Econ 136: Problem Set 1

Franklin She

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Problem 1.

1.

$$\begin{aligned}
Cov(X, Y) &= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] \\
&= \mathbb{E}[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y] \\
&= \mathbb{E}[XY] - \mathbb{E}[X\mu_Y] - \mathbb{E}[Y\mu_X] + \mathbb{E}[\mu_X\mu_Y] \\
&= \mathbb{E}[XY] - \mu_Y\mathbb{E}[X] - \mu_X\mathbb{E}[Y] + \mu_X\mu_Y \\
&= \mathbb{E}[XY] - \mu_X\mu_Y
\end{aligned}$$

2.

$$\begin{aligned}
Cov(X, Y) &= \mathbb{E}[XY] - \mu_X\mu_Y \\
&= \mathbb{E}[X^3] - \mu_X\mu_Y \\
&= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] \\
&= 0 - 0\mathbb{E}[X^2] \\
&= 0
\end{aligned}$$

Problem 2.

$$\begin{aligned}
Var(\mathbf{X}) &= \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T] \\
&= \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X}^T - \mathbb{E}[\mathbf{X}]^T)] \\
&= \mathbb{E}[\mathbf{X}\mathbf{X}^T - \mathbf{X}\mathbb{E}[\mathbf{X}]^T - \mathbb{E}[\mathbf{X}]\mathbf{X}^T + \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T] \\
&= \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \mathbb{E}[\mathbf{X}\mathbb{E}[\mathbf{X}]^T] - \mathbb{E}[\mathbb{E}[\mathbf{X}]\mathbf{X}^T] + \mathbb{E}[\mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T] \\
&= \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T + \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T \\
&= \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]^T.
\end{aligned}$$

Problem 3.

1. First, we express $(a + b \cdot Y + c \cdot Z) - \tilde{m}(X)$ as:

$$(a + b \cdot Y + c \cdot Z) - (a + b \cdot \mathbb{E}[Y|X] + c \cdot \mathbb{E}[Z|X]) = b \cdot (Y - \mathbb{E}[Y|X]) + c \cdot (Z - \mathbb{E}[Z|X]).$$

   Taking the expectation and using the linearity of expectation, we get:

$$\begin{aligned}
\mathbb{E}[((a + b \cdot Y + c \cdot Z) - \tilde{m}(X)) \cdot g(X)] &= \mathbb{E}[b \cdot (Y - \mathbb{E}[Y|X]) \cdot g(X) + c \cdot (Z - \mathbb{E}[Z|X]) \cdot g(X)] \\
&= b \cdot \mathbb{E}[(Y - \mathbb{E}[Y|X]) \cdot g(X)] + c \cdot \mathbb{E}[(Z - \mathbb{E}[Z|X]) \cdot g(X)] \\
&= b \cdot 0 + c \cdot 0 \\
&= 0.
\end{aligned}$$

2. We have shown in part (a) that

$$\mathbb{E}[((a + b \cdot Y + c \cdot Z) - \tilde{m}(X)) \cdot g(X)] = 0 \quad \text{for all } g.$$

   Therefore, by the orthogonality principle proven in class, we can define that $\tilde{m}(X)$ is $\mathbb{E}[a+b \cdot Y+c \cdot Z|X]$, the expected value of $a + b \cdot Y + c \cdot Z$ given X.

Problem 4. '

1. To express $\mathbb{E}[Y|X]$ as a linear function of X, consider the two possible values of X, which are 0 and 1. The conditional expectation of Y given X can thus be written as:

$$\begin{aligned}\mathbb{E}[Y|X=0] &= \beta_0, \\ \mathbb{E}[Y|X=1] &= \beta_0 + \beta_1.\end{aligned}$$

   Hence, for $X = 0$ and $X = 1$, the conditional expectation $\mathbb{E}[Y|X]$ can be represented as $\beta_0 + \beta_1 X$, where:

$$\begin{aligned}\beta_0 &= \mathbb{E}[Y|X=0], \\ \beta_1 &= \mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0].\end{aligned}$$

2. We use the law of iterated expectations and the result of part (a),

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|X]] \\ &= \mathbb{E}[\beta_0 + \beta_1 X] \\ &= \mathbb{E}[\mathbb{E}[Y|X=0] + (\mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0])X] \\ &= \mathbb{E}[\mathbb{E}[Y|X=0]] + \mathbb{E}[(\mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0])X] \\ &= \mathbb{E}[Y|X=0] + (\mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0])\mathbb{E}[X] \\ &= \mathbb{E}[Y|X=0] + (\mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0])p \\ &= p \cdot \mathbb{E}[Y|X=1] + (1-p) \cdot \mathbb{E}[Y|X=0]\end{aligned}$$

3. We use the law of iterated expectations at the result of part (a),

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[\mathbb{E}[XY|X]] \\ &= \sum_x \mathbb{E}[XY|X=x] \cdot \Pr[X=x] \\ &= \mathbb{E}[XY|X=0] \cdot (1-p) + \mathbb{E}[XY|X=1] \cdot p \\ &= p \cdot \mathbb{E}[Y|X=1] \qquad \text{(by choosing } X=0 \text{ or } X=1\text{)}\end{aligned}$$

4. By definition,

$$\begin{aligned}\text{Cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y], \\ \text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2.\end{aligned}$$

   From (c), solving for $\mathbb{E}[Y|X=1]$ gives:

$$\mathbb{E}[Y|X=1] = \frac{\mathbb{E}[XY]}{p}$$

   From (b), solving for $\mathbb{E}[Y|X=0]$ gives:

$$\begin{aligned}\mathbb{E}[Y] - p\mathbb{E}[Y|X=1] &= (1-p)\mathbb{E}[Y|X=0] \\ \mathbb{E}[Y|X=0] &= \frac{\mathbb{E}[Y] - p\mathbb{E}[Y|X=1]}{1-p} \\ &= \frac{\mathbb{E}[Y] - \mathbb{E}[XY]/p}{1-p}\end{aligned}$$

Solving for β_1 gives:

$$\begin{aligned}
\beta_1 &= \mathbb{E}[Y|X=1] - \mathbb{E}[Y|X=0] \\
&= \frac{\mathbb{E}[XY]}{p} - \frac{\mathbb{E}[Y] - \mathbb{E}[XY]/p}{1-p} \\
&= \frac{(1-p)\mathbb{E}[XY] - p\mathbb{E}[Y] + \mathbb{E}[XY]}{p(1-p)} \\
&= \frac{\mathbb{E}[XY] - \mathbb{E}[Y] \cdot p}{p(1-p)} \\
&= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{p(1-p)} \\
&= \frac{\text{Cov}(X,Y)}{\text{Var}(X)}
\end{aligned}$$

Problem 5.

1.

$$\begin{aligned}
\mathbb{E}[X_1] &= \mathbb{E}[r_{B,1} - r_f] \\
&= \mathbb{E}[r_{B,1}] - \mathbb{E}[r_f] \\
&= 0.12 - 0.02 \\
&= 0.10
\end{aligned}$$

$$\begin{aligned}
Var(X_1) &= Var(r_{B,1} - r_f) \\
&= Var(r_{B,1}) + Var(r_f) - 2Cov(r_{B,1}, r_f) \\
&= Var(r_{B,1}) \\
&= 0.01 = 0.1^2
\end{aligned}$$

By the properties of the normal distribution, we have that $\frac{X_1-0.10}{0.1} \sim N(0,1)$.

2.

(a) Since we have shown that $\frac{X_1-0.10}{0.1} \sim N(0,1)$, we can express:

$$\begin{aligned}
Pr[X_1 \leq 0] &= Pr[\frac{X_1 - 0.10}{0.1} \leq \frac{0-0.10}{0.1}] \\
&= Pr[Z \leq -1] \\
&= \Phi(-1)
\end{aligned}$$

(b) R code for calculating the probability:

```
probability <- pnorm(-1)
print(probability)
```

The probability is approximately 0.1587. There is about a 15.87% chance of observing zero excess return or less in one year. The relatively low probability suggests that having a zero or negative excess return in one year is not extremely unlikely.

(c) If we use a significance level of 0.05, the observed zero excess return in one year is not implausible because

$$0.1587 > 0.05$$

3.

(a) The standard error of the mean for $\bar{X}_4$ is given by $\frac{\sigma}{\sqrt{n}}$, where $\sigma = 0.1$ is the standard deviation of X_t and $n = 4$ is the number of years. Thus, the standard error is $\frac{0.1}{\sqrt{4}} = \frac{0.1}{2} = 0.05$. Since $\mathbb{E}[\bar{X}_4] = 0.10$, we can normalize $\bar{X}_4$ as follows:

$$\begin{aligned} Pr[\bar{X}_4 \leq 0] &= Pr\left[\frac{\bar{X}_4 - 0.10}{0.05} \leq \frac{0 - 0.10}{0.05}\right] \\ &= Pr[Z \leq -2] \\ &= \Phi(-2) \end{aligned}$$

(b) Using R, the calculation is performed as follows:

```r
        probability <- pnorm(-2)
        print(probability)
```

The probability is approximately 0.0228. There is about a 2.28% chance of observing zero or negative average excess return across four years. This low probability suggests that observing such an outcome is quite unlikely.

(c) If we use a significance level of 0.05, the observed zero excess return in one year is implausible because

$$0.0228 < 0.05$$

4.

(a) The probability in question can be decomposed into two tails of the distribution:

$$\begin{aligned} Pr[|\bar{X}_4 - 0.10| \geq 0.10] &= Pr[\bar{X}_4 \leq 0] + Pr[\bar{X}_4 \geq 0.20] \\ &= \Phi\left(\frac{0 - 0.10}{0.05}\right) + \left(1 - \Phi\left(\frac{0.20 - 0.10}{0.05}\right)\right) \\ &= \Phi(-2) + (1 - \Phi(2)). \end{aligned}$$

(b) Using R, we calculate this probability as follows:

```r
    probability <- pnorm(-2) + (1 - pnorm(2))
    print(probability)
```

This will yield a probability of approximately 0.0455. There is about a 4.55% chance of observing an average excess return either 0.10 below or above the asserted expected excess return of 0.10 in one year.