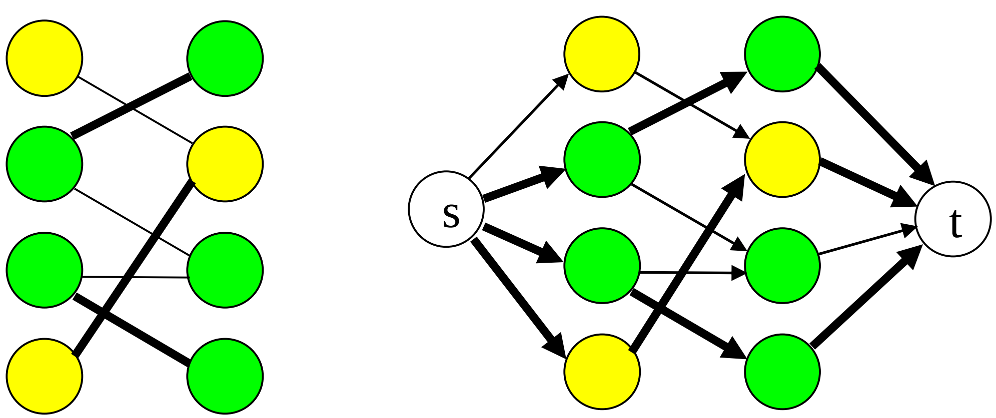
- A bipartite graph $G = (V, E)$
  - V is partitioned into two sets L and R
  - For every edge $(u, v) \in E$, if $u \in L$, then $v \in R$, and vice versa

- A matching
  - A subset of edges $M \subseteq E$
  - For each vertex in V, at most one edge of M is incident on it
  - E.g., 3 thick edges in the bipartite graph



Maximum Bipartite Matching (Cont'd)

- Maximum bipartite matching finds a matching with a maximum number of edges
  - Add a source *s* and link *s* to all nodes in *L*
  - Add a sink *t* and link all nodes in *R* to *t*
  - Every edge has unit capacity
  - Solve the maximum flow problem
- Ford-Fulkerson method solves in $O(VE)$



Heuristic Algorithms

- Applies heuristics, or rules of thumb
- Finds good but not always optimal solutions
- Efficient in time
  - Best for hard (NPC or NP-hard) problems
- Solution quality cannot always be guaranteed
  - Nearest Neighbor for TSP
- Either directly searches the solution space
  - Greedy algorithm, dynamic programming, branch and bound
- Or exerts perturbations on solutions
  - Simulated annealing, genetic algorithms

Greedy Algorithm

- General idea
  - Stages the optimization problem
  - Makes locally optimal choices at each stage
- Real life example
  - Giving change with a minimum number of coins
  - Heuristic: pick the coin with the greatest value
  - 36 cents: quarter → dime → penny: 3 coins
- Two properties make greedy algorithms work
  - Greedy choice
  - Optimal substructure
- Applications: Dijkstra’s, Prim’s algorithms

Greedy Choice Property

- The global optimal solution can be made by making locally optimal, or greedy, choices
- Does not consider the impact of the current choice on future choices
- Counterexamples
  - Nearest Neighbor for TSP
  - Giving change of 40 cents if there were 20-cent coins
    - (Greedy) quarter → dime → nickel: 3 coins
    - (Optimal) 2 20-cent coins: 2 coins

Optimal Substructure Property

- The global optimal solution consists of optimal solutions to its subproblems
  - The problem is divisible into subproblems
  - The combination of optimal solutions to subproblems is globally optimal
- Giving change of 36 cents
  - into 26 cents + 10 cents:
  - (quarter → penny) + dime : global optimal

Dynamic Programming (DP)

- Combines solutions to its dependent subproblems by utilizing the dependency
  - Unlike divide-and-conquer: subproblems are independent
  - Avoids repeatedly solving the same subproblems
- Example: matrix-chain multiplication
  - Find the multiplication sequence with the least number of scalar multiplications
  - Matrices A, B, C: 30 x 100, 100 x 2, 2 x 50
    - (AB)C: #scalar multiplications = 9000
    - A(BC): #scalar multiplications = 160,000

Matrix-Chain Multiplication

- A chain of n matrices $<M_1, M_2, \ldots, M_n>$
  - M_i: dimension v_{i-1} x v_i
- Search all possible multiplication orders
  - Exponential with n: infeasible
- $m[i, j]$: minimum number of scalar multiplications for $M_i M_{i+1} \ldots M_j$
  - Recurrence relation:

$$m[i, j] = \begin{cases} 0 & \textit{if } \; i = j, \\ \min\limits_{i \le k < j} \{ m[i,k] + m[k+1, j] + v_{i-1} v_k v_j \} & \textit{if } \; i < j. \end{cases}$$

  - Target: find $m[1, n]$

Matrix-Chain Multiplication (Cont'd)

```
BottomUpMatrixChain(Vector v)
  n = v.size – 1;
  for (i = 1 to n) m[i, i] = 0;
  for (p = 2 to n) // p: chain length
     for (i = 1 to n – p + 1){
        j = i + p – 1;
        m[i, j] = ∞;
        for (k = i to j - 1){
           temp = m[i, k] + m[k + 1, j]
                  + v_{i-1}v_kv_j;
           if (temp < m[i, j]){
              m[i, j] = temp;
              d[i, j] = k;
           }
        }
     }
  return m and d;
```

- $d[i, j]$
  - where the separation is
- Constructs the $m[i,j]$, ($0 \le i \le j \le n$) table in a bottom-up fashion
- Avoids repeated calculations of $m[i, j]$ in a naive recursive function
- Time complexity: $O(n^3)$

Two Properties for DP

- Overlapping Subproblems
  - The decomposed subproblems are dependent or overlapped
- Optimal Substructure
  - The same as in greedy algorithms
  - DP or greedy?
    - Whether the problem has “overlapping subproblems” or “greedy choice”
- Matrix-chain multiplication has both

Memoization

- Constructs the table in a top-down, recursive fashion

```
TopDownMatrixChain(Vector v)
n = v.size – 1;
for (i = 1 to n)
  for (j = i to n) m[i, j] = ∞;
return Memoize(v, 1, n);

Memoize(Vector v, Index i, Index j)
if (m[i ,j] < ∞) return m[i, j];
if (i = j) m[i, j] = 0;
else for (k = i to j - 1){
        temp = Memoize(v, i, k) +
            Memoize(v, k + 1, j) + v_{i-1}v_kv_j;
        if (temp < m[i, j]) m[i, j] = temp;
     }
return m[i, j];
```

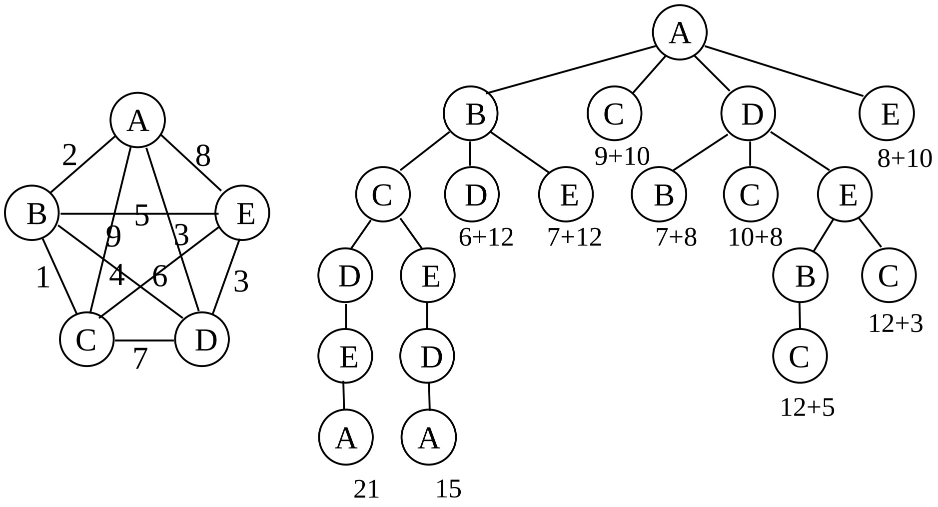
- “Memo”ization
  - Takes “memo”s
  - Records table items along the recursion
- Sometimes better than the bottom-up approach
  - If not all table entries must be visited

Branch and Bound

- Branching
  – Makes several choices at the same point to branch out into the search space
  – The solution space forms a tree-like structure
  – Fully-branched space: too vast to explore
- Bounding and pruning
  – Estimates a lower bound on solution quality to prune out obviously impossible branches
  – Efficiently reduces the solution space made with branching

Branch and Bound for TSP

- Branching: the next node on the route
- Bounding: use MST to estimate the cost lower bound of unvisited route



Simulated Annealing (SA)

- Mimics the process of controlled cooling
  - The heat gives molecules huge energy first
    - Large perturbations
  - Slow cooling gradually deprives the energy
    - Chances to reach stable crystalline configuration
- Analogies
  - Energy → cost function
  - Movements of molecules → perturbations in the solution space
  - Temperature → control parameter *T*

Pseudocode for SA

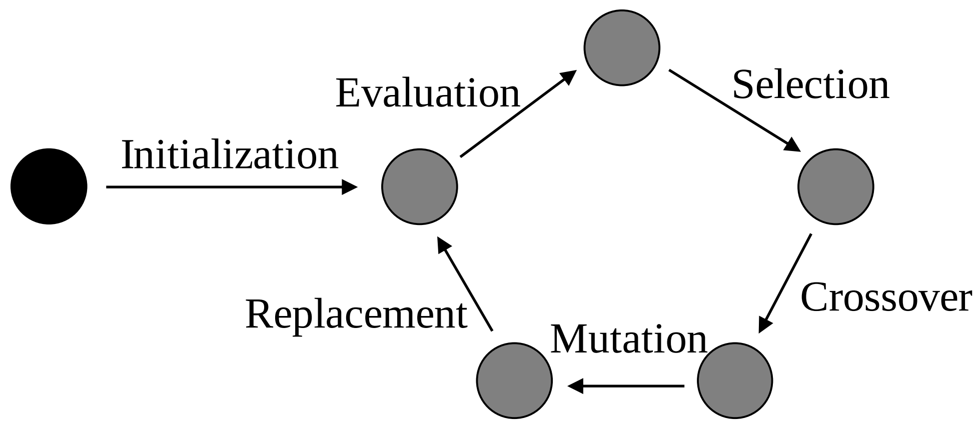
```
Accept(temperature T, cost Δc)
Choose a random number “rand” between 0 and 1;
return (e^(-Δc/T) > rand);

SimulatedAnnealing()
solution sNow, sNext, sBest;
temperature T, endingT;
Initialize sNow, T and endingT;
while (T > endingT){
   while(!ThermalEquilibrium(T)){
      sNext = Perturb(sNow);
      if (cost(sNext) < cost(sNow)){
         sNow = sNext;
         if (cost(sNow) < cost(sBest))
              sBest = sNow;
         }
       else if (Accept(T, cost(sNext)-cost(sNow)))
         sNow = sNext;
    }
    Decrease(T);
  }
  return sBest;
```

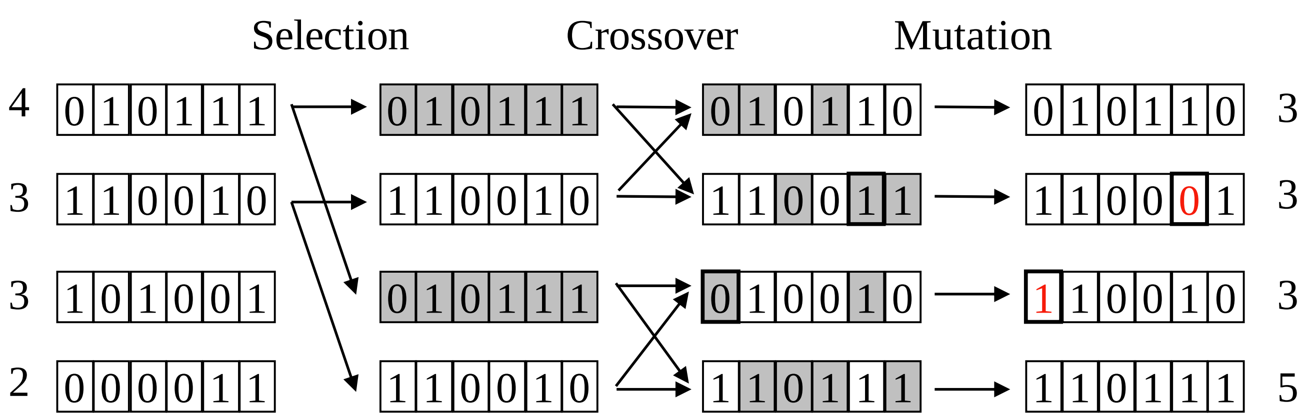
- Cooling schedule
  - Combination of ThermalEquilibrium, Decrease, and endingT
  - Characterization of the SA
- Advantages
  - Can escape local optima
  - Flexible in running time

Genetic Algorithms (GA)

- Inspired by evolutional biology
  - Uses operations like crossover and mutation
  - Operates on a set of feasible solutions
    - “population”
  - Solutions encoded in math symbols like bits
    - “genes”
  - A feasible solution is a bit string
    - a “chromosome”
  - “Survival of the fittest”
    - Solutions are evaluated by a “fitness function”

Simple Genetic Algorithms (SGA)





Average fitness: **3**
Highest fitness: **4**

Average fitness: **3.5**
Highest fitness: **5**

Mathematical Programming

- Problem formulation
  - ⇨ Minimize (or maximize) $f(x)$;
  - ⇨ Subject to $X = \{ x \mid g_i(x) \leq b_i, i = 1...m \}$;

  where
  - $x = (x_1,..., x_n)$ are optimization (or decision) variables,
  - $f : R^n \rightarrow R$ is the objective function
  - $g_i : R^n \rightarrow R$ and $b_i \in R$ form the constraints for the valid values of x.

Categories of Mathematical Programming Problems

1. If $X = R^n$, the problem is unconstrained.
2. If f and all the constraints are linear, the problem is called a linear programming (LP) problem.
   - Constraints can then be represented in the matrix form $Ax \leq B$, where A is an $m \times n$ matrix corresponding to the coefficients in $g_i(x)$.
3. If the problem is linear, and all the variables are constrained to integers, the problem is called an integer linear programming (ILP) problem.
   - If only some of the variables are integers, it is called a mixed integer linear programming (MILP) problem.

Categories of Mathematical Programming Problems (Cont'd)

4. If the constraints are linear, but the objective function f contains some quadratic terms, the problem is called a quadratic programming (QP) problem.
5. If f or any of $g_i(x)$ is not linear, it is called a nonlinear programming (NLP) problem
6. If all the constraints have the following convexity property: $g_i(\alpha x_a + \beta x_b) \leq \alpha\, g_i(x_a) + \beta\, g_i(x_b)$

   where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$

   then the problem is called a convex programming or convex optimization problem.
7. If the set of feasible solutions defined by f and X are discrete, the problem is called a discrete or combinatorial optimization problem.

Linear Programming (LP) Problem

- Intuition: solving LP problems should be simpler than solving the general mathematical optimization problems

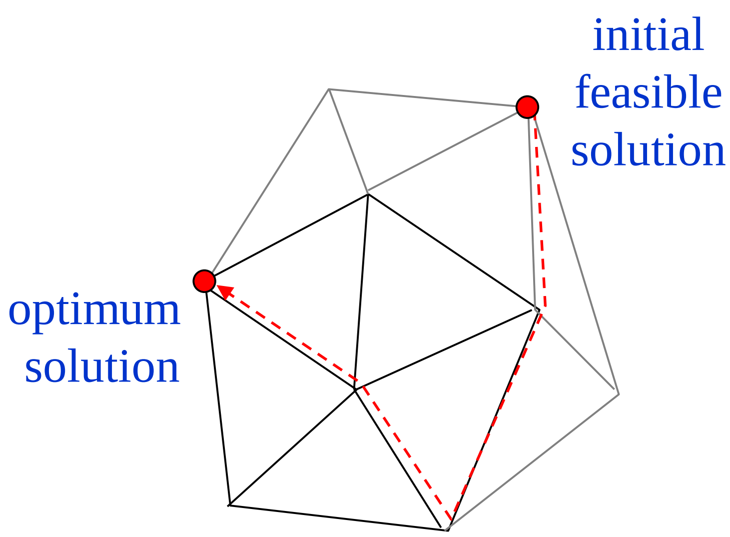
- Fact: A polynomial-time algorithm was not available until 1970’s

LP Formulation

- One linear equation forms a hyper-plane.
- One linear inequality (constraint) forms a hyper half-space.
- The set of linear constraints forms a polyhedron in a multi-dimensional space.
- The optimal value of a linear objective function over a set of linear constraints occurs at an extreme point of the polyhedron.

Simplex Method

- Developed by George Dantzig in 1947
  - First practical procedure used to solve the LP problem
- Finds a basic feasible solution that satisfies all the constraints
  - A basic solution is conceptually a *vertex* (i.e., an extreme point) of the convex polyhedron
- Moves along the edges of the polyhedron in the direction towards finding a better value of the objective function
  - Guaranteed to eventually terminate at the optimal solution



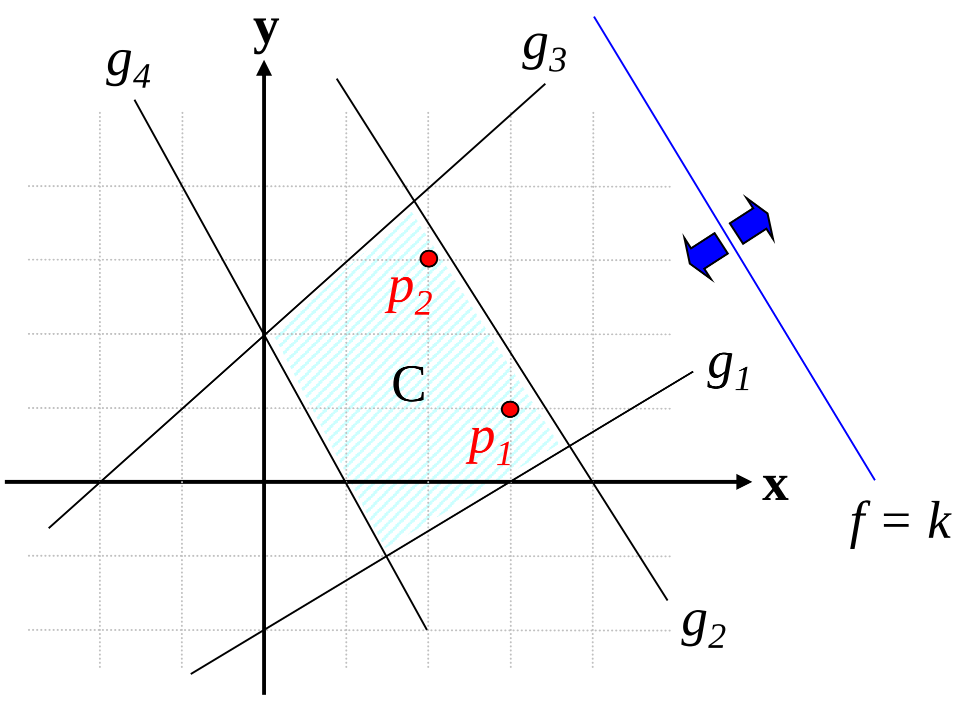
Integer Linear Programming (ILP) Problem

- Fact:
  - Many EDA problems are best formulated with integer variables
    - E.g., signal values in a digital circuit are under a modular number system
    - E.g., problems that need to enumerate the possible cases, or are related to scheduling of certain events
  - In general more difficult than the LP counterpart

An ILP Example

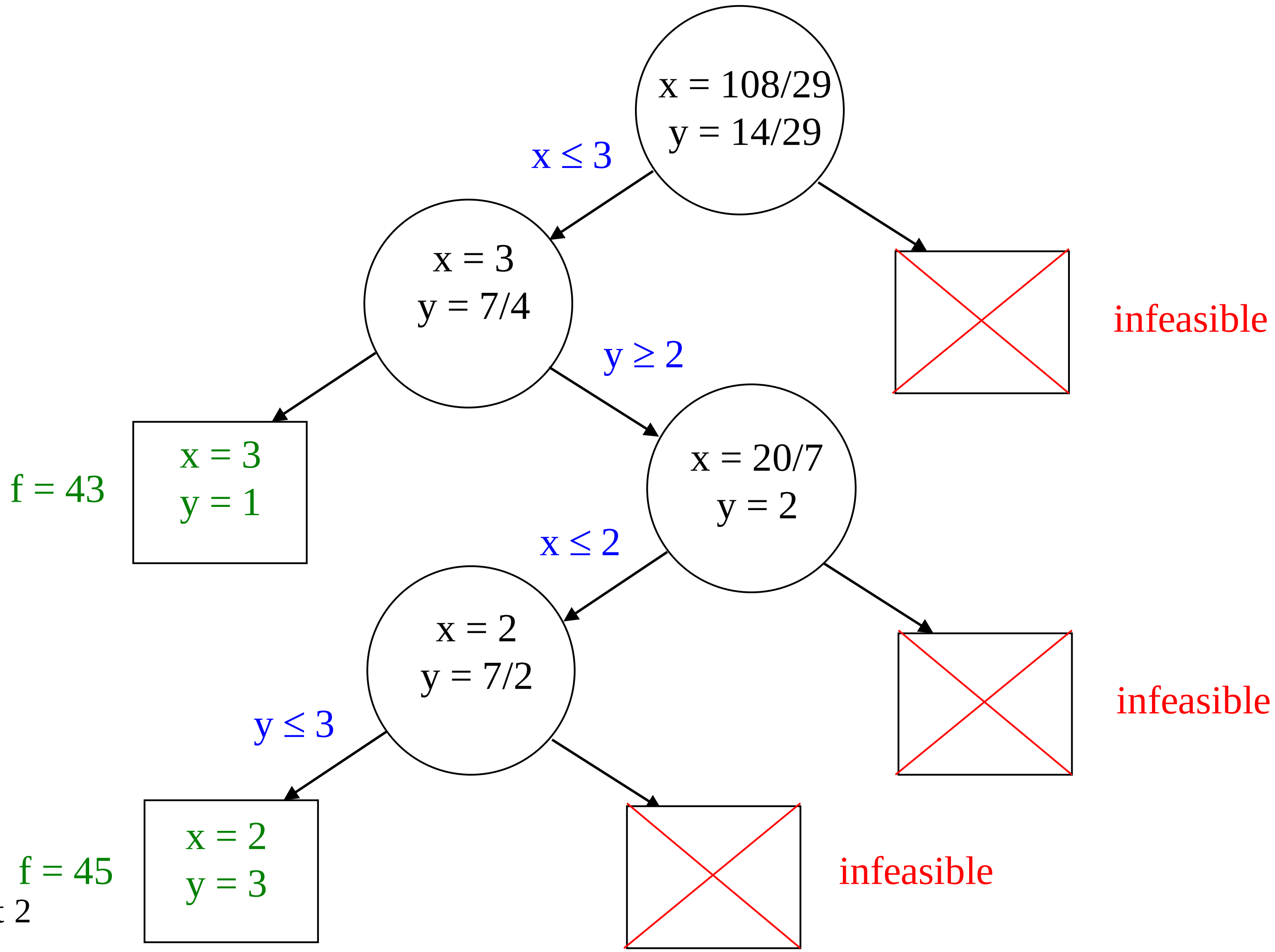
- maximize
  - f: $12x + 7y$
- subject to
  - g_1: $2x - 3y \le 6$
  - g_2: $7x + 4y \le 28$
  - g_3: $-x + y \le 2$
  - g_4: $-2x - y \le 2$

  where $x, y \in Z$



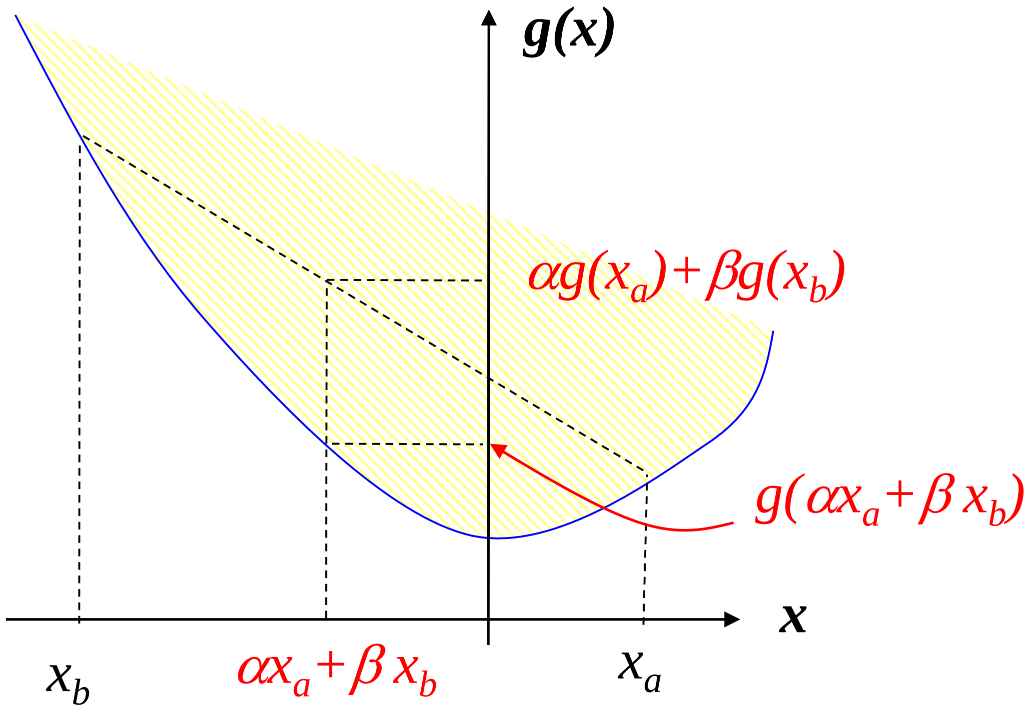
p_1 and p_2 are two possible points for optimal solution

LP Relaxation and Branch-and-Bound Procedure



Convex Optimization Problem

- Convexity property
  - $g_i(\alpha x_a + \beta x_b) \leq \alpha\, g_i(x_a) + \beta\, g_i(x_b)$
  - For a convex function, a local optimal solution is also a global optimal solution



Interior-Point Method

- An effective method that can solve the convex optimization problem in polynomial time within a reasonably small number of iterations
- Idea
  - Obtains an initial feasible solution which is approximated as an interior point
  - Iterates along a path, called a central path, as the approximation improves towards the optimal solution