Franklin Hu, Sunil Pedapudi
CS 194-10 Machine Learning
Fall 2011
Assignment 5

1. Conjugate priors

   (a) Let

$$\text{Likelihood: } \mathbb{P}(x_1,\ldots,x_N) = \prod_i^N \lambda \exp\left(-\lambda x_i\right)$$
$$\text{Prior: gamma}(\lambda|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

Then,

Posterior:

$$\begin{aligned}\mathbb{P}(\lambda|x_1,\ldots,x_N) &= \prod_i^N \lambda \exp\left(-\lambda x_i\right) \cdot \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda} \\ &= \lambda \exp\left(\sum_i^N -\lambda x_i\right)\lambda^{\alpha-1}e^{-\beta\lambda}\frac{\beta^\alpha}{\Gamma(\alpha)} \\ &= \lambda^{\alpha+N-1}\exp\left(\sum_i^N -\lambda x_i - \beta\lambda\right)\frac{\beta^\alpha}{\Gamma(\alpha)} \\ &= \lambda^{\alpha+N-1}\exp\left(-\lambda\sum_i^N x_i + \beta\right)\frac{\beta^\alpha}{\Gamma(\alpha)} \sim \text{gamma}(\alpha+N, \beta+\sum_i^N x_i)\end{aligned}$$

Since the posterior also has a gamma distribution, we find the updates parameters are of the form $\alpha+N$,$\beta+\sum_i^N x_i$. To find the prediction distribution,

$$\begin{aligned}\mathbb{P}(x_N+1|x_1,\ldots,x_N) &\propto \frac{\beta^\alpha}{\Gamma(\alpha)}\int \lambda\exp\left(-\lambda x_{N+1}\right)\cdot\lambda^{\alpha+N-1}\exp\left(-\lambda(\beta+\sum_i^N x_i)\right)d\lambda \\ &\propto \frac{\beta^\alpha}{\Gamma(\alpha)}\int \lambda\cdot\lambda^{\alpha+N-1}\exp\left(-\lambda(\beta+\sum_i^{N+1} x_i)\right)d\lambda \\ &\propto \frac{\beta^\alpha}{\Gamma(\alpha)}\int \lambda\cdot P(\lambda|\alpha+N,\beta+\sum_i^{N+1} x_i)d\lambda\end{aligned}$$

We note that this describes the expectation for λ given a gamma function $\sim \text{gamma}(\lambda|\alpha+N,\beta+\sum_i^{N+1} x_i)$. Therefore,

$$\mathbb{P}(x_N+1|x_1,\ldots,x_N) \propto \frac{\alpha+N}{\beta+\sum_i^{N+1} x_i}$$

   (b) Given the geometric distribution

$$P(X_i = k|\theta) = (1-\theta)^{k-1}\cdot\theta$$

and the beta distribution

$$\beta(\theta|a,b) = \alpha\theta^{a-1}(1-\theta)^{b-1}$$

we prove that the beta distribution is the conjugate prior for a likelihood with a geometric distribution.

$$\begin{aligned} P(\theta|X) &= P(\theta) \cdot P(X|\theta) \\ &= \alpha \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot (1-\theta)^{k-1} \cdot \theta \\ &= \alpha \cdot \theta^{a} \cdot (1-\theta)^{b+k-2} \\ &= \beta(\theta|a+1, b+k-1) \end{aligned}$$

The posterior has the form of a beta distribution so therefore the beta distribution is the conjugate prior for the geometric distribution.
The update procedure for a beta posterior simply involves updating the a and b parameters

$$\begin{aligned} a_{N+1} &\leftarrow a_N + 1 \\ b_{N+1} &\leftarrow b_N + k - 1 \end{aligned}$$

(c) Given

$$\begin{aligned} \text{Likelihood: } &\mathbb{P}(\mathbf{X}|\theta) \\ \text{Mixture prior: } &\mathbb{P}(\theta|\gamma_1, \ldots, \gamma_m) \end{aligned}$$

We wish to find the posterior via

$$\begin{aligned} \mathbb{P}(\theta|\mathbf{X}) &= \mathbb{P}(\theta|\gamma_1, \ldots, \gamma_m) \cdot \mathbb{P}(\mathbf{X}|\theta) \\ &= \sum_{m=1}^{M} w_m \mathbb{P}(\theta|\gamma_m) \prod_{i}^{N} \mathbb{P}(x_i|\theta) \\ &= \sum_{m=1}^{M} w_m \mathbb{P}(\theta|\gamma_m^+) \end{aligned}$$

This is to say that we can find a γ_m^+ that renders $\mathbb{P}(\theta|\gamma_m^+)$ equal to $\mathbb{P}(\theta|\gamma_m) \prod_i^N \mathbb{P}(x_i|\theta)$. The updates to γ may be done iteratively as

$$\begin{aligned} \mathbb{P}(\theta|\gamma_m) \prod_{i}^{N} \mathbb{P}(x_i|\theta) &= \mathbb{P}(\theta|\gamma_m)\mathbb{P}(x_1|\theta) \ldots \mathbb{P}(x_N|\theta) \\ &= \mathbb{P}(\theta|\gamma_m')\mathbb{P}(x_1|\theta) \ldots \mathbb{P}(x_{N-1}|\theta) \\ &= \mathbb{P}(\theta|\gamma_m'')\mathbb{P}(x_1|\theta) \ldots \mathbb{P}(x_{N-1}|\theta) \\ &\vdots \\ &= \mathbb{P}(\theta|\gamma_m^+) \end{aligned}$$

Since $\mathbb{P}(\theta|\gamma)$ is the conjugate prior for $\mathbb{P}(\mathbf{X}|\theta$, we retain the mixture model throughout the update.

(d) Given

$$\begin{aligned} \text{Mixture likelihood: } &\sum_{i=1}^{N} w_i \mathbb{P}(x_i|\theta_i)) \\ \text{Prior: } &\mathbb{P}(\theta_1, \ldots, \theta_N|\gamma) \end{aligned}$$

We find the posterior via

$$\begin{aligned}\mathbb{P}(\theta_1, \ldots, \theta_N|\mathbf{X}) &= \sum_{i=1}^{N} w_i \mathbb{P}(x_i|\theta_i) \cdot \mathbb{P}(\theta_i|\gamma) \\ &= \sum_{i=1}^{N} w_i \mathbb{P}(x_i|\theta_i') \cdot \mathbb{P}(\theta_i|\gamma')\end{aligned}$$

We update the parameter γ indpendently per x_i. In other words, to find γ', we invoke $update(update(update(\gamma, x_1),$

2. Bayesian Naive Bayes

   (a) Maximum likelihood learning chooses the hypothesis with the greatest likelihood where as Bayesian learning computes the weights over all hypotheses and uses a linear combination of their outputs. To use Bayesian learning, let us re-examine the computation of $\mathbb{P}(x_i|class$.

   $$\mathbb{P}(x_i|class) = \int \mathbb{P}(x_i|\lambda_{i,class})\mathbb{P}(\lambda_i|class)d\lambda_{i,class}$$

   We note that $\mathbb{P}(x_i|\lambda_{i,class})$ is an exponential distribution as that is the likelihood that attribute x_i belongs to a certain classification while $\mathbb{P}(\lambda_i|class)$ follows a gamma distribution as that is the prior given a certain classification. This renders

   $$\mathbb{P}(x_i|class) = \int (\lambda_{i,class} \exp{(-\lambda_{i,class}x_i)}) \cdot \left(\frac{\beta_i^{\alpha_i}}{\Gamma(\alpha)}\lambda^{\alpha_i - 1}\exp(-\beta_i\lambda_i)\right) d\lambda_{i,class}$$

   We train across email samples to find the relevant α_i, β_i by

   ```
   for each sample (x, y=class):
       for i = 1 to D:
           α_{i,y} ← α_{i,y} + 1
           β_{i,y} ← β_{i,y} + x_i
   ```

   (b)

3. Logistic regression for credit scoring

   (a) The data structure we chose for logistic regression is simply a class that keeps a set of weights for each of the features, has an update method for updating the weights, and draws predictions using the logit function

   $$\text{Probability} = \frac{1}{1 + e^{-w^T x}}$$

   (b) The likelihood is

   $$\begin{aligned}L(w) &= \frac{1}{1 + e^{-yw^T x}} \\ \text{log likelihood} &= \log\frac{1}{1 + e^{-yw^T x}} \\ &= -\log(1 + e^{-yw^T x}) \\ \text{negative log likelihood} &= \log(1 + e^{-yw^T x})\end{aligned}$$

Now we compute the gradient of the negative log likelihood

$$\begin{aligned}
\nabla \text{log}(1+e^{-yw^Tx}) &= \nabla \text{log}\left(\frac{e^{yw^Tx}+1}{e^{yw^Tx}}\right) \\
&= \nabla\left(\log(e^{yw^Tx}+1) - \log(e^{yw^Tx})\right) \\
&= \left(\frac{1}{e^{yw^Tx}+1}\cdot e^{yw^Tx}\cdot -yx_i\right) - \left(\frac{1}{e^{yw^Tx}}\cdot e^{yw^Tx}\cdot -yx_i\right) \\
&= yx_i - yx_i\cdot\frac{e^{yw^Tx}}{e^{yw^Tx}+1} \\
&= yx_i - yx_i\cdot\left(\frac{e^{yw^Tx}+1}{e^{yw^Tx}}\right)^{-1} \\
&= yx_i - yx_i\cdot(1+e^{-yw^Tx})^{-1} \\
&= yx_i\left(1-\frac{1}{1+e^{-yw^Tx}}\right)
\end{aligned}$$

Therefore our update rule is simply

$$\begin{aligned}
w_{i+1} &= w_i + \alpha\cdot\nabla L \\
&= w_i + \alpha\cdot yx_i\cdot\left(1-\frac{1}{1+e^{-yw^Tx}}\right)
\end{aligned}$$

(c)

(d) With the probability generated by the model, we compute the expectation based on the loan amount, interest rate, and other parameters that go into calcuating whether the loaner (bank) will make money off the loan.