EENG 431 - Homework 2

Franklin She

March 11, 2024

Problem 2.

1.
$$\mathcal{H}^{\mathcal{X}}_{=k} = \{h \in \{0,1\}^{\mathcal{H}} : |\{x : h(x) = 1\}| = k\}$$

First, we can see that $\text{VCdim}(\mathcal{H}_{=k}) \leq k$, because it is impossible for $h \in \mathcal{H}_{=k}$ to correctly label $k+1$ positive points. Secondly, we can see that $\text{VCdim}(\mathcal{H}_{=k}) \leq |\mathcal{X}| - k$, because it is impossible for $h \in \mathcal{H}_{=k}$ to correctly label $|\mathcal{X}| - k + 1$ negative points. Combining the two, $\text{VCdim}(\mathcal{H}_{=k}) \leq \min\{k, |\mathcal{X}| - k + 1\}$.

Now, consider a set C of size $\min\{k, |\mathcal{X}| - k + 1\}$. The total of positive points in C is at most k. We can define h as

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathcal{X} \setminus C \\ 1 & \text{if } x \in C \text{ and has positive label} \\ 0 & \text{if } x \in C \text{ and has negative label} \\ 0 & \text{otherwise} \end{cases}$$

This h shatters this set C, therefore we can conclude that $\text{VCdim}(\mathcal{H}_{=k}) = \min\{k, |\mathcal{X}| - k + 1\}$.

2.
$$\mathcal{H}_{at-most-k} = \{h \in \{0,1\}^{\mathcal{H}} : |\{x : h(x) = 1\}| \leq k \text{ or } |\{x : h(x) = 0\}| \leq k\}$$

First, we can see that $\text{VCdim}(\mathcal{H}_{at-most-k}) \leq k$, because it is impossible for $h \in \mathcal{H}_{at-most-k}$ to correctly label $k+1$ positive points. Now, consider a set C of size k. We define h as

$$h(x) = \begin{cases} 1 & \text{if } x \in C \text{ and has positive label} \\ 0 & \text{if } x \in C \text{ and has negative label} \\ 0 & \text{otherwise} \end{cases}$$

This h shatters this set C, therefore we can conclude that $\text{VCdim}(\mathcal{H}_{at-most-k}) = k$.

Problem 5.

$$\mathcal{H}^d_{\text{rec}} \text{ is the class of axis-aligned rectangles in } \mathbb{R}^d$$

First, we can see that $\text{VCdim}(\mathcal{H}^d_{\text{rec}}) \leq 2d$, because it is impossible for $h \in \mathcal{H}^d_{\text{rec}}$ to shatter a set C with $2d+1$ points. We locate an "inside" point $x \in C$, defined as $\forall j \in [d]$, $\exists x', x'' \in C$ such that $x'_j \leq x_j$ and $x''_j \geq x_j$. Setting the label of x as negative, and all other points as positive is an example of a configuration in which all points cannot be correctly labeled by $h \in \mathcal{H}^d_{\text{rec}}$.

Now considering a set C of size $2d$, positioning each point in a way such that $\forall x, x' \in C$ and $\forall j \in [d]$, $x_j \neq x'_j$, we have a set in which $\mathcal{H}^d_{\text{rec}}$ shatters. Taking any arbitrary subset of C to be positive labeled points and the rest negative, we can still find a $\mathbb{R}_d$ rectangle identifying only and all positive labeled points.

Problem 7.

1. We have seen in class that the set of threshold functions:

$$\mathcal{H} = \{h_a : a \in \mathbb{R}\} \text{ where } h_a(x) = \mathbb{1}_{x \leq a}$$

is infinite, but $\text{VCdim}(\mathcal{H}) = 1$.

2. Consider

$$\mathcal{H} = \{h_1, h_{1/2}\}$$

We have $\lfloor \log_2(|\mathcal{H}|) \rfloor = \lfloor \log_2(2) \rfloor = 1$. We also have $\text{VCdim}(\mathcal{H}) = 1$

Problem 9.

$$\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{-1, 1\}\}$$

where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{if } x \notin [a,b] \end{cases}$$

As this was an example in class, we will show $\mathrm{VCdim}(\mathcal{H}) = 3$. We will approach this explicitly finding a set C with three elements shattered by $\mathcal{H}$ and then showing that any set of four elements cannot be shattered by $\mathcal{H}$. Let $C = \{0, 1, 2\}$. For each of the $2^3 = 8$ configurations, we will define $h_{a,b,s}$ correctly labeling all elements.

- $((0, -1), (1, -1), (2, -1))$: $a = -0.5$, $b = 2.5$, $s = -1$
- $((0, -1), (1, -1), (2, +1))$: $a = 1.5$, $b = 2.5$, $s = +1$
- $((0, -1), (1, +1), (2, -1))$: $a = 0.5$, $b = 1.5$, $s = +1$
- $((0, -1), (1, +1), (2, +1))$: $a = 0.5$, $b = 2.5$, $s = +1$
- $((0, +1), (1, -1), (2, -1))$: $a = -0.5$, $b = 0.5$, $s = +1$
- $((0, +1), (1, -1), (2, +1))$: $a = 0.5$, $b = 1.5$, $s = -1$
- $((0, +1), (1, +1), (2, -1))$: $a = -0.5$, $b = 1.5$, $s = +1$
- $((0, +1), (1, +1), (2, +1))$: $a = -0.5$, $b = 2.5$, $s = +1$

Now consider a set $\{a, b, c, d\}$. WLOG, take $a < b < c < d$. The labeling $((a, +1), (b, -1), (c, +1), (d, -1))$ cannot be correctly labeled by any $h \in \mathcal{H}$.

Problem 11 (part 1). Let $d = \max_i \mathrm{VCdim}(\mathcal{H}_i)$ and $d \geq 3$. Let $\mathrm{VCdim}(\bigcup_i^r \mathcal{H}_i) = k$. Because the VC dimension of the union of hypothesis classes is at most any of the VC dimensions of the individual classes, we know that $\forall 1 \leq i \leq r$, $\mathrm{VCdim}(\mathcal{H}_i) \leq d \leq k$. Let C be a subset of size k in which $\bigcup_i^r \mathcal{H}_i$ shatters, meaning $\bigcup_i^r \mathcal{H}_i$ can produce all 2^k labeling configurations of C. By Sauer's lemma, we have $\tau_{\mathcal{H}_i}(k) \leq \left(\frac{ek}{d}\right)^d$. As $d \geq 3$, we can bound this with $\tau_{\mathcal{H}_i}(k) \leq k^d$. By the definition of the growth function, we also have $\tau_{\mathcal{H}}(k) \leq \sum_{i=1}^r \tau_{\mathcal{H}_i}(k)$. Therefore, we have $\tau_{\mathcal{H}}(k) \leq rk^d \implies 2^k \leq rk^d \implies k \leq d \log k + \log r$. Using Lemma A.2, we have $k < 4d \log(2d) + 2 \log r$.