Math 255: Analysis I Notes

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Contents

1 Sets and functions

Definition 1.1. (Set, naively). A set is an unordered collection of objects (elements) without multiplicity.

Definition 1.2. (Injectivity). A function $f : A \to B$ is injective (or one-to-one) if $\forall x, y \in A$, $f(x) = f(y) \implies x = y$.

Definition 1.3. (Surjectivity). A function $f : A \to B$ is surjective (or onto) if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

2 Natural numbers and the Peano axioms

Definition 2.1. (Natural numbers). A set $\mathbb{N}$ with a successor function $S : \mathbb{N} \to \mathbb{N}$ that assigns to every element $n \in \mathbb{N}$ its successor. It has the following properties:

I. $1 \in \mathbb{N}$.

II. $\forall n \in \mathbb{N}$, $S(n) \in \mathbb{N}$.

III. $\forall n \in \mathbb{N}$, $S(n) \neq 1$.

IV. $\forall n, m \in \mathbb{N}$, $S(n) = S(m) \implies n = m$. (Injectivity of S).

V. Any subset $A \subseteq \mathbb{N}$ such that $1 \in A$ and $\forall n \in A$, $S(n) \in A$ must be equal to $\mathbb{N}$.

Proposition 2.2. $4 \neq 1$.

Proof. By definition of S, $4 = S(3) = S(S(2)) = S(S(S(1)))$. By I and II, $3 \in \mathbb{N}$. Suppose $4 = 1$. $S(3) = 1$. A contradiction to III. □

Proposition 2.3. $6 \neq 2$.

Proof. Assume in contradiction that $6 = 2$. Then $S(5) = S(1)$. By IV, $5 = 1$. A contradiction to III by proof similar to $4 \neq 1$. □

Proposition 2.4. *$\forall n \in \mathbb{N}$, $S(n) \neq n$.*

Proof. By induction on n. For $n = 1$, if $S(1) = 1$, this contradicts III. Assume $S(n) \neq n$, want to show

$$S(S(n)) \neq S(n)$$

Assume by contradiction that $S(S(n)) = S(n)$. By IV, $S(n) = n$. A contradiction of our assumption. □

Remark. This strategy of proof via induction uses property V. We considered the subset

$$A = \{n \in \mathbb{N} \colon S(n) \neq n\} \subseteq \mathbb{N}$$

and showed that $1 \in A$ and $\forall n \in A$, $S(n) \in A$. This implies that $A = \mathbb{N}$ by V.

Axiom 2.5. *There exists a set satisfying I - V (Peano axioms). Such a set is called the set of natural numbers and is denoted by* $\mathbb{N}$.

2.1 Cardinality

Definition 2.6. (Equal cardinality). Two sets A and B have equal cardinality, denoted $|A| = |B|$, if there exists a bijection $f : A \to B$. Denote for $n \in \mathbb{N}$, $\underline{n} = \{1, 2, \ldots, n\}$.

Definition 2.7. (Size n set). A set A is said to have size n, $|A| = n$, if A has equal cardinality to $\underline{n}$.

Proposition 2.8. *The "equal cardinality" relation is an equivalence relation. That is, it is reflexive, symmetric, and transitive.*

Proof. Problem Set 1. □

Definition 2.9. (Finite set). A set A is said to be finite if $|A| = n$ for some $n \in \mathbb{N}$ or $A = \emptyset$ (where $|\emptyset| = 0$). Otherwise, A is said to be infinite.

Theorem 2.10. *(Uniqueness of cardinality). If $|A| = n$ for some $n \in \mathbb{N}$, then $|A| \neq m$ for any $m \in \mathbb{N}$ such that $m \neq n$.*

Lemma 2.11. *If $|X| = n$ for some $n \in \mathbb{N}$, then given any $x \in X$, $|X \setminus \{x\}| = n - 1$. (Where $n - 1 = 0$ if $n = 1$ and $n - 1 = m$ when $n = S(m)$).*

Proof. of lemma. Prove by induction on n.

Base case:

- For $n = 1$, since $|X| = 1$, there exists a bijection $f : X \to \{1\}$.
- Since f is onto, $\exists y \in X$ such that $f(y) = 1$. In particular, $X \neq \emptyset$.
- Since f is injective, there are no other elements in X, because $\forall x \in X$, $f(x) \in \{1\} \implies f(x) = 1 = f(y)$.
- By injectivity, $x = y \implies X = \{y\}$. Hence $\forall x \in X$, $X \setminus \{x\} = \emptyset$.
- This implies $|X \setminus \{x\}| = 0$. Hence the lemma holds for $n = 1$.

Inductive step:

- Assume $|X| = S(n)$ for some $n \in \mathbb{N}$. There exists a bijection
$$f : X \to \underline{S(n)} = \{1, 2, \ldots, S(n)\}$$
- Let $x \in X$ be any element. Define $g : X \setminus \{x\} \to \underline{n}$ by
$$g(y) = \begin{cases} f(y) & \text{if } f(y) < f(x) \\ f(y) - 1 & \text{if } f(y) > f(x) \end{cases}$$
- We want to show that g is a bijection. It is clearly onto because f is onto.
- It is also injective because f is injective. If $g(y_1) = g(y_2)$, then $f(y_1) = f(y_2)$.
- If $f(y_1) < f(x)$, then $f(y_1) = g(y_1) = g(y_2) = f(y_2) = f(y_2) - 1$.
- If $f(y_1) > f(x)$, then $f(y_1) - 1 = g(y_1) = g(y_2) = f(y_2) - 1 \implies f(y_1) = f(y_2)$.
- By injectivity of f, $y_1 = y_2$. Therefore, g is a bijection. This implies that $|X \setminus \{x\}| = n$.

□

Proof. of theorem. Prove by induction on n.

Base case: If $n = 1$. Assume in contradiction that $\exists m \in \mathbb{N}$ such that $m \neq 1$ and $|X| = m$. Since $n \neq m$, there exists $m - 1 \in \mathbb{N}$ such that $m = S(m - 1)$. By the lemma,

1. On one hand, $|X \setminus \{x\}| = 0$ for any $x \in X \implies X \setminus \{x\} = \emptyset$.
2. But on the other hand, $|X \setminus \{x\}| = m - 1 \in \mathbb{N} \implies X \setminus \{x\} \neq \emptyset$.

This is a contradiction, proving the base case.

Inductive step: Assume theorem true for $n \in \mathbb{N}$. We want to show that the theorem holds for $S(n)$.

TODO: Finish this proof. ☐

Corollary 2.12. $\mathbb{N}$ *is infinite.*

Proof. Assume in contradiction that $|\mathbb{N}| = n$ for some $n \in \mathbb{N}$.

- By the lemma, $|\mathbb{N} \setminus \{1\}| = n - 1 \in \mathbb{N}$
- In particular, $\mathbb{N} \setminus \{1\} \neq \emptyset$.
- But the successor function $S : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ is a bijection, so $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$, a contradiction.

☐

Remark. Let X be an infinite set. Does $|X| = |\mathbb{N}|$? No. We will come back to this later.

2.2 Arithmetic

Remark. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and define $S(0) = 1$. Notice one can still induct on $\mathbb{N}_0$.

Lemma 2.13. *For any $A \subseteq \mathbb{N}_0$, if $0 \in A$ and $\forall n \in A$, $S(n) \in A$, then $A = \mathbb{N}_0$.*

Proof. Let A be as above. Denote $A' = A \cap \mathbb{N}$. Since $0 \in A \implies S(0) = 1 \in A \implies 1 \in A'$. For any $n \in A'$, $n \in A$ because $A' \subseteq A$. Hence, $S(n) \in A$. Moreover, $S(n) \in \mathbb{N}$ because $A' \subseteq \mathbb{N}$. This implies that $S(n) \in A'$. By Peano axiom V, $A' = \mathbb{N} \implies A = \mathbb{N}_0$. ☐

Definition 2.14. We'll define the sum in $\mathbb{N}_0$ inductively. $\forall n \in \mathbb{N}_0$:

1. $0 + n = n$.
2. $\forall m \in \mathbb{N}_0$, $S(m) + n = S(m + n)$.

Proposition 2.15. $\forall m \in \mathbb{N}_0$*,* $m + 0 = m$*.*

Proof. By induction of $m \in \mathbb{N}_0$. For $m = 0$, $0 + 0 = 0$ by definition. Assume $m + 0 = m$ for some $m \in \mathbb{N}_0$. Then $S(m) + 0 = S(m + 0)$ by definition. By the induction hypothesis, $S(m + 0) = S(m)$. This implies that $S(m) + 0 = S(m)$. ☐

Proposition 2.16. $\forall m, n \in \mathbb{N}_0$*,* $m + S(n) = S(m + n)$*.*

Proof. By induction on $m \in \mathbb{N}_0$. For $m = 0$, $0 + S(n) = S(0 + n) = S(n)$. Assume $m + S(n) = S(m + n)$. Then $S(m) + S(n) = S(m + S(n))$ by definition. Then $S(m + S(n)) = S(S(m + n))$ by the induction hypothesis. Then $S(S(m + n)) = S(S(m) + n)$ by definition. ☐

Definition 2.17. (Order). For $a, b \in \mathbb{N}_0$, we say that $a \leq b$ if and only if $\exists n \in \mathbb{N}_0$ such that $a + n = b$. $a < b$ if and only if $a \leq b$ and $a \neq b$.

Proposition 2.18. *The order relation satisfies:*

1. *Trichotomy:* $\forall a, b \in \mathbb{N}_0$*, exactly one of* $a < b$*,* $a = b$*, or* $a > b$ *holds.*
2. *Transitivity:* $\forall a, b, c \in \mathbb{N}_0$*, if* $a \leq b$ *and* $b \leq c$*, then* $a \leq c$*.*

Proof. Problem set. ☐

2.3 Integers

Definition 2.19. (Integers). The set of integers $\mathbb{Z}$, is the set of formal expressions of the form $[a-b]$, where $a, b \in N_0$. We identify any two integers $[a-b] = [c-d]$ if and only if $a+d = b+c$. We define addition on $\mathbb{Z}$ by

$$[a-b] + [c-d] = [(a+c) - (b-d)]$$

One can identify $\mathbb{N} \subseteq \mathbb{Z}$ by identifying $n \in \mathbb{N}$ with $[n-0] \in \mathbb{Z}$. This always us to define an order on $\mathbb{Z}$.

Definition 2.20. (Order on $\mathbb{Z}$). $[a-b] \leq [c-d]$ if and only if $[c-d] = [a-b] + [n-0]$ for some $n \in \mathbb{N}_0$.

Proposition 2.21. *The above order relation on $\mathbb{Z}$ satisfies trichotomy and transitivity.*

Proof. Also skipped. □

Definition 2.22. (Negation in $\mathbb{Z}$). The negation of $[a-b] \in \mathbb{Z}$ is defined to be $-[a-b] = [b-a]$.

Definition 2.23. (Subtraction in $\mathbb{Z}$). Subtraction is defined by $[a-b] - [c-d] = [a-b] + (-[c-d])$.

Definition 2.24. (Multiplication in N_0). We define inductively $\forall n \in \mathbb{N}_0$:

1. $0 \times n = 0$.
2. $\forall m \in \mathbb{N}_0$, $S(m) \times n = (m \times n) + n$.

Definition 2.25. (Multiplication extended to $\mathbb{Z}$). We define

$$[a-b] \times [c-d] = [(a \times c + b \times d) - (a \times d + b \times c)]$$

2.4 Rationals

Definition 2.26. The set of rationals $\mathbb{Q}$ is the set of formal expressions of the form $[p//q]$, where $p, q \in \mathbb{Z}$ and $q \neq [0-0]$. We identify any two rationals $[p//q] = [r//s]$ if and only if $p \times s = q \times r$. We can identify $N_0 \subseteq \mathbb{Q}$ by identifying $n \in \mathbb{N}_0$ with $[[n-0]//[1-0] \in \mathbb{Q}$.

Definition 2.27. (Addition in $\mathbb{Q}$). We define

$$[p//q] + [r//s] = [(p \times s + q \times r)//(q \times s)]$$

Definition 2.28. (Multiplication in $\mathbb{Q}$). We define

$$[p//q] \times [r//s] = [(p \times r)//(q \times s)]$$

Definition 2.29. Order on $\mathbb{Q}$. We can define an order on $\mathbb{Q}$ by

1. $0 \leq [p//q]$ iff $p = [n-0], q = [m-0]$ for some $n, m \in \mathbb{N}_0$.
2. $[p//q] \leq [r//s]$ iff $[r//s] = [p//q] + [x//y]$ where $0 \leq [x//y]$.

3 Fields

Definition 3.1. (Field). A field is a set $\mathbb{F}$ with a pair of operations

$$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$
$$\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

satisfying the following properties:

1. Commutativity of addition: $\forall x, y \in \mathbb{F}$, $x + y = y + x$.
2. Associativity of addition: $\forall x, y, z \in \mathbb{F}$, $(x + y) + z = x + (y + z)$.
3. Existence of neutral element for addition: $\exists 0 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}$, $x + 0 = x$.
4. Existence of additive inverse: $\forall x \in \mathbb{F}$, $\exists y \in \mathbb{F}$ such that $x + y = 0$.
5. Commutativity of multiplication: $\forall x, y \in \mathbb{F}$, $x \times y = y \times x$.
6. Associativity of multiplication: $\forall x, y, z \in \mathbb{F}$, $(x \times y) \times z = x \times (y \times z)$.
7. Existence of neutral element for multiplication: $\exists 1 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}$, $x \times 1 = x$.
8. Existence of multiplicative inverse: $\forall x \in \mathbb{F} \setminus \{0\}$, $\exists y \in \mathbb{F}$ such that $x \times y = 1$.
9. Distributivity: $\forall x, y, z \in \mathbb{F}$, $x \times (y + z) = x \times y + x \times z$.

Example 3.2.

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields with the usual operations.
2. $\mathbb{F}_3 = \{0, 1, 2\}$ with addition and multiplication modulo 3.

Anti-examples:

1. F_6 is not a field with addition and multiplication modulo 6.
2. $\mathbb{Z}$ is not a field, no multiplicative inverses. ($\mathbb{Z}$ is a ring).
3. R^2 is not a field, no "natural" multiplication operation.

Proposition 3.3. *(Cancellation law). $\forall x, y, z \in \mathbb{F}$, if $x + y = x + z$, then $y = z$.*

Proof. By 4, $\exists(-x) \in \mathbb{F}$ such that $x+(-x) = 0$. By $+$ being well-defined, $(-x)+(x+y) = (-x)+(x+z) \overset{2}{\Longrightarrow} ((-x) + x) + y = ((-x) + x) + z \overset{4}{\Longrightarrow} 0 + y = 0 + z \overset{3}{\Longrightarrow} y = z$. □

Proposition 3.4. $\forall x \in \mathbb{F}$*,* $x \cdot 0 = 0$*.*

Proof. $x \cdot 0 \overset{3}{=} x \cdot (0 + 0) \overset{9}{=} x \cdot 0 + x \cdot 0 \overset{3}{\Longrightarrow} 0 + x \cdot 0 = x \cdot 0 + x \cdot 0$. By the cancellation law, $0 = x \cdot 0$. □

Proposition 3.5. $0 \in \mathbb{F}$ *does not have a multiplicative inverse.*

Proof. Assume in contradiction $\exists y \in \mathbb{F}$ such that $0 \cdot y = 1$. By proposition above, $\forall x \in \mathbb{F}$, $x \cdot 0 = 0$. This implies that $1 = 0$ by the cancellation law, a contradiction of 7. □

Remark. This is why we disallow taking the multiplicative inverse of 0.

3.1 Ordered sets

Definition 3.6. (Ordered set). An ordered set is a set S with a relation $<$ satisfying:

1. Trichotomy: $\forall a, b, c \in S$, exactly one of $a < b$, $a = b$, or $a > b$ holds.
2. Transitivity: $\forall a, b, c \in S$, if $a < b$ and $b < c$, then $a < c$.

Example 3.7. $\mathbb{Q}$, $\mathbb{N}$, $\{-1, 0, 15\}$, $\{a, aa, b, ba, c\}$ with the lexicographic order.

Definition 3.8. (Maximum). A maximum for an ordered set S is an element $y \in S$ such that $\forall x \in S$, $x \leq y$.

Remark. Not all ordered sets have a maximum. For example, $\mathbb{N}$ does not have a maximum. Also $\{\frac{n-1}{n} : n \in \mathbb{N}\}$ does not have a maximum.

Proposition 3.9. $S = \{\frac{n-1}{n} : n \in \mathbb{N}\}$ *does not have a maximum.*

Proposition 3.10. *Assume in contradiction that* $\exists y \in S$ *such that* $\forall x \in S$, $x \leq y$. *Then* $y = \frac{m-1}{m}$ *for some* $m \in \mathbb{N}$. *But then* $y < \frac{m}{m+1} = \frac{(m+1)-1}{m+1} \in S$, *a contradiction that* y *is the maximum.*

Proposition 3.11. *If a ordered set* S *has a maximum, then it is unique. In such a case, we denote the maximum by* $\max S$.

Proof. Let y and y' be maxima for S. Then by the definition of maximum, $y \leq y'$ and $y' \leq y$. By trichotomy, $y = y'$. □

Proposition 3.12. *Every finite non-empty ordered set has a maximum.*

Proof. Proof by induction on $n = |S|$. For $n = 1$, $S = \{x\}$, then x is the maximum, trivially. Assume claim true for all ordered sets of size $n \in \mathbb{N}$. Let S be an ordered set of size $n+1$. Pick $s_0 \in S$ and set $T = S \setminus \{s_0\}$ with some (restricted) order. By the induction hypothesis, $\exists \max T = t_0$. Now there are two cases.

1. $s_0 \leq t_0$. Then $\forall x \in T \cup \{s_0\}$, $x \leq t_0 \implies t_0 = \max S$
2. $t_0 < s_0$. Then $\forall t \in T$, $t \leq t_0 < s_0 \implies t \leq s_0$. This implies that $\forall x \in S = T \cup \{s_0\}$, $x \leq s_0 \implies s_0 = \max S$.

In either case, $\max S$ exists. □

Definition 3.13. (Upper bound). Let S be an ordered set and let $A \subseteq S$. An upper bound for A in S is an element $z \in S$ such that $\forall a \in A$, $a \leq z$.

Example 3.14. $T = \{\frac{n-1}{n} : n \in \mathbb{N}\}$ has upper bounds in $\mathbb{Q}$, e.g. 1, 4, 1000, etc.

Definition 3.15. (Least upper bound). A least upper bound for $A \subseteq S$ is an upper bound z such that for any other upper bound z', they satisfy $z \leq z'$.

Proposition 3.16. *If* $A \subseteq S$ *has a least upper bound, it is unique and it is called the supremum of* A *(in* S*), denoted* $\sup A$.

Remark. Not all $A \subseteq S$ have a least upper bound. E.g. $\mathbb{N} \subseteq \mathbb{Q}$.

Proposition 3.17. $T = \{\frac{n-1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{Q}$ *has* $\sup T = 1$.

Proof. First $\forall n \in \mathbb{N}$, $\frac{n-1}{n} \leq 1$. This implies 1 is an upper bound for T. Assume there exists an upper bound $z \in T \subseteq \mathbb{Q}$ such that $z < 1$. Then $z = p/q$ for some $p, q \in \mathbb{Z}$, and we may assume $q \in \mathbb{N}$. This implies $z < \frac{q}{q+1} \in T$, a contradiction that z is an upper bound. Hence 1 is the least upper bound for T. □

3.2 Ordered fields

Definition 3.18. (Ordered field). An ordered field is a field $\mathbb{F}$ which is also an ordered set, satisfying:

1. Order respects addition: $\forall x, y, z \in \mathbb{F}$, if $x < y$, then $x + z < y + z$.
2. Order respects multiplication: If $x, y \in \mathbb{F}$ satisfy $x > 0$, $y > 0$, then $x \times y > 0$.

Example 3.19. (Fact). $\mathbb{Q}$ with the order we constructed is an ordered field.

Proposition 3.20. *Let $\mathbb{F}$ be an ordered field. If $x > 0$ and $y < 0$, then $x \times y < 0$.*

Proof. By order respects addition, $y < 0 \implies 0 < -y$. By order respects multiplication, $x \times (-y) > 0$. By order respects addition, we add $x \cdot y$ to both sides. The LHS:

$$\begin{aligned} x \times (-y) + x \times y &= x \times (-y + y) \\ &= x \times 0 \\ &= 0 \end{aligned}$$

This implies $x \times y < 0$, as $x \times y$ is the RHS. □

Proposition 3.21. *Let $\mathbb{F}$ be an ordered field. Then $\forall x \in \mathbb{F}$, $x < x + 1$.*

Proof. Assume in contradiction that $1 < 0$. (This is enough, because of order respects addition). $1 < 0 \implies 0 < -1 \implies 0 < -1 \times -1 = 1$, a contradiction to trichotomy. Therefore $0 \leq 1$. However, $0 \neq 1$ by an axiom of fields. This implies $0 < 1$. □

Proposition 3.22. *There exists no order on the field $\mathbb{F}_3$ making it an ordered field.*

Proof. Assume in contradiction that $\mathbb{F}_3$ has such a structure. Then $0 < 1$ and $1 < 2$ and $2 < 2 + 1 = 0$. By transitivity, $0 < 2$, a contradiction to trichotomy. □

Example 3.23. $\mathbb{C}$ has no structure of an ordered field. Problem set 3.

3.3 A hole in $\mathbb{Q}$

Lemma 3.24. $\sqrt{2} \notin \mathbb{Q}$*. That is, there exists no $x \in \mathbb{Q}$ such that $x^2 = 2$.*

Proof. Assume in contradiction that there exists $x \in \mathbb{Q}$ such that $x^2 = 2$.

- Then $\exists m, n \in \mathbb{Z}$ such that $x = m/n$. (We assume that this is a reduced fraction, i.e. there exists no integer $k \in \mathbb{Z} \setminus \{1\}$ such that k divides both m and n to remainder.)
- This implies $\left(\frac{m}{n}\right)^2 = 2 \implies m^2 = 2n^2$.

Lemma 3.25. *The square power of an odd integer is odd.*

Proof.

- Let $2k + 1 \in \mathbb{Z}$ be any integer, which is odd.
- Then $(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- As $2k^2 + 2k \in \mathbb{Z}$, this implies that $(2k + 1)^2$ is odd.

□

- We know $m^2 = 2n^2$ is an even number (dichotomy). Therefore m is even. Then $m = 2k$ for some $k \in \mathbb{Z}$.
- Then, $(2k)^2 = 4k^2 = 2n^2 \implies 2k^2 = n^2$. This implies that n^2 is even, implying n is also even.

- This is a contradiction to m/n being a reduced fraction.

$\square$

Proposition 3.26. *Consider $A = \{x \in \mathbb{Q} \colon x^2 < 2\}$. Then $y \in \mathbb{Q}$ is an upper bound for A if and only if $y > 0$ and $y^2 > 2$.*

Proof. ($\Longleftarrow$).

- If $y > 0$ and $y^2 > 2$, let $x \in A$ be any element.
- Assume in contradiction $y < x$. Then both $y > 0$ and $x > 0$ (transitivity).
- This implies $x \cdot y > y^2$ and $x^2 > x \cdot y$ (order respects multiplication).
- This implies $x^2 > y^2$ (transitivity). However, we know that $x^2 < 2$ and $2 < y^2$ by $x \in A$ and our assumption.
- This means $x^2 < y^2$, a contradiction to trichotomy.
- Therefore, $\forall x \in A$, $x \leq y$.

($\Longrightarrow$).

- Assume that $y \in \mathbb{Q}$ is an upper bound for A. Since $1 \in A$, we know that $0 < 1 \leq y$.
- Assume in contradiction that $y^2 \leq 2$. Since $y \in \mathbb{Q}$ we know $y^2 \neq 2$ hence $y^2 < 2$.
- The idea: Find $\epsilon \in \mathbb{Q}$, $\epsilon > 0$ for which $(y + \epsilon)^2 < 2$.
- This would imply that $y + \epsilon \in A$ and $y < y + \epsilon$, a contradiction to y being an upper bound.

Draft:

$$\begin{aligned}(y+\epsilon)^2 < 2 &\iff y^2 + 2y\epsilon + \epsilon^2 < 2 \\ &\iff \epsilon(2y+\epsilon) < 2 - y^2 \\ &\iff \epsilon < \frac{2-y^2}{2y+\epsilon} && \text{if } y > 0,\ \epsilon > 0\end{aligned}$$

Assume $\epsilon < 1$. Therefore

$$\epsilon < \frac{2-y^2}{2y+1} < \frac{2-y^2}{2y+\epsilon} \implies (y+\epsilon)^2 < 2$$

Now, continuing the proof. Let's fix:

$$\epsilon = 1/2 \min\left\{1, \frac{2-y^2}{2y+1}\right\}$$

- This implies $0 < \epsilon \leq 1/2 < 1$ and $\epsilon < \frac{2-y^2}{2y+1}$. Therefore, $\epsilon \in \mathbb{Q}$.
- This implies $y + \epsilon \in \mathbb{Q}$ and $(y+\epsilon)^2 < 2 \implies y + \epsilon \in A$.
- This is a contradiction to y being an upper bound for A.

$\square$

Corollary 3.27. *$A \subseteq \mathbb{Q}$ is bounded above but has no supremum.*

Proof. First $2^2 > 2$ and $2 > 0$. This implies 2 is an upper bound for A. (by the proposition above). Let $y \in \mathbb{Q}$ be an upper bound for A. We will show there exists $y' \in \mathbb{Q}$ such that $y' < y$ which is also an upper bound. That would imply A has no least upper bound. Fix some upper bound $y \in \mathbb{Q}$. By the proposition above, $y > 0$ and $y^2 > 2$.

Draft: We are looking for $\epsilon > 0 \in \mathbb{Q}$ such that $y-\epsilon > 0 \iff \epsilon < y$ and $(y-\epsilon)^2 > 2 \iff y^2-2y\epsilon+\epsilon^2 > 2$. It's enough for $y^2 - 2y\epsilon > 2 \iff \epsilon < \frac{y^2-2}{2y}$. So we pick

$$\epsilon = 1/3 \min \left\{ y, \frac{y^2-2}{2y} \right\}$$

This implies $0 < \epsilon \in \mathbb{Q}$ and $\epsilon < y$ and $\epsilon < \frac{y^2-2}{2y}$. This implies $y - \epsilon > 0$ and $(y-\epsilon)^2 > 2$. By previous proposition, $y - \epsilon$ is also an upper bound for A. □

3.4 Least upper bound (LUB) property

Definition 3.28. (Least upper bound property). An ordered set S is said to satisfy the LUB property if any $\emptyset \neq A \subseteq S$ which is bounded above has a supremum.

Theorem 3.29. *There exists an ordered field, containing $\mathbb{Q}$ with the LUB property. Moreover, any two such fields are "isomorphic". We call such a field $\mathbb{R}$.*

Remark. In ordered set S, $A \subseteq S$

- $y \in S$ is a lower bound for A if $\forall x \in A$, $y \leq x$.
- $y = \min A$ if $y \in A$ and y is a lower bound for A.
- $y = \inf A$ if y is a lower bound for A and $\forall z \in S$, if z is a lower bound for A, then $z \leq y$.

Remark. It follows from problem set 3 if $\emptyset \neq A \subseteq \mathbb{R}$ is bounded below, then A has an infimum in $\mathbb{R}$.

3.5 Properties of $\mathbb{R}$

Proposition 3.30. $\forall x, y \in \mathbb{R}$, $x > 0$, $\exists n \in \mathbb{N}$ *such that* $nx > y$.

Proof.
- Assume in contradiction that y is an upper bound for $\emptyset \neq A = \{nx \colon n \in \mathbb{N}\}$.
- By the LUB property, there exists $z = \sup A \in \mathbb{R}$. In particular $z - x$ is not an upper bound for A.
- This implies $\exists n_0 \in \mathbb{N}$ such that $n_0 x \in A$ satisfies $n_0 x > z - x$.
- This implies $(n_0 + 1)x > z$. This is a contradiction to z being an upper bound for A.

□

Corollary 3.31. *(Archimedean property of $\mathbb{R}$).*

1. $\forall y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ *such that* $n > y$.
2. $\forall \epsilon > 0 \in \mathbb{R}$, $\exists n \in \mathbb{N}$ *such that* $1/n < \epsilon$.

That is, $\mathbb{R}$ does not contain an infinitely large element nor an infinitesimally small element.

Proof. For the first part, take $x = 1$ in the proposition above. For the second part, take $y = 1$ and $x = \epsilon$ in the proposition above. This implies $\exists n \in \mathbb{N}$ such that $n\epsilon > 1 \implies 1/n < \epsilon$. □

Lemma 3.32. *Any $\emptyset \neq A \subseteq \mathbb{N}$ has a minimum.*

Proof. Let $\emptyset \neq A \subseteq \mathbb{N}$. Consider $1 \in L = \{l \in \mathbb{N} : \forall a \in A, l \leq a\}$. L is the set of lower bounds for A. Either

1. $\exists n \in \mathbb{N}$ such that $n \in L$ but $n+1 \notin L \implies \exists a \in A$ such that $n \leq a < n+1 \implies a = n \implies a = \min A$.

2. $\forall n \in L$, $n+1 \in L \implies L = \mathbb{N}$ by induction. By corollary, $A = \emptyset$, a contradiction.

□

Corollary 3.33. *Every $\emptyset \neq A \subseteq \mathbb{Z}$ which is bounded below has a minimum.*

Proof. If $\exists n \in \mathbb{N}$ such that n is a lower bound for A, then $\emptyset \neq A \subseteq \mathbb{N}$. By the lemma, this implies $\exists \min A$. Otherwise, the set $A \cap \{-n \colon n \in \mathbb{N}_0\}$ is finite. This implies $A \cap \{-n \colon n \in \mathbb{N}_0\}$ has a minimum. That will also be a minimum for A. (Exercise, fill in details.) □

Definition 3.34. (Ceiling). The ceiling of $x \in \mathbb{R}$ is $\lceil x \rceil := \min\{k \in \mathbb{Z} : x \geq k\}$.

Proposition 3.35. *(Denseness of $\mathbb{Q}$ in $\mathbb{R}$). $\forall x, y \in \mathbb{R}$, $x < y$, $\exists q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. Let $x < y \in \mathbb{R}$. Since $y - x > 0$. There exists $m \in \mathbb{N}$ such that $m(y-x) > 1$, i.e. $my - mx > 1$. (think rescaling distance between x and y). It'll suffice to show $\exists k \in \mathbb{Z}$ with $mx < k < my$. Then $x < k/m < y$ with $k/m \in \mathbb{Q}$. In fact, take $k = \lceil mx \rceil + 1$. Indeed,

- $k < my$ because $k < \min\{l \in \mathbb{Z} : l \geq my\} = \lceil my \rceil$.
- $k - mx = \lceil mx \rceil + 1 - mx \geq my - 1 - mx = (my - mx) - 1 > 0$. This implies $mx < k$.

Therefore, we have found $k \in \mathbb{Z}$ such that $mx < k < my$. This implies $\exists q = k/m \in \mathbb{Q}$ such that $x < q < y$. □

3.6 Roots and exponents

Proposition 3.36. $\forall n \in \mathbb{N}$ *and* $\forall x > 0$*, there exists a unique $y > 0$ such that $y^n = x$. We denote this number $y = x^{1/n} = \sqrt[n]{x}$.*

Proof. **Uniqueness**: $\forall 0 < y_1 < y_2$, $y_1^n < y_2^n$. This implies $y_1^n \neq x \neq y_2^n$.

Existence: Consider $E = \{z \in \mathbb{R} : z^n < x\}$. First, if $t = \frac{x}{1+x}$, then $t < 1$ and $t < x$. This implies $t^n \leq t < x \implies E \neq \emptyset$. Second, $\forall t > 1 + x$ $t^n \geq t > x \implies t \notin E$. This implies if $z \in E$, then $z \leq 1 + x$, so E is bounded above.

Proposition 3.37. *$y = \sup E$ satisfies $y^n = x$.*

Proof. Proof for all n by induction, an exercise. □

Proof of existence continues. TODO.

□

4 Countability

Proposition 4.1. *If S is countable and $A \subseteq S$ is infinite, then A is countable.*

Proof.

- S is countable, so $\exists g : \mathbb{N} \to S$ that is a bijection.
- We can write $S = \{x_1, x_2, \ldots\}$ where $x_n = g(n)$.
- We define $m_1 = \min\{l \in \mathbb{N} : x_l \in A\}$.
- For all $n \in \mathbb{N}$, we define $m_{n+1} = \min\{l \in \mathbb{N} : l > m_n, x_l \in A\}$.
- Denote $f : \mathbb{N} \to A$ by $f(n) = x_{m_n}$. Since A is infinite, f is well-defined, i.e. $f(n)$ is unique for all n.
- f is injective since by definition $m_{n+1} > m_n \implies m_n < m_k$ for all $k > n$. Since g is injective, if $n < k$, $x_{m_n} \neq x_{m_k}$.
- f is surjective. Let $a \in A$. Since g is surjective, $\exists N \in \mathbb{N}$ such that $a = x_N$. Consider $n = \min\{l \in \mathbb{N} : m_l \geq N\}$. We want to show $a = x_{m_n}$, showing that $f(n) = a$. By definition $m_n \geq N$ and $m_{n-1} < N$. By construction of m_n from m_{n-1}, we know $m_n \leq N$ because $x_N = \in A$. This implies $m_n = N$ and $a \in f(\mathbb{N})$.

□

Corollary 4.2. *$\mathbb{Q}$ is countable.*

Proof. Consider the following function $h : \mathbb{Q} \to \mathbb{Z}^2$ sending $q \in \mathbb{Q}$ to $(m, n) \in \mathbb{Z}^2$ where (m, n) is the unique pair satisfying $q = m/n$ is reduced and $n \in \mathbb{N}$. Denote $A = h(\mathbb{Q}) \subseteq \mathbb{Z}^2$. Then A is infinite. (e.g. $(m, 1) \in A$ for all $m \in \mathbb{Z}$). This implies that A is countable by the proposition above ($\mathbb{Z}^2$ is countable). Since h is injective, we deduce that $|\mathbb{Q}| = |A|$. □

Corollary 4.3. *The set of prime numbers is countable.*

Lemma 4.4. *The union of a countable collection of countable sets, i.e., given $\{S_1, S_2, \ldots\} = \{S_n : n \in \mathbb{N}\}$ where $|S_n| = |\mathbb{N}|$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} S_n$ is countable.*

Proof. (Sketch). Assume disjoint, that $S_{n_1} \cap S_{n_2} = \emptyset$ for all $n_1 \neq n_2$. There exists a bijection $f_n : \mathbb{N} \to S_n$ for all $n \in \mathbb{N}$. Construct $F : \mathbb{N} \times \mathbb{N} \to \bigcup_{n\in\mathbb{N}} S_n$ by $F(n, m) = f_n(m)$. F is a bijection. □

Corollary 4.5. *Any union of countably many finite sets is at most countable. That is, it is either finite or countable.*

Theorem 4.6. *(Cantor).* $(0, 1) \subseteq \mathbb{R}$ *is uncountable.*

Proof. Assume in contradiction that $(0, 1)$ is countable. Hence, we can write $(0, 1) = \{x_1, x_2, \ldots\}$.

1. Pick $I_1 = [a_1, b_1] \subseteq (0, 1)$ such that $a_1 < b_1$ and which satisfies $x_1 \notin I_1$.
2. For $n = 2$ pick, $I_2 = [a_2, b_2] \subseteq I_1$ that $x_2 \notin I_2$.
3. Once we have chosen $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$, such that $x_k \notin I_k$ for all $1 \leq k \leq n$.
4. Pick $I_{n+1} = [a_{n+1}, b_{n+1}] \subseteq I_n$ such that $x_{n+1} \notin I_{n+1}$.

Denote $A = \{a_n : n \in \mathbb{N}\} \subseteq (0, 1)$, then $A \neq \emptyset$ is bounded above by b_1 because $A \subseteq I_1$. By the lower bound property, $\exists z = \sup A \in \mathbb{R}$. (Actually, $\forall k \in \mathbb{N}$, b_k is an upper bound for A, verify). This implies $z \leq b_k$ for all $k \in \mathbb{N}$ and $a_k \leq z$. $\implies z \in I_k$ for all $k \in \mathbb{N} \implies z \in \bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$. However, $x_n \notin \cap_{k\in\mathbb{N}} I_k$ for all $n \in \mathbb{N}$, a contradiction of our assumption that $(0, 1) = \{x_1, x_2, \ldots\}$ and in particular that z is in this list. □

Corollary 4.7. *$\mathbb{R}$ is uncountable.*

Theorem 4.8. *(Cantor's diagonal argument, sketch).*

Proof. Assume in contradiction that $(0,1)$ is countable. Hence, we can write $(0,1) = \{x_1, x_2, \ldots\}$. E.g. $x_1 = 0.1246789\ldots$, $x_2 = 0.9876543\ldots$, etc. Let's construct $y \in (0,1)$ such that $y \neq x_n$ for all $n \in \mathbb{N}$. We do this by choosing y such that its $n-th$ digit is not equal to the $n-th$ digit of x_n. Therefore, $y \neq x_n$ for all $n \in \mathbb{N}$, a contradiction. □

Corollary 4.9. *There are uncountably many irrational numbers in $\mathbb{R}$. I.e., $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.*

Proof. If $\mathbb{R} \setminus \mathbb{Q}$ were countable, then $\mathbb{R}$ would be countable, a contradiction. □

4.1 Power sets

Definition 4.10. (Power set). The power set of a set A is the set of all subsets of A. We denote the power set of A by $\mathcal{P}(A) = \{E : E \subseteq A\}$.

Theorem 4.11. *(Cantor, again). For any set A, $|A| \neq |\mathcal{P}(A)|$.*

Proof. Assume in contradiction that there exists a bijection $\Phi : A \to \mathcal{P}(A)$. Consider $E = \{y \in A : y \notin \Phi(y)\} \in \mathcal{P}(A)$. Since Φ is onto, $\exists e \in A$ such that $\Phi(e) = E$.

1. If $e \in E$, then $e \in \Phi(e) = E$, implying $e \notin E$, a contradiction.
2. If $e \notin E$, then $e \notin \Phi(e) = E$, implying $e \in E$, a contradiction.

□

Corollary 4.12. $\mathcal{P}(\mathbb{N})$ *is uncountable.*

5 Metric spaces

Definition 5.1. (Metric space). A metric space is a set X together with a distance function $d : X \times X \to \mathbb{R}$ satisfying:

1. Positivity: $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle-inequality: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Example 5.2.

1. $\mathbb{R}$ with the standard metric $d(x, y) = |x - y|$.
2. $\mathbb{R}^k$ with the standard metric $d(x, y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$.
3. Any set X with the discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$.

Definition 5.3. (r-neighborhood). Let (X, d) be a metric space. The r-neighborhood of a point $p \in X$ is the set $N_r(p) = \{x \in X : d(x, p) < r\}$ with $r > 0$.

Example 5.4. In $\mathbb{R}$, the interval $(a, b) = N_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$.

Definition 5.5. (Interior point). Let (X, d) be a metric space and $E \subseteq X$. A point $p \in X$ is an interior point in E if there exists $\epsilon > 0$ such that $N_\epsilon(p) \subseteq E$.

Definition 5.6. (Open set). A set $E \subseteq X$ is open if every point in E is an interior point.

Example 5.7. $(0, 1) \subseteq \mathbb{R}$ is open because $\forall x \in (0, 1)$, x is an interior point by taking $\epsilon = \min\{x, 1 - x\}$.

Lemma 5.8. *Let (X, d) be a metric space. $\forall p \in X, r > 0$, $N_r(p)$ is open.*

Proof. Let $q \in N_r(p)$. We want to find an $\epsilon > 0$ such that $N_\epsilon(q) \subseteq N_r(p)$. Let $\epsilon < r - d(p, q)$. $r - d(p, q) > 0$ because $q \in N_r(p)$. For any $x \in N_\epsilon(q)$,

$$\begin{aligned} d(x, p) &\leq d(x, q) + d(q, p) && \text{(Triangle inequality)} \\ &< \epsilon + d(q, p) && (x \in N_\epsilon(q)) \\ &= r && (\epsilon < r - d(p, q)) \end{aligned}$$

This implies that $x \in N_r(p)$, so $N_\epsilon(q) \subseteq N_r(p)$, so q is an interior point of $N_r(p)$. ☐

Proposition 5.9. *Let (X, d) be a metric space.*

1. *Let $\{G_\alpha\}_{\alpha \in I}$ be any collection of open sets in X. Then $\bigcup_{\alpha \in I} G_\alpha$ is open.*

   Proof.

   - Let $p \in \bigcup_{\alpha \in I} G_\alpha$.
   - Then $\exists \alpha_0 \in I$ such that $p \in G_{\alpha_0}$.
   - Since G_{α_0} is open, $\exists \epsilon > 0$ such that $N_\epsilon(p) \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in I} G_\alpha$.
   - Therefore, p is an interior point of $\bigcup_{\alpha \in I} G_\alpha$, so $\bigcup_{\alpha \in I} G_\alpha$ is open.

   ☐

2. *If $G_1, G_2, \ldots, G_n$ are open, then $\bigcap_{i=1}^{n} G_i$ is open.*

   Proof.

- Let $p \in \bigcap_{i=1}^n G_i$.
- Since G_i is open for all $1 \leq i \leq n$, there exists $\epsilon_1, \ldots, \epsilon_n$ such that $N_{\epsilon_i}(p) \subseteq G_i$ for all $1 \leq i \leq n$.
- Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$.
- Then $N_\epsilon(p) \subseteq N_{\epsilon_i}(p) \subseteq G_i$ for all $1 \leq i \leq n$, so $N_\epsilon(p) \subseteq \bigcap_{i=1}^n G_i$.
- Therefore, p is an interior point of $\bigcap_{i=1}^n G_i$, so $\bigcap_{i=1}^n G_i$ is open.

□

Example 5.10. Counterexample for an infinite intersection. Consider $\left\{G_n = \left(-\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Then $\bigcap_{n=1}^\infty G_n = \{0\}$, which is not open.

5.1 Limit points and closed sets

Definition 5.11. (Limit point, isolated point, closed set). Let (X, d) be a metric space. Let $E \subseteq X$.

1. A point $p \in X$ is a limit point of E if $\forall \epsilon > 0$, $N_\epsilon(p) \cap (E \setminus \{p\}) \neq \emptyset$. In other words, $\forall \epsilon > 0$, $N_\epsilon(p) \cap E$ contains $q \neq p$.
2. A point $p \in E$ is called isolated if it is not a limit point. Equivalently, $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap E = \{p\}$.
3. A set E is closed if it contains all of its limit points.

Example 5.12. All examples in $\mathbb{R}$.

1. $E = (0, 1)$. The set of limit points of E is $[0, 1]$, so E is not closed.
2. $E = [0, 1]$. The set of limit points of E is $[0, 1]$, so E is closed.
3. $\emptyset$ and $\mathbb{R}$ are both open and closed.
4. $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. All elements in this set are isolated, so E is not closed. But set of limit points of E is $\{0\}$.
5. $E = \mathbb{Q}$. The set of all limit points of E is $\mathbb{R}$ (density of $\mathbb{Q}$ in $\mathbb{R}$).

Proposition 5.13. *Any neighborhood of a limit point p of $E \subseteq X$ contains infinitely many points of E.*

Proof. We want to show that if $p \in X$ is a limit point of E and $\epsilon > 0$, then $N_\epsilon(p) \cap E$ is infinite. Assume in contradiction that $\exists \epsilon_0 > 0$ such that $N_{\epsilon_0}(p) \cap E = \{p_1, \ldots, p_n\}$ is finite. Let $\delta = \min\{d(p, p_i) : 1 \leq i \leq n \text{ and } p_i \neq p\}$. Then $N_{\frac{\delta}{2}}(p) \cap E = \{p\}$ (verify). Therefore, p is not a limit point of E, which is a contradiction. □

Corollary 5.14. *Finite sets have no limit points.*

Corollary 5.15. *All finite sets are closed.*

Proposition 5.16. *E is open iff $E^c := X \setminus E$ is closed.*

Proof. ($\Longrightarrow$) Assume E is open.

- Let p be a limit point of E^c. This means that $\forall \epsilon > 0$, $N_\epsilon(p)$ contains elements of E^c.
- In particular, $N_\epsilon(p) \not\subseteq E$, so p is not an interior point of E. This implies that p cannot be in E because E is open.
- Therefore, $p \in E^c$, so E^c is closed.

($\Longleftarrow$) Assume E^c is closed.

- Hence, $\forall p \in E$, p is not a limit point of E^c.

- This implies $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap E^c = \emptyset$.
- This means that $N_\epsilon(p) \subseteq E$, so p is an interior point of E.
- Therefore, E is open.

$\square$

Remark.

1. Since $(E^c)^c = E$, we have that E is closed iff E^c is open.
2. Some metric spaces have non-empty clopen subsets that are not X. For example $X = [0,1] \cup [2,3]$ with $d(x,y) = |x-y|$. Check with $[0,1]$ is open and closed.

Corollary 5.17. *Let (X,d) be a metric space.*

1. *Let $\{F_\alpha\}_{\alpha \in I}$ be any collection of closed sets. Then $\bigcap_{\alpha \in I} F_\alpha$ is closed.*
2. *If $F_1, F_2, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed.*

Proof.

- Notice that $\left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c$ and $\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c$.
- To prove the first statement, since F_α is closed $\forall \alpha \in I$, F_α^c is open.
- This implies that $\bigcup_{\alpha \in I} F_\alpha^c$ is open (arbitrary union of open sets are open).
- This implies $\left(\bigcap_{\alpha \in I} F_\alpha\right)^c$ is open, so $\bigcap_{\alpha \in I} F_\alpha$ is closed.
- The second statement follows similarly.

$\square$

Remark. There exists sets in $\mathbb{R}$ that are neither open nor closed. For example, $\mathbb{Q}$, $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Definition 5.18. (Closure). The closure of a set $E \subseteq X$ is the set

$$\overline{E} := \{x \in X : x \in E \text{ or } x \text{ is a limit point of } E\}$$

.

Example 5.19.

1. $\overline{(0,1)} = [0,1]$.
2. $\overline{\mathbb{Q}} = \mathbb{R}$.
3. $\overline{\left\{\frac{1}{n} : n \in \mathbb{N}\right\}} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Remark. A set $A \subseteq B$ is said to be dense in B if $\overline{A} = B$.

Proposition 5.20. *For any $E \subseteq X$, $\overline{E}$ is the smallest closed set containing E. That is, $\overline{E}$ is closed and for any closed set $F \subseteq X$ with $E \subseteq F$, $\overline{E} \subseteq F$.*

Proof. First we show $\overline{E}^c$ is open. Let $p \in \overline{E}^c$. Since $p \notin \overline{E}$, we know that $p \notin E$ and p is not a limit point of E. This implies that $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap E =$ FINISH THIS PROOF.

Let $F \subseteq X$ be closed with $E \subseteq F$ and let p be a limit point of E. This means that $\forall \epsilon > 0$, $N_\epsilon(p) \cap (E \setminus \{p\}) \neq \emptyset$. Because $N_\epsilon(p) \cap (E \setminus \{p\}) \subseteq N_\epsilon(p) \cap (F \setminus \{p\})$, we have that p is a limit point of F. This implies that $p \in F$, so $\overline{E} \subseteq F$. $\square$

Remark. Consider $\mathbb{F} = \{F \subseteq X : E \subseteq F \text{ and } F \text{ is closed}\}$. Then $\overline{E} = \bigcap_{F \in \mathbb{F}} F$.

Proof. $\overline{E} \in \mathbb{F}$ implies $\bigcap_{F\in\mathbb{F}} F \subseteq \overline{E}$. $\forall F \in \mathbb{F}$, $\overline{E} \subseteq F$ implies $\overline{E} \subseteq \bigcap_{F\in\mathbb{F}} F$. $\square$

Proposition 5.21. *If $\emptyset \neq E \subseteq \mathbb{R}$ is bounded above then* $\sup E \in \overline{E}$*.*

Proof. Let $z = \sup E$. If $z \in E$, then $z \in \overline{E}$. Otherwise, we show that z is a limit point of E. $\forall \epsilon > 0$, $z - \epsilon$ is not an upper bound of E. This implies that $\exists q \in E$ such that $z - \epsilon < q \leq z$. Since $z \notin E$, we have the strict inequality $z - \epsilon < q < z$. Or in other words, $q \in N_\epsilon(z) \cap E \setminus \{z\}$. Therefore, z is a limit point of E, so $z \in \overline{E}$. $\square$

5.2 Bounded sets

Definition 5.22. (Bounded set). A set $E \subseteq X$ is bounded if $\exists M > 0$ such that $\forall p, q \in E$, $d(p, q) \leq M$.

Proposition 5.23. *For any $\emptyset \neq E \subseteq \mathbb{R}$, E is bounded if and only if $\exists \tilde{M} > 0$ and $p_0 \in X$ such that $E \subseteq N_{\tilde{M}}(p_0)$.*

Proof. ($\Longrightarrow$) Assume E is bounded. Fix $p_0 \in E$. Then $\exists M > 0$ such that $\forall q \in E$, $d(p_0, q) \leq M$. This implies that $E \subseteq N_M(p_0)$.

($\Longleftarrow$) Assume $\exists \tilde{M} > 0$ and $p_0 \in X$ such that $E \subseteq N_{\tilde{M}}(p_0)$. Then $\forall p, q \in E$,

$$d(p, q) \leq d(p, p_0) + d(p_0, q) \leq \tilde{M} + \tilde{M} = 2\tilde{M}.$$

So E is bounded by a constant $2\tilde{M}$. $\square$

Corollary 5.24. *$E \subseteq \mathbb{R}$ is bounded if any only if it is bounded both above and below.*

5.3 Connected sets

Definition 5.25. $E \subseteq X$ is disconnected if there exists two non-empty subsets $A, B \subseteq X$ such that $E = A \cup B$ and both $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. A set is called connected if it is not disconnected.

Example 5.26.

1. $\{0, 1\}$ is disconnected by taking $A = \{0\}$ and $B = \{1\}$.
2. $[-1, 0) \cup (0, 1]$ is disconnected by taking $A = [-1, 0)$ and $B = (0, 1]$.
3. $\mathbb{Q}$ is disconnected by taking $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$.
4. $[-1, 1]$, $(-1, 1)$, and $\mathbb{R}$ are all connected.

Proposition 5.27. *$E \subseteq \mathbb{R}$ is connected if and only if $\forall x < y \in E$, then $[x, y] \subseteq E$.*

Proof. ($\Longrightarrow$)

- Assume E is connected and assume in contradiction that $\exists x < z < y$ such that $x, y \in E$ but $z \notin E$.
- Consider $E = A \cup B$ where $A = E \cap (-\infty, z)$ and $B = E \cap (z, \infty)$.
- Since $x \in A$, $y \in B$, we have $A \neq \emptyset$ and $B \neq \emptyset$.
- Next $\overline{A} \subseteq \overline{(-\infty, z)} = (-\infty, z]$.
- Hence, $\overline{A} \cap B \subseteq (-\infty, z] \cap (z, \infty) = \emptyset$.
- Similarly, $A \cap \overline{B} \subseteq (-\infty, z) \cap [z, \infty) = \emptyset$.
- This implies that E is disconnected, which is a contradiction.

($\Longleftarrow$)

- Assume $\forall x < y \in E$, then $[x, y] \subseteq E$.

- Let $E = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap B = \emptyset$.
- Without loss of generality, let $x \in A$, $y \in B$ such that $x < y$.
- Denote $z = \sup\{t \in A \colon t < y\} = \sup(A \cap (-\infty, y))$.
- Then $z \in \overline{A}$ by a previous proposition (because $z \in \overline{(A \cap (-\infty, y))} \subseteq \overline{A}$).
- We have two cases:
  1. If $z \in B$, then $\overline{A} \cap B \neq \emptyset$. Hence, $E = A \cup B$ is not a witness to E being disconnected.
  2. If $z \notin B$, then in particular $z \neq y$ and hence $x \leq z < y \implies [z, y] \subseteq [x, y] \subseteq E \implies (z, y) \subseteq E \setminus A \subseteq B \implies z \in \overline{(z, y)} \subseteq \overline{B}$. Also, since $z \notin B \implies z \in A$, so we conclude that $z \in A \cap \overline{B} \neq \emptyset$. Again showing $E = A \cup B$ is not a witness to E being disconnected.
- Therefore, E is connected.

□

Proposition 5.28. *X is disconnected if and only if $X = A \cup B$ where A and B are non-empty and disjoint open sets.*

Proof. (Idea). If $X = A \cup B$ and $A \cap \overline{B} = \emptyset$, then $\overline{B} = A^c$ is closed. □

Remark. The induced metric on a subset $Y \subseteq X$ is just the restriction of d to $Y \times Y$, i.e. the distance in Y between any two points in Y is the same as their distance in X. A set $U \subseteq Y \subseteq X$ is open if it is open as a subset $U \subseteq Y$ with respect to the induced metric on Y. $Y \subseteq X$ is disconnected if and only if $Y = A \cup B$ where A and B are non-empty and disjoint open sets in Y.

5.4 Compact sets

Definition 5.29. (Open cover, finite subcover). A collection $\{G_\alpha\}_{\alpha \in I}$ is called an open cover of a subset $E \subseteq X$ if each G_α, $\alpha \in I$, is an open set and $E \subseteq \bigcup_{\alpha \in I} G_\alpha$

An open cover is called finite if it contains finitely many open sets. It is called infinite otherwise.

Given an open cover $\{G_\alpha\}_{\alpha \in I}$ of a subset $E \subseteq X$, a subcover is a subcollection $\{V_\beta\}_{\beta \in J} \subseteq \{G_\alpha\}_{\alpha \in I}$ that is still an open cover of E. That is, still satisfying $E \subseteq \bigcup_{\beta \in J} V_\beta$.

Example 5.30.

1. $E = [0, 1]$ with
$$\left\{G_x = \left(x - \frac{1}{10}, x + \frac{1}{10}\right)\right\}_{x \in [0,1]}$$
has a finite sub-cover. E.g.
$$\left\{G_0, G_{\frac{1}{10}}, \ldots, G_{\frac{9}{10}}, G_1\right\}$$
2. $A = (0, 1)$ with
$$\left\{G_x = \left(x - \frac{1}{10}, x + \frac{1}{10}\right)\right\}_{x \in [0,1]}$$
has a finite sub-cover.
3. $\left\{W_n = \left(\frac{1}{n}, 2\right)\right\}_{n \in \mathbb{N}}$ is also an open cover of A without a finite subcover. This is because any finite subcollection
$$\{W_{n_1}, \ldots, W_{n_k}\} \subseteq \{W_n\}_{n \in \mathbb{N}}$$
would satisfy
$$\bigcup_{i=1}^{k} W_{n_i} \subseteq \left(\frac{1}{M} 2\right)$$
where $M = \max\{n_1, \ldots, n_k\}$.

Definition 5.31. (Compact set). A subset $K \subseteq X$ is called compact if every open cover of K has a finite subcover.

Example 5.32.

1. In $\mathbb{R}$, $A = (0, 1]$ is not compact. See above example 2.
2. In $\mathbb{R}$, $\mathbb{Z}$ is not compact.
3. Every finite set in X is compact.
4. In $\mathbb{R}$, $E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is compact.

   Proof. Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover of E. Then, $\exists \alpha_0$ such that $0 \in G_{\alpha_0}$. Since G_{α_0} is open, there exists $\epsilon > 0$ such that $N_\epsilon(0) = (-\epsilon, \epsilon) \subseteq G_{\alpha_0}$. Take $m = \max\left\{n : \frac{1}{n} \geq \epsilon\right\}$. Then $E \setminus G_{\alpha_0} \subseteq \left\{1, \frac{1}{2}, \ldots, \frac{1}{m}\right\}$. So we can pick m more elements of $\{G_\alpha : G_{\alpha_1}, \ldots, G_{\alpha_m}\}$ such that $E \subseteq \bigcup_{i=0}^{m} G_{\alpha_i}$. $\square$

Proposition 5.33. *If $K \subseteq X$ is compact then K is closed.*

Proof. We'll show K^c is open. Let $p \in K^c$. For any $q \in K$, since $p \neq q$, $d(p, q) > 0$ and

$$p \notin N_{\frac{d(p,q)}{2}}(q) =: V_q$$

Consider $\{V_q\}_{q \in K}$. Then $\{V_q\}_{q \in K}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{V_{q_1}, \ldots, V_{q_n}\} \subseteq \{V_q\}_{q \in K}$ such that $K \subseteq \bigcup_{i=1}^{n} V_{q_i}$ Denote

$$r = \min\left\{\frac{d(p, q_i)}{2} : 1 \leq i \leq n\right\} > 0$$

. Then

$$N_r(p) \cap \bigcup_{i=1}^{n} V_{q_i} = \emptyset$$

Otherwise, if

$$x \in N_r(p) \cap \bigcup_{i=1}^{n} V_{q_i}$$

then $\exists 1 \leq j \leq n$ such that $x \in N_r(p) \cap V_{q_j}$. This implies

$$d(p, q_j) \leq d(p, x) + d(x, q_j) < r + \frac{d(p, q_j)}{2} \leq d(p, q_j)$$

A contradiction. This implies that $N_r(p) \subseteq \left(\bigcup_{i=1}^{n} V_{q_i}\right)^c \subset K^c$. This implies p is an interior point of K^c. $\square$

Proposition 5.34. *If $K \subseteq X$ is compact, then K is bounded.*

Proof. Consider the following open cover of K:

$$\{N_1(q)\}_{q \in K}$$

Since K is compact, $\exists q_1, \ldots, q_n \in K$ such that

$$\forall p \in k \exists 1 \leq i \leq n \text{ such that } d(p, q_i) < 1$$

Denote $D = \max\{d(q_i, q_j) : 1 \leq i, j \leq n\}$. Let $x, y \in K$. Then $\exists 1 \leq i \leq n$ such that $d(x, q_i) < 1$ and $\exists 1 \leq j \leq n$ such that $d(y, q_j) < 1$. Using the triangle inequality, we have

$$d(x, y) \leq d(x, q_i) + d(q_i, q_j) + d(q_j, y) < 1 + D + 1 = D + 2 =: M$$

Hence, we've shown that $\forall x, y \in K$, $d(x, y) \leq M$. Therefore K is bounded. $\square$

Proposition 5.35. *Let $K \subseteq X$ be compact. Any infinite subset $E \subseteq K$ has a limit point in K.*

Proof.

- Let $E \subseteq K$ be a set without a limit point in K. We will show E is finite (or empty).
- Let $q \in K$. Since q is not a limit point of E we know there exists a neighborhood V_q of q satisfying $E \cap V_q \subseteq \{q\}$
- Consider the open cover $\{V_q\}_{q \in K}$. Since K is compact, there exists a finite subcover $\{V_{q_1}, \ldots, V_{q_n}\} \subseteq \{V_q\}_{q \in K}$ such that $K \subseteq \bigcup_{i=1}^n V_{q_i}$.

$$E = E \cap \left(\bigcup_{i=1}^{n} V_{q_i} \right) = \bigcup_{i=1}^{n} (E \cap V_{q_i}) \subseteq \bigcup_{i=1}^{n} \{q_i\}$$

- Therefore, E is finite.

□

Proposition 5.36. *For any $a < b$, $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof.

- Assume in contradiction that there exists an open cover $\{G_\alpha\}_\alpha$ of $I = [a, b]$ which does not contain a finite sub-cover.
- Notation: Given an interval $J = [c, d]$. Denote $J^L = [c, \frac{c+d}{2}]$ and $J^R = [\frac{c+d}{2}, d]$.
- At most one of $\{I^L, I^R\}$ can be covered by finitely many G_α. (Since if $\exists \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that $I^L \subseteq \bigcup_{i=1}^n G_{\alpha_i}$ and $I^R \subseteq \bigcup_{i=1}^m G_{\beta_i}$, then
- $\{G_{\alpha_1}, \ldots, G_{\alpha_n}, G_{\beta_1}, \ldots, G_{\beta_m}\}$ is a finite subcover of I.)
- Denote $I_1 = I^L$ if I^L cannot be covered by finitely many G_α and set $I_1 = I^R$ otherwise.
- For any $n \geq 1$, if I_n cannot be covered by finitely many G_α, then at least one of
- $\{I_n^L, I_n^R\}$ also cannot be covered by finitely many G_α.
- Denote $I_{n+1} = I_n^L$ if it cannot be covered by finitely many G_α and set $I_{n+1} = I_n^R$ otherwise.
- Notice that the length of I^n is $2^{-n}(b - a)$.
- Also notice that $I \supseteq I_1 \supseteq I_2 \supseteq \ldots$
- We've seen before (in the proof of Cantor's Theorem) that in this situation in $\mathbb{R}$, $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.
- Let $z \in \bigcap_{n \in \mathbb{N}} I_n$. In particular, $z \in [a, b]$.
- Since $\{G_\alpha\}$ covers $[a, b]$, $\exists \alpha_0$ is open, $\exists \epsilon > 0$ for which $N_\epsilon(z) \subseteq G_{\alpha_0}$
- Since $\exists n_0 \in \mathbb{N}$ with $2^{-n_0}(b - a) < \epsilon$, and since $z \in I_{n_0}$, we have that $I_{n_0} \subseteq N_\epsilon(z) \subseteq G_{\alpha_0}$.
- This is a contradiction of our construction which ensured that I_{n_0} cannot be covered by finitely many G_α.

□

Corollary 5.37. *Every infinite subset of $[a, b] \in \mathbb{R}$ has a limit point.*

Remark. A similar statement holds for k-cells in $\mathbb{R}^n$.

$$[a_1, b_1] \times \ldots \times [a_k, b_k]$$

These are all compact in $\mathbb{R}^n$.

Proposition 5.38. *If $K \subseteq X$ is compact and $F \subseteq K$ is closed, then F is compact.*

Proof.

- Let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of F.
- Then $\{G_\alpha\}_{\alpha \in I} \cup \{F^c\}$ is an open cover of K. ($\{F^c\}$ is open because F is closed).
- Since K is compact, there exists a finite subcover $\{G_{\alpha_1}, \ldots, G_{\alpha_n}, F^c\}$ of K.
- $\implies \{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ is a finite subcover of F.
- Since $\{G_\alpha\}_{\alpha \in I}$ was arbitrary, this implies that F is compact.

□

Corollary 5.39. *If $K \subseteq X$ is compact and $F \subseteq X$ is closed then $F \cap K$ is compact.*

Proof. $F \cap K$ is closed in K and K is compact. □

Theorem 5.40. *In R^n, the following are equivalent (TFAE) for $E \subseteq \mathbb{R}^n$:*

1. *E is bounded and closed.*
2. *E is compact.*
3. *Every infinite subset of E has a limit point in E.*

Proof. We'll show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

1. **(1) $\implies$ (2).**
   - Assume E is bounded and closed. Then there exists a cell $C = [a_1, b_1] \times \ldots \times [a_n, b_n]$ such that $E \subseteq C$. (E.g. $R^n = \cup_{k \in \mathbb{N}}[-k, k] \times \ldots \times [-k, k]$).
   - $\implies E \subseteq C$ is a closed subset of a compact set, so E is compact.
2. **(2) $\implies$ (3).** Shown in previous proposition.
3. **(3) $\implies$ (1).** Prove by contrapositive.
   - Assume E is unbounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in E$ such that $d(x_n, x_0) \geq n$ for some $x_0 \in R^n$ (e.g. the origin).
   - Consider $J_1 = \{x_n\}_{n \in \mathbb{N}}$. Then J_1 is infinite (verify). If J_1 were finite, then it would have been bounded.
   - J_1 is without a limit point (verify). If it had a limit point, it would see infinite amount of points in its neighborhood, which is impossible.
   - Assume E is not closed. Then $\exists q \in X \setminus E$ such that q is a limit point of E.
   - Since q is a limit point of E, $\forall n \in \mathbb{N}$, $\exists y_n \in E$ such that $y_n \in N_{1/n}(q)$
   - Consider $J_2 = \{y_n\}_{n \in \mathbb{N}}$. Then J_2 is infinite (verify).
   - We claim that the only limit point of J_2 is q. Hence, J_2 is without a limit point in E.
   - Assume q' is a limit point of J_2. For any $n \in \mathbb{N}$, there exists infinitely many elements of J_2 in $N_{1/n}(q')$.
   - In particular, $\exists k > n$ such that $y_k \in N_{1/n}(q')$. $\implies$

$$\begin{aligned} d(q, q') &\leq d(q, y_k) + d(y_k, q') \\ &< \frac{1}{k} + \frac{1}{n} \leq \frac{2}{n} \end{aligned}$$

- Since this holds for all $n \in \mathbb{N} \implies d(q, q') = 0 \implies q' = q$.

$\square$

Remark. The equivalence of (2) $\iff$ (3) holds in any metric space. So does (2) $\implies$ (1). The structure of $\mathbb{R}^n$ is used to show (1) $\implies$ (2). (1) $\implies$ (2) does not hold in general, e.g. if X is infinite with the discrete metric. (Every subset of X are closed and bounded, but only finite sets are compact)

Example 5.41. Middle-$\frac{1}{3}$ Cantor set.

- $C_0 = [0, 1]$.
- $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.
- $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
- This process continues.
- The middle-$\frac{1}{3}$ Cantor set is defined as $C_{\frac{1}{3}} = \bigcap_{n \in \mathbb{N}} C_n$.
- For each n: C_n is the union of 2^n closed intervals of length 3^{-n}. C_{n+1} is given by removing the middle $\frac{1}{3}$ of each of these intervals in C_n.

Facts:

1. $C_{\frac{1}{3}} \neq \emptyset$ and $C_{\frac{1}{3}}$ is compact.
2. $C_{\frac{1}{3}}$ is uncountable.
3. The "length" of $C_{\frac{1}{3}}$ is 0.
4. Every point in $C_{\frac{1}{3}}$ is a limit point. (No isolated points).
5. There are no interior points in $C_{\frac{1}{3}}$.

6 Sequences

6.1 Convergence

Definition 6.1. (Sequence). A sequence $(p_n)_{n=1}^{\infty}$ in X is a function $p\colon \mathbb{N} \to X$.

Remark. We're allowed repetitions in sequences. Order matters.

Example 6.2. in $\mathbb{R}$:

1. $a_n \equiv 0\ \forall n \in \mathbb{N}$.
2. $b_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n^2} & \text{if } n \text{ is even} \end{cases}$.
3. q_n some enumeration of $\mathbb{Q}$. ($q : \mathbb{N} \to \mathbb{Q}$ is a bijection).

Definition 6.3. (Convergence). A sequence $(p_n)_n$ in X is said to converge to $q \in X$, denoted $p_n \to q$ or $\lim_{n\to\infty} p_n = q$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, p_n \in N_\epsilon(q)$$

Example 6.4.

- $a_n \equiv 0$ converges to 0.
- $b_n = \frac{1}{n^2}$ converges to 0. $\forall \epsilon > 0$, take $N > \frac{1}{\sqrt{\epsilon}} \implies \forall n \geq N$, $|b_n| = \frac{1}{n^2} < \frac{1}{N} < \epsilon$. $\implies b_n \in N_\epsilon(0)$.

Definition 6.5. (Convergent sequence). A sequence $(p_n)_n$ in X is called convergent if $\exists q \in X$ such that $p_n \to q$. Otherwise, $(p_n)_n$ is called divergent.

Example 6.6. Divergent sequences:

- $a_n = n$ in $\mathbb{R}$ is divergent.
- $b_n = \frac{1}{n}$ in $(0, 1]$ is divergent.
- $c_n = (-1)^n = \{-1, 1, -1, \ldots\}$ in $\mathbb{R}$ is divergent.
- $q_n =$ some enumeration of $\mathbb{Q}$ in $\mathbb{R}$ is divergent.

Proposition 6.7. *(Uniqueness of limit). If $p_n \to q$ and $p_n \to q'$, then $q = q'$.*

Proof.

- Let $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $p_n \in N_\epsilon(q)$.
- Also, there exists $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, $p_n \in N_\epsilon(q')$.
- So, for all $n \geq \max\{N_1, N_2\}$, $p_n \in N_\epsilon(q) \cap N_\epsilon(q')$.
- This implies $d(q, q') \leq d(q, p_n) + d(p_n, q') < 2\epsilon$.
- But, $\epsilon > 0$ was arbitrary. So, $d(q, q') = 0 \implies q = q'$.

$\square$

Definition 6.8. (Subsequence). A subsequence $(p_{n_k})_k$ of a sequence $(p_n)_n$ in X is given by a function $\mathbb{N} \to \mathbb{N}$ sending $k \mapsto n_k$ where $n_{k+1} > n_k$ for all $k \in \mathbb{N}$. That is, $(p_{n_k})_k$ is the function $k \mapsto p_{n_k}$.

Remark. The idea is that $(p_1, p_2, p_3, \ldots)$ is a sequence and $(p_{n_1}, p_{n_2}, p_{n_3}, \ldots)$ is a subsequence, where we never pick the same element twice and we never backtrack.

Example 6.9. Let $b_n = \frac{1}{n}$ in $\mathbb{R}$ be our sequence

- $a_k = b_{k^2}$ is a subsequence where $n_k = k^2$. Explicitly, $a_k = \frac{1}{k^2}$.

Proposition 6.10. *(Limits are hereditary). If $p_n \to q$, then any subsequence $(p_{n_k})_k$ of $(p_n)_n$ also converges to q.*

Proof.

- Notice that for all $k \in \mathbb{N}$, $n_k \geq k$.
- Let $\epsilon > 0$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $p_n \in N_\epsilon(q)$.
- In particular for all $k \geq N$, $n_k \geq k \geq N \implies p_{n_k} \in N_\epsilon(q)$.
- Hence, $p_{n_k} \to q$ as $k \to \infty$ by definition.

□

Example 6.11. $c_n = (-1)^n$ in $\mathbb{R}$ is divergent. Because otherwise, there exists $x \in \mathbb{R}$ such that $a_k = c_{2k} = 1 \to x$ and $b_k = c_{2k+1} = -1 \to x$. This implies $1 = x = -1$, which is a contradiction.

Proposition 6.12. *If $K \subseteq X$ is compact and $(p_n)_n$ is a sequence in K, then $(p_n)_n$ has a convergent subsequence with a limit in K.*

Proof.

- Denote $E = \{p_n \in K : n \in \mathbb{N}\}$.
- If E is finite, $E = \{p_1, \dots, p_m\}$, then $\exists 1 \leq i < m$ such that $p_{n_k} = p_i$ for some sequence $n_k \in \mathbb{N}$ with $n_{k+1} > n_k$.
- In particular, $p_{n_k} \to p_i \in K$ as $k \to \infty$.
- If E is infinite, then by a previous proposition we've shown, E has a limit point $q \in K$.
- Recall that this means that $\forall \epsilon > 0$, $N_\epsilon(q) \setminus \{q\}$ contains infinitely many elements of E.
- For all $k \in \mathbb{N}$, pick $p_{n_1} = p_1$. For each $k \in \mathbb{N}$, pick $n_{k+1} > n_k$ satisfying that $p_{n_k+1} \in N_{\frac{1}{k+1}}(q)$.
- We can always pick such an n_{k+1} because $E \cap N_{\frac{1}{k+1}}(q) \setminus \{p_i : 1 \leq i \leq n_k\}$ is infinite.
- By construction, $(p_{n_k})_k$ is a subsequence of $(p_n)_n$.
- And, $\forall \epsilon > 0$ take $N \geq 2$ such that $\frac{1}{N} < \epsilon$.
- Then $\forall k \geq N$, $p_{n_k} \in N_{\frac{1}{k}}(q) \subseteq N_{\frac{1}{N}}(q) \subseteq N_\epsilon(q)$.
- $\implies \lim_{k\to\infty} p_{n_k} = q$.

□

Corollary 6.13. *Any sequence $(a_n)_n$ in $[a, b] \subseteq \mathbb{R}$ has a convergent subsequence.*

Definition 6.14. (Bounded sequence). A sequence $(p_n)_n$ is called bounded if $E = \{p_n : n \in \mathbb{N}\} \subseteq X$ is a bounded set.

Proposition 6.15. *If $(p_n)_n$ is convergent, then it is bounded.*

Proof.

- Denote $q = \lim_{n\to\infty} p_n$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $p_n \in N_1(q)$.
- Set $M = \max\{1, d(p_1, q), \dots, d(p_{N-1}, q)\} \implies 0 < M < \infty$.
- Then $\forall n \in \mathbb{N}$, $p_n \in N_{M+\frac{1}{2}}(q)$.
- Then $(p_n)_n$ is bounded.

□

6.2 Cauchy sequences

Example 6.16. Consider the sequence:

$$a_n = \sum_{p \text{ prime, } p \leq n} 2^{-p}$$

For example, $a_{10} = 2^{-2} + 2^{-3} + 2^{-5} + 2^{-7}$. Notice that $\forall n > m$

$$a_n - a_m = \sum_{p \text{ prime, } m < p \leq n} 2^{-p} \leq \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m}$$

For example, $a_{10^9} - a_{10^6} \leq 2^{-10^6}$.

Definition 6.17. (Cauchy sequence). A sequence $(p_n)_n$ in X is called Cauchy if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N, d(p_n, p_m) < \epsilon$$

Example 6.18. $b_n = \frac{1}{n}$ in $(0, 1]$ is divergent, but it is Cauchy. $\forall \epsilon > 0$, take $N > \frac{1}{\epsilon}$. Then $\forall n, m \geq N$, $|b_n - b_m| = \frac{1}{n} + \frac{1}{m} < \frac{2}{N} < \epsilon$.

Definition 6.19. (Complete metric space). A metric space X is called complete if every Cauchy sequence in X is convergent.

Proposition 6.20. *If $(x_n)_n$ is a convergent sequence in X, then it is Cauchy.*

Proof.

- Denote $q = \lim_{n \to \infty} x_n$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, q) < \epsilon/2$.
- Hence, $\forall n, m \geq N$, $d(x_n, x_m) \leq d(x_n, q) + d(q, x_m) < \epsilon$.

$\square$

Remark. In a general metric space, there might be Cauchy sequences that are not convergent (divergent).

- In $X = \mathbb{R} \setminus \{0\}$, $a_n = \frac{1}{n}$ is Cauchy but not convergent.
- In $X = \mathbb{Q}$, take $b_1 = 1$, $b_2 = 1.4$, $b_3 = 1.41$, $b_4 = 1.414$, ..., $b_n = \frac{\lfloor 10^{n-1}\sqrt{2} \rfloor}{10^{n-1}}$. where $\lfloor t \rfloor = \max\{k \in \mathbb{Z} : k \leq t\}$. Then $(b_n)_n$ is Cauchy but not convergent.

Remark. It is not enough to have $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ for a sequence to be Cauchy.

- For example, in $\mathbb{R}$, $a_n = \sqrt{n}$ is not Cauchy.
- But $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{2\sqrt{n+1}} \to 0$ as $n \to \infty$.

Proposition 6.21. *If $(x_n)_n$ is a Cauchy sequence, then it is bounded.*

Proof.

- $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x_N) < 1$.
- Denote $M = \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} \implies \{x_n : n \in \mathbb{N}\} \subseteq N_{M+1}(x_N)$.

$\square$

Proposition 6.22. *If $(x_n)_n$ is Cauchy and there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ that converges to $q \in X$, then $(x_n)_n$ converges to q.*

Proof.

- Let $\epsilon > 0$, then $\exists N_1 \in N$ such that $\forall n, m \geq N_1$, $d(x_n, x_m) < \frac{\epsilon}{2}$. ($(x_n)_n$ is Cauchy).
- Also $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $d(x_{n_k}, q) < \frac{\epsilon}{2}$. ($(x_{n_k})_k$ converges to q).
- Pick some $l \geq K$ for which $n_l \geq N_1$. Set $N = n_l$.
- For any $n \geq N$, $d(x_n, q) \leq d(x_n, x_{n_l}) + d(x_{n_l}, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
- Because $n_l \geq N_1$ and $l \geq K$.

□

Corollary 6.23. *If $K \subseteq X$ is a compact set and $(x_n)_n$ is a Cauchy sequence in K, then $(x_n)_n$ is convergent in K.*

Proof.

- We've shown that any sequence in a compact set has a convergent subsequence.
- If $(x_n)_n \subseteq K$ is Cauchy, then by the previous proposition it is convergent.

□

Corollary 6.24. *Every Cauchy sequence in $\mathbb{R}^d$ is convergent.*

Proof.

- Let $(x_n)_n$ be a Cauchy sequence in $\mathbb{R}^d$.
- Then $(x_n)_n$ is bounded.
- Then $(x_n)_n \subseteq [-M, M]^d$ for some $M > 0$, which is compact.
- Then $(x_n)_n$ is convergent.

□

Theorem 6.25. *$\mathbb{R}^d$ is complete.*

6.3 Monotonic sequences in $\mathbb{R}$

Definition 6.26. (Monotonic sequence). A sequence $(a_n)_n \subseteq \mathbb{R}$ is called monotonically increasing if $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$. Similarly, $(a_n)_n$ is called monotonically decreasing if $\forall n \in \mathbb{N}$, $a_n \geq a_{n+1}$. In general, $(a_n)_n$ is called monotonic if it is either monotonically increasing or monotonically decreasing.

Proposition 6.27. *If $(a_n)_n$ is monotonic in $\mathbb{R}$, then it is convergent if and only if it is bounded.*

Proof.

- Let $(a_n)_n$ be monotonically increasing.
- ($\implies$) We've shown this.
- ($\impliedby$) Assume $(a_n)_n$ is bounded. $\exists \sup\{a_n : n \in \mathbb{N}\} = A \in \mathbb{R}$.
- We claim that $a_n \to A$ as $n \to \infty$.
- Indeed, $\forall \epsilon > 0$, since $A - \epsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, $\exists N \in \mathbb{N}$ such that $a_N > A - \epsilon$.
- Hence, $\forall n \geq N$, $A - \epsilon < a_N \leq a_n \leq A$. In particular $|a_n - A| < \epsilon$.
- Similarly, for monotonically decreasing sequences (with infimum).

□

7 The extended real line

Definition 7.1. (Extended real line). The extended real line, denoted $\mathbb{R}_{ext}$ or $[-\infty, \infty]$, is the ordered set $\mathbb{R} \cup \{-\infty, \infty\}$ with the natural order for any $x, y \in \mathbb{R}$ and with $\forall x \in \mathbb{R}$, $-\infty < x < \infty$.

Remark. Warning!

1. The metric on $\mathbb{R}$ does not extend to a metric on $[-\infty, \infty]$. We are *not* thinking of $[-\infty, \infty]$ as a metric space.
2. $[-\infty, \infty]$ is *not* a field. Be careful with arithmetic in $[-\infty, \infty]$. Some operations are natural
   - $\forall x \in \mathbb{R}$, $x + \infty = \infty$
   - $x + (-\infty) = -\infty$.
   - $-\infty + \infty$ is undefined.

Proposition 7.2. *Every set in $[-\infty, \infty]$ has a supremum and an infimum.*

Proof.

- Let $E \subseteq [-\infty, \infty]$.
- If $\exists M \in \mathbb{R}$ such that $e \leq M$ for all $e \in E$, then E is bounded above in $\mathbb{R}$, $\implies$ it has a supremum in $\mathbb{R}$.
- Otherwise ∞ is the only upper bound for E in $[-\infty, \infty]$, $\implies \sup E = \infty$.
- Similarly, for the infimum.

□

Definition 7.3. (Convergence to ∞). A sequence $(a_n)_n$ in $\mathbb{R}$ is said to converge to ∞ (in the extended sense), if $\forall M > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n \geq M$. Similarly, $(a_n)_n$ is said to converge to $-\infty$ if $\forall M > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n \leq -M$.

Proposition 7.4. *Any sequence $(a_n)_n$ in $\mathbb{R}$ has a convergent subsequence in the extended sense, i.e. converging to a limit in $[-\infty, \infty]$, i.e. a subsequence converging to a limit in $[-\infty, \infty]$.*

Proposition 7.5. *Every sequence $(x_n)_n$ in $\mathbb{R}$ has a convergent subsequence in the extended sense (i.e. converging to a limit in $[-\infty, \infty]$).*

Proof.

- Let $(x_n)_n$ be any sequence in $\mathbb{R}$. If $(x_n)_n$ is unbounded, then $\forall M > 0$, $\exists n \in \mathbb{N}$ such that $x_n \notin [-M, M]$.
- In particular, this would imply that $\forall M > 0$, $\exists$ infinitely many elements of $(x_n)_n$ in $[-M, M]^c = (-\infty, M) \cup (M, \infty)$
- If $\forall M > 0$, $\exists$ infinitely many x_n's in $(M, \infty) \implies \exists$ subsequence $(x_{n_k})_k$ converging to ∞.
- Otherwise $\exists$ a subsequence converging to $-\infty$.
- If $(x_n)_n$ is bounded, then $\exists M > 0$ such that $(x_n)_n \subseteq [-M, M]$
- Since $[-M, M]$ is compact, we are ensured that there exists a convergent subsequence.

□

Definition 7.6. (Accumulation points). Given $(x_n)_n \subseteq \mathbb{R}$, we define the set of accumulation points of $(x_n)_n$ in $[-\infty, \infty]$ as:

$$\text{Accum}(x_n)_n := \{z \in [-\infty, \infty] : \exists \text{ a subsequence } (x_{n_k})_k \text{ of } (x_n)_n \text{ s.t. } x_{n_k} \to z \text{ as } k \to \infty\}$$

Note that the previous proposition implies that $\text{Accum}(x_n)_n \neq \emptyset$.

Definition 7.7. (Limit superior and limit inferior). The limit superior and limit inferior of $(x_n)_n$ are defined as:

$$\limsup_{n\to\infty} x_n := \sup \mathrm{Accum}(x_n)_n$$
$$\liminf_{n\to\infty} x_n := \inf \mathrm{Accum}(x_n)_n$$

Proposition 7.8. *$Accum(x_n)_n \cap \mathbb{R}$ is closed. Moreover, if $Accum(x_n)_n$ is not bounded above (resp. below) then $\infty \in Accum(x_n)_n$ (resp. $-\infty \in Accum(x_n)_n$).*

Proof.

- Let $z_0 \in \mathbb{R}$ be a limit point of $\mathrm{Accum}(x_n)_n \cap \mathbb{R}$.
- $\implies \exists$ a sequence $(z_l)_l$ in $\mathrm{Accum}(x_n)_n \cap \mathbb{R}$ such that $z_l \to z_0$ as $l \to \infty$.
- Since $\forall l \in \mathbb{N}$, $z_l \in \mathrm{Accum}(x_n)_n$, $\exists$ a subsequence $(x_{n_{l,k}})_k$ of $(x_n)_n$ such that $x_{n_{l,k}} \to z_l$ as $k \to \infty$.
- Pick $x_{n_{0,1}} = x_{n_{1,1}}$. (arbitrarily set $n_{0,1} = n_{1,1}$).
- For every $m \geq 2$, $\exists N_1 \in \mathbb{N}$ such that $\forall l \geq N_1$, $|z_l - z_0| < \frac{1}{2m}$
- Pick such an $l \geq N_1$. Since $x_{n_{l,k}} \to z_l$ as $k \to \infty$, $\exists N_2 \in \mathbb{N}$ such that $\forall k \geq N_2$, $|x_{n_{l,k}} - z_l| < \frac{1}{2m}$.
- Pick $k \geq N_2$ for which $n_{l,k} \geq n_{0,n-1}$ ($\leftarrow$ already constructed).
- Set $n_{0,m} = n_{l,k}$.
- Hence, $|x_{n_{0,m}} - z_0| = |x_{n_{l,k}} - z_0| \leq |x_{n_{l,k}} - z_l| + |z_l - z_0| < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}$.
- Hence, $(x_{n_{0,m}})_m$ is a subsequence of $(x_n)_n$ which satisfies $\forall m \geq 2$, $|x_{n_{0,m}} - z_0| < \frac{1}{m}$.
- $\implies x_{n_{0,m}} \to z_0$ as $m \to \infty \implies z_0 \in \mathrm{Accum}(x_n)_n \implies \mathrm{Accum}(x_n)_n \cap \mathbb{R}$ is closed.

A similar argument works for $z_l \to \infty$ or $z_l \to -\infty$, which would prove the moreover. ☐

Corollary 7.9. $\limsup_{n\to\infty} x_n \in$ *Accum*$(x_n)_n$ *and* $\liminf_{n\to\infty} x_n \in$ *Accum*$(x_n)_n$*. I.e. there exists a subsequence realizing the limit superior and limit inferior.*

Proposition 7.10. *If $\alpha \in \mathbb{R}$ satisfies that $\limsup_{n\to\infty} x_n < \alpha$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n < \alpha$. Similarly, if $\beta \in \mathbb{R}$ satisfies that $\liminf_{n\to\infty} x_n > \beta$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n > \beta$.*

Remark. The strict inequality is crucial since e.g. $a_n = 1/n$, then since $a_n \to 0$ so do all its subsequences. This implies that $\limsup_{n\to\infty} a_n = 0$. Hence, $\limsup_{n\to\infty} a_n \leq 0$ but $a_n \not\leq 0$ for all $n \in \mathbb{N}$.

Proof.

- Assume $\limsup_{n\to\infty} x_n < \alpha$. Assume in contradiction that $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $x_n \geq \alpha$.
- Note that $(x_n)_n$ is bounded above because otherwise there exists a subsequence $x_{n_k} \to \infty$ as $k \to \infty \implies \limsup_{n\to\infty} x_n = \infty$, a contradiction.
- Hence, $\exists M > 0$ s.t. $x_n \leq M$ for all $n \in \mathbb{N}$.
- Both conditions imply that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ contained in $[\alpha, M]$.
- Since $[\alpha, M]$ is compact, $\implies$ there exists a "sub-subsequence" $(x_{n_{k,l}})_l$ of $(x_{n_k})_k$ converging to a limit in $[\alpha, M]$.
- But all "sub-subsequences" are subsequences. In particular, $(x_{n_{k,l}})_l$ is a subsequence of $(x_n)_n$.
- This contradicts the fact that $\sup \mathrm{Accum}(x_n)_n = \limsup_{n\to\infty} x_n < \alpha$.

A similar proof holds for the $\liminf_{n\to\infty} x_n > \beta$ case. ☐

Corollary 7.11. *A sequence $(x_n)_n$ in $\mathbb{R}$ converges to a limit $L \in \mathbb{R}$ if and only if* $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L$. *Equivalently, Accum$(x_n)_n = \{L\}$.*

Proof. ($\Longrightarrow$) Hereditary property of limits.

($\Longleftarrow$) Assume $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L$.

- $\forall \epsilon > 0$ since $\limsup_{n\to\infty} x_n < L + \epsilon$ and $\liminf_{n\to\infty} x_n > L - \epsilon$,
- Then $\exists N_1$ such that $\forall n \geq N_1$, $x_n < L + \epsilon$ and $\exists N_2$ such that $\forall n \geq N_2$, $x_n > L - \epsilon$.
- By taking $N = \max\{N_1, N_2\}$, we have that $\forall n \geq N$, $|x_n - L| < \epsilon$.

□

Proposition 7.12. *If $(a_n)_n$ and $(b_n)_n$ are sequences in $\mathbb{R}$, and if $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n \leq b_n$, then* $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$ *and* $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Proof.

- Assume in contradiction that $\liminf_{n\to\infty} a_n > \beta > \liminf_{n\to\infty} b_n$.
- By the previous proposition, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > \beta$.
- OTOH, $\exists$ a subsequences $(b_{n_k})_k$ of $(b_n)_n$ converging to $\liminf_{n\to\infty} b_n$.
- $\Longrightarrow$ $\exists N_2 \in \mathbb{N}$ such that $\forall k \geq N_2$, $b_{n_k} < \beta$.
- $\Longrightarrow$ $\exists k \geq \max\{N, N_2\}$ such that $b_{n_k} < \beta < a_{n_k}$, a contradiction.

A similar proof holds for the $\limsup$. □

Corollary 7.13. *(Squeeze theorem/sandwich theorem). If $(a_n)_n$, $(b_n)_n$, and $(c_n)_n$ are sequences in $\mathbb{R}$ such that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n \leq c_n \leq b_n$, and if $a_n \to L \in \mathbb{R}$ and $b_n \to L$, then $c_n \to L$ as $n \to \infty$.*

Proof.

- $L = \liminf a_n \leq \liminf x_n \leq \limsup x_n \leq \limsup b_n = L$.
- This implies that $\liminf x_n = \limsup x_n = L$.
- Then $x_n \to L$ as $n \to \infty$.

□

Proposition 7.14. *If $x_n \to x$ and $y_n \to y$, then*

1. $x_n + y_n \to x + y$.
2. $\forall c \in \mathbb{R}$, $c \cdot x_n \to \lambda x$.
3. $x_n \cdot y_n \to x \cdot y$.
4. *For* $x \neq 0$, $\frac{y_n}{x_n} \to \frac{y}{x}$.

Proof. Proof for the first point.

- $x_n \to x$ means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n - x| < \epsilon$.
- $y_n \to y$ means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|y_n - y| < \epsilon$.
- We want to show that $|(x_n + y_n) - (x + y)| < \epsilon$.
- By triangle inequality, $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y|$.
- Take N_1 such that $\forall n \geq N_1$, $|x_n - x| < \frac{\epsilon}{2}$. Take N_2 such that $\forall n \geq N_2$, $|y_n - y| < \frac{\epsilon}{2}$.

- Take $N = \max\{N_1, N_2\}$.
- Then $\forall n \geq N$, $|(x_n + y_n) - (x + y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Proof for the third point.

$$\begin{aligned}
|x_n \cdot y_n - x \cdot y| &= |x_n \cdot y_n - x \cdot y_n + x \cdot y_n - x \cdot y| \\
&\leq |x_n y_n - x y_n| + |x y_n - x y| \\
&= |x_n - x||y_n| + |x||y_n - y| \\
&\leq \frac{\epsilon}{2M} M + M \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

- Take N_1 such that $\forall n \geq N_1$, $|x_n - x| < \frac{\epsilon}{2M}$.
- Take N_2 such that $\forall n \geq N_2$, $|y_n - y| < M$.
- For n large enough, the above inequality holds.

Remark that the fourth point.

- We can't divide by x_n if $x_n = 0$.
- However, because $x \neq 0$ we know that $x_n \neq 0$ for n large enough.

Notice that the second point follows from the third point. □

Proposition 7.15.

1. *$p > 0$, then $\frac{1}{n^p} \to 0$.*
2. *$x > 0$, then $x^{1/n} \to 1$.*
3. *$|x| < 1$, then $x^n \to 0$.*
4. *$n^{1/n} \to 1$.*

Proof. Proof for the second point.

- Let $x > 1$. Then $y_n = x^{1/n} - 1 > 0$.
- It suffices to show that $y_n \to 0$. This is because $1 \to 1$ and if $y_n \to 0$, then $x^{1/n} \to 1$.
- $$\begin{aligned}
y_n &= x^{1/n} - 1 \\
\implies 1 + y_n &= x^{1/n} \\
\implies (1 + y_n)^n &= x \\
\implies 1 + n y_n + \binom{n}{2} y_n^2 + \ldots + y_n^n &= x \\
\implies 1 + n y_n &\leq x \\
n y_n &\leq x - 1 \\
0 \leq y_n &\leq \frac{x - 1}{n} \to 0 \\
\implies y_n &\to 0
\end{aligned}$$
- Let $0 < x < 1$. Then $1/x > 1$.
- Then $1/x^{1/n} = \left(\frac{1}{x}\right)^{1/n} \to 1$.
- $\frac{1}{1/x^{1/n}} = x^{1/n} \to 1$.

□

8 Continuity

Remark. The main idea of continuity is that small changes in the input should result in small changes in the output.

Definition 8.1. (Limit of a function at a point). Let $f : X \to Y$ be a function and $P \in X$ be a limit point. Let $q \in Y$. We say $\lim_{x \to P} f(x) = q$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$0 < |x - P| < \delta \implies |f(x) - q| < \epsilon$$

Example 8.2. $f(x) = 6x$ and $f : \mathbb{R} \to \mathbb{R}$. We claim that $\lim_{x \to p} f(x) = 6p$.

- Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{6}$.
- Assume $0 < |x - p| < \delta = \frac{\epsilon}{6}$.
- Then $|6x - 6p| = \epsilon$.
- Then $|f(x) - 6p| < \epsilon$.

Example 8.3. $f(x) = x^2$ and $f : \mathbb{R} \to \mathbb{R}$. We claim that $\lim_{x \to p} f(x) = p^2$.

- We want to show that $|f(x) - p^2| < \epsilon$ for some $\delta > 0$.
- We know $|x^2 - p^2| = |x - p||x + p|$.
- Let $\delta = \min\{1, \frac{\epsilon}{r}$ where $r = \max |2p + 1|, |2p - 1|\}$.
- If $0 < |x - p| < \delta$, then $|f(x) - p^2| = |x - p||x + p| \le |x - p|r \le \frac{\epsilon}{r} \cdot r = \epsilon$.

Example 8.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

We claim that $\lim_{x \to 0} f(x) = 0$. (Any $\delta > 0$ works).
The limit behavior is what is around the point, not at the point.

Example 8.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$.

We claim that $\lim_{x \to 0} f(x)$ does not exist.

- Take $\epsilon < 1/2$. For any $\delta > 0$
$$0 < |x| < \delta \not\Longrightarrow |f(x) - q| < \epsilon$$

Proposition 8.6. *Let $f : X \to Y$ be a function and $P \in X$ be a limit point. Let $q \in Y$. Then* $\lim_{x \to P} f(x) = q$ *if and only if* $\forall P_n \to P$, $P_n \neq P \implies f(P_n) \to q$.

Proof. ($\implies$) Assume $\lim_{x \to P} f(x) = q$.

- Also assume $P_n \to P$ and $P_n \neq P$. We want to show that $f(P_n) \to q$.
- by definition of limit function, let $\epsilon > 0$. Take $\delta > 0$ such that $0 < |x - P| < \delta \implies |f(x) - q| < \epsilon$.
- By definition of convergence, For $\delta > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \ge N$, $|P_n - P| < \delta$.
- Then $|f(P_n) - q| < \epsilon$.

□

Proof.

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□