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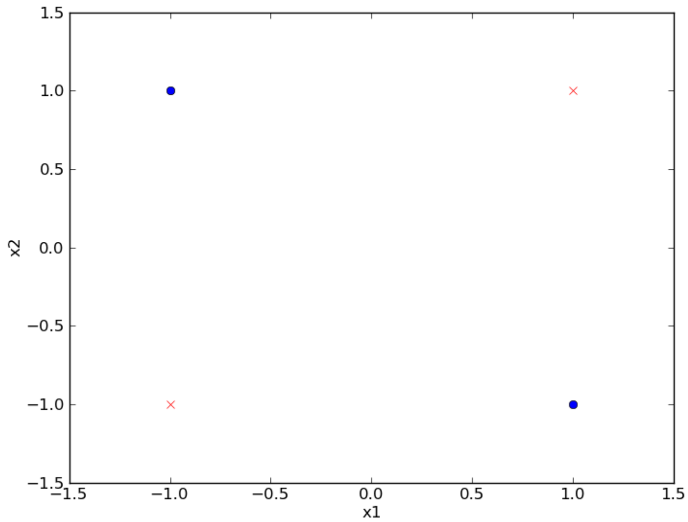
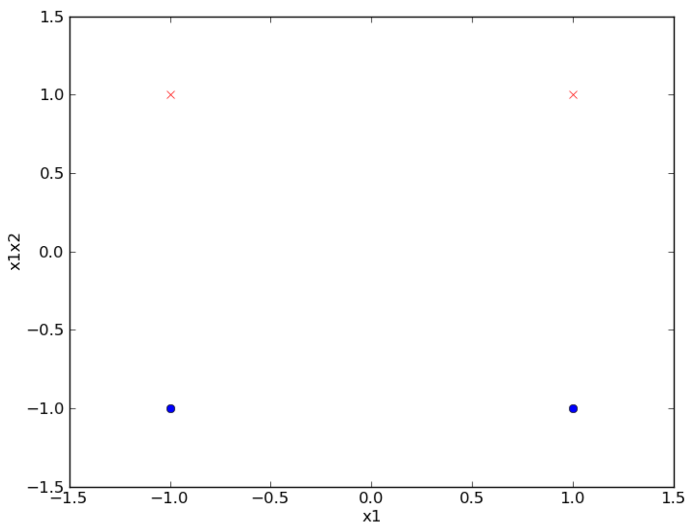
CS 194-10
2011-09-19
Assignment 2

1. Kernels

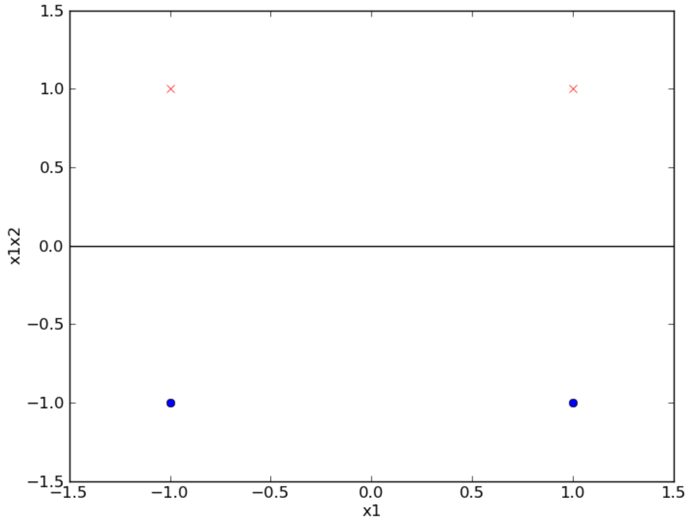
   (a) Original input



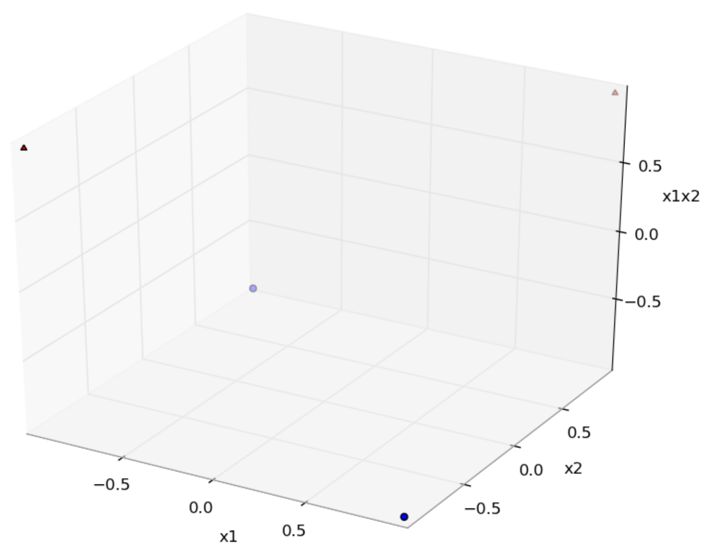
Input mapped onto space consisting of x_1 and $x_1 x_2$:

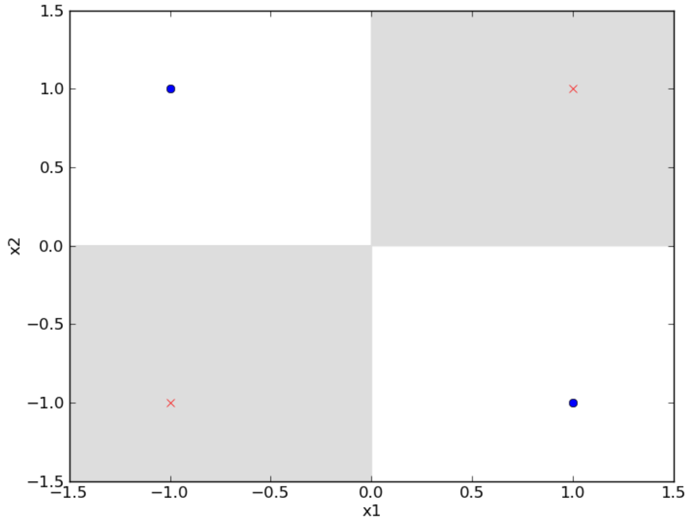


The maximum margin separator is the line $x_1 x_2 = 0$.



The separating line on the original input space is a plane that rests at $x_1 x_2 = 0$.





(b) Given

$$(x_1 - a)^2 + (x_2 - b)^2 - r^2 = 0$$
$$x_1^2 - 2ax_1 + a^2 + x_2^2 - 2bx_2 + b^2 - r^2 = 0$$

let us pose the following:

$$\mathbf{w} = [-2a, -2b, 1, 1]$$
$$\mathbf{x} = [x_1, x_2, x_1^2, x_2^2]$$
$$\beta = a^2 + b^2 - r^2$$
$$\mathbf{w^T x} + \beta > 0, \text{ if } \mathbf{x} \text{ escapes the circle region}$$
$$\mathbf{w^T x} + \beta < 0, \text{ if } \mathbf{x} \text{ occupies the circle region}$$
$$\mathbf{w^T x} + \beta = 0, \text{ if } \mathbf{x} \text{ demarcates the circle region}$$

We then let $y_i = -1$ if $\mathbf{x}$ occupies the region inside the circle; $y_i = 1$ otherwise. Then, to satisfy the separability constraint, we note that

$$y_i(\mathbf{w^T x} + \beta) > 0, \forall i$$

Thus, we show that in feature space (x_1, x_2, x_1^2, x_2^2) , the region defined by $(x_1-a)^2+(x_2-b)^2-r^2 = 0$ is linearly separable.

2. Logistic Regression
Given:

$$L(w) = -\sum_{i=1}^{N} log(\frac{1}{1 + e^{y_i(w^T x_i + b)}}) + \lambda \|w\|_2^2$$

(a)

$$\begin{aligned}
\frac{\partial L}{\partial w_j} &= -\sum_{i=1}^{N}(1+e^{y_i(w^T x_i+b)})\cdot -1\cdot(1+e^{y_i(w^T x_i+b)})^{-2}(e^{y_i(w^T x_i+b)})\cdot x_{ij}y_i + \frac{\partial}{\partial w_j}(\lambda\|w\|_2^2)\\
&= -\sum_{i=1}^{N}\frac{-e^{y_i(w^T x_i+b)}}{(1+e^{y_i(w^T x_i+b)})}\cdot x_{ij}y_i + 2\lambda w_j\\
&= \sum_{i=1}^{N}\frac{e^{y_i(w^T x_i+b)}}{(1+e^{y_i(w^T x_i+b)})}\cdot x_{ij}y_i + 2\lambda w_j
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial^2 L}{\partial w_j \partial w_k} &= \frac{\partial L}{\partial w_k}(\sum_{i=1}^{N}\frac{e^{y_i(w^T x_i+b)}}{(1+e^{y_i(w^T x_i+b)})}\cdot x_{ij}y_i + 2\lambda w_j)\\
&= \sum_{i=1}^{N}\frac{x_{ij}y_i\cdot(1+e^{y_i(w^T x_i+b)})\cdot\frac{\partial L}{\partial w_k}(e^{y_i(w^T x_i+b)}) - e^{y_i(w^T x_i+b)}\cdot\frac{\partial L}{\partial w_k}(1+e^{y_i(w^T x_i+b)})}{(1+e^{y_i(w^T x_i+b)})^2}\\
&= \sum_{i=1}^{N}\frac{x_{ij}y_i\cdot(1+e^{y_i(w^T x_i+b)})\cdot\frac{\partial L}{\partial w_k}(e^{y_i(w^T x_i+b)}) - e^{y_i(w^T x_i+b)}\cdot\frac{\partial L}{\partial w_k}(e^{y_i(w^T x_i+b)})}{(1+e^{y_i(w^T x_i+b)})^2}\\
&= \sum_{i=1}^{N}\frac{x_{ij}y_i\cdot(1+e^{y_i(w^T x_i+b)} - e^{y_i(w^T x_i+b)})\cdot\frac{\partial L}{\partial w_k}(e^{y_i(w^T x_i+b)})}{(1+e^{y_i(w^T x_i+b)})^2}\\
&= \sum_{i=1}^{N}\frac{x_{ij}y_i\cdot\frac{\partial L}{\partial w_k}(e^{y_i(w^T x_i+b)})}{(1+e^{y_i(w^T x_i+b)})^2}\\
&= \sum_{i=1}^{N}\frac{x_{ij}y_i\cdot e^{y_i(w^T x_i+b)}x_{ik}y_i}{(1+e^{y_i(w^T x_i+b)})^2}\\
&= \sum_{i=1}^{N}\frac{x_{ij}x_{ik}y_iy_i\cdot e^{y_i(w^T x_i+b)}}{(1+e^{y_i(w^T x_i+b)})^2}
\end{aligned}$$

Since $y_i^2 = 1$, we simply rewrite this as

$$\sum_{i=1}^{N} x_{ij}x_{ik}\cdot\frac{e^{y_i(w^T x_i+b)}}{(1+e^{y_i(w^T x_i+b)})^2} \tag{1}$$

(c) Then, we wish to show

$$\mathbf{a^T H a} \equiv \sum_{j,k} a_j a_k H_{j,k} \geq 0$$

Note that summation (1) indicates the j,k^{th} element of the Hessian which allows us to rewrite the the above inequality as

$$
\begin{aligned}
\sum_{j,k} a_j a_k H_{j,k} &= \sum_{j,k} a_j a_k \sum_{i=1}^{N} x_{ij} x_{ik} \cdot \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \\
&= \sum_{j,k} a_j a_k \sum_{i=1}^{N} x_{ij} x_{ik} \cdot \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \\
&= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \sum_{j,k} a_j a_k x_{ij} x_{ik} \\
&= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \sum_{j,k} \mathbf{a^T x} a_k x_{ik} \\
&= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \sum_{j,k} \mathbf{a^T x a^T x} \\
&= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \sum_{j,k} \mathbf{a^T x a^T x} \\
&= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \cdot (\mathbf{a^T x})^2 \geq 0
\end{aligned}
$$

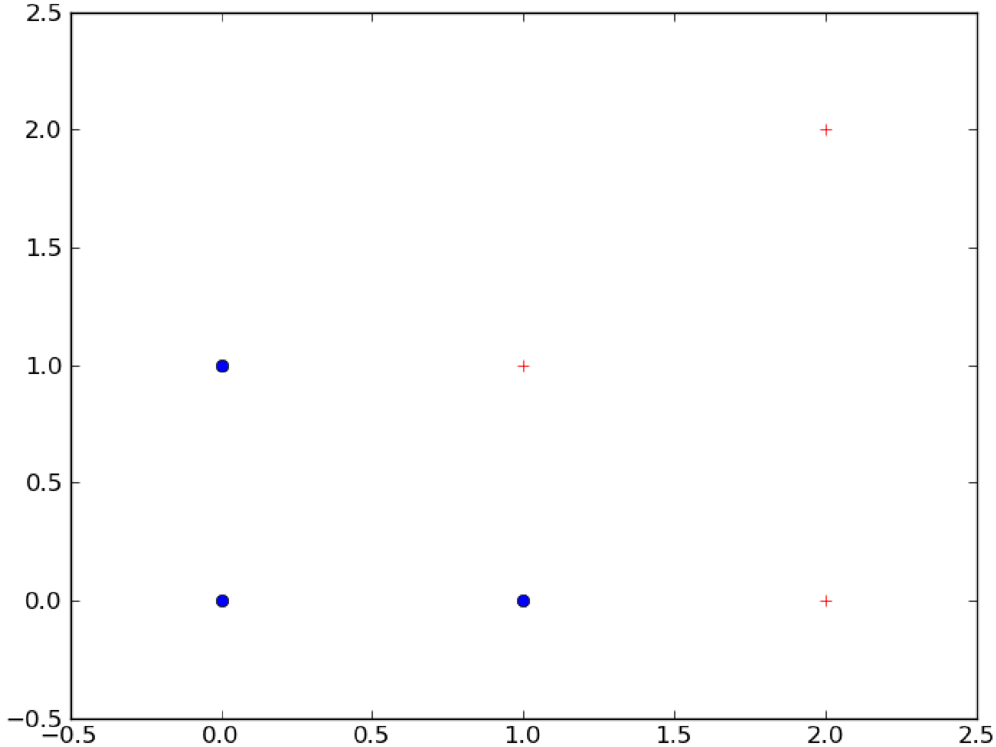
We show this summation is non-negative by showing each component of the summation is non-negative. Consider

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{\alpha}{\beta} \cdot \epsilon &= \sum_{i=1}^{N} \frac{e^{y_i(w^T x_i + b)}}{(1 + e^{y_i(w^T x_i + b)})^2} \cdot (\mathbf{a^T x})^2 \geq 0 \text{ Then, we realize that} \\
\alpha &= e^{y_i(w^T x_i + b)} > 0, \text{since } e^z \text{ is always positive} \\
\beta &= (1 + e^{y_i(w^T x_i + b)})^2 > 0 \\
\epsilon &= (\mathbf{a^T x})^2 \geq 0
\end{aligned}
$$

Therefore, L is convex.

3. Training data

   (a) Yes the classes {+,-} are linearly separable. The - class is represented by circles in the graph below.



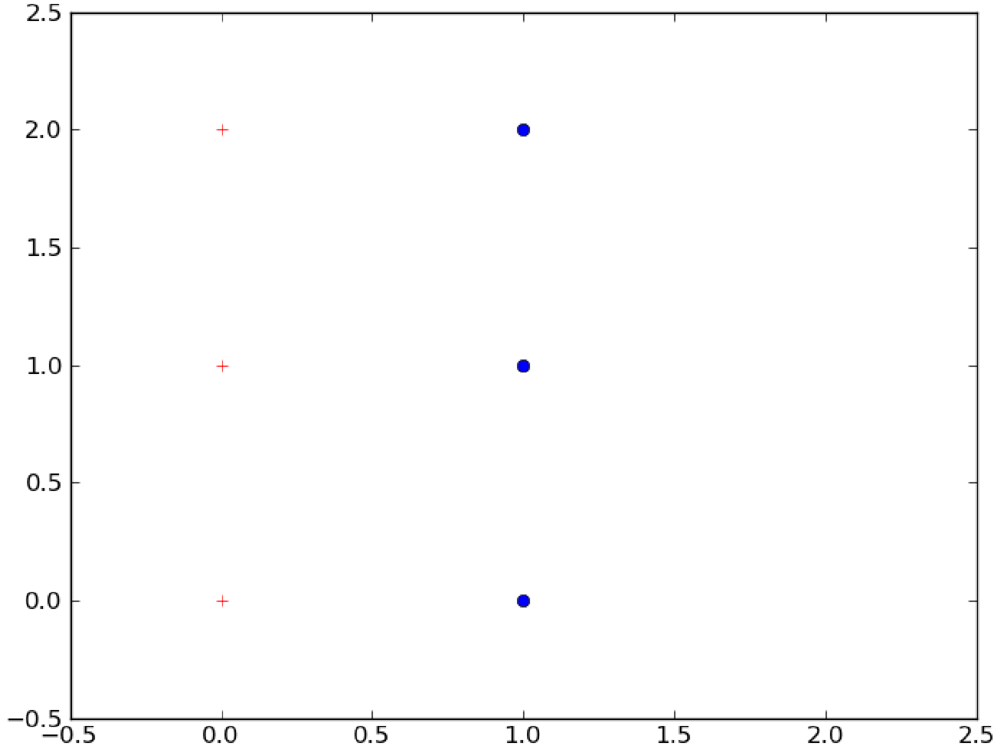
(b) The best hyperplane by inspection is:

$$
\begin{aligned}
x_2 &= -x_1 + 1.5 \\
x_1 + x_2 - 1.5 &= 0 \\
\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 1.5 &= 0
\end{aligned}
$$

So therefore $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b = -1.5$. The support vectors are $(1,0), (0,1), (2,0), (1,1)$.

(c) If we remove a support vector, then the optimal margin will increase since there are fewer constraints.

(d) The answer for (c) is not always true. Consider if we have a class + with points $(0,0), (0,1), (0,2)$ and a class - with points $(1,0), (1,1), (1,2)$. If we remove either $(0,1)$ or $(1,1)$, the best hyperplane does not change and thus the optimal margin remains the same.



4. 3 point dataset

    (a) No

    (b)

$$\phi(x_1) = [1, 0, 0]^T$$
$$\phi(x_2) = [1, -\sqrt{2}, 1]^T$$
$$\phi(x_3) = [1, \sqrt{2}, 1]^T$$

    Yes, this is linearly separable with the hyperplane $x^2 = \frac{1}{2}$

    (c) Let

$$\begin{aligned}
x_1 &= 0 \\
x_2 &= -1 \\
x_3 &= 1 \\
y_1 &= 1 \\
y_2 &= -1 \\
y_3 &= -1 \\
\Lambda(w_1, w_2, w_3, b, \lambda, \mu, \varepsilon) &= \frac{1}{2}\|w\|_2^2 \\
&+ \lambda(y_1(w_1 + b) - 1) \\
&+ \mu(y_2(w_1 - \sqrt{2}w_2 + w_3 + b) - 1) \\
&+ \varepsilon(y_3(w_1 + \sqrt{2}w_2 + w_3 + b) - 1)
\end{aligned}$$

Then, using the method of Lagrange multipliers,

$$\frac{\partial \Lambda}{\partial w_1} = \frac{1}{2}w_1^2 + \lambda - \mu - \varepsilon = 0 \tag{2}$$
$$\frac{\partial \Lambda}{\partial w_2} = \frac{1}{2}w_2^2 + \sqrt{2}\mu - \sqrt{2}\varepsilon = 0 \tag{3}$$
$$\frac{\partial \Lambda}{\partial w_3} = \frac{1}{2}w_3^2 - \mu - \varepsilon = 0 \tag{4}$$
$$\frac{\partial \Lambda}{\partial b} = \lambda - \mu - \varepsilon = 0 \tag{5}$$
$$\frac{\partial \Lambda}{\partial \lambda} = w_1 + b - 1 = 0 \tag{6}$$
$$\frac{\partial \Lambda}{\partial \mu} = -(w_1 - \sqrt{2}w_2 + w_3 + b) - 1 = 0 \tag{7}$$
$$\frac{\partial \Lambda}{\partial \varepsilon} = -(w_1 + \sqrt{2}w_2 + w_3 + b) - 1 = 0 \tag{8}$$

We inspect these equations to arrive at the following conclusions:
From (4), we know $\lambda - \mu - \varepsilon = 0$ so in (1), we realize that $\frac{1}{2}w_1^2 + \lambda - \mu - \varepsilon = \frac{1}{2}w_1^2 = 0$, therefore $w_1 = 0$. Then, in (5), $w_1 + b - 1 = 0 + b - 1 = 0$, therefore $b = 1$. Then, (6) and (7) render a system of simple equations.

$$-(0 - \sqrt{2}w_2 + w_3 + 1) - 1 = 0$$
$$-(0 + \sqrt{2}w_2 + w_3 + 1) - 1 = 0$$

Solving this system of equations renders $w_3 = -1$ and $w_2 = 0$.

(d) Generalizing the solution to 4c. renders that $b = \rho$ from (5). Given ρ_1 and ρ_2 , let us say that 4c. expresses $b, \mathbf{w}$ for some ρ_1 . Then, for some ρ_2 , we find $b = \rho_2, \mathbf{w} = [0, 0, -2\rho_2]$. We realize that our function classifies according to the sign of $\rho(\mathbf{w}^\mathbf{T}\mathbf{x} + b)$ instead of simply $\mathbf{w}^\mathbf{T}\mathbf{x} + b$. Knowing that $\rho \geq 1$, we realize that $sign(\mathbf{w}^\mathbf{T}\mathbf{x} + b) = sign(\rho(\mathbf{w}^\mathbf{T}\mathbf{x} + b))$ so the classification remains the same for all such ρ .

5. Seismic waves

(a) phase, iphase frequencies

- phase

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phase	absolute frequency	relative frequency
Lg	1594	0.017811
P	61779	0.690322
PKP	5974	0.066754
Pg	403	0.004503
Pn	10762	0.120255
Rg	11	0.000123
S	4685	0.052350
Sn	4285	0.047881

- iphase

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iphase	absolute frequency	relative frequency
Lg	2171	0.024259
N	10683	0.119372
P	50815	0.567810
Pg	5291	0.059122
Pn	12610	0.140905
Px	365	0.004079
Rg	444	0.004961
Sn	318	0.003553
Sx	4179	0.046696
tx	2617	0.029243

(b) Confusion matrix (empty cells are zero)

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		phase										
		Lg	PKP	P	S	Rg	Sn	Pn	Pg		Total	Accuracy (%)
iphase	Lg	293	2	114	860	5	859	34	4		2171	13.496
	Sx	297	61	971	1257	3	1191	393	6		4179	58.579
	tx	17	383	2039	26		18	111	23		2617	0
	Px	30	13	101	46		68	61	46		365	60.548
	N	431	564	6097	1278	1	1133	1149	30		10683	0
	P	105	4586	42600	336		153	2993	42		50815	83.834
	Rg	83		8	182	2	169				444	0.450
	Pg	218	120	2716	318		303	1509	107		5291	2.022
	Pn	95	244	7123	243	256		4504	145		12610	35.717
	Sn	25	1	10	139		135	8			318	42.453

(c) Top stations

i. 7: 8751 detections
ii. 24: 5794 detections
iii. 3: 2677 detections
iv. 80: 2528 detections
v. 19: 2478 detections
vi. 38: 2429 detections
vii. 63: 2411 detections
viii. 12: 2343 detections
ix. 74: 2265 detections
x. 65: 2227 detections

(d) Data munging

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station	iphase accuracy (%)	classifier accuracy (%)
7	97.75	88.08
24	87.23	92.15
3	83.86	92.02
80	95.64	88.10
19	67.56	88.87
38	94.74	90.69
63	91.09	88.18
12	82.33	88.77
74	81.44	89.56
65	81.56	92.51

(e) Optimal c

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station	c	accuracy
7	0.42	93.23
24	0.5	86.64
	0.1	87.43
	0.2	87.03
	0.05	87.98
	0.01	89.79
	0.001	92.04
	0.0001	92.20
	0	92.15