Problem Set 3

Math 255: Analysis I

Due: Thursday, Feb 8th at 11:59pm EST

Problem 1. Prove that the complex field $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, with $i^2 = -1$, does not support an order relation which makes it an ordered field.

Proof. Assume in contradiction that $\mathbb{C}$ supports an order relation which makes it an ordered field. Consider $i \in \mathbb{C}$. By definition, $i \neq 0$. Then, by trichotomy, either $i > 0$ or $i < 0$.

- If $i > 0$, then $i \cdot i = i^2 = -1 > 0$ by order respects multiplication. This is a contradiction, as $-1 < 0$.
- If $i < 0$, then we have $-i > 0$. Then, $-i \cdot -i = i^2 = -1 > 0$ by order respects multiplication. This is a contradiction, as $-1 < 0$.

Both cases lead to a contradiction, so $\mathbb{C}$ does not support an order relation which makes it an ordered field. $\square$

Problem 2. In this problem we will discuss a construction of $\mathbb{R}$ using Dedekind cuts.
A *cut set* in $\mathbb{Q}$ is a subset $\alpha \subset \mathbb{Q}$ satisfying:

- $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$;
- if $q \in \alpha$ and $p < q$ then $p \in \alpha$; and
- α has no maximum, that is, for all $q \in \alpha$ there exists $q' \in \alpha$ with $q < q'$.

Consider
$$R = \{\alpha \subset \mathbb{Q} : \alpha \text{ is a cut set}\},$$
the set of all cut sets of $\mathbb{Q}$. Moreover, we define the relation $<$ on R by
$$\alpha < \beta \quad \text{if and only if} \quad \alpha \subsetneq \beta \qquad \text{for all } \alpha, \beta \in R.$$

1. Prove that R with this relation is an ordered set (i.e. show this relation satisfies the trichotomy and transitivity properties).

   Proof. **Trichotomy:** Let $\alpha, \beta \in R$.

   (a) If $\alpha < \beta$, then $\alpha \subsetneq \beta$. This implies that $\exists x \in \beta$ such that $x \notin \alpha$. Then, $\beta \not\subseteq \alpha$, so $\alpha \not> \beta$ and $\alpha \neq \beta$.

   (b) If $\alpha = \beta$, then $\alpha \not\subsetneq \beta$ and $\beta \not\subsetneq \alpha$. Therefore, $\alpha \not< \beta$ and $\alpha \not> \beta$.

   (c) If $\alpha > \beta$, then $\beta < \alpha$. Then, $\beta \subsetneq \alpha$. This implies that $\exists x \in \alpha$ such that $x \notin \beta$. Then, $\alpha \not\subseteq \beta$, so $\alpha \not> \beta$ and $\alpha \neq \beta$.

   Transitivity: Let $\alpha, \beta, \gamma \in R$ such that $\alpha < \beta$ and $\beta < \gamma$. Then, $\alpha \subsetneq \beta$ and $\beta \subsetneq \gamma$. This implies that $\alpha \subsetneq \gamma$, so $\alpha < \gamma$. $\square$

2. Prove that R with this order has the least upper bound property. That is, prove that any non-empty $A \subset R$ having an upper bound also has a supremum.

   Proof. Let $A \subset R$ be non-empty and have an upper bound. Let
$$S = \bigcup_{\alpha \in A} \alpha.$$
   In other words, S is the set of all rationals that belong to at least one cut set in A. We will show that S is a cut set and that it is the supremum of A.

   S is a cut set:

   - $S \neq \emptyset$: Since A is non-empty, there exists $\alpha \in A$. α is not empty by definition of cut set. Then, $\exists q \in \alpha$, so $q \in S$.
   - $S \neq \mathbb{Q}$: Since A has an upper bound, there exists $\beta \in R$ such that $\alpha \leq \beta$ for all $\alpha \in A$. Because β is a cut set, $\beta \neq \mathbb{Q}$, so $\exists q \in \mathbb{Q}$ where $q \notin \beta$. We also know that q is greater than any element in β, and by transitivity, q is greater than any element of any set A. Then, $q \notin S$, so $S \neq \mathbb{Q}$.
   - If $q \in S$ and $p < q$, then $p \in S$: Let $q \in S$ and $p < q$. Then, $q \in \alpha$ for some $\alpha \in A$. Since α is a cut set, $p \in \alpha$, so $p \in S$.
   - S has no maximum: For any $q \in S$, there exists $\alpha \in A$ such that $q \in \alpha$. Since α is a cut set, there exists $q' \in \alpha$ such that $q < q'$. Then, $q' \in S$.

   S is the least upper bound of A:

   By our construction of S, we have that for all $\alpha \in A$, $\alpha \subset S$, so S is an upper bound of A. Now, we will show that S is the least upper bound of A. Suppose in contradiction that there exists T such that $T < S$ and T is an upper bound of A. Then, $T \subsetneq S$. This implies that there exists $q \in S$ such that $q \notin T$. Since $q \in S$, there exists $\alpha \in A$ such that $q \in \alpha$. Since T is an upper bound of A, $\alpha \subset T$. Then, $q \in T$, a contradiction. Thus, S is the least upper bound of A. $\square$

3. Let $\alpha, \beta \in R$ be two cuts. Prove that

$$\alpha + \beta = \{p + q : p \in \alpha, q \in \beta\}$$

is also a cut set of $\mathbb{Q}$. Hence we can define an addition operation on R.

Proof. Let us call $\alpha + \beta = \gamma$. We will show that γ is a cut set of $\mathbb{Q}$.

- $\gamma \neq \emptyset$: Since α and β are cut sets, there exists $p \in \alpha$ and $q \in \beta$. Then, $p + q \in \gamma$.
- $\gamma \neq \mathbb{Q}$: Since α and β are cut sets, $\alpha \neq \mathbb{Q}$ and $\beta \neq \mathbb{Q}$. Then, there exists $p', q' \in \mathbb{Q}$ such that $p' \notin \alpha$ and $q' \notin \beta$. Now consider some $p \in \alpha$, and $q \in \beta$. By definition of a cut set, $p < p'$ and $q < q'$, which implies $p + q < p' + q'$. Thus, $p' + q' \notin \gamma$.
- If $r \in \gamma$ and $s < r$, then $s \in \gamma$: For $r = p + q$ with $p \in \alpha$ and $q \in \beta$, if $s < r$, then $s < p + q$. Since α and β are cut sets, if $p' < p$ for some $p' \in \mathbb{Q}$, then $p' \in \alpha$; similarly for $q' < q$. Therefore, we can always find $p' \in \alpha$ and $q' \in \beta$ such that $s = p' + q'$, thus $s \in \gamma$.
- γ has no maximum: Suppose $r \in \gamma$, where $r = p + q$ for some $p \in \alpha$ and $q \in \beta$. Because α and β have no maximum, there exists $p' > p$ in α and $q' > q$ in β. Thus, $r' = p' + q' > r$ is also in γ, proving γ has no maximum.

$\square$

4. How could you identify $\mathbb{Q}$ as a subset of R?

Proof. For any rational number $q \in \mathbb{Q}$, we define the cut set

$$\alpha_q = \{p \in \mathbb{Q} : p < q\}.$$

This is a cut set because

- $\alpha_q \neq \emptyset$: $q - 1 \in \alpha_q$. ($\mathbb{Q}$ has no minimum).
- $\alpha_q \neq \mathbb{Q}$: There exists $p \in \mathbb{Q}$ such that $p > q$, so $p \notin \alpha_q$.
- If $p \in \alpha_q$ and $r < p$, then $r \in \alpha_q$: If $r < p$, then $r < q$, so $r \in \alpha_q$.
- α_q has no maximum: We apply denseness of $\mathbb{Q}$ in $\mathbb{R}$ (proven in class) to use the denseness of $\mathbb{Q}$ in $\mathbb{Q}$ (proof is identical) to state that for any $p \in \alpha_q$, we can find an $r \in \mathbb{Q}$, such that $p < r < q$. Thus, α_q has no maximum.

Then, we can identify $\mathbb{Q}$ as a subset of R by the map $q \mapsto \alpha_q$. $\square$

5. *(Not for submission)* How would you define multiplication in R? Try to convince yourselves that these operations define an ordered field structure on R, yielding a construction of $\mathbb{R}$.

Problem 3. Let $A, B \subset \mathbb{R}$ be two subsets which are bounded above. Prove:

1. $\sup(A \cup B) = \max(\sup A, \sup B)$.

   Proof. We will show that $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$ and that it is the least upper bound. This is sufficient to show that $\sup(A \cup B) = \max(\sup A, \sup B)$, because we have shown in class that the least upper bound is unique.

   Let $a \in A \cup B$. Then, $a \in A$ or $a \in B$. If $a \in A$, then $a \leq \sup A \leq \max(\sup A, \sup B)$. If $a \in B$, then $a \leq \sup B \leq \max(\sup A, \sup B)$. Thus, $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$.

   Now, let u be an upper bound for $A \cup B$. Then, for all $a \in A \cup B$, $a \leq u$. If $a \in A$, then $a \leq \sup A \leq u$. If $a \in B$, then $a \leq \sup B \leq u$. Thus, $\sup A \leq u$ and $\sup B \leq u$. Then, $\max(\sup A, \sup B) \leq u$. Thus, $\max(\sup A, \sup B)$ is the least upper bound for $A \cup B$.

   □

2. If $A \cap B \neq \emptyset$ then $\sup(A \cap B) \leq \min(\sup A, \sup B)$. Give an example where equality does not hold.

   Proof. Let $A = \{\frac{n-1}{n} : n \in \mathbb{N}\}$ and $B = \{\frac{n-1}{2n} : n \in \mathbb{N}\}$. Then, $A \cap B = \{0\}$. We have $\sup A = 1$ and $\sup B = 1/2$. Then, $\sup(A \cap B) = 0 < 1/2 = \min(\sup A, \sup B)$. □

3. The set $-A = \{-x : x \in A\}$ is bounded below and satisfies $\inf(-A) = -\sup A$.

   Proof. First, we show that $-A$ is bounded below. By assumption, A is bounded above, meaning $\exists u \in \mathbb{R}$ such that $\forall a \in A$, $a \leq u$. Then, $\forall -a \in -A$, $-a \geq -u$. Thus, $-A$ is bounded below by $-u$.

   Next, we show that $\inf(-A) = -\sup A$ by showing that $-\sup A$ is a lower bound for $-A$ and there exists lower bound of $-A$ in $\mathbb{R}$ greater than $-\sup A$. Let $s = \sup A$, which we obtain by the LUB property of $\mathbb{R}$. By s being an upper bound, we reason similarly to the previous paragraph that $-s$ is a lower bound for $-A$.

   Assume in contradiction there exists $y > -s$ that is a lower bound for $-A$. This implies that for all $x \in -A$, we have $-x \geq y$. Thus, $x \leq -y$, which means $-y$ is an upper bound for A. However, since $y > -s$, we have $-y < s$, a contradiction of our assumption that $s = \sup A$. Therefore, no such $y > -s$ can exist, meaning that $-s$ is the greatest lower bound of A, i.e. $inf(-A) = -\sup A$. □

Problem 4. For each of the following subsets in $\mathbb{R}$

$$A = \left\{ \frac{(-1)^n \cdot n}{n+1} : n \in \mathbb{N} \right\} \quad , \quad B = \left\{ \frac{(-1)^n}{n+1} : n \in \mathbb{N} \right\} \quad , \text{ and } \quad C = \left\{ \frac{1}{(q-1)^2} : q \in \mathbb{Q}, q \neq 1 \right\}$$

answer the following (justify your claims):

1. Is the set bounded above[1]? bounded below?
2. If bounded above what is its supremum? If bounded below what is its infimum?
3. Does the set have a maximum? minimum?

1. A
   (a) A is bounded above and bounded below.
   (b) $\sup A = 1$ and $\inf A = -1$.
   (c) A has no maximum and no minimum.
2. B
   (a) B is bounded above and bounded below.
   (b) $\sup B = 1/3$ and $\inf B = -1/2$.
   (c) $\max B = 1/3$ and $\min B = -1/2$.
3. C
   (a) C is only bounded below.
   (b) $\inf C = 0$.
   (c) C has no maximum and no minimum.

[1] i.e. has an upper bound.