Thermodynamic properties of cavity-assisted many-body atomic systems

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1 Model Hamiltonian and Partition Function

We consider an ensemble of N two-level atoms coupled by two photon Raman transition, and atoms are placed in a cavity. Atoms and photons are expected to interact in a highly nonlinear fashion, though atom themselves are considered non-interacting. We don't consider cavity pump and decay in this work.

Using bare atomic psedo-spin operater $\Psi_\sigma(\mathbf{r})$ and photon field operater $\tilde{c}$ (without explicit time dependence), we can write the atom-cavity Hamiltonian as,

$$H = \sum_{j=1}^{N}\sum_{\sigma}\int d\mathbf{r}\left[\Psi^\dagger_{j\sigma}(\mathbf{r})\left(\frac{\hbar^2\hat{\mathbf{k}}^2}{2m}+\epsilon^0_\sigma\right)\Psi_{j\sigma}(\mathbf{r})\right]+\frac{\Omega}{2}\frac{1}{\sqrt{V}}\sum_{j=1}^{N}\int d\mathbf{r}\left(e^{2i\hbar k_r z}\Psi^\dagger_{j\uparrow}(\mathbf{r})\Psi_{j\downarrow}(\mathbf{r})ce^{i\omega_R t}+\text{c.c}\right)+\hbar\omega_c\tilde{c}^\dagger\tilde{c}$$

We work in rotating frame $c=\tilde{c}e^{i\omega_R t}$ (which amounts to performing an unitary transformation $U=e^{i\omega_R\tilde{c}^\dagger\tilde{c}t}$ to the Hamiltonian by $H'=UHU^{-1}+i\frac{dU}{dt}U^{-1}$) and gauge transformation $\tilde{\psi}_{j\uparrow}=\Psi_{j\uparrow}e^{-i\hbar k_r z}$, $\tilde{\psi}_{j\downarrow}=\Psi_{j\downarrow}e^{+i\hbar k_r z}$. It is helpful to make use of Baker-Hausdorff lemma to compute UHU^{-1} term. For instance, $e^{i\omega_R\tilde{c}^\dagger\tilde{c}t}ce^{i\omega_R t}e^{-i\omega_R\tilde{c}^\dagger\tilde{c}t}=e^{i\omega_R t}(\tilde{c}+i(\omega_R t)[\tilde{c}^\dagger\tilde{c},\tilde{c}]+\frac{(i\omega_R t)^2}{2!}[\tilde{c}^\dagger\tilde{c},[\tilde{c}^\dagger\tilde{c},\tilde{c}]]+\ldots)=e^{i\omega_R t}(\tilde{c}-i(\omega_R t)\tilde{c}+\frac{(i\omega_R t)^2}{2!}\tilde{c}-\ldots)=e^{i\omega_R t}\tilde{c}e^{-i\omega_R t}=\tilde{c}$ and conjugate term follows similarly. After unitary transformations of the original Hamiltonian, we can write

$$H = \sum_{j=1}^{N}\int d\mathbf{r}\left[\begin{pmatrix}\tilde{\psi}^\dagger_{j\uparrow}(\mathbf{r}) & \tilde{\psi}^\dagger_{j\downarrow}(\mathbf{r})\end{pmatrix}\left[\frac{\hat{\mathbf{k}}^2}{2}+k_r\hat{k_z}\sigma_{jz}+\delta\sigma_{jz}\right]\begin{pmatrix}\tilde{\psi}_{j\uparrow}(\mathbf{r})\\ \tilde{\psi}_{j\downarrow}(\mathbf{r})\end{pmatrix}+\frac{\Omega}{2\sqrt{V}}\left(\tilde{\psi}^\dagger_{j\uparrow}(\mathbf{r})\tilde{\psi}_{j\downarrow}(\mathbf{r})c+c.c\right)\right]+\delta_R c^\dagger c \quad (1)$$

where $\delta_R=\omega_c-\omega_R$ which comes from time derivative of U and we have neglected constant energy $\frac{\hbar^2k_r^2}{2m}$ and incorprated energy shift of two-photon detunning into δ. Writing the operator in momentum space, we write $\tilde{\psi}_{j\sigma}(\mathbf{r})=\frac{V}{(2\pi)^3}\int d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\psi}_{j\sigma}(\mathbf{k})$ and Eq. 1 is reduced to,

$$H = \delta_R c^\dagger c+\sum_{j=1}^{N}\sum_{\mathbf{k}}\left[\left(h_1(\mathbf{k})\hat{\psi}^\dagger_{j\uparrow}(\mathbf{k})\hat{\psi}_{j\uparrow}(\mathbf{k})+h_2(\mathbf{k})\hat{\psi}^\dagger_{j\downarrow}(\mathbf{k})\hat{\psi}_{j\downarrow}(\mathbf{k})\right)+\frac{\Omega}{2}\frac{1}{\sqrt{V}}\left(\hat{\psi}^\dagger_{j\uparrow}(\mathbf{k})\hat{\psi}_{j\downarrow}(\mathbf{k})c+c^\dagger\hat{\psi}^\dagger_{j\downarrow}(\mathbf{k})\hat{\psi}_{j\uparrow}(\mathbf{k})\right)\right] \quad (2)$$

where $h_1(\mathbf{k})=\frac{\hbar^2\mathbf{k}^2}{2m}+\frac{\hbar^2}{m}k_rk_z+\delta$ and $h_2(\mathbf{k})=\frac{\hbar^2\mathbf{k}^2}{2m}-\frac{\hbar^2}{m}k_rk_z-\delta$. The thermodynamic functions can be calculated from the canonical partition function, $Z(N,T)=\text{Tr}[e^{-\beta H}]$. A convenient basis to calculate the trace of the partition function is the Glauber's coherent state $|\alpha\rangle$ for the photon field, then we have

$$Z(N,T)=\sum_{s_1=\uparrow,\downarrow}\ldots\sum_{s_N=\uparrow,\downarrow}\frac{V}{(2\pi)^3}\int d\mathbf{k}_1\ldots\frac{V}{(2\pi)^3}\int d\mathbf{k}_N\int\frac{d^2\alpha}{\pi}\langle\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N|\langle\alpha|e^{-\beta H}|\alpha\rangle|\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N\rangle \quad (3)$$

where atomic field is denoted by atom index j and momentum $\mathbf{k}$ and spin $\sigma=\uparrow,\downarrow$. It follows then that the expectation value becomes,

$$\langle\alpha|e^{-\beta H}|\alpha\rangle=\exp\left\{-\beta\delta_R|\alpha|^2-\beta\sum_{j=1}^{N}\sum_{\mathbf{k}}h_j(\mathbf{k})\right\} \quad (4)$$

$$h_j(\mathbf{k})=\left(h_1(\mathbf{k})\hat{\psi}^\dagger_{j\uparrow}(\mathbf{k})\hat{\psi}_{j\uparrow}(\mathbf{k})+h_2(\mathbf{k})\hat{\psi}^\dagger_{j\downarrow}(\mathbf{k})\hat{\psi}_{j\downarrow}(\mathbf{k})\right)+\frac{\Omega}{2}\frac{1}{\sqrt{V}}\left(\hat{\psi}^\dagger_{j\uparrow}(\mathbf{k})\hat{\psi}_{j\downarrow}(\mathbf{k})\alpha+\hat{\psi}^\dagger_{j\downarrow}(\mathbf{k})\hat{\psi}_{j\uparrow}(\mathbf{k})\alpha^*\right) \quad (5)$$

and using the property $[h_i(\mathbf{k}),h_j(\mathbf{k}')]=0$, we can reduce integrand of Eq. 3 to

$$\begin{aligned}\langle\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N|\langle\alpha|e^{-\beta H}|\alpha\rangle|\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N\rangle &= e^{-\beta\delta_R|\alpha|^2}\langle\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N|e^{-\beta\sum_{j=1}^{N}\sum_{\mathbf{k}}h_j(\mathbf{k})}|\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N\rangle & (6)\\ &= e^{-\beta\delta_R|\alpha|^2}\langle\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N|\prod_{j=1}^{N}\prod_{\mathbf{k}=-\infty}^{\infty}e^{-\beta h_j(\mathbf{k})}|\mathbf{k}_1s_1;\ldots;\mathbf{k}_Ns_N\rangle & (7)\\ &= e^{-\beta\delta_R|\alpha|^2}\prod_{j=1}^{N}\langle\mathbf{k}_js_j|e^{-\beta h_j(\mathbf{k}_j)}|\mathbf{k}_js_j\rangle & (8)\end{aligned}$$

From Eq. 3 and Eq. 8, we have

$$Z(N,T) = \sum_{s_1=\uparrow,\downarrow}\dots\sum_{s_N=\uparrow,\downarrow}\frac{V}{(2\pi)^3}\int d\mathbf{k}_1\dots\frac{V}{(2\pi)^3}\int d\mathbf{k}_N\int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left(\prod_{j=1}^{N}\langle\mathbf{k}_j s_j|e^{-\beta h_j(\mathbf{k}_j)}|\mathbf{k}_j s_j\rangle\right) \tag{9}$$

$$= \int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left\{\frac{V}{(2\pi)^3}\int d\mathbf{k}_j\mathrm{Tr}_\sigma\exp\left[-\beta\begin{pmatrix} h_1(\mathbf{k}_j) & \frac{\Omega}{2}\frac{1}{\sqrt{V}}\alpha \\ \frac{\Omega}{2}\frac{1}{\sqrt{V}}\alpha^* & h_2(\mathbf{k}_j)\end{pmatrix}\right]\right\}^N \tag{10}$$

where the Trace is only for spin degrees of freedom. The eigenvalue of the 2×2 matrix is given by

$$\epsilon_j^{\pm}(\mathbf{k}_j) = \frac{\hbar^2\mathbf{k}_j^2}{2m}\pm\sqrt{\left(\frac{\hbar^2}{m}k_r k_{jz}+\delta\right)^2+\left(\frac{\Omega}{2}\frac{1}{\sqrt{V}}\right)^2|\alpha|^2}\equiv\frac{\hbar^2\mathbf{k}_j^2}{2m}\pm|\mu(\mathbf{k}_j)| \tag{11}$$

Then from Eq. 10 we have,

$$Z(N,T) = \int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left[\frac{V}{(2\pi)^3}\int d\mathbf{k}_j\left(e^{-\beta\epsilon_j^+(\mathbf{k}_j)}+e^{-\beta\epsilon_j^-(\mathbf{k}_j)}\right)\right]^N \tag{12}$$

$$= \int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left[\frac{V}{(2\pi)^3}\int d\mathbf{k}\exp(-\beta\frac{\hbar^2\mathbf{k}^2}{2m})\left(e^{-\beta|\mu(\mathbf{k})|}+e^{\beta|\mu(\mathbf{k})|}\right)\right]^N \tag{13}$$

$$= \int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left[\frac{V}{(2\pi)^3}\int d\mathbf{k}\exp(-\beta\frac{\hbar^2\mathbf{k}^2}{2m})2\cosh\beta\sqrt{\left(\frac{\hbar^2}{m}k_r k_z+\delta\right)^2+\left(\frac{\Omega}{2}\frac{1}{\sqrt{V}}\right)^2|\alpha|^2}\right]^N \tag{14}$$

$$= \left[\frac{V}{(2\pi)^3}\int dk_x\int dk_y\exp(-\beta\frac{\hbar^2(k_x^2+k_y^2)}{2m})\right]^N \tag{15}$$

$$\times \int\frac{d^2\alpha}{\pi}e^{-\beta\delta_R|\alpha|^2}\left(\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})2\cosh(\beta\sqrt{\left(\frac{\hbar^2}{m}k_r k_z+\delta\right)^2+\left(\frac{\Omega}{2}\frac{1}{\sqrt{V}}\right)^2|\alpha|^2})\right)^N \tag{16}$$

$$= \left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)^N 2\int_0^\infty re^{-\beta\delta_R r^2}dr\left(\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})2\cosh(\beta\sqrt{\left(\frac{\hbar^2}{m}k_r k_z+\delta\right)^2+\left(\frac{\Omega}{2}\frac{1}{\sqrt{V}}\right)^2 r^2})\right)^N \tag{17}$$

In canonical ensemble and thermodynamic limit, free energy per particle is given by

$$f(T,\Omega,\omega_c,\delta) = -\frac{1}{\beta}\lim_{N\to\infty}\frac{\log Z(N,T)}{N} \tag{18}$$

and we need to seek phase transition property by studying saddle point solution to F energy landscapes.

Although we could take the limit of $N\to\infty$ numerically, we shall make use of Laplace's method to further reduce the integrals. The formal statement of Lapace's method is:

Assume that $f(x)$ is a twice differentiable function on $[a,b]$ with $x_0\in[a,b]$ the unique point such that $f(x_0) = \max_{[a,b]}f(x)$. Assume additionally that $f''(x_0)<0$, then

$$\lim_{N\to+\infty}\left(\frac{\int_a^b e^{Nf(x)}dx}{e^{Nf(x_0)}\sqrt{\frac{2\pi}{-Nf''(x_0)}}}\right) = 1 \tag{19}$$

We denote,

$$\mathcal{S} = \int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})2\cosh(\beta\sqrt{\left(\frac{\hbar^2}{m}k_r k_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2\frac{|\alpha|^2}{N}}) \tag{20}$$

Then, partition function is formally written as

$$Z(N,T) = \left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)^N 2\int_0^\infty re^{-N\frac{\beta\delta_R r^2}{N}}e^{N\log\mathcal{S}}dr \tag{21}$$

$$= N\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)^N\int_0^\infty dy\exp\left\{N\left[-\beta\delta_R y+\log\mathcal{S}\right]\right\} \tag{22}$$

where we have denoted $y = \frac{|\alpha|^2}{N}$. By Laplace's method, the integral is given by

$$Z(N,T) = N\frac{1}{\sqrt{N}}\sqrt{\frac{2\pi}{-\phi''(y_0)}}\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)^N \max_{0\le y\le\infty}\exp\{N\phi(y)\} \tag{23}$$

where y_0 is the point that gives maximum and $\phi(y) = -\beta\delta_R y + \log\mathcal{S}$. Then

$$\phi'(y) = -\beta\delta_R + \frac{\beta\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\sinh\left(\beta\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}\right)\frac{\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2}{\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}}}{\mathcal{S}} \tag{24}$$

Putting $\phi'(y) = 0$ we get an integral equation of $\eta = \sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}$, and we shall denote $\tilde{\Omega} = \sqrt{N}\Omega\frac{1}{\sqrt{V}} = \sqrt{\rho}\Omega$ is the Tavis-Cummings coupling constant, an enhancement of coupling strength which scales as $\sqrt{N}$, but remains finite in the thermodynamic limit that $N\to\infty$, $V\to\infty$ such that $\rho = \frac{N}{V}$ is finite.

$$\delta_R\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})2\cosh(\beta\eta) = \int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\sinh(\beta\eta)\frac{\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2}{\eta} \tag{25}$$

where k_z is yet to be integrated out. Then,

$$\begin{aligned}
-\beta f(\beta,\tilde{\Omega},\delta_R,\delta) &= \lim_{N\to\infty}\frac{1}{N}\log\left[N\frac{1}{\sqrt{N}}\sqrt{\frac{2\pi}{-\phi''(y_0)}}\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)^N\exp\{N\phi(y_0)\}\right] &(26)\\
&= \lim_{N\to\infty}\frac{\log\left(\sqrt{N}\sqrt{\frac{2\pi}{-\phi''(y_0)}}\right)}{N}+\log\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)+\phi(y_0) &(27)\\
&= \log\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)+\phi(y_0) &(28)\\
&= \log\left(\frac{V}{(2\pi)^3}\frac{2\pi m}{\hbar^2\beta}\right)-\beta\delta_Ry_0+\log\int dk_z\exp(-\beta\frac{k_z^2}{2})2\cosh(\beta\sqrt{(k_rk_z+\delta)^2+\left(\frac{\tilde{\Omega}}{2}\right)^2y_0}) &(29)
\end{aligned}$$

Second order derivative can be checked straightforwardly,

$$\phi''(y) = \frac{\frac{\partial Q}{\partial y}\mathcal{S}-Q^2}{\mathcal{S}^2} \tag{30}$$

where $Q \equiv \frac{\partial\mathcal{S}}{\partial y} = \beta\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\sinh\left(\beta\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}\right)\frac{\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2}{\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}}$ and

$$\begin{aligned}
\frac{\partial Q}{\partial y} &= \beta\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\partial_y\left(\frac{\sinh\left(\beta\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}\right)}{\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}}\right)\\
&= \beta\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^4\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\frac{\frac{\beta}{2}\cosh\left(\beta\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}\right)-\frac{\sinh\left(\beta\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}\right)}{2\sqrt{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y}}}{\left(\frac{\hbar^2}{m}k_rk_z+\delta\right)^2+\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^2y} &(31)\\
&= \beta\left(\frac{\sqrt{N}\Omega}{2}\frac{1}{\sqrt{V}}\right)^4\int dk_z\exp(-\beta\frac{\hbar^2k_z^2}{2m})\frac{\frac{\beta}{2}\cosh(\beta\eta)-\frac{\sinh(\beta\eta)}{2\eta}}{\eta^2} &(32)
\end{aligned}$$

In numerical calculations, we use dimensionless unit, by setting $\hbar = m = k_B = 1$ and choose a typical frequency $\omega = 1\text{MHz}$. Then it follows from the estimate, $k_r = 2\pi/\lambda$ and in dimensionless unit

$$k_r\sqrt{\hbar/(m\omega)} = \frac{2\pi}{773\times10^{-9}m}\sqrt{\frac{1.05\times10^{-34}J\cdot s}{(85\times1.67\times10^{-27}kg)\times(10^6Hz)}} \approx 0.22$$

for a rubidium atom. Temperature T is measured in $1/k_B$, $\tilde{\Omega} = \sqrt{\rho}\Omega$ and δ_R are measured in ω.

Note, although $\delta_R = \omega_c - \omega_R$ can be either positive or negative, partition function Eq. 3 can only be convergent when $\delta_R > 0$ (by analogy to positive electronic energy states) in the thermodynamic limit.

2 $k_r = 0$ physics: BEC and super-radiant phase

When we manually set $k_r = 0$, we have kinetic energy term on top of a Dicke (Tavis-Cummings) Hamiltonian. We shall investigate the effect of kinetic energy term, and study potential competing phases of Bose-Einstein condensation and super-radiant phase.

1. With the partition function, it is easy to compute the average photon number,

$$\begin{aligned}
\langle \frac{a^\dagger a}{N} \rangle &= \frac{\mathrm{Tr}[a^\dagger a e^{-\beta H}/N]}{Z(N,T)} && (33) \\
&= \frac{|\alpha|^2}{N} && (34) \\
&= y_0 && (35)
\end{aligned}$$

   Thus, super-radiant phase occurs for $y_0 > 0$ where $|\alpha|^2$ would be an infinitely large value in the thermodynamic limit $N \to \infty$.

2. If we consider the Hamiltonian without kinetic and SOC term, then

$$\begin{aligned}
Z(N,T) &= \int \frac{d^2\alpha}{\pi} e^{-\beta\delta_R|\alpha|^2} \left\{ \mathrm{Tr}_\sigma \exp\left[-\beta \begin{pmatrix} \delta & \frac{\Omega}{2}\frac{1}{\sqrt{V}}\alpha \\ \frac{\Omega}{2}\frac{1}{\sqrt{V}}\alpha^* & -\delta \end{pmatrix} \right] \right\}^N \\
&= \int \frac{d^2\alpha}{\pi} e^{-\beta\delta_R|\alpha|^2} \left[2\cosh\left(\beta\sqrt{\delta^2 + (\frac{\tilde{\Omega}}{2})^2 \frac{|\alpha|^2}{N}} \right) \right]^N \\
&= \int_0^\infty 2r dr e^{-\beta\delta_R r^2} \left[2\cosh\left(\beta\sqrt{\delta^2 + (\frac{\tilde{\Omega}}{2})^2 \frac{r^2}{N}} \right) \right]^N \\
&= N \int_0^\infty dy e^{-\beta N \delta_R y} \left[2\cosh\left(\beta\sqrt{\delta^2 + (\frac{\tilde{\Omega}}{2})^2 y} \right) \right]^N \\
&= N \int_0^\infty dy e^{N\left\{ -\beta\delta_R y + \log\left[2\cosh\left(\beta\sqrt{\delta^2 + (\frac{\tilde{\Omega}}{2})^2 y} \right) \right] \right\}} && (36)
\end{aligned}$$

   Let's denote $\phi(y) = -\beta\delta_R y + \log\left[2\cosh\left(\beta\delta\sqrt{1 + (\frac{\tilde{\Omega}}{2\delta})^2 y} \right) \right]$ and use Laplace's method to calculate Eq. 36. Putting $\phi'(y) = 0$, it is easy to see that the max is taken at $\frac{\delta_R\delta}{(\tilde{\Omega}/4)^2}\eta = \tanh(\beta\delta\eta)$ where $\eta = \sqrt{1 + (\frac{\tilde{\Omega}}{2\delta})^2 y}$. If $\tilde{\Omega}^2 < 16\delta_R\delta$, there is no phase transition even at zero temperature; however, if $\tilde{\Omega}^2 \geq 16\delta_R\delta$, there is a critical temperature T_c which is given by $\frac{\delta_R\delta}{(\tilde{\Omega}/4)^2} = \tanh(\frac{\delta}{k_B T_c})$. For $\beta > \beta_c$, the unique solution is given by a nonzero y_0 value that is determined by $\frac{\delta_R\delta}{(\tilde{\Omega}/4)^2}\sqrt{1 + (\frac{\tilde{\Omega}}{2\delta})^2 y_0} = \tanh(\beta\delta\sqrt{1 + (\frac{\tilde{\Omega}}{2\delta})^2 y_0})$.

3. If we consider average Boson number on a particular momentum state $\mathbf{q}$, we need to consider grand canonical ensemble and the expectation value is given by,

$$\begin{aligned}
\langle \hat{n}_\mathbf{q} \rangle &= \frac{\mathrm{Tr}[\hat{n}_\mathbf{q} e^{-\beta\hat{H} + \alpha\hat{N}}]}{\mathrm{Tr}[e^{-\beta\hat{H} + \alpha\hat{N}}]} && (37) \\
&= \frac{\partial}{\partial\alpha} \log \mathrm{Tr}[e^{-\beta\hat{H} + \alpha\hat{n}_\mathbf{q}}] && (38)
\end{aligned}$$

3 $k_r \neq 0$ physics: Effect of Spin-Orbit Coupling