Problem Set 3

Math 255: Analysis I

Due: Thursday, Feb 8th at 11:59pm EST

Problem 1. Prove that the complex field $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, with $i^2 = -1$, does not support an order relation which makes it an ordered field.

Proof. Assume in contradiction that $\mathbb{C}$ supports an order relation which makes it an ordered field. Consider $i \in \mathbb{C}$. By definition, $i \neq 0$. Then, by trichotomy, either $i > 0$ or $i < 0$.

- If $i > 0$, then $i \cdot i = i^2 = -1 > 0$ by order respects multiplication. This is a contradiction, as $-1 < 0$.
- If $i < 0$, then we have $-i > 0$. Then, $-i \cdot -i = i^2 = -1 > 0$ by order respects multiplication. This is a contradiction, as $-1 < 0$.

Both cases lead to a contradiction, so $\mathbb{C}$ does not support an order relation which makes it an ordered field. $\square$

Problem 2. In this problem we will discuss a construction of $\mathbb{R}$ using Dedekind cuts.
A *cut set* in $\mathbb{Q}$ is a subset $\alpha \subset \mathbb{Q}$ satisfying:

- $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$;
- if $q \in \alpha$ and $p < q$ then $p \in \alpha$; and
- α has no maximum, that is, for all $q \in \alpha$ there exists $q' \in \alpha$ with $q < q'$.

Consider

$$R = \{\alpha \subset \mathbb{Q} : \alpha \text{ is a cut set}\},$$

the set of all cut sets of $\mathbb{Q}$. Moreover, we define the relation $<$ on R by

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \subsetneq \beta \qquad \text{for all } \alpha, \beta \in R.$$

1. Prove that R with this relation is an ordered set (i.e. show this relation satisfies the trichotomy and transitivity properties).

   Proof. **Trichotomy:** Let $\alpha, \beta \in R$.

   (a) If $\alpha < \beta$, then $\alpha \subsetneq \beta$. This implies that $\exists x \in \beta$ such that $x \notin \alpha$. Then, $\beta \not\subsetneq \alpha$, so $\alpha \not> \beta$ and $\alpha \neq \beta$.

   (b) If $\alpha = \beta$, then $\alpha \not\subsetneq \beta$ and $\beta \not\subsetneq \alpha$. Therefore, $\alpha \not< \beta$ and $\alpha \not> \beta$.

   (c) If $\alpha > \beta$, then $\beta < \alpha$. Then, $\beta \subsetneq \alpha$. This implies that $\exists x \in \alpha$ such that $x \notin \beta$. Then, $\alpha \not\subsetneq \beta$, so $\alpha \not> \beta$ and $\alpha \neq \beta$.

   Transitivity: Let $\alpha, \beta, \gamma \in R$ such that $\alpha < \beta$ and $\beta < \gamma$. Then, $\alpha \subsetneq \beta$ and $\beta \subsetneq \gamma$. This implies that $\alpha \subsetneq \gamma$, so $\alpha < \gamma$. □

2. Prove that R with this order has the least upper bound property. That is, prove that any non-empty $A \subset R$ having an upper bound also has a supremum.

   Proof. Let $A \subset R$ be non-empty and have an upper bound. Let

   $$S = \bigcup_{\alpha \in A} \alpha.$$

   In other words, S is the set of all rationals that belong to at least one cut set in A. We will show that S is a cut set and that it is the supremum of A.

   S is a cut set:

   - $S \neq \emptyset$: Since A is non-empty, there exists $\alpha \in A$. α is not empty by definition of cut set. Then, $\exists q \in \alpha$, so $q \in S$.
   - $S \neq \mathbb{Q}$: Since A has an upper bound, there exists $\beta \in R$ such that $\alpha \leq \beta$ for all $\alpha \in A$. Because β is a cut set, $\beta \neq \mathbb{Q}$, so $\exists q \in \mathbb{Q}$ where $q \notin \beta$. We also know that q is greater than any element in β, and by transitivity, q is greater than any element of any set A. Then, $q \notin S$, so $S \neq \mathbb{Q}$.
   - If $q \in S$ and $p < q$, then $p \in S$: Let $q \in S$ and $p < q$. Then, $q \in \alpha$ for some $\alpha \in A$. Since α is a cut set, $p \in \alpha$, so $p \in S$.
   - S has no maximum: For any $q \in S$, there exists $\alpha \in A$ such that $q \in \alpha$. Since α is a cut set, there exists $q' \in \alpha$ such that $q < q'$. Then, $q' \in S$.

   S is the least upper bound of A:

   By our construction of S, we have that for all $\alpha \in A$, $\alpha \subset S$, so S is an upper bound of A. Now, we will show that S is the least upper bound of A. Suppose in contradiction that there exists T such that $T < S$ and T is an upper bound of A. Then, $T \subsetneq S$. This implies that there exists $q \in S$ such that $q \notin T$. Since $q \in S$, there exists $\alpha \in A$ such that $q \in \alpha$. Since T is an upper bound of A, $\alpha \subset T$. Then, $q \in T$, a contradiction. Thus, S is the least upper bound of A. □

3. Let $\alpha, \beta \in R$ be two cuts. Prove that

$$\alpha + \beta = \{p + q : p \in \alpha, q \in \beta\}$$

is also a cut set of $\mathbb{Q}$. Hence we can define an addition operation on R.

Proof. Let us call $\alpha + \beta = \gamma$. We will show that γ is a cut set of $\mathbb{Q}$.

- $\gamma \neq \emptyset$: Since α and β are cut sets, there exists $p \in \alpha$ and $q \in \beta$. Then, $p + q \in \gamma$.
- $\gamma \neq \mathbb{Q}$: Since α and β are cut sets, $\alpha \neq \mathbb{Q}$ and $\beta \neq \mathbb{Q}$. Then, there exists $p', q' \in \mathbb{Q}$ such that $p' \notin \alpha$ and $q' \notin \beta$. Now consider some $p \in \alpha$, and $q \in \beta$. By definition of a cut set, $p < p'$ and $q < q'$, which implies $p + q < p' + q'$. Thus, $p' + q' \notin \gamma$.
- If $r \in \gamma$ and $s < r$, then $s \in \gamma$: For $r = p + q$ with $p \in \alpha$ and $q \in \beta$, if $s < r$, then $s < p + q$. Since α and β are cut sets, if $p' < p$ for some $p' \in \mathbb{Q}$, then $p' \in \alpha$; similarly for $q' < q$. Therefore, we can always find $p' \in \alpha$ and $q' \in \beta$ such that $s = p' + q'$, thus $s \in \gamma$.
- γ has no maximum: Suppose $r \in \gamma$, where $r = p + q$ for some $p \in \alpha$ and $q \in \beta$. Because α and β have no maximum, there exists $p' > p$ in α and $q' > q$ in β. Thus, $r' = p' + q' > r$ is also in γ, proving γ has no maximum.

□

4. How could you identify $\mathbb{Q}$ as a subset of R?

Proof. For any rational number $q \in \mathbb{Q}$, we define the cut set

$$\alpha_q = \{p \in \mathbb{Q} : p < q\}.$$

This is a cut set because

- $\alpha_q \neq \emptyset$: $q - 1 \in \alpha_q$. ($\mathbb{Q}$ has no minimum).
- $\alpha_q \neq \mathbb{Q}$: There exists $p \in \mathbb{Q}$ such that $p > q$, so $p \notin \alpha_q$.
- If $p \in \alpha_q$ and $r < p$, then $r \in \alpha_q$: If $r < p$, then $r < q$, so $r \in \alpha_q$.
- α_q has no maximum: We apply denseness of $\mathbb{Q}$ in $\mathbb{R}$ (proven in class) to use the denseness of $\mathbb{Q}$ in $\mathbb{Q}$ (proof is identical) to state that for any $p \in \alpha_q$, we can find an $r \in \mathbb{Q}$, such that $p < r < q$. Thus, α_q has no maximum.

Then, we can identify $\mathbb{Q}$ as a subset of R by the map $q \mapsto \alpha_q$. □

5. *(Not for submission)* How would you define multiplication in R? Try to convince yourselves that these operations define an ordered field structure on R, yielding a construction of $\mathbb{R}$.

Problem 3. Let $A, B \subset \mathbb{R}$ be two subsets which are bounded above. Prove:

1. $\sup(A \cup B) = \max(\sup A, \sup B)$.

   Proof. We will show that $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$ and that it is the least upper bound. This is sufficient to show that $\sup(A \cup B) = \max(\sup A, \sup B)$, because we have shown in class that the least upper bound is unique.

   Let $a \in A \cup B$. Then, $a \in A$ or $a \in B$. If $a \in A$, then $a \leq \sup A \leq \max(\sup A, \sup B)$. If $a \in B$, then $a \leq \sup B \leq \max(\sup A, \sup B)$. Thus, $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$.

   Now, let u be an upper bound for $A \cup B$. Then, for all $a \in A \cup B$, $a \leq u$. If $a \in A$, then $a \leq \sup A \leq u$. If $a \in B$, then $a \leq \sup B \leq u$. Thus, $\sup A \leq u$ and $\sup B \leq u$. Then, $\max(\sup A, \sup B) \leq u$. Thus, $\max(\sup A, \sup B)$ is the least upper bound for $A \cup B$.

   □

2. If $A \cap B \neq \emptyset$ then $\sup(A \cap B) \leq \min(\sup A, \sup B)$. Give an example where equality does not hold.

   Proof. Let $s_A = \sup A$ and $s_B = \sup B$, given by the LUB property. Since s_A and s_B are the least upper bounds of A and B, respectively, it follows that for all $a \in A$, $a \leq s_A$, and for all $b \in B$, $b \leq s_B$.

   Elements of $A \cap B$: For any element $x \in A \cap B$, x is both an element of A and an element of B. Therefore, $x \leq s_A$ and $x \leq s_B$. This means that x is less than or equal to both s_A and s_B, and hence $x \leq \min(s_A, s_B)$.

   Supremum of $A \cap B$: Since every element $x \in A \cap B$ satisfies $x \leq \min(s_A, s_B)$, it follows that $\min(s_A, s_B)$ is an upper bound for $A \cap B$. To prove that $\sup(A \cap B) \leq \min(s_A, s_B)$, we observe that the supremum of $A \cap B$ is the least upper bound of $A \cap B$. Since $\min(s_A, s_B)$ is an upper bound for $A \cap B$, and by definition of supremum, no number greater than $\sup(A \cap B)$ can serve as an upper bound, it follows that $\sup(A \cap B)$ must be less than or equal to this common upper bound, $\min(s_A, s_B)$.

   Example where equality does not hold: Let $A = \{\frac{n-1}{n} : n \in \mathbb{N}\}$ and $B = \{\frac{n-1}{2n} : n \in \mathbb{N}\}$. Then, $A \cap B = \{0\}$. We have $\sup A = 1$ and $\sup B = 1/2$. Then, $\sup(A \cap B) = 0 < 1/2 = \min(\sup A, \sup B)$. □

3. The set $-A = \{-x : x \in A\}$ is bounded below and satisfies $\inf(-A) = -\sup A$.

   Proof. First, we show that $-A$ is bounded below. By assumption, A is bounded above, meaning $\exists u \in \mathbb{R}$ such that $\forall a \in A$, $a \leq u$. By definition of inequality, there exists a x such that $a + x = u$ for all $a \in A$. Then, we can use the additive inverses and the cancellation law to arrive at $-a = -u + x$ for all $a \in A$. Therefore, $\forall -a \in -A$, $-a \geq -u$. Thus, $-A$ is bounded below by $-u$.

   Next, we show that $\inf(-A) = -\sup A$ by showing that $-\sup A$ is a lower bound for $-A$ and there exists lower bound of $-A$ in $\mathbb{R}$ greater than $-\sup A$. Let $s = \sup A$, which we obtain by the LUB property of $\mathbb{R}$. By s being an upper bound, we reason similarly to the previous paragraph that $-s$ is a lower bound for $-A$.

   Assume in contradiction there exists $y > -s$ that is a lower bound for $-A$. This implies that for all $x \in -A$, we have $-x \geq y$. Thus, $x \leq -y$, which means $-y$ is an upper bound for A. However, since $y > -s$, we have $-y < s$, a contradiction of our assumption that $s = \sup A$. Therefore, no such $y > -s$ can exist, meaning that $-s$ is the greatest lower bound of A, i.e. $inf(-A) = -\sup A$. □

4. The set $A + B = \{a + b : a \in A, b \in B\}$ is bounded above and satisfies $\sup(A + B) = \sup A + \sup B$.

   Proof. Given that A and B are both bounded above, there exist real numbers M and N such that for all $a \in A$, $a \leq M$, and for all $b \in B$, $b \leq N$. Consider any element $x \in A + B$, where $x = a + b$ for some $a \in A$ and $b \in B$. It follows that $x \leq M + N$, demonstrating that $A + B$ is bounded above by $M + N$.

$\sup(A+B)$ is an upper bound for $A+B$: Let $s_A = \sup A$ and $s_B = \sup B$. For any $a \in A$ and $b \in B$, we have $a \leq s_A$ and $b \leq s_B$. Therefore, for any element $x = a+b \in A+B$, we have $x = a+b \leq s_A+s_B$. This shows that $s_A + s_B$ is an upper bound for $A+B$.

$\sup(A+B)$ is the least upper bound: Suppose there exists $L < \sup A + \sup B$ such that for all $a \in A$ and $b \in B$, $a+b \leq L$. Given $s_A = \sup A$ and $s_B = \sup B$, by definition of supremum, for any $\epsilon > 0$, there exist $a' \in A$ and $b' \in B$ such that:

- $a' > s_A - \frac{\epsilon}{2}$
- $b' > s_B - \frac{\epsilon}{2}$

Choose $\epsilon = \sup A + \sup B - L > 0$. Such an ϵ exists because we assumed $L < \sup A + \sup B$. Then, for the chosen a' and b', we have:

$$a' + b' > s_A - \frac{\epsilon}{2} + s_B - \frac{\epsilon}{2} = s_A + s_B - \epsilon = L$$

This is a contradiction because we assumed L is an upper bound for $A+B$, which would mean $a'+b' \leq L$ for all $a' \in A$ and $b' \in B$. However, we have found a', b' such that $a' + b' > L$. Therefore, our initial assumption that there exists an upper bound $L < \sup A + \sup B$ for $A+B$ is false. $\square$

Problem 4. For each of the following subsets in $\mathbb{R}$

$$A = \left\{ \frac{(-1)^n \cdot n}{n+1} : n \in \mathbb{N} \right\} \quad , \quad B = \left\{ \frac{(-1)^n}{n+1} : n \in \mathbb{N} \right\} \quad , \text{and} \quad C = \left\{ \frac{1}{(q-1)^2} : q \in \mathbb{Q}, q \neq 1 \right\}$$

answer the following (justify your claims):

1. Is the set bounded above[1]? bounded below?
2. If bounded above what is its supremum? If bounded below what is its infimum?
3. Does the set have a maximum? minimum?

Set A

1. **Boundedness:** The set $A = \left\{ \frac{(-1)^n \cdot n}{n+1} : n \in \mathbb{N} \right\}$ is bounded above by 1 and below by -1. For any $n \in \mathbb{N}$, we indeed have $-1 < \frac{(-1)^n \cdot n}{n+1} < 1$. This is because the magnitude of the numerator is always less than the magnitude of the denominator, ensuring the fraction's absolute value is strictly less than 1.

2. **Supremum and Infimum:** To prove $\sup A = 1$ and $\inf A = -1$, assume for contradiction there exists an upper bound $U < 1$ or a lower bound $L > -1$. For any $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \epsilon$. Then, $\frac{n}{n+1} > 1 - \epsilon$, contradicting U as an upper bound. Similarly, we can show $L > -1$ leads to a contradiction, thus $\sup A = 1$ and $\inf A = -1$.

3. **Maximum and Minimum:** Since for every $n \in \mathbb{N}$, there exists $n+1$ such that $\frac{(-1)^{n+1} \cdot (n+1)}{n+2}$ is closer to either 1 or -1, A does not contain its supremum or infimum as elements. Hence, A has no maximum or minimum.

Set B

1. **Boundedness:** The set $B = \left\{ \frac{(-1)^n}{n+1} : n \in \mathbb{N} \right\}$ is bounded above by 1 and below by -1 for similar reasons as A, with the absolute value of all elements bounded by 1 due to the denominator being larger than the numerator in absolute value.

2. **Supremum and Infimum:** $\sup B = \frac{1}{3}$ and $\inf B = -\frac{1}{2}$ are demonstrated by directly evaluating the sequence for $n = 1$ and $n = 0$, respectively. No element for $n > 1$ can exceed these bounds due to the increasing denominator, establishing these as the supremum and infimum.

3. **Maximum and Minimum:** As $\frac{1}{3}$ and $-\frac{1}{2}$ are explicitly present in B for $n = 1$ and $n = 0$ respectively, they are the maximum and minimum of B.

Set C

1. **Boundedness:** The set $C = \left\{ \frac{1}{(q-1)^2} : q \in \mathbb{Q}, q \neq 1 \right\}$ is unbounded above because as q approaches 1, the denominator approaches 0, causing the fraction to increase without bound. However, C is bounded below by 0 since the fraction is always positive.

2. **Infimum:** We show $\inf C = 0$ using an ϵ-δ argument. For any $\epsilon > 0$, choose q such that $0 < |q - 1| < \sqrt{\frac{1}{\epsilon}}$. Then, $\frac{1}{(q-1)^2} > \epsilon$, indicating the fraction approaches 0 as q approaches 1, but never reaches 0.

3. **Maximum and Minimum:** C has no maximum or minimum. The minimum value of 0 is approached but never reached because the numerator is always 1 and the denominator is always positive, ensuring the fraction is always positive. The lack of an upper bound directly implies no maximum.

[1] i.e. has an upper bound.