

Quantum version of the model in a trap

Lin Dong, July 10, 2015

We can write the 1D atom-cavity Hamiltonian in a hamonic trap as,

$$\begin{aligned}\mathcal{H}_{\text{eff}} &= \int dz \begin{pmatrix} \psi_{\uparrow}^{\dagger}(z) & \psi_{\downarrow}^{\dagger}(z) \end{pmatrix} \left[\frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 + \frac{\hbar^2}{m} q_r k_z \sigma_z + \delta \sigma_z \right] \begin{pmatrix} \psi_{\uparrow}(z) \\ \psi_{\downarrow}(z) \end{pmatrix} \\ &+ \int dz \frac{\Omega}{2} \psi_{\uparrow}^{\dagger}(z) \psi_{\downarrow}(z) c + \int dz \frac{\Omega}{2} c^{\dagger} \psi_{\downarrow}^{\dagger}(z) \psi_{\uparrow}(z) \\ &+ i \varepsilon_p (c^{\dagger} - c) - \hbar \delta_c c^{\dagger} c.\end{aligned}\tag{1}$$

Dissipation process is modeled by Liouvillean term \mathcal{L} appearing in the master equation,

$$\dot{\rho} = \frac{1}{i\hbar} [\mathcal{H}_{\text{eff}}, \rho] + \mathcal{L}\rho\tag{2}$$

where

$$\mathcal{L}\rho = \kappa(2c\rho c^{\dagger} - c^{\dagger}c\rho - \rho c^{\dagger}c).\tag{3}$$

Then, we write the commutator explicitly as,

$$\begin{aligned}[\mathcal{H}_{\text{eff}}, \rho] &= \int dz \psi_{\uparrow}^{\dagger}(z) \left(\frac{k_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 + \frac{q_r k_z}{m} + \delta \right) \psi_{\uparrow}(z) \rho - \int dz \rho \psi_{\uparrow}^{\dagger}(z) \left(\frac{k_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 + \frac{q_r k_z}{m} + \delta \right) \psi_{\uparrow}(z) \\ &+ \int dz \psi_{\downarrow}^{\dagger}(z) \left(\frac{k_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 - \frac{q_r k_z}{m} - \delta \right) \psi_{\downarrow}(z) \rho - \int dz \rho \psi_{\downarrow}^{\dagger}(z) \left(\frac{k_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 - \frac{q_r k_z}{m} - \delta \right) \psi_{\downarrow}(z) \\ &+ \frac{\Omega}{2} \int dz \left(\psi_{\uparrow}^{\dagger}(z) \psi_{\downarrow}(z) c \rho + c^{\dagger} \psi_{\downarrow}^{\dagger}(z) \psi_{\uparrow}(z) \rho - \rho \psi_{\uparrow}^{\dagger}(z) \psi_{\downarrow}(z) c - \rho c^{\dagger} \psi_{\downarrow}^{\dagger}(z) \psi_{\uparrow}(z) \right) \\ &+ i \varepsilon_p (c^{\dagger} \rho - c \rho - \rho c^{\dagger} + \rho c) - \delta_c (c^{\dagger} c \rho - \rho c^{\dagger} c).\end{aligned}$$

To this end, we choose our basis states as $|n; q, \sigma\rangle$ where $n = 0, 1, 2, \dots, N$ and N is the truncation number of photon, $q = 1, 2, \dots, Q$ and Q is the truncation number of harmonic oscillator energy levels and $\sigma = \uparrow, \downarrow$. Our goal is to calculate matrix elements of density operator under this basis $\langle m; p, \sigma | \rho | n; q, \sigma' \rangle \equiv \rho_{mn}^{p\sigma q\sigma'}$. Rules for creation and annihilation operators are

$$\begin{aligned}c|n; q, \sigma'\rangle &= \sqrt{n}|n-1; q, \sigma'\rangle, \quad c^{\dagger}|n; q, \sigma'\rangle = \sqrt{n+1}|n+1; q, \sigma'\rangle \\ \langle m; q, \sigma | c &= \sqrt{m+1}\langle m+1; q, \sigma |, \quad \langle m; q, \sigma | c^{\dagger} = \sqrt{m}\langle m-1; q, \sigma |.\end{aligned}$$

We write field operator $\psi_{\sigma}(z) = \sum_{q=1}^Q \varphi_q(z) a_{q\sigma}$ in second quantization, where $\varphi_q(z)$ is the eigenstate wavefunction of harmonic oscillator. Also, $k_z = -i \frac{\partial}{\partial z}$ serves as first quantization and only operates on wavefunction $\varphi_p(z)$. For arbitrary state, we have (where we have chosen trap unit by setting $\hbar = m = \omega = 1$)

$$\begin{aligned}\text{FirstTerm} &\equiv \langle m; p, \sigma | \int dz \psi_{\uparrow}^{\dagger}(z) \left(\frac{k_z^2}{2} + \frac{1}{2} z^2 + q_r k_z + \delta \right) \psi_{\uparrow}(z) \rho | n; q, \sigma' \rangle \\ &= \langle m; p, \sigma | \int dz \sum_{p'} \varphi_{p'}^*(z) a_{p'\uparrow}^{\dagger} (H_{\text{osc}} + \delta - i q_r \frac{\partial}{\partial z}) \sum_{q'} \varphi_{q'}(z) a_{q'\uparrow} \rho | n; q\sigma' \rangle \\ &= \sum_{p'q'} \left[\int dz \varphi_{p'}^*(z) (H_{\text{osc}} + \delta - i q_r \frac{\partial}{\partial z}) \varphi_{q'}(z) \right] \langle m; p, \sigma | a_{p'\uparrow}^{\dagger} a_{q'\uparrow} \rho | n; q\sigma' \rangle \\ &= \sum_{p'q'} \left[(q' + \frac{1}{2} + \delta) \int dz \varphi_{p'}^*(z) \varphi_{q'}(z) - i q_r \int dz \varphi_{p'}^*(z) \frac{\partial}{\partial z} \varphi_{q'}(z) \right] \langle m; p, \sigma | a_{p'\uparrow}^{\dagger} a_{q'\uparrow} \rho | n; q\sigma' \rangle\end{aligned}$$

Easy to have $\int dz \varphi_{p'}^*(z) \varphi_{q'}(z) = \delta_{p'q'}$ but more steps to follow for $-i q_r \int dz \varphi_{p'}^*(z) \frac{\partial}{\partial z} \varphi_{q'}(z)$. From

$$\varphi_q(z) = \mathcal{A}_q H_q(z) e^{-\frac{z^2}{2}}, \quad \mathcal{A}_q = \frac{1}{\sqrt{2^q q!}} \frac{1}{\pi^{1/4}}.$$

where $H_q(z)$ is the Hermite polynomials, we have

$$\begin{aligned}
-i\frac{\partial}{\partial z}\varphi_q(z) &= -i\mathcal{A}_qe^{-\frac{z^2}{2}}(H'_q(z) - zH_q(z)) \\
&= -i\mathcal{A}_qe^{-\frac{z^2}{2}}(2qH_{q-1}(z) - zH_q(z)) \\
&= -i\mathcal{A}_qe^{-\frac{z^2}{2}}\left(2qH_{q-1}(z) - \frac{1}{2}(H_{q+1}(z) + 2qH_{q-1}(z))\right) \\
&= -i\mathcal{A}_qe^{-\frac{z^2}{2}}\left(qH_{q-1}(z) - \frac{1}{2}H_{q+1}(z)\right) \\
&= -i\left(q\frac{\mathcal{A}_q}{\mathcal{A}_{q-1}}\varphi_{q-1}(z) - \frac{1}{2}\frac{\mathcal{A}_q}{\mathcal{A}_{q+1}}\varphi_{q+1}(z)\right) \\
&= -i\left(\sqrt{\frac{q}{2}}\varphi_{q-1}(z) - \sqrt{\frac{q+1}{2}}\varphi_{q+1}(z)\right)
\end{aligned}$$

where we have used recurrence relation for Hermite polynomials, including $H'_n(x) = 2nH_{n-1}(x)$ or $H_{n+1}(x) = 2xH_n(x) - H'_n(x)$ and $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$. The above result is equivalent to replace $k_z = \frac{1}{\sqrt{2i}}(a - a^\dagger)$ and operate it on the wavefunction $\varphi_p(z)$, namely in one line we can prove,

$$\begin{aligned}
k_z\varphi_{q'}(z) &= \frac{1}{\sqrt{2i}}(a - a^\dagger)\varphi_{q'}(z) \\
&= -i\frac{1}{\sqrt{2}}\left(\sqrt{q'}\varphi_{q'-1}(z) - \sqrt{q'+1}\varphi_{q'+1}(z)\right)
\end{aligned}$$

Then, we should have $-iq_r \int dz \varphi_{p'}^*(z) \frac{\partial}{\partial z} \varphi_{q'}(z) = -iq_r \sqrt{\frac{1}{2}} \int dz \varphi_{p'}^*(z) (\sqrt{q'}\varphi_{q'-1}(z) - \sqrt{q'+1}\varphi_{q'+1}(z)) = -iq_r \sqrt{\frac{1}{2}} (\sqrt{q'}\delta_{p',q'-1} - \sqrt{q'+1}\delta_{p',q'+1})$. Also, note that

$$\begin{aligned}
\langle m; p, \sigma | a_{p'\uparrow}^\dagger a_{q'\uparrow} \rho | n; q\sigma' \rangle &= \delta_{p',p} \delta_{\sigma,\uparrow} \langle m; \text{vac} | a_{q'\uparrow} \rho | n; q\sigma' \rangle \\
&= \delta_{p',p} \delta_{\sigma,\uparrow} \langle m; q' \uparrow | \rho | n; q\sigma' \rangle \\
&= \delta_{p',p} \delta_{\sigma,\uparrow} \rho_{mn}^{q'\sigma q\sigma'}
\end{aligned}$$

To avoid confusion, we have to point out one subtle difference between a and $a_{q\sigma}$, where a is the lowering operator for the harmonic oscillator state and $a_{q\sigma}$ is the atom number annihilation operator at oscillator state q and spin σ . Thus $a\varphi_q(z) = \sqrt{q}\varphi_{q-1}(z)$, but $a_{q\sigma}|n; p\sigma'\rangle = \delta_{q,p}\delta_{\sigma,\sigma'}|n; \text{vac}\rangle$ and there is no \sqrt{p} term as a prefactor! Wrapping it up, the first term is massaged into

$$\begin{aligned}
&\delta_{\sigma,\uparrow} \sum_{p'q'} \left[\left(q' + \frac{1}{2} + \delta \right) \delta_{p'q'} - \frac{iq_r}{\sqrt{2}} \left(\sqrt{q'}\delta_{p',q'-1} - \sqrt{q'+1}\delta_{p',q'+1} \right) \right] \delta_{p',p} \rho_{mn}^{q'\sigma q\sigma'} \\
&= \left(p + \frac{1}{2} + \delta \right) \rho_{mn}^{p\uparrow q\sigma'} - \frac{iq_r}{\sqrt{2}} \left(\sqrt{p+1}\rho_{mn}^{(p+1)\uparrow q\sigma'} - \sqrt{p}\rho_{mn}^{(p-1)\uparrow q\sigma'} \right)
\end{aligned}$$

and the rest of terms could be followed similarly.

$$\begin{aligned}
&-\langle m; p, \sigma | \int dz \rho \psi_\uparrow^\dagger(z) \left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \frac{q_r k_z}{m} + \delta \right) \psi_\uparrow(z) | n; q, \sigma' \rangle = -\left(q + \frac{1}{2} + \delta \right) \rho_{mn}^{p\sigma q\uparrow} + \frac{iq_r}{\sqrt{2}} \left(\sqrt{q}\rho_{mn}^{p\sigma(q-1)\uparrow} - \sqrt{q+1}\rho_{mn}^{p\sigma(q+1)\uparrow} \right) \\
&\langle m; p, \sigma | \int dz \psi_\downarrow^\dagger(z) \left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \frac{q_r k_z}{m} - \delta \right) \psi_\downarrow(z) \rho | n; q, \sigma' \rangle = \left(p + \frac{1}{2} - \delta \right) \rho_{mn}^{p\downarrow q\sigma'} + \frac{iq_r}{\sqrt{2}} \left(\sqrt{p+1}\rho_{mn}^{(p+1)\downarrow q\sigma'} - \sqrt{p}\rho_{mn}^{(p-1)\downarrow q\sigma'} \right) \\
&-\langle m; p, \sigma | \int dz \rho \psi_\downarrow^\dagger(z) \left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \frac{q_r k_z}{m} - \delta \right) \psi_\downarrow(z) | n; q, \sigma' \rangle = -\left(q + \frac{1}{2} - \delta \right) \rho_{mn}^{p\sigma q\downarrow} - \frac{iq_r}{\sqrt{2}} \left(\sqrt{q}\rho_{mn}^{p\sigma(q-1)\downarrow} - \sqrt{q+1}\rho_{mn}^{p\sigma(q+1)\downarrow} \right) \\
&\langle m; p, \sigma | \frac{\Omega}{2} \int dz \left(\psi_\uparrow^\dagger(z) \psi_\downarrow(z) c \rho + c^\dagger \psi_\downarrow^\dagger(z) \psi_\uparrow(z) \rho - \rho \psi_\uparrow^\dagger(z) \psi_\downarrow(z) c - \rho c^\dagger \psi_\downarrow^\dagger(z) \psi_\uparrow(z) \right) | n; q, \sigma' \rangle = \sum_{p'q'} \int dz \varphi_{p'}^*(z) \varphi_{q'}(z) \times \\
&\left[\langle m; p, \sigma | a_{p'\uparrow}^\dagger a_{q'\downarrow} c \rho | n; q\sigma' \rangle + \langle m; p, \sigma | c^\dagger a_{p'\downarrow}^\dagger a_{q'\uparrow} \rho | n; q\sigma' \rangle - \langle m; p, \sigma | \rho a_{p'\uparrow}^\dagger a_{q'\downarrow} c | n; q\sigma' \rangle - \langle m; p, \sigma | \rho c^\dagger a_{p'\downarrow}^\dagger a_{q'\uparrow} | n; q\sigma' \rangle \right] = \\
&\sum_{p'q'} \delta_{p'q'} \left[\sqrt{m+1}\delta_{pp'}\delta_{\sigma\uparrow}\rho_{(m+1)n}^{q'\downarrow q\sigma'} + \sqrt{m}\delta_{pp'}\delta_{\sigma\downarrow}\rho_{(m-1)n}^{q'\uparrow q\sigma'} - \sqrt{n}\delta_{qq'}\delta_{\sigma'\downarrow}\rho_{m(n-1)}^{p\sigma p'\uparrow} - \sqrt{n+1}\delta_{qq'}\delta_{\sigma'\uparrow}\rho_{m(n+1)}^{p\sigma p'\downarrow} \right] = \\
&\sqrt{m+1}\delta_{\sigma\uparrow}\rho_{(m+1)n}^{p\downarrow q\sigma'} + \sqrt{m}\delta_{\sigma\downarrow}\rho_{(m-1)n}^{p\uparrow q\sigma'} - \sqrt{n}\delta_{\sigma'\downarrow}\rho_{m(n-1)}^{p\sigma q\uparrow} - \sqrt{n+1}\delta_{\sigma'\uparrow}\rho_{m(n+1)}^{p\sigma q\downarrow}
\end{aligned}$$

$$\begin{aligned}
& \langle m; p, \sigma | i\varepsilon_p (c^\dagger \rho - c\rho - \rho c^\dagger + \rho c) - \delta_c (c^\dagger c\rho - \rho c^\dagger c) | n; q, \sigma' \rangle = \\
& i\varepsilon_p \left(\sqrt{m} \rho_{(m-1)n}^{p\sigma q\sigma'} - \sqrt{m+1} \rho_{(m+1)n}^{p\sigma q\sigma'} - \sqrt{n+1} \rho_{m(n+1)}^{p\sigma q\sigma'} + \sqrt{n} \rho_{m(n-1)}^{p\sigma q\sigma'} \right) - \delta_c (m-n) \rho_{mn}^{p\sigma q\sigma'} \\
& \kappa \langle m; p, \sigma | (2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c) | n; q, \sigma' \rangle = \kappa \left(2\sqrt{m+1}\sqrt{n+1} \rho_{(m+1)(n+1)}^{p\sigma q\sigma'} - (m+n) \rho_{mn}^{p\sigma q\sigma'} \right)
\end{aligned}$$

With above preparations, we write master equation Eq. 2 as,

$$\begin{aligned}
\frac{d}{dt} \rho_{mn}^{p\sigma q\sigma'} &= \frac{1}{i} \left[\left(p + \frac{1}{2} + \delta \right) \rho_{mn}^{p\uparrow q\sigma'} - \frac{iq_r}{\sqrt{2}} \left(\sqrt{p+1} \rho_{mn}^{(p+1)\uparrow q\sigma'} - \sqrt{p} \rho_{mn}^{(p-1)\uparrow q\sigma'} \right) \right] \\
&+ \frac{1}{i} \left[- \left(q + \frac{1}{2} + \delta \right) \rho_{mn}^{p\sigma q\uparrow} + \frac{iq_r}{\sqrt{2}} \left(\sqrt{q} \rho_{mn}^{p\sigma(q-1)\uparrow} - \sqrt{q+1} \rho_{mn}^{p\sigma(q+1)\uparrow} \right) \right] \\
&+ \frac{1}{i} \left[\left(p + \frac{1}{2} - \delta \right) \rho_{mn}^{p\downarrow q\sigma'} + \frac{iq_r}{\sqrt{2}} \left(\sqrt{p+1} \rho_{mn}^{(p+1)\downarrow q\sigma'} - \sqrt{p} \rho_{mn}^{(p-1)\downarrow q\sigma'} \right) \right] \\
&+ \frac{1}{i} \left[- \left(q + \frac{1}{2} - \delta \right) \rho_{mn}^{p\sigma q\downarrow} - \frac{iq_r}{\sqrt{2}} \left(\sqrt{q} \rho_{mn}^{p\sigma(q-1)\downarrow} - \sqrt{q+1} \rho_{mn}^{p\sigma(q+1)\downarrow} \right) \right] \\
&+ \frac{1}{i} \left[\sqrt{m+1} \delta_{\sigma\uparrow} \rho_{(m+1)n}^{p\downarrow q\sigma'} + \sqrt{m} \delta_{\sigma\downarrow} \rho_{(m-1)n}^{p\uparrow q\sigma'} - \sqrt{n} \delta_{\sigma'\downarrow} \rho_{m(n-1)}^{p\sigma q\uparrow} - \sqrt{n+1} \delta_{\sigma'\uparrow} \rho_{m(n+1)}^{p\sigma q\downarrow} \right] \\
&+ \varepsilon_p \left(\sqrt{m} \rho_{(m-1)n}^{p\sigma q\sigma'} - \sqrt{m+1} \rho_{(m+1)n}^{p\sigma q\sigma'} - \sqrt{n+1} \rho_{m(n+1)}^{p\sigma q\sigma'} + \sqrt{n} \rho_{m(n-1)}^{p\sigma q\sigma'} \right) \\
&+ i\delta_c (m-n) \rho_{mn}^{p\sigma q\sigma'} + \frac{\kappa}{i} \left(2\sqrt{m+1}\sqrt{n+1} \rho_{(m+1)(n+1)}^{p\sigma q\sigma'} - (m+n) \rho_{mn}^{p\sigma q\sigma'} \right)
\end{aligned} \tag{4}$$