Problem Set 1

Math 255: Analysis I, Franklin She

Due: Thursday, Jan 25th at 11:59pm EST

Problem 1. Let A, B, C be any sets. Show the following:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity)

   Proof. For the first statement, first we show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$. If $x \in C$, then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$. Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

   Next we show that $A \cap (B \cup C) \supset (A \cap B) \cup (A \cap C)$. Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$, so $x \in A \cap (B \cup C)$. Thus, $A \cap (B \cup C) \supset (A \cap B) \cup (A \cap C)$. Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   For the second statement, first we show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Thus, $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

   Next we show that $A \cup (B \cap C) \supset (A \cup B) \cap (A \cup C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \in B$, then $x \in B \cap C$, so $x \in A \cup (B \cap C)$. If $x \in C$, then $x \in B \cap C$, so $x \in A \cup (B \cap C)$. Thus, $A \cup (B \cap C) \supset (A \cup B) \cap (A \cup C)$. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. $\square$

2. If $A \subset X$ then $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$

   Proof. For the first statement, first we show that $A \cup (X \setminus A) \subset X$. Let $x \in A \cup (X \setminus A)$. Then $x \in A$ or $x \in X \setminus A$. If $x \in A$, then $x \in X$. If $x \in X \setminus A$, then $x \in X$. Thus, $A \cup (X \setminus A) \subset X$.

   Next we show that $A \cup (X \setminus A) \supset X$. Let $x \in X$. Then $x \in A$ or $x \in X \setminus A$ by our assumption $A \subset X$. If $x \in A$, then $x \in A \cup (X \setminus A)$. If $x \in X \setminus A$, then $x \in A \cup (X \setminus A)$. Thus, $A \cup (X \setminus A) \supset X$. Therefore, $A \cup (X \setminus A) = X$.

   For the second statement, we will show that no element can exist in this set. Let $x \in A \cap (X \setminus A)$. Then $x \in A$ and $x \in X \setminus A$. If $x \in A$, then $x \notin X \setminus A$, so $x \notin A \cap (X \setminus A)$. Thus, we have shown that no element can exist in $A \cap (X \setminus A)$. Therefore, this set is the empty set.

   $\square$

3. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

   Proof. First we show that $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$. Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. If $x \notin B$, then $x \in A \setminus B$, so $x \in (A \setminus B) \cup (A \setminus C)$. If $x \notin C$, then $x \in A \setminus C$, so $x \in (A \setminus B) \cup (A \setminus C)$. Thus, $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$.

   Next we show that $A \setminus (B \cap C) \supset (A \setminus B) \cup (A \setminus C)$. Let $x \in (A \setminus B) \cup (A \setminus C)$. Then $x \in A \setminus B$ or $x \in A \setminus C$. If $x \in A \setminus B$, then $x \in A$ and $x \notin B$, so $x \notin B \cap C$, so $x \in A \setminus (B \cap C)$. If $x \in A \setminus C$, then $x \in A$ and $x \notin C$, so $x \notin B \cap C$, so $x \in A \setminus (B \cap C)$. Thus, $A \setminus (B \cap C) \supset (A \setminus B) \cup (A \setminus C)$. Therefore, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. $\square$

Problem 2. Let $f : A \to B$ and $g : B \to C$ be functions.

1. Show that if $g \circ f$ is injective then f is injective. Is g also necessarily injective?

   Proof. Assume $g \circ f$ is injective. This means for all x_1, x_2 in the domain of f (say, set A), if $g(f(x_1)) = g(f(x_2))$, then $x_1 = x_2$. To show f is injective, assume for contradiction that f is not injective. Then, there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Apply g to both sides of $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is injective by assumption, this implies $x_1 = x_2$, which contradicts our assumption that $x_1 \neq x_2$. Therefore, f must be injective.

   To see why g is not necessarily injective, consider a case where g is not injective but $g \circ f$ is. Let $f : A \to B$ be an injective function and $g : B \to C$ a non-injective function. In this scenario, $g \circ f$ can still be injective if f maps elements of A to distinct elements of B such that the application of g does not create any collisions. □

2. Show that if $g \circ f$ is surjective then g is surjective. Is f also necessarily surjective?

   Proof. Assume $g \circ f$ is surjective. This means for all $y \in C$, there exists $x \in A$ such that $g(f(x)) = y$. To show g is surjective, we must show that for all $y \in C$, there exists $b \in B$ such that $g(b) = y$. Let $y \in C$. Since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = y$. Since $f(x) \in B$, let $b = f(x)$. Then $g(b) = g(f(x)) = y$. Therefore, g is surjective.

   However, the surjectivity of f is not necessarily implied by the surjectivity of $g \circ f$. For example, consider a situation where $f : A \to B$ is not surjective, but $g : B \to C$ is such that it maps some elements in B not in the range of f to elements in C that are also images of elements in the range of f. In this case, every element in C would have a pre-image in A through $g \circ f$, despite f not being surjective. □

3. If both f and g are bijective then so is $g \circ f$.

   Proof. We need to show that $g \circ f$ is bijective, meaning it must be both injective and surjective.

   Assume $x_1, x_2 \in A$ and $g(f(x_1)) = g(f(x_2))$. Since g is injective, from $g(f(x_1)) = g(f(x_2))$ it follows that $f(x_1) = f(x_2)$. Furthermore, since f is injective, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Therefore, $g \circ f$ is injective.

   To show surjectivity, let $y \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = y$. Also, as f is surjective, there exists $x \in A$ such that $f(x) = b$. Thus, $g(f(x)) = g(b) = y$, implying that $g \circ f$ is surjective.

   Since $g \circ f$ is both injective and surjective, it is bijective. This completes the proof. □

Problem 3. Prove that a function $f : A \to B$ is a bijection if and only if there exists a function $h : B \to A$ satisfying

$$h \circ f = \mathrm{Id}_A \quad \text{and} \quad f \circ h = \mathrm{Id}_B,$$

where $\mathrm{Id}_A : A \to A$ is the identity function mapping every element of A to itself, and similarly Id_B for B. Such a function h is called the *inverse function* to f and typically denoted as f^{-1}.

Proof. To prove this, we must show that f is a bijection if and only if it has an inverse h as described.

($\Rightarrow$) Assume f is a bijection, meaning f is both injective and surjective.

Injectivity of f implies that for any $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ only if $x_1 = x_2$. This uniqueness allows us to define a function $h : B \to A$ where for each $y \in B$, $h(y)$ is the unique $x \in A$ such that $f(x) = y$.

Surjectivity of f ensures that for each $y \in B$, there exists an $x \in A$ such that $f(x) = y$. This guarantees that h is well-defined for every $y \in B$.

The function h thus defined satisfies $h(f(x)) = x$ for all $x \in A$ and $f(h(y)) = y$ for all $y \in B$, making $h \circ f = \mathrm{Id}_A$ and $f \circ h = \mathrm{Id}_B$.

($\Leftarrow$) Conversely, assume the existence of a function $h : B \to A$ such that $h \circ f = \mathrm{Id}_A$ and $f \circ h = \mathrm{Id}_B$.

To show f is injective, consider $x_1, x_2 \in A$ with $f(x_1) = f(x_2)$. Applying h to both sides gives $h(f(x_1)) = h(f(x_2))$. By $h \circ f = \mathrm{Id}_A$, we get $x_1 = x_2$.

To demonstrate f's surjectivity, let $y \in B$. The element $x = h(y) \in A$ satisfies $f(x) = f(h(y)) = y$, by $f \circ h = \mathrm{Id}_B$. Thus, every $y \in B$ is the image of some $x \in A$ under f.

Therefore, f is both injective and surjective, making it a bijection.

Hence, f is a bijection if and only if there exists an inverse h satisfying the given conditions. $\square$

Problem 4. Let $f : A \to B$ be a function. The *image* of a subset $E \subset A$ by f, denoted $f(E)$, is the set of images of all elements of E. That is,

$$f(E) = \{f(x) : x \in E\}.$$

The *pre-image*, or *inverse image*, of a subset $F \subset B$ by f, denoted $f^{-1}(F)$,[1] is the set of all elements of A mapped into F. That is,

$$f^{-1}(F) = \{x \in A : f(x) \in F\}.$$

Prove the following

1. For any subsets $U, V \subset B$

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \ , \ f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \text{ and } f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V).$$

   (a) *Proof.* We first show that $f^{-1}(U \cap V) \subset f^{-1}(U) \cap f^{-1}(V)$. Let $x \in f^{-1}(U \cap V)$. Then $f(x) \in U \cap V$. This implies $f(x) \in U$ and $f(x) \in V$, so $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. Therefore, $x \in f^{-1}(U) \cap f^{-1}(V)$, and so $f^{-1}(U \cap V) \subset f^{-1}(U) \cap f^{-1}(V)$.

   Next we show that $f^{-1}(U) \cap f^{-1}(V) \subset f^{-1}(U \cap V)$. Let $x \in f^{-1}(U) \cap f^{-1}(V)$. Then $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. This implies $f(x) \in U$ and $f(x) \in V$, so $f(x) \in U \cap V$. Therefore, $x \in f^{-1}(U \cap V)$, and so $f^{-1}(U) \cap f^{-1}(V) \subset f^{-1}(U \cap V)$. Thus, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. □

   (b) *Proof.* We first show that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$. Let $x \in f^{-1}(U \cup V)$. Then $f(x) \in U \cup V$. This implies $f(x) \in U$ or $f(x) \in V$, so $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Therefore, $x \in f^{-1}(U) \cup f^{-1}(V)$, and so $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

   Next we show that $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$. Let $x \in f^{-1}(U) \cup f^{-1}(V)$. Then $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. This implies $f(x) \in U$ or $f(x) \in V$, so $f(x) \in U \cup V$. Therefore, $x \in f^{-1}(U \cup V)$, and so $f^{-1}(U) \cup f^{-1}(V) \subset f^{-1}(U \cup V)$. Thus, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. □

   (c) *Proof.* We first show that $f^{-1}(U \setminus V) \subset f^{-1}(U) \setminus f^{-1}(V)$. Let $x \in f^{-1}(U \setminus V)$. Then $f(x) \in U \setminus V$. This implies $f(x) \in U$ and $f(x) \notin V$, so $x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$. Therefore, $x \in f^{-1}(U) \setminus f^{-1}(V)$, and so $f^{-1}(U \setminus V) \subset f^{-1}(U) \setminus f^{-1}(V)$.

   Next we show that $f^{-1}(U) \setminus f^{-1}(V) \subset f^{-1}(U \setminus V)$. Let $x \in f^{-1}(U) \setminus f^{-1}(V)$. Then $x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$. This implies $f(x) \in U$ and $f(x) \notin V$, so $f(x) \in U \setminus V$. Therefore, $x \in f^{-1}(U \setminus V)$, and so $f^{-1}(U) \setminus f^{-1}(V) \subset f^{-1}(U \setminus V)$. Thus, $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$. □

2. For any subsets $U, V \subset A$

$$f(U \cap V) \subset f(U) \cap f(V) \quad , \quad f(U \cup V) \subset f(U) \cup f(V) \quad \text{and} \quad f(U \setminus V) \subset f(U) \setminus f(V).$$

   (a) *Proof.* We show that $f(U \cap V) \subset f(U) \cap f(V)$. Let $y \in f(U \cap V)$. Then there exists $x \in U \cap V$ such that $f(x) = y$. This implies $x \in U$ and $x \in V$, so $y \in f(U)$ and $y \in f(V)$. Therefore, $y \in f(U) \cap f(V)$, and so $f(U \cap V) \subset f(U) \cap f(V)$.

   □

   (b) *Proof.* We show that $f(U \cup V) \subset f(U) \cup f(V)$. Let $y \in f(U \cup V)$. Then there exists $x \in U \cup V$ such that $f(x) = y$. This implies $x \in U$ or $x \in V$, so $y \in f(U)$ or $y \in f(V)$. Therefore, $y \in f(U) \cup f(V)$, and so $f(U \cup V) \subset f(U) \cup f(V)$.

   □

   (c) *Proof.* We show that $f(U \setminus V) \subset f(U) \setminus f(V)$. Let $y \in f(U \setminus V)$. Then there exists $x \in U \setminus V$ such that $f(x) = y$. This implies $x \in U$ and $x \notin V$, so $y \in f(U)$ and $y \notin f(V)$. Therefore, $y \in f(U) \setminus f(V)$, and so $f(U \setminus V) \subset f(U) \setminus f(V)$.

   □

3. If for all $S \subset A$ we have $f^{-1}(f(S)) = S$ then f is injective.

[1] Note that even though we are using the notation f^{-1}, we are not assuming that the function f is a bijection and hence has an inverse. This notation refers to a set (subset of the domain) and not an element.

Proof. Assume f is not injective. Then there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. Let $S = \{x_1\}$. Then $f(S) = \{f(x_1)\} = \{f(x_2)\}$, and so $x_2 \in f^{-1}(f(S))$. However, $x_2 \notin S$ since $S = \{x_1\}$ and $x_1 \neq x_2$. This contradicts the assumption that $f^{-1}(f(S)) = S$ for all $S \subset A$. Therefore, f must be injective. $\square$

4. If for all $T \subset B$ we have $f(f^{-1}(T)) = T$ then f is surjective.

   Proof. Let $y \in B$. Consider the set $T = \{y\} \subset B$. By assumption, $f(f^{-1}(T)) = T$, which implies that $y \in f(f^{-1}(T))$. Therefore, there exists some $x \in f^{-1}(T)$ such that $f(x) = y$. Since this holds for any $y \in B$, it follows that f is surjective. $\square$

Problem 5. Recall that a set X has cardinality equal to the set Y if there exists a bijection $f : X \to Y$. Prove the following properties of the equal cardinality relation:

1. (Reflexivity) For any set X, X has the same cardinality as itself.

   Proof. For any set X, consider the identity function $\mathrm{Id}_X : X \to X$ defined by $\mathrm{Id}_X(x) = x$ for all $x \in X$. This function is clearly a bijection as it is both injective and surjective. Therefore, X has the same cardinality as itself. □

2. (Symmetry) For any two sets X and Y, if X has the same cardinality as Y then Y has the same cardinality as X.

   Proof. Assume X and Y are sets such that there exists a bijection $f : X \to Y$. This means that f is both injective and surjective. We want to show that the inverse function $f^{-1} : Y \to X$ (defined as $\forall x \in X, y \in Y$ such that $f(x) = y$, $f^{-1}(y) = x$) exists and is also a bijection.

   Suppose $y_1, y_2 \in Y$ and $f^{-1}(y_1) = f^{-1}(y_2)$. Since f is a bijection, for y_1 and y_2, there exist unique $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By the definition of inverse function, $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Since $f^{-1}(y_1) = f^{-1}(y_2)$, it follows that $x_1 = x_2$. Applying f to both sides, we get $f(x_1) = f(x_2)$, thus $y_1 = y_2$. Therefore, f^{-1} is injective.

   Let $x \in X$. Since f is surjective, there exists a $y \in Y$ such that $f(x) = y$. Now, $f^{-1}(y)$ must equal x because $f(x) = y$. Therefore, for every $x \in X$, there is a $y \in Y$ such that $f^{-1}(y) = x$, proving that f^{-1} is surjective.

   Since f^{-1} is both injective and surjective, it is a bijection. Therefore, if X has the same cardinality as Y, then Y has the same cardinality as X. □

3. (Transitivity) For any three sets X, Y, Z, if X has the same cardinality as Y and Y has the same cardinality as Z then X has the same cardinality as Z.

   Proof. Let X, Y, and Z be sets. Assume there are bijections $f : X \to Y$ and $g : Y \to Z$. The composition $g \circ f : X \to Z$ is a bijection as the composition of two bijections is a bijection, proved in problem 2, part 3. Hence, if X has the same cardinality as Y and Y has the same cardinality as Z, then X has the same cardinality as Z. □

A relation satisfying the reflexivity, symmetry and transitivity properties is called an *equivalence relation.*