Quantum version of the model in a trap

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We can write the 1D atom-cavity Hamiltonian in a hamonic trap as,

$$\begin{aligned}
\mathcal{H}_{\text{eff}} &= \int dz \begin{pmatrix} \psi_\uparrow^\dagger(z) & \psi_\downarrow^\dagger(z) \end{pmatrix} \left[\frac{\hbar^2 k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \frac{\hbar^2}{m} q_r k_z \sigma_z + \delta\sigma_z\right] \begin{pmatrix} \psi_\uparrow(z) \\ \psi_\downarrow(z) \end{pmatrix} \\
&+ \int dz \frac{\Omega}{2}\psi_\uparrow^\dagger(z)\psi_\downarrow(z)c + \int dz \frac{\Omega}{2} c^\dagger \psi_\downarrow^\dagger(z)\psi_\uparrow(z) \\
&+ i\varepsilon_p(c^\dagger - c) - \hbar\delta_c c^\dagger c. \qquad (1)
\end{aligned}$$

Dissipation process is modeled by Liouvillean term $\mathcal{L}$ appearing in the master equation,

$$\dot{\rho} = \frac{1}{i\hbar}[\mathcal{H}_{\text{eff}}, \rho] + \mathcal{L}\rho \qquad (2)$$

where

$$\mathcal{L}\rho = \kappa(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c). \qquad (3)$$

Then, we write the commutator explicitly as,

$$\begin{aligned}
[\mathcal{H}_{\text{eff}}, \rho] &= \int dz \psi_\uparrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \frac{q_r k_z}{m} + \delta\right)\psi_\uparrow(z)\rho - \int dz \rho\psi_\uparrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \frac{q_r k_z}{m} + \delta\right)\psi_\uparrow(z) \\
&+ \int dz \psi_\downarrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \frac{q_r k_z}{m} - \delta\right)\psi_\downarrow(z)\rho - \int dz \rho\psi_\downarrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \frac{q_r k_z}{m} - \delta\right)\psi_\downarrow(z) \\
&+ \frac{\Omega}{2}\int dz \left(\psi_\uparrow^\dagger(z)\psi_\downarrow(z)c\rho + c^\dagger\psi_\downarrow^\dagger(z)\psi_\uparrow(z)\rho - \rho\psi_\uparrow^\dagger(z)\psi_\downarrow(z)c - \rho c^\dagger\psi_\downarrow^\dagger(z)\psi_\uparrow(z)\right) \\
&+ i\varepsilon_p\left(c^\dagger\rho - c\rho - \rho c^\dagger + \rho c\right) - \delta_c\left(c^\dagger c\rho - \rho c^\dagger c\right).
\end{aligned}$$

To this end, we choose our basis states as $|n; q, \sigma\rangle$ where $n = 0, 1, 2, ...N$ and N is the truncation number of photon, $q = 0, 1, 2, ...Q$ and Q is the truncation number of harmonic oscillator energy levels and $\sigma =\uparrow, \downarrow$. Our goal is to calculate matrix elements of density operator under this basis $\langle m; p, \sigma|\rho|n; q, \sigma'\rangle \equiv \rho_{mn}^{p\sigma q\sigma'}$. Rules for creation and annilation operators are

$$\begin{aligned}
c|n; q, \sigma'\rangle &= \sqrt{n}|n-1; q, \sigma'\rangle, \quad c^\dagger|n; q, \sigma'\rangle = \sqrt{n+1}|n+1; q, \sigma'\rangle \\
\langle m; q, \sigma|c &= \sqrt{m+1}\langle m+1; q, \sigma|, \langle m; q, \sigma|c^\dagger = \sqrt{m}\langle m-1; q, \sigma|.
\end{aligned}$$

We write field operator $\psi_\sigma(z) = \sum_{q=1}^{Q}\varphi_q(z)a_{q\sigma}$ in second quantization, where $\varphi_q(z)$ is the eigenstate wavefunction of harmonic oscillator. Also, $k_z = -i\frac{\partial}{\partial z}$ serves as first quantization and only operates on wavefunction $\varphi_p(z)$. For arbitrary state, we have (where we have chosen trap unit by setting $\hbar = m = \omega = 1$)

$$\begin{aligned}
\text{FirstTerm} \equiv &\ \langle m; p, \sigma| \textstyle\int dz \psi_\uparrow^\dagger(z)\left(\frac{k_z^2}{2} + \frac{1}{2}z^2 + q_r k_z + \delta\right)\psi_\uparrow(z)\rho|n; q, \sigma'\rangle \\
= &\ \langle m; p, \sigma| \textstyle\int dz \sum_{p'}\varphi_{p'}^*(z)a_{p'\uparrow}^\dagger\left(H_{\text{osc}} + \delta - iq_r\frac{\partial}{\partial z}\right)\sum_{q'}\varphi_{q'}(z)a_{q'\uparrow}\rho|n; q\sigma'\rangle \\
= &\ \textstyle\sum_{p'q'}\left[\int dz \varphi_{p'}^*(z)\left(H_{\text{osc}} + \delta - iq_r\frac{\partial}{\partial z}\right)\varphi_{q'}(z)\right]\langle m; p, \sigma|a_{p'\uparrow}^\dagger a_{q'\uparrow}\rho|n; q\sigma'\rangle \\
= &\ \textstyle\sum_{p'q'}\left[(q' + \frac{1}{2} + \delta)\int dz\varphi_{p'}^*(z)\varphi_{q'}(z) - iq_r\int dz\varphi_{p'}^*(z)\frac{\partial}{\partial z}\varphi_{q'}(z)\right]\langle m; p, \sigma|a_{p'\uparrow}^\dagger a_{q'\uparrow}\rho|n; q\sigma'\rangle
\end{aligned}$$

Easy to have $\int dz\varphi_{p'}^*(z)\varphi_{q'}(z) = \delta_{p'q'}$ but more steps to follow for $-iq_r\int dz\varphi_{p'}^*(z)\frac{\partial}{\partial z}\varphi_{q'}(z)$. From

$$\varphi_q(z) = \mathcal{A}_q H_q(z)e^{-\frac{z^2}{2}}, \quad \mathcal{A}_q = \frac{1}{\sqrt{2^q q!}}\frac{1}{\pi^{1/4}}.$$

where $H_q(z)$ is the Hermite polynomials, we have

$$
\begin{aligned}
-i\frac{\partial}{\partial z}\varphi_q(z) &= -i\mathcal{A}_q e^{-\frac{z^2}{2}}\left(H_q'(z) - zH_q(z)\right) \\
&= -i\mathcal{A}_q e^{-\frac{z^2}{2}}\left(2qH_{q-1}(z) - zH_q(z)\right) \\
&= -i\mathcal{A}_q e^{-\frac{z^2}{2}}\left(2qH_{q-1}(z) - \frac{1}{2}(H_{q+1}(z) + 2qH_{q-1}(z))\right) \\
&= -i\mathcal{A}_q e^{-\frac{z^2}{2}}\left(qH_{q-1}(z) - \frac{1}{2}H_{q+1}(z)\right) \\
&= -i\left(q\frac{\mathcal{A}_q}{\mathcal{A}_{q-1}}\varphi_{q-1}(z) - \frac{1}{2}\frac{\mathcal{A}_q}{\mathcal{A}_{q+1}}\varphi_{q+1}(z)\right) \\
&= -i\left(\sqrt{\frac{q}{2}}\varphi_{q-1}(z) - \sqrt{\frac{q+1}{2}}\varphi_{q+1}(z)\right)
\end{aligned}
$$

where we have used recurrence relation for Hermite polynomials, including $H_n'(x) = 2nH_{n-1}(x)$ or $H_{n+1}(x) = 2xH_n(x) - H_n'(x)$ and $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$. The above result is equivalent to replace $k_z = \frac{1}{\sqrt{2}i}(a - a^\dagger)$ and operate it on the wavefunction $\varphi_p(z)$, namely in one line we can prove,

$$
\begin{aligned}
k_z\varphi_{q'}(z) &= \frac{1}{\sqrt{2}i}(a - a^\dagger)\varphi_{q'}(z) \\
&= -i\frac{1}{\sqrt{2}}\left(\sqrt{q'}\varphi_{q'-1}(z) - \sqrt{q'+1}\varphi_{q'+1}(z)\right)
\end{aligned}
$$

Then, we should have $-iq_r\int dz\varphi_{p'}^*(z)\frac{\partial}{\partial z}\varphi_{q'}(z) = -iq_r\sqrt{\frac{1}{2}}\int dz\varphi_{p'}^*(z)(\sqrt{q'}\varphi_{q'-1}(z) - \sqrt{q'+1}\varphi_{q'+1}(z)) = -iq_r\sqrt{\frac{1}{2}}(\sqrt{q'}\delta_{p',q'-1} - \sqrt{q'+1}\delta_{p',q'+1})$. Also, note that

$$
\begin{aligned}
\langle m; p, \sigma|a_{p'\uparrow}^\dagger a_{q'\uparrow}\rho|n; q\sigma'\rangle &= \delta_{p',p}\delta_{\sigma,\uparrow}\langle m; \mathrm{vac}|a_{q'\uparrow}\rho|n; q\sigma'\rangle \\
&= \delta_{p',p}\delta_{\sigma,\uparrow}\langle m; q'\uparrow|\rho|n; q\sigma'\rangle \\
&= \delta_{p',p}\delta_{\sigma,\uparrow}\rho_{mn}^{q'\sigma q\sigma'}
\end{aligned}
$$

To avoid confusion, we have to point out one subtle difference between a and $a_{q\sigma}$, where a is the lowering operator for the harmonic oscillator state and $a_{q\sigma}$ is the atom number annhilation operator at oscillator state q and spin σ. Thus $a\varphi_q(z) = \sqrt{q}\varphi_{q-1}(z)$, but $a_{q\sigma}|n; p\sigma'\rangle = \delta_{q,p}\delta_{\sigma,\sigma'}|n; \mathrm{vac}\rangle$ and there is no $\sqrt{p}$ term as a prefactor! Wrapping it up, the first term is massaged into

$$
\begin{aligned}
&\delta_{\sigma,\uparrow}\textstyle\sum_{p'q'}\left[\left(q' + \frac{1}{2} + \delta\right)\delta_{p'q'} - \frac{iq_r}{\sqrt{2}}\left(\sqrt{q'}\delta_{p',q'-1} - \sqrt{q'+1}\delta_{p',q'+1}\right)\right]\delta_{p',p}\rho_{mn}^{q'\sigma q\sigma'} \\
= &\left(p + \tfrac{1}{2} + \delta\right)\rho_{mn}^{p\uparrow q\sigma'} - \frac{iq_r}{\sqrt{2}}\left(\sqrt{p+1}\rho_{mn}^{(p+1)\uparrow q\sigma'} - \sqrt{p}\rho_{mn}^{(p-1)\uparrow q\sigma'}\right)
\end{aligned}
$$

and the rest of terms could be followed similarly.

$$
-\langle m; p, \sigma|\int dz\rho\psi_\uparrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2z^2 + \frac{q_rk_z}{m} + \delta\right)\psi_\uparrow(z)|n; q, \sigma'\rangle = -\left(q + \frac{1}{2} + \delta\right)\rho_{mn}^{p\sigma q\uparrow} + \frac{iq_r}{\sqrt{2}}\left(\sqrt{q}\rho_{mn}^{p\sigma(q-1)\uparrow} - \sqrt{q+1}\rho_{mn}^{p\sigma(q+1)\uparrow}\right)
$$

$$
\langle m; p, \sigma|\int dz\psi_\downarrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2z^2 - \frac{q_rk_z}{m} - \delta\right)\psi_\downarrow(z)\rho|n; q, \sigma'\rangle = \left(p + \frac{1}{2} - \delta\right)\rho_{mn}^{p\downarrow q\sigma'} + \frac{iq_r}{\sqrt{2}}\left(\sqrt{p+1}\rho_{mn}^{(p+1)\downarrow q\sigma'} - \sqrt{p}\rho_{mn}^{(p-1)\downarrow q\sigma'}\right)
$$

$$
-\langle m; p, \sigma|\int dz\rho\psi_\downarrow^\dagger(z)\left(\frac{k_z^2}{2m} + \frac{1}{2}m\omega^2z^2 - \frac{q_rk_z}{m} - \delta\right)\psi_\downarrow(z)|n; q, \sigma'\rangle = -\left(q + \frac{1}{2} - \delta\right)\rho_{mn}^{p\sigma q\downarrow} - \frac{iq_r}{\sqrt{2}}\left(\sqrt{q}\rho_{mn}^{p\sigma(q-1)\downarrow} - \sqrt{q+1}\rho_{mn}^{p\sigma(q+1)\downarrow}\right)
$$

$$
\begin{aligned}
&\langle m; p, \sigma|\frac{\Omega}{2}\int dz\left(\psi_\uparrow^\dagger(z)\psi_\downarrow(z)c\rho + c^\dagger\psi_\downarrow^\dagger(z)\psi_\uparrow(z)\rho - \rho\psi_\uparrow^\dagger(z)\psi_\downarrow(z)c - \rho c^\dagger\psi_\downarrow^\dagger(z)\psi_\uparrow(z)\right)|n; q, \sigma'\rangle = \frac{\Omega}{2}\sum_{p'q'}\int dz\varphi_{p'}^*(z)\varphi_{q'}(z)\times \\
&\left[\langle m; p, \sigma|a_{p'\uparrow}^\dagger a_{q'\downarrow}c\rho|n; q\sigma'\rangle + \langle m; p, \sigma|c^\dagger a_{p'\downarrow}^\dagger a_{q'\uparrow}\rho|n; q\sigma'\rangle - \langle m; p, \sigma|\rho a_{p'\uparrow}^\dagger a_{q'\downarrow}c|n; q\sigma'\rangle - \langle m; p, \sigma|\rho c^\dagger a_{p'\downarrow}^\dagger a_{q'\uparrow}|n; q\sigma'\rangle\right] = \\
&\frac{\Omega}{2}\sum_{p'q'}\delta_{p'q'}\left[\sqrt{m+1}\delta_{pp'}\delta_{\sigma\uparrow}\rho_{(m+1)n}^{q'\downarrow q\sigma'} + \sqrt{m}\delta_{pp'}\delta_{\sigma\downarrow}\rho_{(m-1)n}^{q'\uparrow q\sigma'} - \sqrt{n}\delta_{qq'}\delta_{\sigma'\downarrow}\rho_{m(n-1)}^{p\sigma p'\uparrow} - \sqrt{n+1}\delta_{qq'}\delta_{\sigma'\uparrow}\rho_{m(n+1)}^{p\sigma p'\downarrow}\right] = \frac{\Omega}{2} \\
&\sqrt{m+1}\delta_{\sigma\uparrow}\rho_{(m+1)n}^{p\downarrow q\sigma'} + \sqrt{m}\delta_{\sigma\downarrow}\rho_{(m-1)n}^{p\uparrow q\sigma'} - \sqrt{n}\delta_{\sigma'\downarrow}\rho_{m(n-1)}^{p\sigma q\uparrow} - \sqrt{n+1}\delta_{\sigma'\uparrow}\rho_{m(n+1)}^{p\sigma q\downarrow}
\end{aligned}
$$

$$\langle m;p,\sigma|i\varepsilon_p\left(c^\dagger\rho-c\rho-\rho c^\dagger+\rho c\right)-\delta_c\left(c^\dagger c\rho-\rho c^\dagger c\right)|n;q,\sigma'\rangle \quad = \quad i\varepsilon_p\left(\sqrt{m}\rho^{p\sigma q\sigma'}_{(m-1)n}-\sqrt{m+1}\rho^{p\sigma q\sigma'}_{(m+1)n}-\sqrt{n+1}\rho^{p\sigma q\sigma'}_{m(n+1)}+\sqrt{n}\rho^{p\sigma q\sigma'}_{m(n-1)}\right)-\delta_c\left(m-n\right)\rho^{p\sigma q\sigma'}_{mn}$$

$$\kappa\langle m;p,\sigma|(2c\rho c^\dagger-c^\dagger c\rho-\rho c^\dagger c)|n;q,\sigma'\rangle=\kappa\left(2\sqrt{m+1}\sqrt{n+1}\rho^{p\sigma q\sigma'}_{(m+1)(n+1)}-(m+n)\rho^{p\sigma q\sigma'}_{mn}\right)$$

With above preparations, we write master equation Eq. 2 as,

$$\begin{aligned}
\frac{d}{dt}\rho^{p\sigma q\sigma'}_{mn} \quad &= \quad \frac{1}{i}\left[\left(p+\frac{1}{2}+\delta\right)\rho^{p\uparrow q\sigma'}_{mn}-\frac{iq_r}{\sqrt{2}}\left(\sqrt{p+1}\rho^{(p+1)\uparrow q\sigma'}_{mn}-\sqrt{p}\rho^{(p-1)\uparrow q\sigma'}_{mn}\right)\right] \\
&+ \quad \frac{1}{i}\left[-\left(q+\frac{1}{2}+\delta\right)\rho^{p\sigma q\uparrow}_{mn}+\frac{iq_r}{\sqrt{2}}\left(\sqrt{q}\rho^{p\sigma(q-1)\uparrow}_{mn}-\sqrt{q+1}\rho^{p\sigma(q+1)\uparrow}_{mn}\right)\right] \\
&+ \quad \frac{1}{i}\left[\left(p+\frac{1}{2}-\delta\right)\rho^{p\downarrow q\sigma'}_{mn}+\frac{iq_r}{\sqrt{2}}\left(\sqrt{p+1}\rho^{(p+1)\downarrow q\sigma'}_{mn}-\sqrt{p}\rho^{(p-1)\downarrow q\sigma'}_{mn}\right)\right] \\
&+ \quad \frac{1}{i}\left[-\left(q+\frac{1}{2}-\delta\right)\rho^{p\sigma q\downarrow}_{mn}-\frac{iq_r}{\sqrt{2}}\left(\sqrt{q}\rho^{p\sigma(q-1)\downarrow}_{mn}-\sqrt{q+1}\rho^{p\sigma(q+1)\downarrow}_{mn}\right)\right] \\
&+ \quad \frac{\Omega}{2}\frac{1}{i}\left[\sqrt{m+1}\delta_{\sigma\uparrow}\rho^{p\downarrow q\sigma'}_{(m+1)n}+\sqrt{m}\delta_{\sigma\downarrow}\rho^{p\uparrow q\sigma'}_{(m-1)n}-\sqrt{n}\delta_{\sigma'\downarrow}\rho^{p\sigma q\uparrow}_{m(n-1)}-\sqrt{n+1}\delta_{\sigma'\uparrow}\rho^{p\sigma q\downarrow}_{m(n+1)}\right] \\
&+ \quad \varepsilon_p\left(\sqrt{m}\rho^{p\sigma q\sigma'}_{(m-1)n}-\sqrt{m+1}\rho^{p\sigma q\sigma'}_{(m+1)n}-\sqrt{n+1}\rho^{p\sigma q\sigma'}_{m(n+1)}+\sqrt{n}\rho^{p\sigma q\sigma'}_{m(n-1)}\right) \\
&+ \quad i\delta_c\left(m-n\right)\rho^{p\sigma q\sigma'}_{mn}+\kappa\left(2\sqrt{m+1}\sqrt{n+1}\rho^{p\sigma q\sigma'}_{(m+1)(n+1)}-(m+n)\rho^{p\sigma q\sigma'}_{mn}\right)
\end{aligned} \tag{4}$$

Following the previous work, we define

$$[\rho^{p\sigma q\sigma'}_{mn}]_{(2N+2)(Q+1)\times(2N+2)(Q+1)}=\begin{pmatrix} [\rho^{p\uparrow q\uparrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} & [\rho^{p\uparrow q\downarrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \\ [\rho^{p\downarrow q\uparrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} & [\rho^{p\downarrow q\downarrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \end{pmatrix} \tag{5}$$

We columnize the matrix array by array. For instance, in the column of $[\rho^{p\uparrow q\uparrow}_{mn}]_{(N+1)^2Q^2\times 1}$, the kth element is accessed as $k=m(N+1)(Q+1)^2+n(Q+1)^2+p(Q+1)+q+1$ where $m,n=0,1,2,...N$ and $p,q=0,1,2,...Q$. We then further write EOM of density matrix as

$$[\rho]=\begin{pmatrix} [\rho^{p\uparrow q\uparrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \\ [\rho^{p\uparrow q\downarrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \\ [\rho^{p\downarrow q\uparrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \\ [\rho^{p\downarrow q\downarrow}_{mn}]_{(N+1)(Q+1)\times(N+1)(Q+1)} \end{pmatrix}_{(2N+2)^2(Q+1)^2\times 1}$$

$$\frac{d}{dt}[\rho]=\begin{pmatrix} [M^{p\uparrow q\uparrow}_{mn}] & [S^{1pq}_{mn}] & [S^{2pq}_{mn}] & 0 \\ [S^{3pq}_{mn}] & [M^{p\uparrow q\downarrow}_{mn}] & 0 & [S^{4pq}_{mn}] \\ [S^{5pq}_{mn}] & 0 & [M^{p\downarrow q\uparrow}_{mn}] & [S^{6pq}_{mn}] \\ 0 & [S^{7pq}_{mn}] & [S^{8pq}_{mn}] & [M^{p\downarrow q\downarrow}_{mn}] \end{pmatrix}_{(2N+2)^2(Q+1)^2\times(2N+2)^2(Q+1)^2} [\rho] \tag{6}$$

To benchmark the results, we first consider the case without pumping and decay, with zero photon inside the cavity and atom being the excited state.

- If we further set $q_r = 0$, then orbital degree becomes a good quantum number. In this simplified case, we have the small subspace $\{|n, q, \sigma\rangle\}$ where $n = 0, 1$ and $\sigma =\uparrow, \downarrow$, q is only a constant number. We use simplified notation of density operator and write the master equation as,

$$\begin{aligned}\frac{d}{dt}\rho_{mn}^{q\sigma q\sigma'} &= \frac{1}{i}\left[\left(q+\frac{1}{2}+\delta\right)\rho_{mn}^{q\uparrow q\sigma'} - \left(q+\frac{1}{2}+\delta\right)\rho_{mn}^{q\sigma q\uparrow} + \left(q+\frac{1}{2}-\delta\right)\rho_{mn}^{q\downarrow q\sigma'} - \left(q+\frac{1}{2}-\delta\right)\rho_{mn}^{q\sigma q\downarrow}\right] \\ &+ \frac{\Omega}{2}\frac{1}{i}\left[\sqrt{m+1}\delta_{\sigma\uparrow}\rho_{(m+1)n}^{p\downarrow q\sigma'} + \sqrt{m}\delta_{\sigma\downarrow}\rho_{(m-1)n}^{p\uparrow q\sigma'} - \sqrt{n}\delta_{\sigma'\downarrow}\rho_{m(n-1)}^{p\sigma q\uparrow} - \sqrt{n+1}\delta_{\sigma'\uparrow}\rho_{m(n+1)}^{p\sigma q\downarrow}\right]\end{aligned}$$

  Due to conservation of excitation number, the Hilbert space is further reduced to the smaller subspace. For instance, if we have $|N-1, q, \uparrow\rangle$ as initial state, then we would only have population in another state $|N, q, \downarrow\rangle$ and these two states lead to four elements of density matrix and in solving the master equation, we are dealing with 4 by 4 ODE time evolution for $N = 1$,

$$\frac{d}{dt}\begin{pmatrix}\rho_{00}^{q\uparrow q\uparrow} \\ \rho_{01}^{q\uparrow q\downarrow} \\ \rho_{10}^{q\downarrow q\uparrow} \\ \rho_{11}^{q\downarrow q\downarrow}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{\Omega}{2i} & \frac{\Omega}{2i} & 0 \\ -\frac{\Omega}{2i} & \frac{2\delta}{i} & 0 & \frac{\Omega}{2i} \\ \frac{\Omega}{2i} & 0 & -\frac{2\delta}{i} & -\frac{\Omega}{2i} \\ 0 & \frac{\Omega}{2i} & -\frac{\Omega}{2i} & 0\end{pmatrix}\begin{pmatrix}\rho_{00}^{q\uparrow q\uparrow} \\ \rho_{01}^{q\uparrow q\downarrow} \\ \rho_{10}^{q\downarrow q\uparrow} \\ \rho_{11}^{q\downarrow q\downarrow}\end{pmatrix} \tag{7}$$

  and for arbitrary $N \geq 1$,

$$\frac{d}{dt}\begin{pmatrix}\rho_{N-1,N-1}^{q\uparrow q\uparrow} \\ \rho_{N-1,N}^{q\uparrow q\downarrow} \\ \rho_{N,N-1}^{q\downarrow q\uparrow} \\ \rho_{N,N}^{q\downarrow q\downarrow}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{\Omega}{2i}\sqrt{N} & \frac{\Omega}{2i}\sqrt{N} & 0 \\ -\frac{\Omega}{2i}\sqrt{N} & \frac{2\delta}{i} & 0 & \frac{\Omega}{2i}\sqrt{N} \\ \frac{\Omega}{2i}\sqrt{N} & 0 & -\frac{2\delta}{i} & -\frac{\Omega}{2i}\sqrt{N} \\ 0 & \frac{\Omega}{2i}\sqrt{N} & -\frac{\Omega}{2i}\sqrt{N} & 0\end{pmatrix}\begin{pmatrix}\rho_{N-1,N-1}^{q\uparrow q\uparrow} \\ \rho_{N-1,N}^{q\uparrow q\downarrow} \\ \rho_{N,N-1}^{q\downarrow q\uparrow} \\ \rho_{N,N}^{q\downarrow q\downarrow}\end{pmatrix} \tag{8}$$

- Also, we can consider case when $q_r \neq 0$ but $\Omega = 0$. In this case, excitation number is still a good quantum number but different oscillator q is coupled. The master equation is written as,

$$\begin{aligned}\frac{d}{dt}\rho_{00}^{p\uparrow q\uparrow} &= \frac{1}{i}\left[\left(p+\frac{1}{2}+\delta\right)\rho_{00}^{p\uparrow q\uparrow} - \frac{iq_r}{\sqrt{2}}\left(\sqrt{p+1}\rho_{00}^{(p+1)\uparrow q\uparrow} - \sqrt{p}\rho_{00}^{(p-1)\uparrow q\uparrow}\right)\right] \\ &+ \frac{1}{i}\left[-\left(q+\frac{1}{2}+\delta\right)\rho_{00}^{p\uparrow q\uparrow} + \frac{iq_r}{\sqrt{2}}\left(\sqrt{q}\rho_{00}^{p\uparrow(q-1)\uparrow} - \sqrt{q+1}\rho_{00}^{p\uparrow(q+1)\uparrow}\right)\right]\end{aligned}$$

  where we only need to consider Hilbert space spanned by $|0, q, \uparrow\rangle$ and $q = 0, 1, 2, ..., Q$ (if we choose initial state as zero photon and up spin and lowest oscillator state)