Problem Set 5

Math 255: Analysis I

Due: Thursday, Feb 22nd at 11:59pm EST

Problem 1. Show that the following subset of $\mathbb{R}^2$ is open[1]:

$$E = \{(x, y) \in \mathbb{R}^2 \mid x < y\}.$$

Proof. We will prove that for every point in E, there exists an ϵ-neighborhood of that point contained within E. An ϵ-neighborhood of a point $p = (a, b) \in E$ is the set

$$N_\epsilon(p) = \{q \in R^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}$$

We will find an ϵ such that for all $(x, y) \in N_\epsilon(p)$, $x < y$. Let $\epsilon = \frac{b-a}{3} > 0$, because $b > a$. Then, every point $(x, y) \in N_\epsilon(p)$ satisfies

$$\sqrt{(x-a)^2 + (y-b)^2} < \frac{b-a}{3}.$$

Notice, that this expression implies the following two inequalities:

$$|x - a| < \frac{b-a}{3}$$
$$|y - b| < \frac{b-a}{3}$$

This in turn implies:

$$a - \frac{b-a}{3} < x < a + \frac{b-a}{3}$$
$$b - \frac{b-a}{3} < y < b + \frac{b-a}{3}$$

To prove that $x < y$, we will take the highest possible value of x and the lowest possible value of y and show that $y - x > 0$. Let $x = a + \frac{b-a}{3}$ and $y = b - \frac{b-a}{3}$. Then,

$$\begin{aligned} y - x &= b - \frac{b-a}{3} - (a + \frac{b-a}{3}) \\ &= b - a - \frac{b-a}{3} - \frac{b-a}{3} \\ &= \frac{b-a}{3} > 0. && \text{(since } b > a\text{)} \end{aligned}$$

Hence, $x < y$ for every point $(x, y) \in N_\epsilon(p)$, implying $N_\epsilon(p) \subseteq E$. This applies to every point $p \in E$. Therefore, every point in E is an interior point, proving E is open in $\mathbb{R}^2$. ☐

[1] Whenever not explicitly stated otherwise, $\mathbb{R}^n$ is considered with the "natural" Euclidean metric $d(v, w) = [\sum_{i=1}^{n}(v_i - w_i)^2]^{\frac{1}{2}}$.

Problem 2. For each of the following subsets of $\mathbb{R}$ describe all of their interior points and all of their limit points. Determine whether these sets are open, closed or neither. Justify your answers.

1. $A = \mathbb{N} \cup \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$

   Proof. First, let's enumerate some elements of A. Notice that A doesn't contain any irrational numbers.

   $$A = \{1, 2, 2 + \frac{1}{2}, 3, 3 + \frac{1}{3}, 4, 4 + \frac{1}{4}, \dots\}$$

   Interior Points: A has no interior points. We will show that $\forall a \in A$, $\forall \epsilon > 0$, $\exists x \in N_\epsilon(a)$, such that $x \notin A$(implying that $N_\epsilon(c) \nsubseteq C$). After choosing any ϵ for any a, we examine the case where ϵ is rational and the case where ϵ is irrational. In both cases, we will find an $x \in N_\epsilon(a)$, but $x \notin A$.

   (a) ϵ is rational: Let $x = a + \frac{\sqrt{2}}{2} \cdot \epsilon$
   (b) ϵ is irrational: Let $x = a + \frac{1}{2} \cdot \epsilon$

   We can see that $d(a, x) < \epsilon$, so $x \in N_\epsilon(a)$, but x is irrational, so $x \notin A$. Therefore, A has no interior points.

   Limit Points: A has no limit points, that is, A is composed entirely of isolated points. We will show $\forall a \in A$, $\exists \epsilon > 0$, such that $N_\epsilon(a) \cap A = \{a\}$.

   Let $a \in A$. Either $a = n \in \mathbb{N}$ or $a = n + \frac{1}{n}$ for some $n \in \mathbb{N}$. Notice that the closest point to a in A which is not a is always a distance $\frac{1}{n}$ away. Let's take $\epsilon = \frac{1}{2n}$. Therefore, the ϵ-neighborhood of point a will see no points other points in A, showing that a is an isolated point with $\epsilon = \frac{1}{2n}$.

   Open/Closed: A is not open because A has no interior points. A is closed as it contains all its limits points, which is none. ☐

2. $B = [-1, 1] \setminus \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$

   Proof. **Interior Points:** The set of interior points of B is $(-1, 1) \setminus \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$. B is a closed interval with "holes" at $\left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$. Therefore, the interior points of B are all the points in $(-1, 1)$ that are not in $\left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$.

   Limit Points: The set of limit points of B is $[-1, 1]$. It is trivial that all points in B are limit points. To see why all points of $\left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$ are limit points, any point p in this set exists in the interval $(-1, 1)$. Therefore, any ϵ-neighborhood around p will contain (infinitely many) elements of $(-1, 1) \subseteq B$.

   Open/Closed: B is not open because -1 and 1 are not interior points but are in B. B is not closed because $\left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$ are limit points of B not contained in B.

   ☐

3. $C = \left\{\frac{1}{q^2} : q \in \mathbb{Q}, q \neq 0\right\}$

   Proof. First, let's rewrite the set C. Notice that C doesn't contain any irrational numbers.

   $$C = \left\{\frac{1}{\left(\frac{p}{q}\right)^2} : p, q \in \mathbb{Z}, p, q \neq 0\right\} = \left\{\frac{q^2}{p^2} : p, q \in \mathbb{Z}, p, q \neq 0\right\}$$

   Interior Points: C has no interior points. We will show that $\forall c \in C$, $\forall \epsilon > 0$, $\exists x \in N_\epsilon(c)$, such that $x \notin C$ (implying that $N_\epsilon(c) \nsubseteq C$). After choosing any ϵ for any c, we examine the case where ϵ is rational and the case where ϵ is irrational. In both cases, we will find an $x \in N_\epsilon(c)$, but $x \notin C$.

   (a) ϵ is rational: Let $x = c + \frac{\sqrt{2}}{2} \cdot \epsilon$

(b) ϵ is irrational: Let $x = c + \frac{1}{2} \cdot \epsilon$

We can see that $d(c, x) < \epsilon$, so $x \in N_\epsilon(c)$, but x is irrational, so $x \notin C$. Therefore, C has no interior points.

Limit Points: The set of limit points of C are the positive real numbers and 0. This is similar to how the set of limit points of $\mathbb{Q}$ is $\mathbb{R}$. However C is contained to positive squared rationals. Still, we can get arbitrarily close to any positive point in $\mathbb{R}$. For any positive real number or 0, you can find fractions $\frac{q^2}{p^2}$ approaches that real number as closely as desired.

Open/Closed: C is not open because C has no interior points. C is not closed because it does not contain all its limit points, e.g. 0. $\square$

Problem 3. Let A_1, A_2, ... be subsets of a metric space X.

1. If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^n \overline{A}_i$, for any $n \in \mathbb{N}$.

   Proof. Remember that the closure of a set A is defined as the set of all points in X that are either in A or are limit points of A.

   First, we show $\bigcup_{i=1}^n \overline{A}_i \subseteq \overline{B}_n$.

   Let $x \in \bigcup_{i=1}^n \overline{A}_i$. This means that there exists some $j \in 1, 2, \ldots, n$ such that $x \in \overline{A}_j$. Since $x \in \overline{A}_j$, x is either in A_j or is a limit point of A_j.

   (a) If x is in A_j, then clearly x is in $B_n = \bigcup_{i=1}^n A_i$, and thus in $\overline{B}_n$.

   (b) If x is a limit point of A_j, then every neighborhood of x contains at least one point of A_j different from x. Since $A_j \subseteq B_n$, every neighborhood of x also contains at least one point of B_n different from x, making x a limit point of B_n. Thus, $x \in \overline{B}_n$.

   This shows that every element of $\bigcup_{i=1}^n \overline{A}_i$ is also an element of $\overline{B}_n$.

   Second, we show $\overline{B}_n \subseteq \bigcup_{i=1}^n \overline{A}_i$

   Let $x \in \overline{B}_n$. This means x is either in B_n or is a limit point of B_n.

   (a) If x is in B_n, then x is in at least one A_i for some $i \in 1, 2, \ldots, n$, and hence x is in at least one $\overline{A}_i$ since $A_i \subseteq \overline{A}_i$.

   (b) If x is a limit point of B_n, then every neighborhood of x contains at least one point of B_n different from x. Since $B_n = \bigcup_{i=1}^n A_i$, this point must be in at least one A_i. Therefore, x must also be a limit point at least this particular A_i, and thus $x \in \overline{A}_i$ for some i.

   This shows that every element of $\overline{B}_n$ is also an element of $\bigcup_{i=1}^n \overline{A}_i$. □

2. If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A}_i$. Show, by an example, that this inclusion can be proper, i.e. it may happen that $\overline{B} \neq \bigcup_{i=1}^\infty \overline{A}_i$.

   Proof. Let $x \in \bigcup_{i=1}^\infty \overline{A}_i$. This means that $x \in \overline{A}_j$ for some j. This implies that x is either in A_j or is a limit point of A_j.

   (a) If x is in A_j, then clearly x is in $B = \bigcup_{i=1}^\infty A_i$, and thus in $\overline{B}$.

   (b) If x is a limit point of A_j, then every neighborhood of x contains at least one point of A_j different from x. Since $A_j \subseteq B$, every neighborhood of x also contains at least one point of B different from x, making x a limit point of B. Thus, $x \in \overline{B}$.

   This shows that every element of $\bigcup_{i=1}^\infty \overline{A}_i$ is also an element of $\overline{B}$.

   Proper Inclusion Example:

   Consider the metric space $X = \mathbb{R}$ with the usual metric, and let $A_n = \left\{\frac{1}{n}\right\}$ for each $n \in \mathbb{N}$. Thus, we have $A_1 = 1, A_2 = \left\{\frac{1}{2}\right\}$, and so on. Then,

   $$B = \bigcup_{i=1}^{\infty} A_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$$

   . The closure of each A_n is $\overline{A}_n = A_n$ since each A_n contains a single point and is therefore closed. So,

   $$\bigcup_{i=1}^{\infty} \overline{A}_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$$

   However, $\overline{B} = B \cup 0$, because 0 is a limit point of B and it is not contained in any A_i. Thus, $\overline{B}$ is strictly larger than $\bigcup_{i=1}^\infty \overline{A}_i$, demonstrating that the inclusion can be proper.

   □

Problem 4. Given two sets A and B in $\mathbb{R}$, consider the set

$$A - B := \{a - b : a \in A,\ b \in B\}.^2$$

1. Prove that if A is open then so is $A - B$, for any B.

   Proof. Let z be a point in $A - B$. By definition, there exist elements $a \in A$ and $b \in B$ such that $z = a - b$.

   Since A is open and $a \in A$, a is an interior point of A. Therefore, there exists an $\epsilon > 0$ such that the r-neighborhood $N_\epsilon(a) \subseteq A$.

   We aim to show there exists a $\delta > 0$ such that $N_\delta(z) \subseteq A - B$. Consider $\delta = \epsilon$ and let $z' \in N_\delta(z)$. This means $|z' - z| < \delta$.

   For $z' = a' - b$, we want to find an a' such that $a' \in A$ (which implies $z' \in A - B$). Since $z' - z = (a' - b) - (a - b) = a' - a$, we have $|a' - a| < \delta = \epsilon$. This implies that $a' \in N_\epsilon(a)$. A is open, meaning $N_\epsilon(a) \subseteq A$, so we have $a' \in A$. This means for every $z' \in N_\delta(z)$, we can find such an a' in A. Therefore, $z' = a' - b \in A - B$, showing that $N_\delta(z) \subseteq A - B$.

   Since we have found a δ-neighborhood of z that is entirely contained within $A - B$, z is an interior point of $A - B$. Because our choice of z in $A - B$, every point in $A - B$ is an interior point of $A - B$. Therefore, $A - B$ is open. ☐

2. Give an example of a closed set A and a set B for which $A - B$ is not closed.

   Proof. Let $A = \{0\}$ and $B = (0, 1)$. Then

   $$A - B = \{0 - b \mid b \in B\} = \{-b \mid b \in (0, 1)\} = (-1, 0)$$

   Since $(-1, 0)$ is not closed, we have found an example of a closed set A and a set B for which $A - B$ is not closed. ☐

3. Prove that if the set A has a limit point then 0 is a limit point of $A - A$.

   Proof. Assume A has a limit point, say c. By definition, for every $\epsilon > 0$, there exists a point $a \in A$ such that $a \neq c$ and $|a - c| < \epsilon$. This means that within any arbitrarily small distance around c, we can find a distinct point a from A.

   To show 0 is a limit point of $A - A$, we must show that for every $\epsilon > 0$, there exists a $y \in A - A$ such that $y \neq 0$ and $|y - 0| < \epsilon$.

   Since c is a limit point of A, for any given $\epsilon > 0$, there exists $a_1, a_2 \in A$ such that $|a_1 - c| < \frac{\epsilon}{2}$ and $|a_2 - c| < \frac{\epsilon}{2}$, with $a_1 \neq a_2$ (c being a limit point guarantees the presence of infinitely many points of A in every neighborhood of c).

   Consider $y = a_1 - a_2$, which is an element of $A - A$. We want to show that $|y| = |a_1 - a_2| < \epsilon$. Since both a_1 and a_2 are within $\frac{\epsilon}{2}$ of c, we can use the triangle inequality to deduce that:

   $$|y| = |a_1 - a_2| = |(a_1 - c) - (a_2 - c)| \leq |a_1 - c| + |c - a_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   This shows that $y = a_1 - a_2$ is within ϵ of 0 and $y \neq 0$ since $a_1 \neq a_2$. Thus, for every $\epsilon > 0$, there exists a $y \in A - A$ such that $|y| < \epsilon$ and $y \neq 0$, proving that 0 is a limit point of $A - A$. ☐

[2] For example, if $A = \{0, 1\}$ and $B = \{1, 2\}$ then $A - B = \{-2, -1, 0\}$ and $A - A = \{-1, 0, 1\}$.

Problem 5. Provide an example of a subset E in $\mathbb{R}^2$ satisfying that

- E has no interior points;
- E is not closed; but
- $E \cup \{0\}$ is closed.

You do not need to provide a rigorous proof, but do explain your answers (an illustration may help!).

Proof. Let E be the set of points in the plane that lie on the x-axis, excluding the origin. That is,

$$E = \{(x, 0) \in \mathbb{R}^2 \mid x \neq 0\}$$

E **has no interior points:** An interior point of a set E in $\mathbb{R}^2$ requires that there exists some $\epsilon > 0$ such that an ϵ-neighborhood around that point is entirely contained within E. For any point $(x, 0) \in E$, every ϵ-neighborhood will include points that do not lie on the x-axis (since neighborhoods in $\mathbb{R}^2$ are circular), thus including points not in E. Therefore, no point of E is an interior point, as you cannot find an ϵ-neighborhood around any point in E that lies completely within E.

E **is not closed:** A set is closed if it contains all its limit points. The origin $(0, 0)$ is a limit point of E because, for any $\epsilon > 0$, the ϵ-neighborhood of $(0, 0)$ includes points of E (any point $(x, 0)$ with $0 < |x| < \epsilon$). Since $(0, 0)$ is not included in E, E does not contain all its limit points, making it not closed.

$E \cup \{0\}$ **is closed:** By adding the origin $(0, 0)$ to E, we form the set $E \cup \{0\} = (x, 0) \in \mathbb{R}^2$, which is the entire x-axis. This set includes all its limit points, including the origin. Any limit point of this set would have to lie on the x-axis, and since the set now includes every point on the x-axis, it is closed. There are no points in $\mathbb{R}^2$ that are limit points of $E \cup 0$ without being included in the set itself.

$\square$

Good luck!