Problem Set 7

Math 255: Analysis I

Due: Thursday, March 28th at 11:59pm EST

Problem 1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X. For each of the following statements determine whether it implies the convergence of x_n to q and whether the convergence of x_n to q implies the statement. If an implication holds then prove it, otherwise provide a counter example.

1. $\exists N \in \mathbb{N} \quad \text{s.t. } \forall \varepsilon > 0 \;\; \forall n \geq N \quad d(x_n, q) < \varepsilon$

   Proof.

   1. $\implies x_n \to q$:

   - For any $\epsilon > 0$, we always that N given by assumption such that $d(x_n, q) < \epsilon$ for all $n \geq N$ and for all $\epsilon > 0$.
   - Then, for all $n \geq N$, we have $d(x_n, q) < \epsilon$.
   - Thus, $x_n \to q$.

   $q \to x_n \;\;\not\Longrightarrow\;\; 1$:

   - Let $X = \mathbb{R}$ and $x_n = \frac{1}{n}$. Then, $x_n \to 0$.
   - However, there is no single N that works for all $\epsilon > 0$. For example, if one chooses an N based on $\epsilon = 1$, say $N = 2$, this N does not suffice for a smaller ϵ, say $\epsilon = \frac{1}{10}$, for which we would need $N > 10$.

   □

2. $\forall k \in \mathbb{N} \;\; \exists N \in \mathbb{N} \quad \text{s.t. } \forall n \geq N \quad d(x_n, q) < \frac{1}{k}$

   Proof.

   2. $\implies x_n \to q$:

   - For any $\epsilon > 0$, we choose k such that $\frac{1}{k} < \epsilon$.
   - By the statement, we have N such that $d(x_n, q) < \frac{1}{k} < \epsilon$ for all $n \geq N$.
   - Therefore, $x_n \to q$.

   $x_n \to q \implies 2$:

   - Let $k \in \mathbb{N}$ be arbitrary.
   - Because $x_n \to q$, we can choose N such that $d(x_n, q) < \frac{1}{k}$ for all $n \geq N$.
   - Therefore, for any $k \in \mathbb{N}$, we can choose N such that $d(x_n, q) < \frac{1}{k}$.

   □

3. $\forall 0 < \varepsilon < 1 \;\; \exists N \in \mathbb{N} \quad \text{s.t. } \forall n \geq N \quad d(x_n, q) < \varepsilon$

   Proof.

   $x_n \to q \implies 3$:

- By definition, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, q) < \epsilon$.
- Therefore, for any $0 < \epsilon < 1$, we can choose N such that $d(x_n, q) < \epsilon$.

$3 \implies x_n \to q$:

- For any $0 < \epsilon < 1$, we can choose N such that $d(x_n, q) < \epsilon$.
- In particular, let N' be the N corresponding to $\epsilon = 1/2$. That is, $d(x_n, q) < 1/2$ for all $n \geq N'$.
- For any $\epsilon \geq 1$, we can choose this N' such that $d(x_n, q) < 1/2 < \epsilon$ for all $n \geq N'$.
- Therefore, $x_n \to q$.

$\square$

4. $\forall \varepsilon > 1 \;\; \exists N \in \mathbb{N} \;\; \text{s.t.} \; \forall n \geq N \quad d(x_n, q) < \varepsilon$

Proof. $x_n \to q \implies 4$:

- By definition, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, q) < \epsilon$.
- Therefore, for any $\epsilon > 1$, we can choose N such that $d(x_n, q) < \epsilon$.

$4 \not\Longrightarrow x_n \to q$:

- Let $X = \mathbb{R}$ and $x_n = \frac{(-1)^n}{2}$. x_n does not converge.
- The condition is satisfied for all $\epsilon > 1$ because $d(x_n, 0) = 1/2 < \epsilon$ for all n.
- However, x_n does not converge.

$\square$

Problem 2. For each of the following pairs of sequences, determine whether $(b_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. If it is, find the increasing sequence of indices $(n_k)_{k\in\mathbb{N}}$ describing the subsequence.

1. $a_n = n$ and $b_n = 2^n$

   Proof. $(b_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   - The first few terms of $(b_n)_{n\in\mathbb{N}}$ are $(2, 4, 8, 16, \ldots)$.
   - The first few terms of $(a_{n_k})_{k\in\mathbb{N}}$ are $(2, 4, 8, 16, \ldots)$.
   - $(b_n)_{n\in\mathbb{N}} = (a_{n_k})_{k\in\mathbb{N}}$ where $n_k = 2^k$.
   - We can verify that $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of indices. The first few terms of $(n_k)_{k\in\mathbb{N}}$ are $(2, 4, 8, 16, \ldots)$.

   $\square$

2. $a_n = n$ and $b_n = 2 + (-1)^n$

   Proof. $(b_n)_{n\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   - Notice that $b_1 = 1$ and $b_3 = 1$.
   - However, the only index n such that $a_n = 1$ is $n = 1$.
   - Therefore, $(b_n)_{n\in\mathbb{N}}$ is not a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   $\square$

3. $a_n = n^{\sqrt{n}/2}$ and $b_n = n^n$

   Proof. $(b_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   - The first few terms of $(a_n)_{n\in\mathbb{N}}$ are $(1^{\sqrt{1}/2}, 2^{\sqrt{2}/2}, 3^{\sqrt{3}/2}, 4^{\sqrt{4}/2}, \ldots)$.
   - The first few terms of $(b_n)_{n\in\mathbb{N}}$ are $(1, 2^2, 3^3, 4^4, \ldots)$.
   - $(b_n)_{n\in\mathbb{N}} = (a_{n_k})_{k\in\mathbb{N}}$ where $n_k = k^2$.
   - We can verify that $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of indices. The first few terms of $(n_k)_{k\in\mathbb{N}}$ are $(1, 4, 9, 16, \ldots)$.

   $\square$

4. $a_n = (-1)^{\lfloor \frac{n}{2} \rfloor}$ and $b_n = (-1)^n$, where $\lfloor t \rfloor = \max\{k \in \mathbb{Z} : k \leq t\}$.

   Proof. $(b_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   - The first few terms of $(a_n)_{n\in\mathbb{N}}$ are $(1, -1, -1, 1, 1, -1, -1, 1, \ldots)$.
   - The first few terms of $(b_n)_{n\in\mathbb{N}}$ are $(-1, 1, -1, 1, \ldots)$.
   - $(b_n)_{n\in\mathbb{N}} = (a_{n_k})_{k\in\mathbb{N}}$ where $n_k = 2k$.
   - We can verify that $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of indices. The first few terms of $(n_k)_{k\in\mathbb{N}}$ are $(2, 4, 6, 8, \ldots)$.

   $\square$

Problem 3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Recall the following set defined in class

$$\text{Accum}(x_n)_{n\in\mathbb{N}} = \left\{ z \in [-\infty, \infty] : \exists \text{ a subsequence } (x_{n_k})_{k\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ with } \lim_{k\to\infty} x_{n_k} = z \right\}.$$

1. Let $K \subseteq \mathbb{R}$ be any compact subset[1]. Prove that for any $m \in \mathbb{N}$ there exist finitely many points $p_1, ..., p_{N_m} \in K$ satisfying that for every $x \in K$ there exists $1 \leq i \leq N_m$ with $|x - p_i| < \frac{1}{m}$.

   Use this to construct a sequence $(p_n)_{n\in\mathbb{N}}$ in K which satisfies that $\text{Accum}(p_n)_{n\in\mathbb{N}} = K$.

   Proof.

   - Let K be a compact set in $\mathbb{R}$ and let $m \in \mathbb{N}$ be arbitrary.
   - Consider the open cover $\bigcup_{p\in K} N_{\frac{1}{m}}(p)$ that covers K.
   - Because K is compact, there exists a finite subcover $\bigcup_{i=1}^{N_m} N_{\frac{1}{m}}(p_i)$ that still covers K.
   - $p_1, p_2, \ldots, p_{N_m} \in K$ is the finite set of points that correspond to this finite subcover.
   - Let $x \in K$. Because $x \in K$, there exists $1 \leq i \leq N_m$ such that $x \in N_{\frac{1}{m}}(p_i)$.
   - This implies $|x - p_i| < \frac{1}{m}$.
   - Therefore, for every $x \in K$, there exists $1 \leq i \leq N_m$ such that $|x - p_i| < \frac{1}{m}$.

   □

   To construct a sequence $(p_n)_{n\in\mathbb{N}}$ in K such that $\text{Accum}(p_n)_{n\in\mathbb{N}} = K$:

   Proof.

   - For each $m \in \mathbb{N}$, choose N_m distinct points $p_1^{(m)}, p_2^{(m)}, \ldots, p_{N_m}^{(m)} \in K$ as discussed above.
   - Let $p_1 = p_1^{(1)}, p_2 = p_1^{(2)}, \ldots, p_{N_1} = p_1^{(N_1)}, p_{N_1+1} = p_2^{(1)}, \ldots, p_{N_1+N_2} = p_2^{(N_2)}, \ldots$.
   - Define the sequence $(p_n)_{n\in\mathbb{N}}$ as $p_n = p_n^{(m)}$
   - For any $x \in K$, there exists a subsequence of $(p_n)n \in \mathbb{N}$ converging to x.
   - Therefore, $\text{Accum}(p_n)_{n\in\mathbb{N}} = K$.

   □

2. Let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection[2], and define the sequence $q_n = f(n)$. Prove $\text{Accum}(q_n)_{n\in\mathbb{N}} = [-\infty, \infty]$. [3]

   Proof.

   - Such a bijection exists because $\mathbb{N}$ and $\mathbb{Q}$ is countable.
   - To prove $\text{Accum}(q_n)_{n\in\mathbb{N}} = [-\infty, \infty]$, we will show that for every real number and the points $-\infty$ and ∞, there exists a subsequence of $(q_n)_{n\in\mathbb{N}}$ that converges to x or $-\infty$ or ∞, respectively.
   - Let $x \in \mathbb{R}$ be arbitrary. We will show that there construct a subsequence of $(q_n)_{n\in\mathbb{N}}$ that converges to x.
   - Pick the subsequence indexes k_n such that $x - \frac{1}{n} < f(k_n) < x$ and $k_n < k_{n+1}$.
   - Such a k_n exists because f is a bijection. Such a $f(k_n)$ exists because of the denseness of $\mathbb{Q}$ in $\mathbb{R}$.
   - This subsequence converges to x. (For any $\epsilon > 0$, choose N such that $1/N < \epsilon$.)
   - Let $x = -\infty$. We can choose the subsequence indexes k_n such that $f(k_n) < -n$ and $k_n < k_{n+1}$.
   - This subsequence converges to $-\infty$.

[1] E.g. the middle–$\frac{1}{3}$ Cantor.
[2] Recall - why does there exist one?
[3] Be careful to construct the subsequences of $(q_n)_{n\in\mathbb{N}}$ properly.

- Let $x = \infty$. We can choose the subsequence indexes k_n such that $f(k_n) > n$ and $k_n < k_{n+1}$.
- This subsequence converges to ∞.
- Therefore, $\text{Accum}(q_n)_{n\in\mathbb{N}} = [-\infty, \infty]$.

□

3. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{R}$ where $a_n \to a$ and $b_n \to b$, for $a, b \in \mathbb{R}$. Define

$$c_n = \begin{cases} a_n & n \text{ is odd} \\ b_n & n \text{ is even} \end{cases}.$$

What is $\text{Accum}(c_n)_{c\in\mathbb{N}}$? Prove your claim.

Proof. We will prove that $\text{Accum}(c_n)_{n\in\mathbb{N}} = \{a, b\}$.

- Consider the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ where $n_k = 2k-1$. This is also a subset of $(c_n)_{n\in\mathbb{N}}$.
- Because $a_n \to a$, we have $a_{n_k} \to a$ as limits are hereditary.
- Consider the subsequence $(b_{n_k})_{k\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$ where $n_k = 2k$. This is also a subset of $(c_n)_{n\in\mathbb{N}}$.
- Because $b_n \to b$, we have $b_{n_k} \to b$ as limits are hereditary.
- Therefore, $\{a, b\} \subseteq \text{Accum}(c_n)_{n\in\mathbb{N}}$.
- Now, let $x \in \text{Accum}(c_n)_{n\in\mathbb{N}}$. We will show that $x \in \{a, b\}$.
- Because $x \in \text{Accum}(c_n)_{n\in\mathbb{N}}$, there exists a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ of $(c_n)_{n\in\mathbb{N}}$ that converges to x.

□

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Problem 4. Let $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Briefly prove or refute with a counterexample, the following statements:

1. If $a_n \to L$ then $|a_n| \to |L|$

   Proof.

   - By definition, $a_n \to L$ means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$.
   - Recall the triangle inequality, taking $z = 0$, $|x - y| \leq |x| + |y|$.
   - Taking $x = a_n - L$ and $y = 0$
   - Then, $|a_n - L| < \epsilon$ implies $||a_n| - |L|| \leq |a_n - L| < \epsilon$.
   - This implication is clear by considering all possibilities of a_n and L.
     - If a_n and L are both positive, then $|a_n| = a_n$ and $|L| = L$ so $||a_n| - |L|| = |a_n - L| < \epsilon$.
     - If a_n and L are both negative, then $|a_n| = -a_n$ and $|L| = -L$ so $||a_n| - |L|| = |-a_n - (-L)| = |a_n - L| < \epsilon$.
     - If a_n is positive and L is negative, then $|a_n| = a_n$ and $|L| = -L$ so $||a_n| - |L|| = |a_n - (-L)| = |a_n + L| < |a_n - L| < \epsilon$.
     - If a_n is negative and L is positive, then $|a_n| = -a_n$ and $|L| = L$ so $||a_n| - |L|| = |-a_n - L| < |a_n - L| < \epsilon$.
   - Therefore, $|a_n| \to |L|$.

   $\square$

2. If $|a_n| \to L$ then $a_n \to L$

   Proof. Counterexample:

   - Let $L = 1$ and $a_n = (-1)^n$. Then, $|a_n| = 1$ for all n.
   - It is clear that $|a_n| \to 1$. However, a_n does not converge.

   $\square$

3. If $|a_n| \to 0$ then $a_n \to 0$

   Proof.

   - By definition, $|a_n| \to 0$ means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $||a_n| - 0| < \epsilon$.
   - $||a_n| - 0| < \epsilon$ implies that $|a_n - 0| < \epsilon$ because $||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0|$.
   - Therefore, $a_n \to 0$.

   $\square$

4. If $a_n \to L$ and for all $n \in \mathbb{N}$ we know $a_n > -2$, then $L > -2$

   Proof. Counterexample:

   - Let $L = -2$ and $a_n = -2 + \frac{1}{n}$. It is clear for all $n \in \mathbb{N}$, $a_n > -2$ because $\frac{1}{n} > 0$.
   - To show that $a_n \to -2$, we need to show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - (-2)| < \epsilon$.
   - Let $\epsilon > 0$ be arbitrary. Choose N such that $N > \frac{1}{\epsilon}$.
   - Then, for all $n \geq N$, $|a_n - (-2)| = |(-2 + \frac{1}{n}) - (-2)| = \frac{1}{n} < \epsilon$.
   - Therefore, $a_n \to -2$ and for all $n \in \mathbb{N}$, $a_n > -2$ but $L \not> -2$.

$\square$

5. If $a_n \to 0$ then $a_n^n \to 0$

   Proof.

   - By definition, $a_n \to 0$ means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| = |a_n| < \epsilon$.
   - In particular take $\epsilon = 1$. Denote N' as the N corresponding to $\epsilon = 1$. That is, $|a_n| < 1$ for all $n \geq N'$.
   - To show that $a_n^n \to 0$, we need to show that for any $\delta > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $|a_n^n - 0| = |a_n^n| < \delta$.
   - For $0 < \delta < 1$, we can choose M corresponding to $\epsilon = \delta$. That is, $M = N$ such that $|a_n| < \delta$ for all $n \geq N$.
   - Because $|a_n^n| < |a_n|$ when $0 < \delta < 1$, we have $|a_n^n| < \delta$ for all $n \geq M$.
   - For $\delta \geq 1$, we can choose $M = N'$. Then, $|a_n| < 1$ for all $n \geq N'$.
   - Therefore, as $|a_n| < \delta$, we also have $|a_n^n| < \delta$ for all $n \geq N'$.
   - Therefore, $a_n^n \to 0$.

$\square$

6. If $|a_n| < 1$ for all n then $a_n^n \to 0$

   Proof. Counterexample:

   - Let $a_n = \left(\frac{1}{2}\right)^{\frac{1}{n}}$. It is clear that $|a_n| < 1$ for all n.
   - However, $a_n^n = \left(\frac{1}{2}\right)^{\frac{1}{n}^n} = \frac{1}{2}$ for all n.
   - So a_n^n does not converge to 0.

$\square$

Good luck!