CSCI 3022

intro to data science with probability & statistics

Lecture 19
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Hypothesis testing for variance or SD
The Bootstrap

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Inference for *variances*

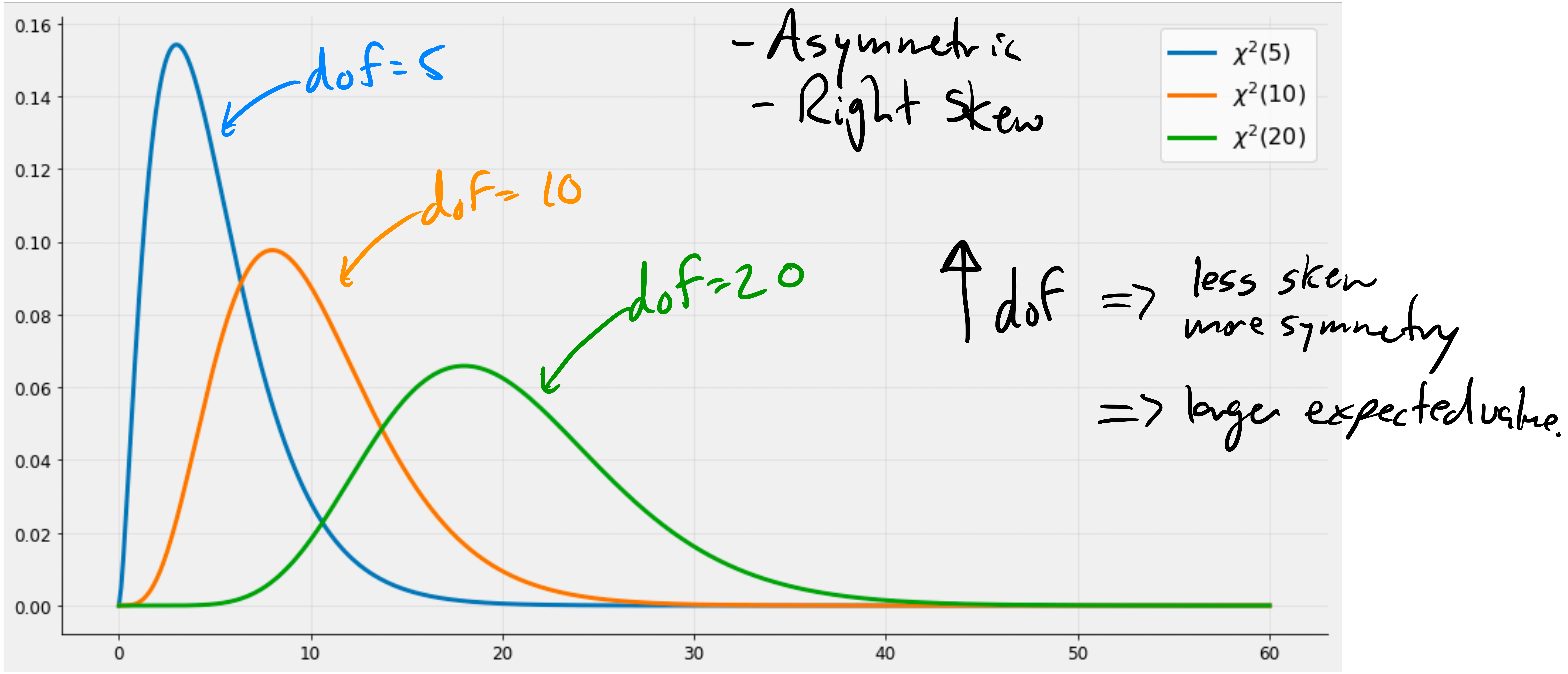
- **Question**: What does the sampling distribution of the variance look like when the population is **normally distributed**?

Reminder: So far all of our hypothesis tests have been for

- mean
- proportion
- diff in means
- diff in proportions.

The Chi-Squared Distribution \nu

- The chi-squared distribution (χ^2_ν) is also parameterized by degrees of freedom $\nu = n - 1$
- The pdfs of the family of χ^2_ν distributions are gross, so lets just draw them!



A confidence interval for the variance

- Let $X_1, X_2, \dots X_n$ be IID samples from a normal distribution with mean μ and standard deviation σ. Define the *sample variance* in the usual way as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

- Then the random variable $(n-1)S^2/\sigma^2$ follows the distribution χ^2_{n-1}.

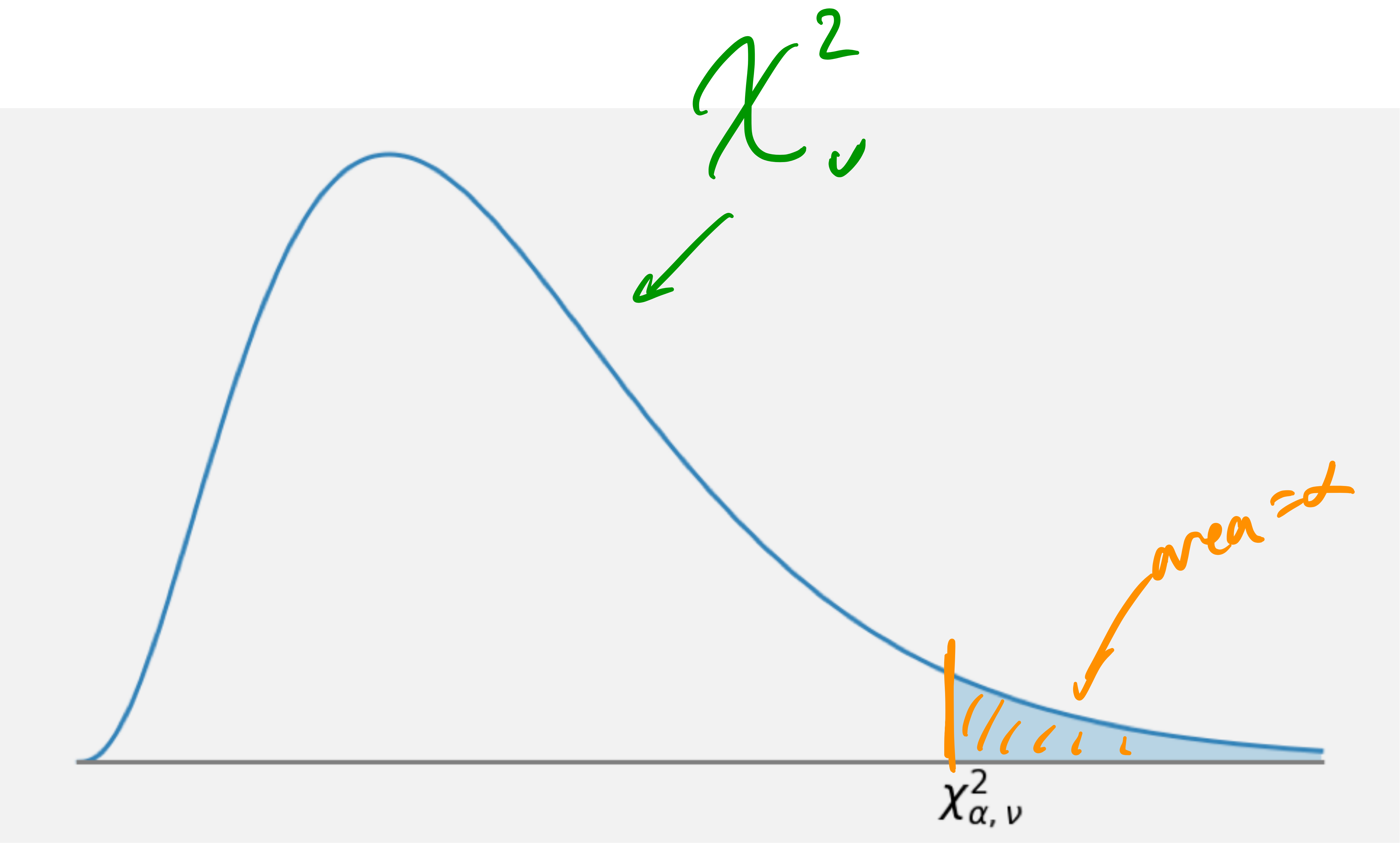
note: $(n-1) S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$

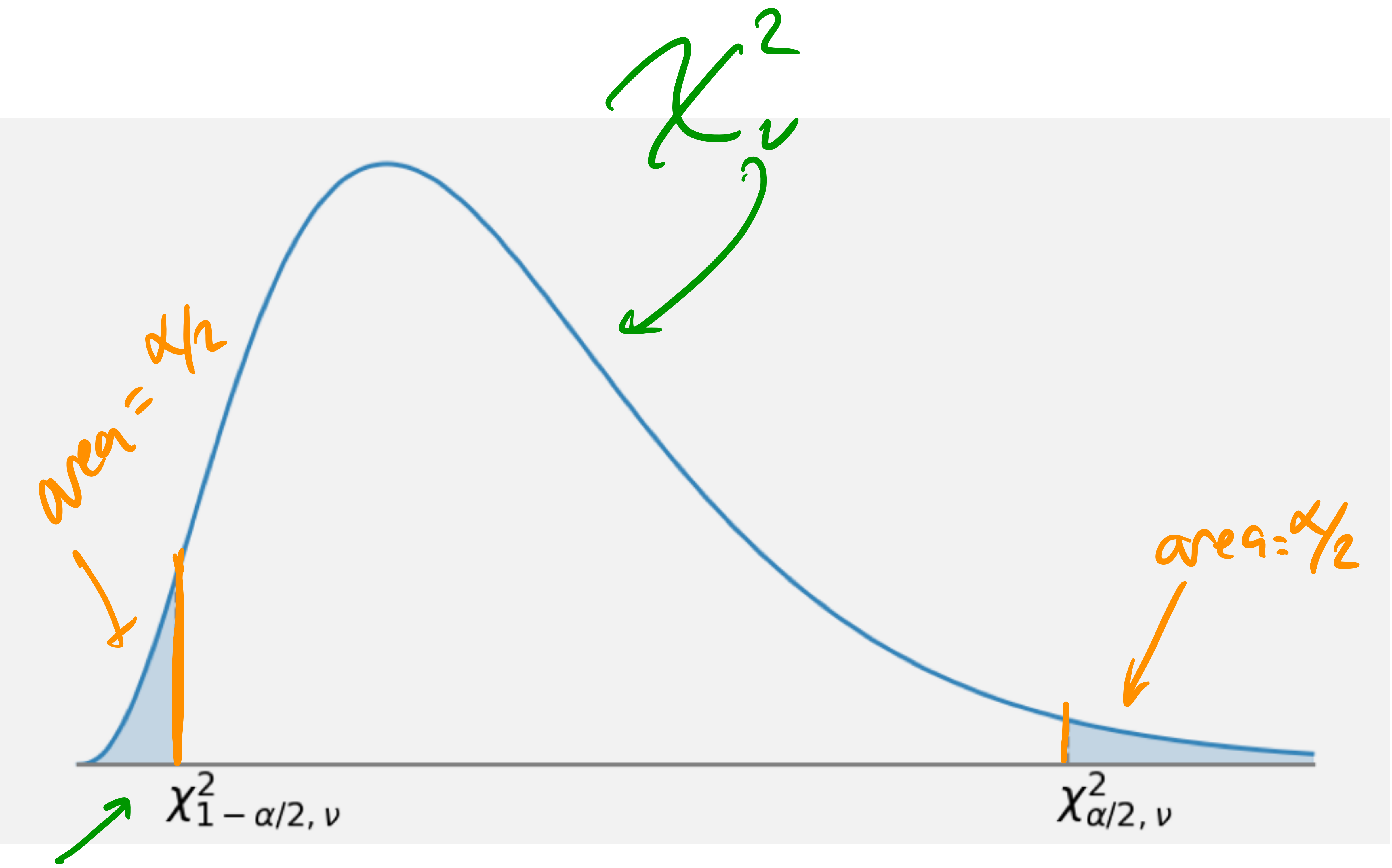
- Then it follows that

$$P\left(\chi^2_{1-\alpha/2, n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2, n-1} \right) = 1-\alpha$$

The Chi-Squared Dist is Non-Symmetric

- Because the distribution is non-symmetric, we need to use two different critical values.





stats.chi2.ppf($\alpha/2$, ν) stats.chi2.ppf($1-\alpha/2$, ν)

A confidence interval for the variance

- For a $100(1-\alpha)\%$ confidence interval we choose the two critical values $X^2_{1-\alpha/2,n-1}$ and $X^2_{\alpha,n-1}$ which puts $\alpha/2$ probability in each tail. Then, with $100(1-\alpha)\%$ confidence we can say that

$$P\left(\chi^2_{1-\alpha/2,n-1} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

solve for this!

$$\frac{1}{\chi^2_{\alpha/2,n-1}} < \frac{\sigma^2}{(n-1)s^2} < \frac{1}{\chi^2_{1-\alpha/2,n-1}} \longrightarrow \boxed{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}$$

A confidence interval for the variance

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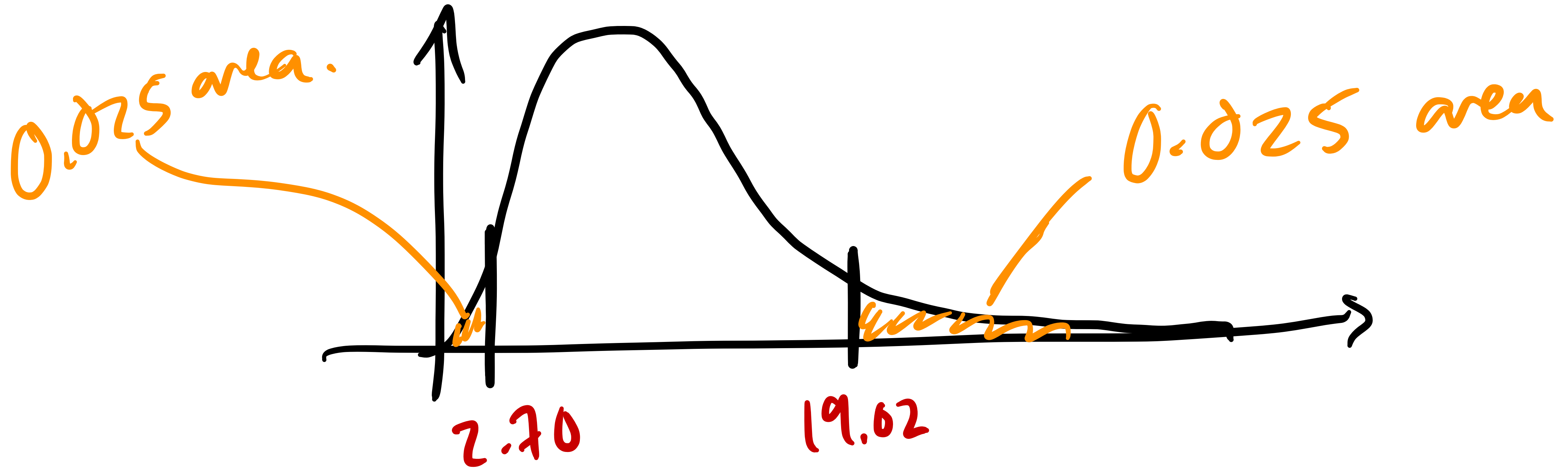
$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Question: How can we use this to get a $100(1-\alpha)\%$ confidence interval for the standard deviation?

$$\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}$$



useless!

- Example: A large candy manufacturer produces packages of candy targeted to weight 52g. The weight of the packages of candy is known to be normally distributed, but a QC engineer is concerned that the variation in the produced packages is larger than acceptable. In an attempt to estimate the variance she selects n=10 bags at random and weighs them. The sample yields a sample variance of 4.2g. Find a 95% confidence interval for the variance and a 95% confidence interval for the standard deviation.

$\sigma^2 : [1.99, 14.0]$

$\sigma : [1.41, 3.74]$

$S^2 = 4.2$

$n = 10$ $\quad n-1 = 9$

$\alpha = 0.05$

$\alpha/2 = 0.025$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \text{ and } \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

$\chi^2_{0.975,9}$ = stats.chi2.ppf(0.025, 9) = 2.70

$\chi^2_{0.025,9}$ = stats.chi2.ppf(0.975, 9) = 19.02

The Bootstrap

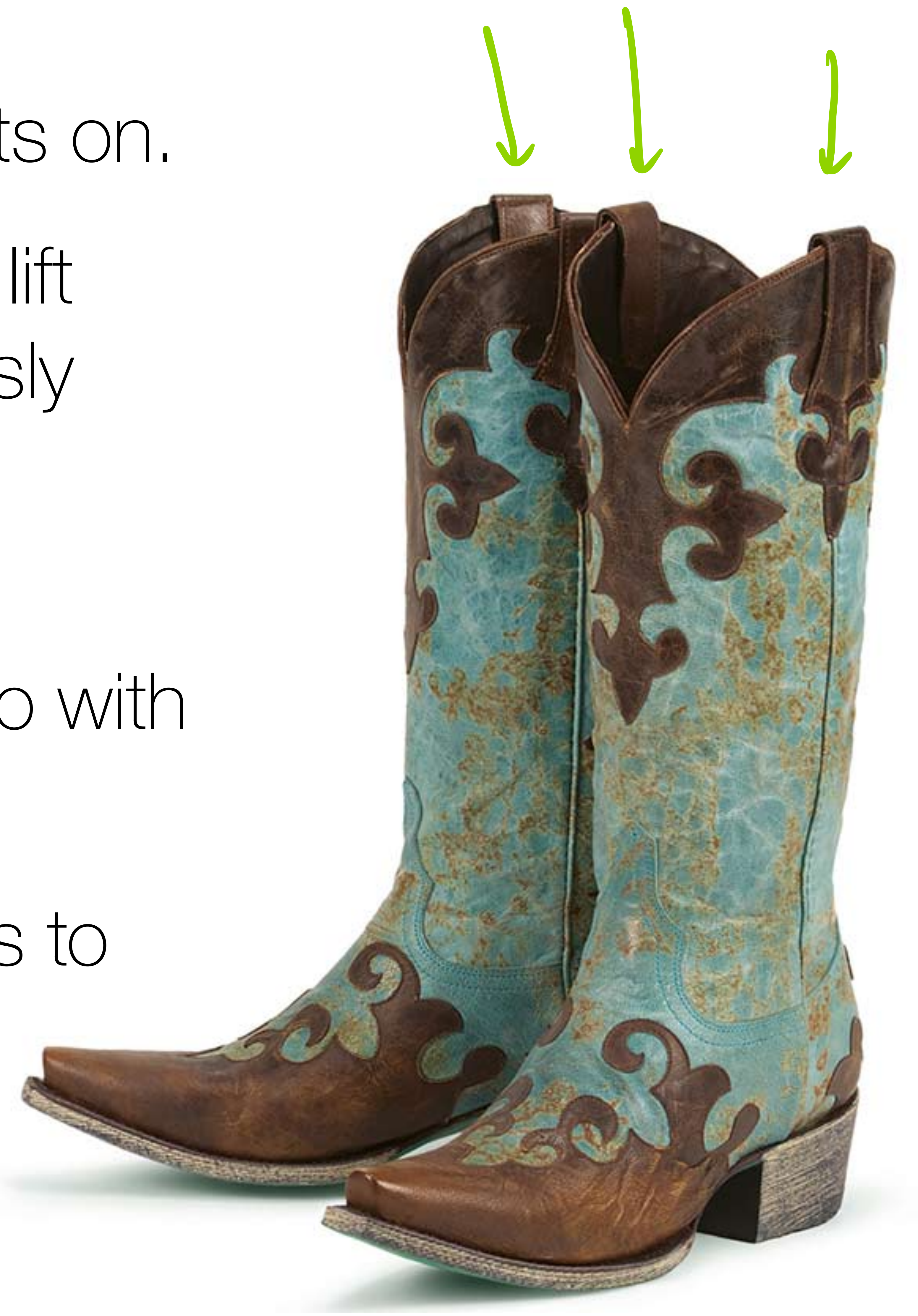
Not all datapoints come cheap…

- In real scenarios, **data can be expensive**…
  - in **money**. For example, data from an aircraft in a wind tunnel.
  - in **time**. For example, polling people in surveys is time consuming.
  - in **privacy tradeoffs**. For example, storing another person's genome in the database incurs ethical risk or cost, even when it does not cost much time or money.

- Today, we'll learn a technique that enables us to learn from small amounts of data to compute confidence intervals: **the bootstrap**

What are bootstraps?

- Bootstraps are the straps that you use to pull your boots on.
- To "pull yourself up by your bootstraps" is to somehow lift yourself upward by pulling on your own shoes. Obviously impossible.
- Now, however, bootstrapping means to accomplish something without aid. To accomplish what you need to with what you've got.
- The statistical bootstrap is in this last sense. It allows us to really **make the most of a small dataset** without sacrificing statistical rigor or collecting more $ samples.



A confidence interval for the mean

- **Recall**: if we have *n* samples from a distribution that is normal *or* non-normal, then by the Central Limit Theorem, the confidence interval for the mean is given by $\bar{X} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$ or for an unknown variance $\bar{X} \pm z_{\alpha/2}\sqrt{\frac{s^2}{n}}$

- The bootstrap is a different approach. Consider the same set of samples as above, $X_1, X_2, \ldots, X_n$, but instead of computing a CI analytically from this sample, instead *re*-sample your sample many times and examine (?) those!

- **Definition**: a bootstrapped resample is a set of *n* draws from the original set, sampled *with replacement*.

A confidence interval for the mean

- **Definition**: a bootstrapped resample is a set of n draws from the original dataset (drawn IID from X), sampled *with replacement*.
  our data, what we collected.
- **Example**: suppose we have the data [2,4,6,7,9] $n = 5$
  - Resample 1 might be: [4, 6, 7, 4, 9]
  - Resample 2 might be: [6, 4, 2, 2, 2]
  - Resample 3 might be: [7, 9, 6, 2, 4]

- Given the example above, what does "*sample with replacement*" mean?

A confidence interval for the mean

From Prev. Slide

- **Definition**: a bootstrapped resample is a set of n draws from the original dataset (drawn IID from X), sampled *with replacement*.

- **Proposition**: a suitable estimate of the 95% confidence interval for the mean of the distribution X is given by $[a,b]$, where a and b are the 2.5 percentile and 97.5 percentile of the means of a large number of bootstrapped resamples.

- **In plain English**: resample your original data many times. Compute the mean for each resample. Compute the 2.5 and 97.5 percentiles of those means.

Magic.

Bootstrap: why we like it

- The bootstrap for a confidence interval around the mean is convenient, particularly when there are **not enough samples** to use the CLT.
- Of course, if we *can* use the CLT, we should. So why is the bootstrap so exciting?

The story so far, for means

- Thus far, we've talked about Hypothesis Testing & Confidence Intervals for the mean of a population in the following cases:

	$n \geq 30$	$n < 30$
Normal Data / Known σ		
Normal Data / Unknown σ use s		
Non-Normal Data / Known σ		
Non-Normal Data / Unknown σ		

not mean, p, σ^2

z test

t test

bootstrap

Bootstrap: why we like it

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We can bootstrap CIs for things other than the mean!

- Median.
- Standard Deviation.
- Other statistical measures that we don't have a theory for.

- Analytical bootstrap for max → practicum?

Bootstrap for the median

7% 86% 7%

100 - 86 = 14

86%

- Let's write down the recipe for how we would bootstrap a CI for the median:

1. Resample. Create M resampled datasets (with replacement). Each dataset has n elements, just like the original.

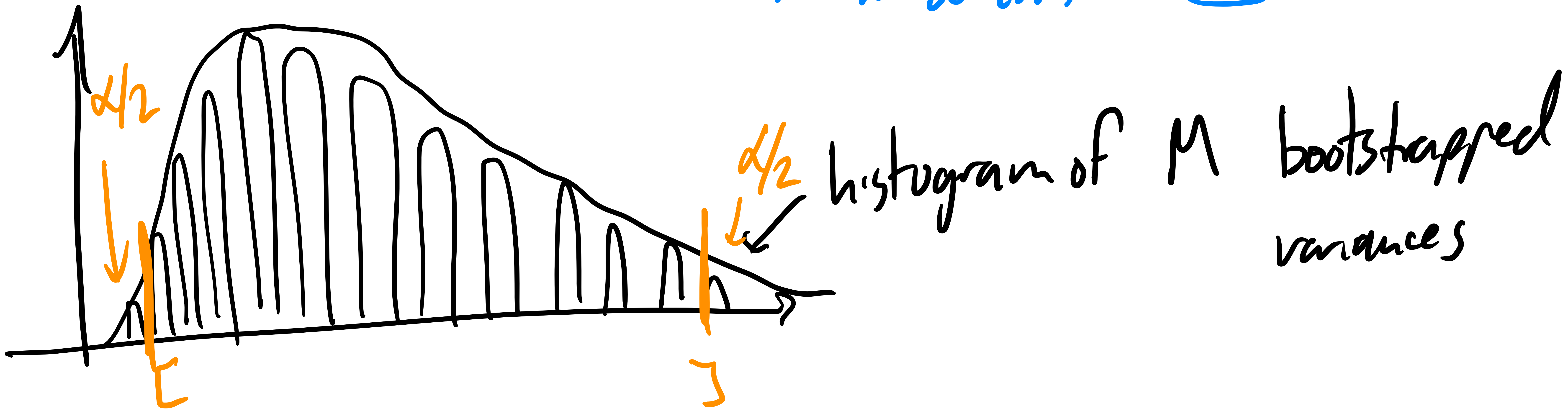
2. For each of the M resampled datasets, compute median. Result is M different medians.

3. Take that distribution of M medians, compute the 7th percentile and the 93rd percentile.

Bootstrap for the variance

- Let's write down the recipe for how we would bootstrap a CI for the variance:

See prev slide. Compute variance M times instead of median! :)



The Non-Parametric Bootstrap

- In the literature—your book, the Wikipedia, etc—you may read about a "non-parametric bootstrap." What is this?

The Non-Parametric Bootstrap

- In the literature—your book, the Wikipedia, etc—you may read about a "non-parametric bootstrap." What is this?
- Let's decode this word, "non-parametric"
- **Definition**: *parametric statistics* assumes that sample data comes from a population that follows a probability distribution based on a fixed set of parameters.
- Can you name some **examples** of distributions with parameters?

$\mathrm{Pois}(\lambda)$ $N(\mu, \sigma^2)$ $\mathrm{Bin}(n, p)$

- Can you name a *non*-parametric distribution we've talked about in class?

Let X be a r.v. s.t. $P(X=-1) = 0.2$, $P(X=0) = 0.5$, $P(X=19) = 0.3$

The Parametric Bootstrap

- We call the bootstrap discussed in class today the non-parametric bootstrap because it doesn't assume any parametric distribution. What you resample is what you get.

- **Definition**: the parametric bootstrap estimates a CI for a desired property in two steps: (1) repeatedly estimate the parameter(s) of the known distribution, and then (2) compute a CI for the desired property by sampling from the ~~known~~ known distribution using the parameters that you inferred.

Arkaive!

Imagine: M resampled data sets.

Compute $\hat{\lambda}$ for each of the M data sets.

Use these $\hat{\lambda}$ to get variable/quantity of interest.

The Parametric Bootstrap

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{M resamples} \arrow[r] \arrow[d, "\text{non-parametric}"] & \text{parameter estimates} \arrow[dl, "\text{parametric}"] \\
\text{quantity of interest} &
\end{tikzcd}$$

- We call the bootstrap discussed in class today the non-parametric bootstrap because it doesn't assume any parametric distribution. What you resample is what you get.
- **Definition**: the parametric bootstrap estimates a CI for a desired property in two steps: (1) repeatedly estimate the parameter(s) of the known distribution, and then (2) compute a CI for the desired property by sampling from the ~~known~~ known distribution using the parameters that you inferred.
- **Why**? The parametric bootstrap can be shown to do a better job than the non-parametric bootstrap in various scenarios.
- Why not use the parametric bootstrap all the time?

You might not know the correct parametric distribution!