CSCI 3022

intro to data science with probability & statistics

October 1, 2018

Expected Values

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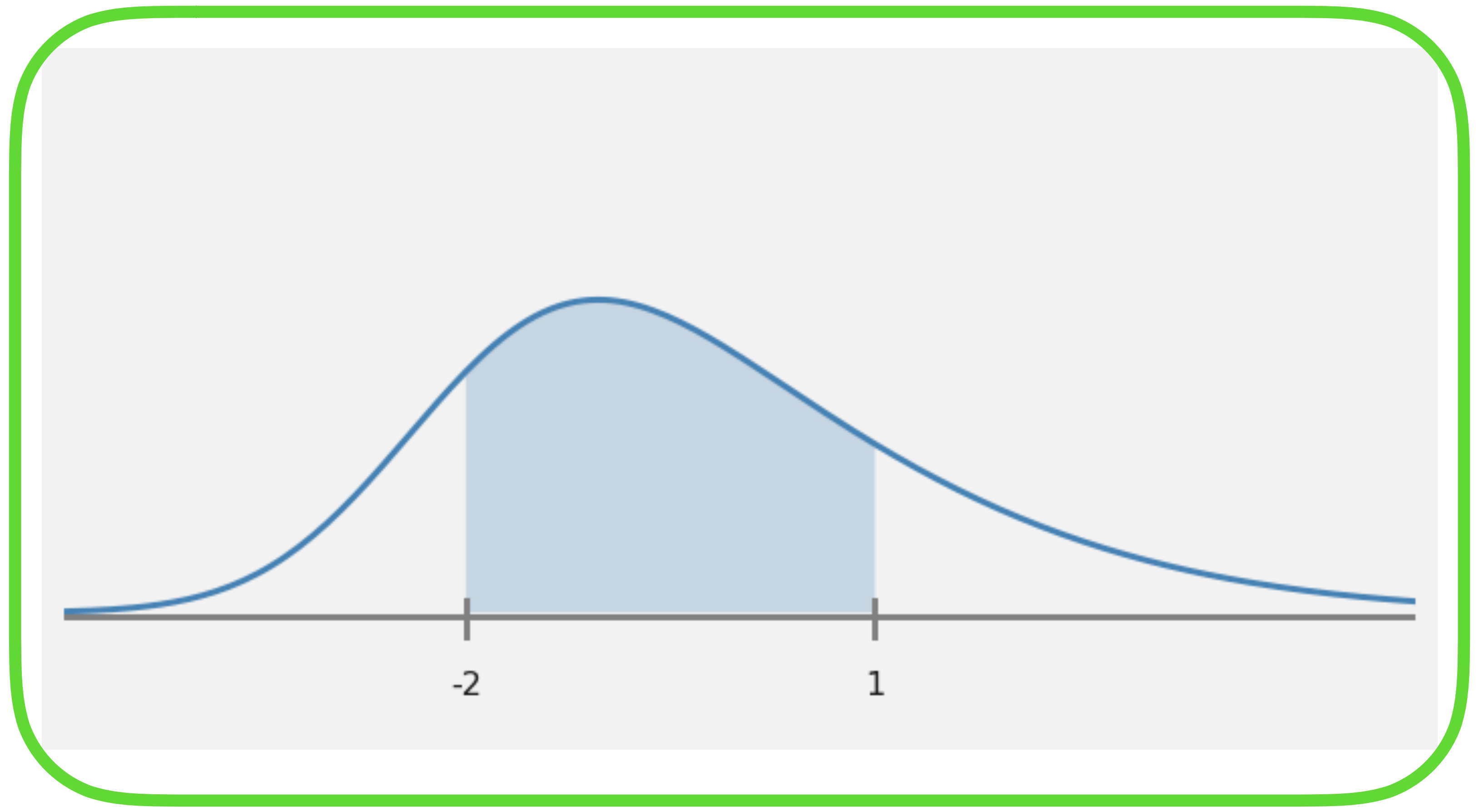
Stuff & Things

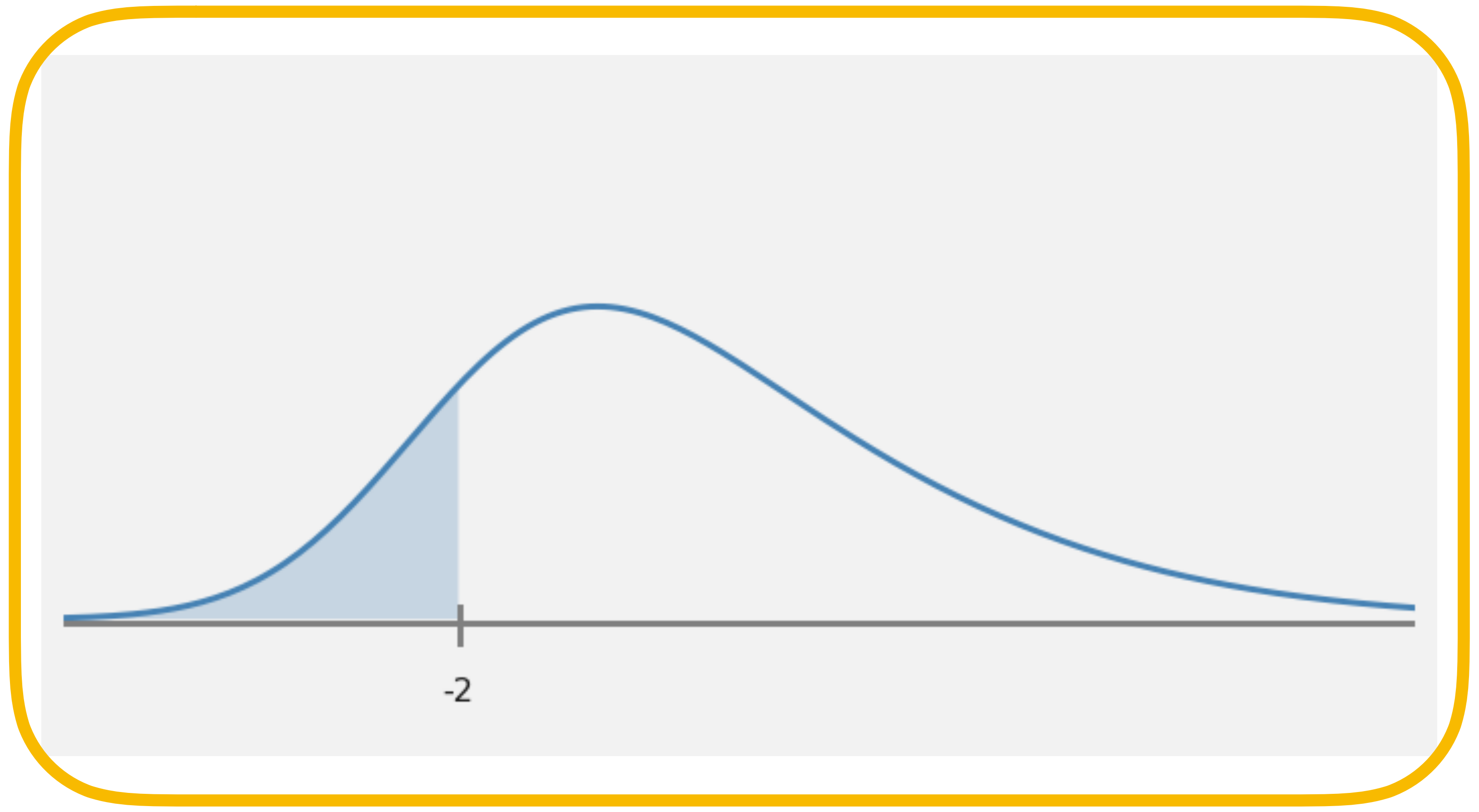
- **Homework 3** due next Friday. Suggested **milestones**:
    - Probs 1, 2, 3 done before the end of the week.
    - Probs 4, 5 done next week.
- **Midterm** next week. Weds, 6:30-8:00 PM. SEE PIAZZA for details..
- Midterm review *in class* next Monday.

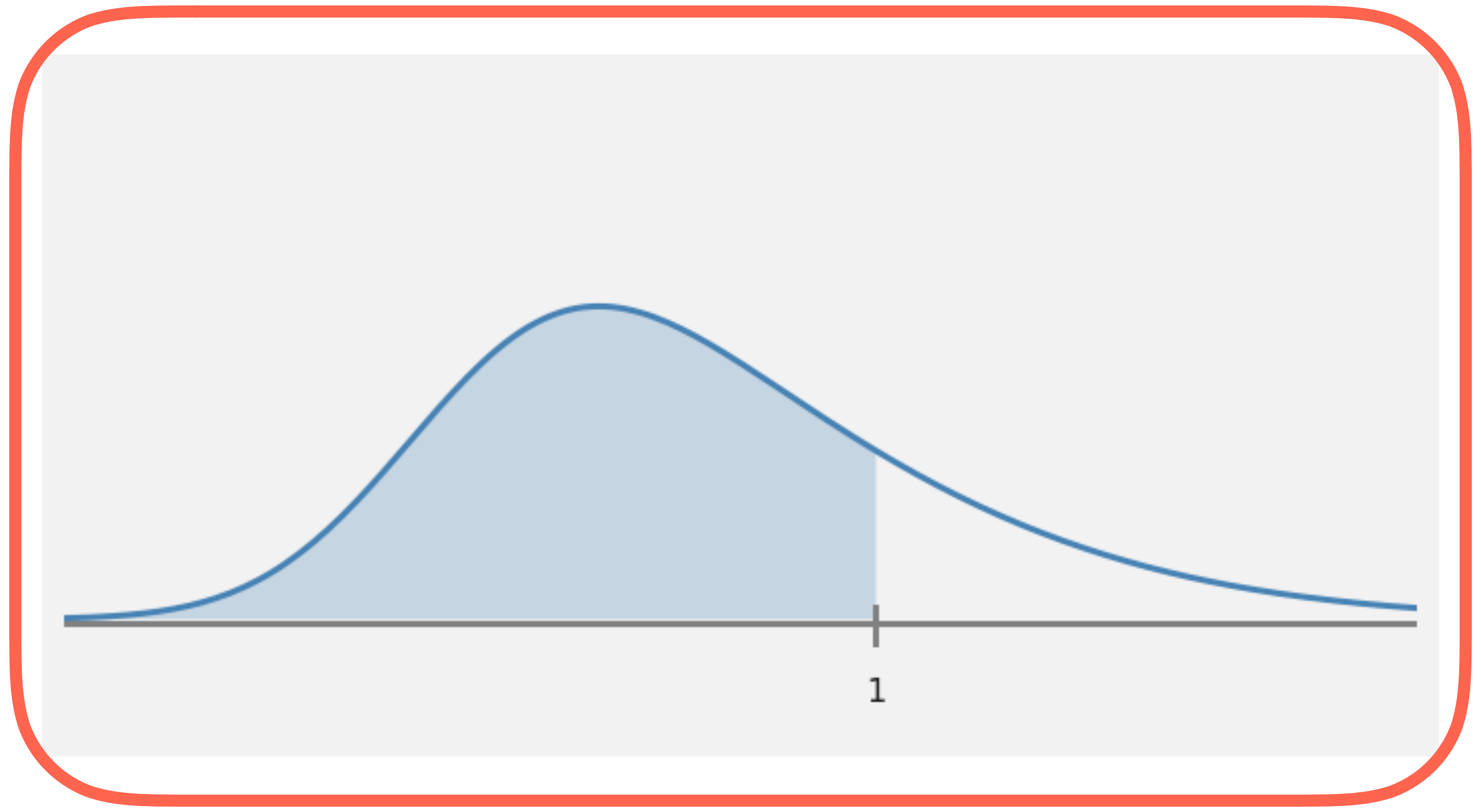
Last time on CSCI 3022:

- **Continuous random variables**:

$$P(a \leq X \leq b) = \boxed{\int_a^b f(t)dt} = \boxed{F(b)} - \boxed{F(a)}$$







- New distributions! Uniform, Exponential, Normal.

Homework Planning

- Suppose *hypothetically* that I write the homework questions as either: easy (takes 10 mins), medium (30 mins), or hard (60 mins).
- The probability that each question is easy, medium, or hard, is: 0.4, 0.35, 0.25, respectively.
- If a homework consists of 5 questions, what's the average time it takes to do the homework? answer in minutes

\mu

weighted average

$$(0.4)10 + (0.35)30 + (0.25)60 = \mu \qquad \boxed{5\mu}$$

$$Pr(easy) \times \text{Time if Easy} + Pr(med) \text{ Time if Med} + Pr(hard) \text{ Time if Hard}$$

Expected Value

mins if problem is hard

Pr(hard)

$$\frac{d}{dp}\left(-\frac{1}{p}\right) = +\frac{+1}{p^2}$$

- **Definition**: The *expectation* or *expected value* of a discrete random variable X that takes the values $a_1, a_2, \ldots$ and with PMF p is given by:

$$p(k) = p(1-p)^{k-1}$$

means Exp. Value of X

$$\mathrm{E}[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

- **Exercise**: What is the expected value of the geometric distribution?

$$E[X] = \sum_i a_i \Pr(X = a_i)$$

$$= \sum_{i=1}^{\infty} i\, p(1-p)^{i-1}$$

$$= p \sum_{i=1}^{\infty} i (1-p)^{i-1}$$

$$= p \sum_{n=0}^{\infty} (n+1)(1-p)^n$$

$$= p\left[\sum_{n=0}^{\infty} (1-p)^n + \sum_{n=0}^{\infty} n(1-p)^n\right]$$

geom series

$$= p\left[\frac{1}{p} + \sum_{n=0}^{\infty} n(1-p)^n\right]$$

$$= 1 + p(1-p)\sum_{n=0}^{\infty} n(1-p)^{n-1}$$

$$= 1 + p(1-p)\frac{d}{dp}\left[-\sum_{n=0}^{\infty}(1-p)^n\right]$$

$$= 1 + p(1-p)\frac{d}{dp}\left(-\frac{1}{p}\right)$$

$$= 1 + \frac{\cancel{p}(1-p)}{p^{\cancel{2}}}$$

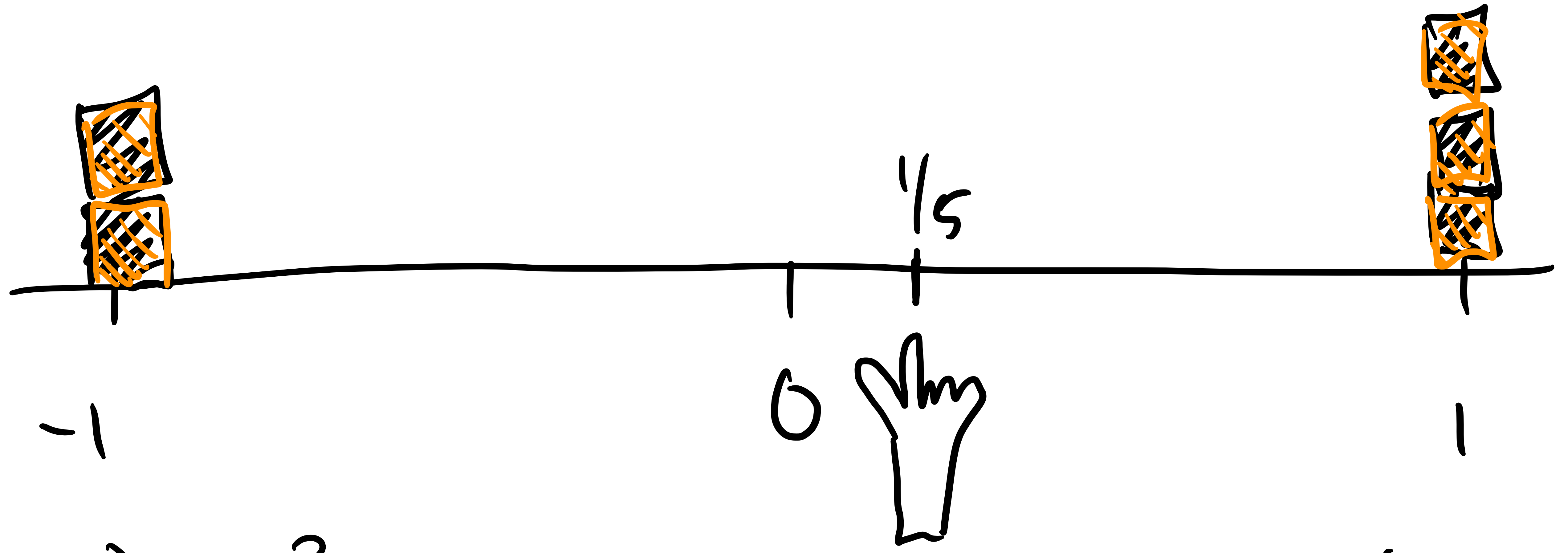
$$= 1 + \frac{1}{p} - \frac{p}{p} = \boxed{\frac{1}{p}}$$

:)

Expected Value: center of gravity

- **Note**: the expected value is the *center of gravity*.

- **Example**: suppose I stack 2 boxes at x=-1 and 3 boxes at position x=1. What is the expected value of this distribution of boxes?



$$P(X=-1) = \frac{2}{5} \qquad P(X=1) = \frac{3}{5}$$

$$E[X] = a_{-1}\,p(a_{-1}) + a_1\,p(a_1)$$

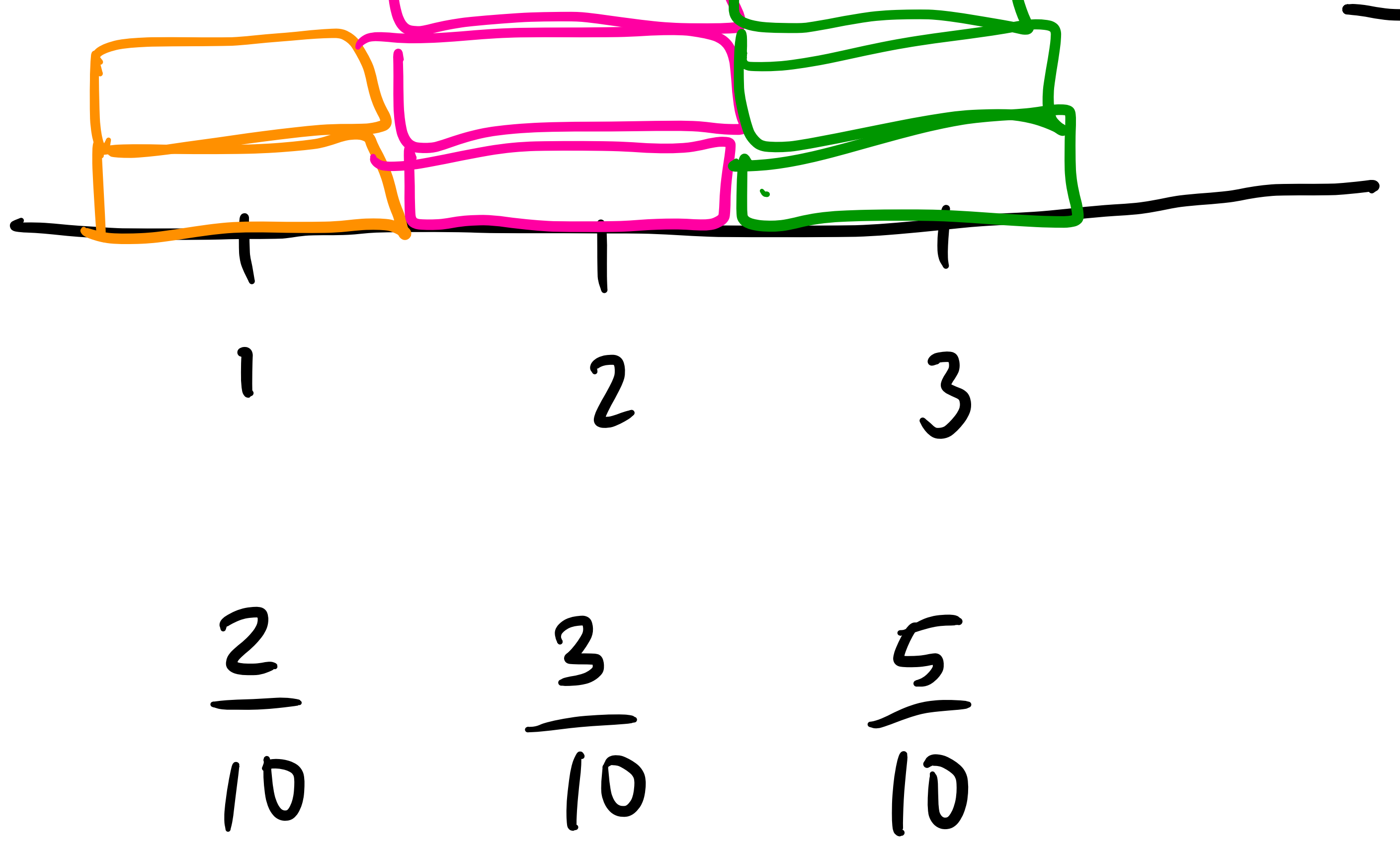
$$-1 \cdot \frac{2}{5} + 1\,\frac{3}{5} = -\frac{2}{5} + \frac{3}{5} = \boxed{\frac{1}{5}}$$

What about continuous RVs?

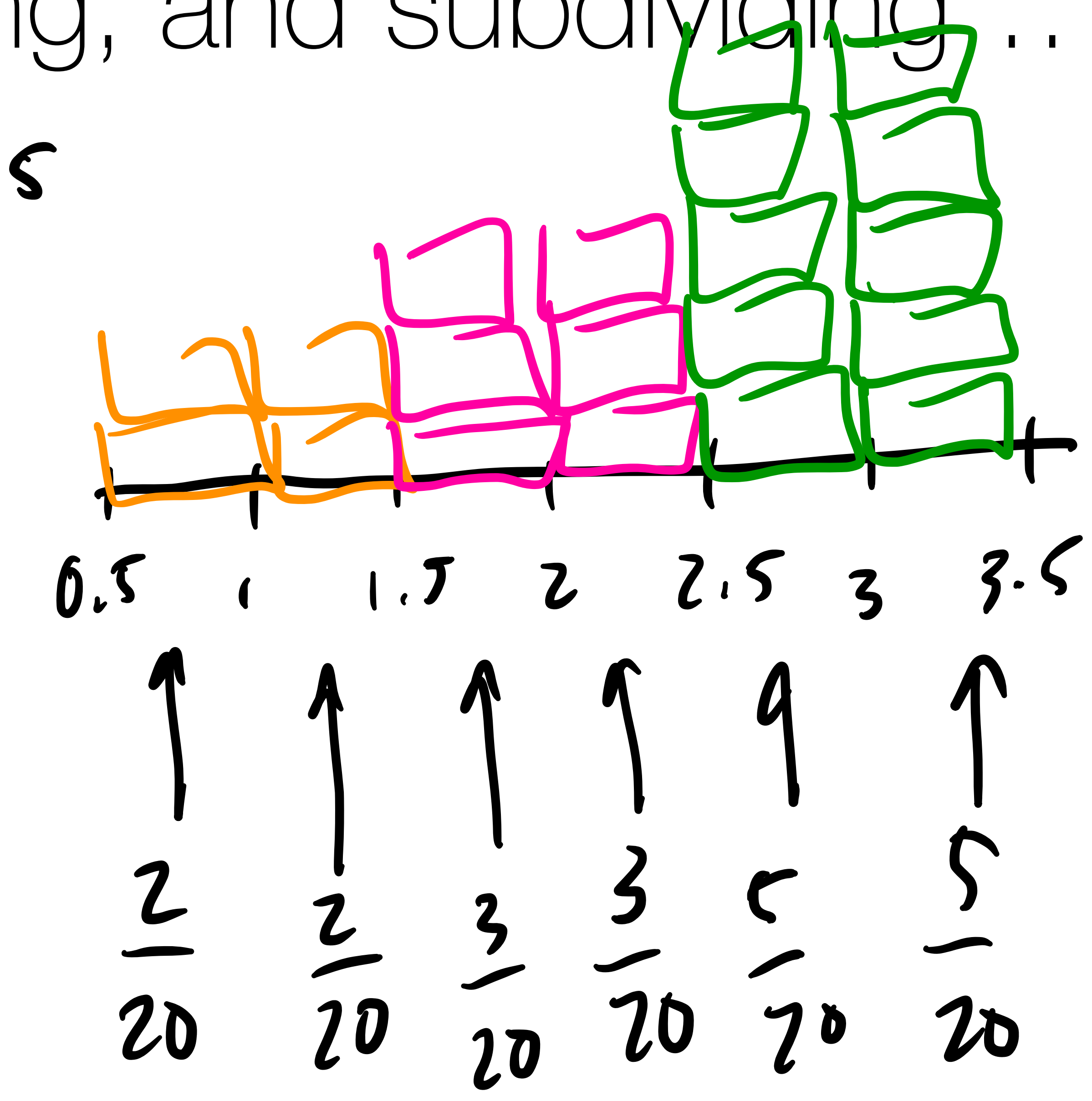
before: $E[X] = \sum_i a_i \, p(a_i)$

- Back to box stacking. What if I have 2, 3, and 5 *physical* IRL boxes at positions 1, 2, and 3, respectively.

- The masses are not concentrated at a point. We can change our discretization, subdividing, and subdividing . . .



$1 \to 0.5$



take it to the limit

from PMF to a PDF

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$\sum_i$ a_i $p(a_i)$

What about continuous RVs?

- Back to box stacking. What if I have 2, 3, and 5 *physical* IRL boxes at positions 1, 2, and 3, respectively.

- The masses are not concentrated at a point. We can change our discretization, subdividing, and subdividing…

- **Definition**: The *expectation* or *expected value* of a continuous random variable X with PDF f is the number:

$$\boxed{\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx}$$

Expected value: average, c. of. g

- The expected value E[X] is also the average of a large number of draws of the random variable X.
- Even in the continuous case, E[X] is the center of gravity.
- **Example**: What is the expectation of an exponential distribution?

$X \sim \exp(\lambda)$

$f(x) = \lambda e^{-\lambda x} \quad x \geq 0$

$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^{\infty} x \lambda e^{-\lambda x}\, dx$

integration by parts :)

⋮

$= \boxed{\frac{1}{\lambda}}$

recall geometric

$\boxed{E[X] = \frac{1}{p}}$

Expected value of a normal

- Let $X \sim N(\mu, \sigma^2)$
- Then: $\mathrm{E}[X] = \int_{-\infty}^{\infty} x \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{f(x)} dx = \mu$

① change of variables.

$z = x - \mu$

$x = z + \mu$

$$\underbrace{\int_{-\infty}^{\infty} z \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2} dz}_{0} + \underbrace{\int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2} dz}_{1}$$

because it's odd

$+ \boxed{\mu}$

Change of variable trick

- Let X and Y be random variables and let $g : \mathcal{R} \rightarrow \mathcal{R}$ be a function.

- For X discrete and Y continuous, then:

$$\mathrm{E}[g(X)] = \sum_i g(a_i) f(a_i)$$

apply function to values of outcomes

keep probs. the same

$$\mathrm{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y) dy$$

same here

If $g(x) = x$

$$E[g(X)] = \sum_i a_i f(a_i) = E[X]$$

Change of variable trick

- Let X and Y be random variables and let $g : \mathcal{R} \rightarrow \mathcal{R}$ be a function.

- For X discrete and Y continuous, then:

$$E[g(X)] = \sum_i g(a_i) f(a_i) \qquad E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- What happens if g(x) = x?

const

const.

- What happens if g(x) = rX +s?

$$E[rX+s] = \sum_i \underbrace{(ra_i+s)}_{\text{distribute}} f(a_i) = r\Big(\sum_i a_i f(a_i)\Big)^{E[X]} + s\Big(\sum_i f(a_i)\Big)^{1} = r\,E[X] + s$$

linearity of expectation.

Linearity of Expectations

Fact: $E[rX + s] = rE[X] + s$

Proof: See prev. slide.

Summary

- **Definition**: The expectation or expected value of a discrete random variable *X* that takes the values $a_1, a_2, \ldots$ and with PMF *p* is given by:

$$\mathrm{E}[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

- **Definition**: The *expectation* or *expected value* of a continuous random variable *X* with PDF *f* is the number:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

- **Change of Variables**: Let *X* be a RV and let $g : \mathcal{R} \rightarrow \mathcal{R}$ be a function

$$\mathrm{E}[g(X)] = \sum_i g(a_i) f(a_i) \qquad \mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$