Problem Set 6

Math 255: Analysis I

Due: Thursday, March 7th at 11:59pm EST

Problem 1. Let (X, d) be a metric space.

1. Prove that X is disconnected if and only if there exists a clopen (closed and open) set $\emptyset \neq Y \subsetneq X$.

   Proof. ($\Longrightarrow$) Assume X is disconnected.

   - By the proposition in class, X is disconnect if and only if $X = A \cup B$ where A and B are non-empty disjoint open sets.
   - Let A and B be the non-empty disjoint open sets that witness X being disconnected.
   - Thus, we can write $A = X \setminus B = B^c$ and $B = X \setminus A = A^c$.
   - Consider A, an open set. As its complement B is open, A is closed.
   - Therefore, A is a clopen non-empty set in X.

   ($\Longleftarrow$) Assume there exists a clopen set $\emptyset \neq Y \subsetneq X$.

   - Let $A = Y$ and $B = X \setminus Y$. We will show that $X = A \cup B$ and $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.
   - Both A and B are nonempty because $\emptyset \neq Y \subsetneq X$.
   - By definition of set minus, we immediately have $X = A \cup B$.
   - As B is the complement of Y, and Y is clopen, B is also clopen.
   - As both A and B are closed, we have $\overline{A} = A$ and $\overline{B} = B$.
   - As B is the complement of A, $A \cap B = \emptyset$.
   - This implies that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.
   - Therefore, A and B are witnesses to X being disconnected.

   Hence, we have shown that X is disconnected if and only if there exists a non-empty clopen set $Y \subsetneq X$. □

2. Prove that if $E \subseteq X$ is connected then so is $\overline{E}$.

   Proof.

   - Assume in contradiction that $\overline{E}$ is disconnected.
   - Then there exists two sets A and B such that $\overline{E} = A \cup B$ and $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.
   - Since $E \subseteq \overline{E}$, we have $E = E \cap (A \cup B)$
   - This implies $E = (E \cap A) \cup (E \cap B)$. (Distributivity of $\cap$ shown in Problem Set 1).
   - Let us see if $(E \cap A)$ and $(E \cap B)$ can be a witness to E being disconnected, which would contradict our assumption.
   - Consider $(E \cap A) \cap \overline{(E \cap B)}$.
   - Notice that $\overline{E \cap B} \subseteq \overline{B}$. We are also given $A \cap \overline{B} = \emptyset$. This implies $A \cap \overline{(E \cap B)} = \emptyset$.
   - By associativity, $(E \cap A) \cap \overline{(E \cap B)} = E \cap (A \cap \overline{(E \cap B)}) = E \cap \emptyset = \emptyset$.

- Symmetrically, we can show that $\overline{(E \cap A)} \cap (E \cap B) = \emptyset$.
- Therefore, E is disconnected, which is a contradiction.

$\square$

3. Give an example of a disconnected set $W \subseteq \mathbb{R}$ for which $\overline{W}$ is connected.

Proof.

- $\mathbb{Q} \subseteq \mathbb{R}$ is disconnected by taking $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$.
- $\overline{\mathbb{Q}} = \mathbb{R}$ by the density of $\mathbb{Q}$ in $\mathbb{R}$ and $\mathbb{R}$ is connected.

$\square$

Problem 2. Let (X, d) be a metric space.

1. Let $K_1, ..., K_N$ be a finite collection of compact sets in X. Prove that $\bigcup_{n=1}^N K_n$ is also compact.

   Proof.

   - Let $\{G_\alpha\}_\alpha$ be an open cover of $\bigcup_{n=1}^N K_n$.
   - Notice that $\{G_\alpha\}_\alpha$ is also an open cover for each K_n.
   - Since each K_n is compact, there exists a finite sub-cover $\{G_{n,1}, G_{n,2}, ..., G_{n,m_n}\}$ for each K_n, $1 \leq n \leq N$.
   - This implies that $\{G_{n,j} : 1 \leq n \leq N, 1 \leq j \leq m_n\} \subseteq \{G_\alpha\}_\alpha$ is a finite sub-cover of $\bigcup_{n=1}^N K_n$.

   □

2. Prove that any compact set $K \subseteq X$ has at most finitely many isolated points (i.e. points in K which are not limit points of K).

   Proof.

   - Assume in contradiction that K has infinitely many isolated points. We will find an open cover of K such that cannot have a finite subcover.
   - Let us call the infinite set of isolated points $I \subseteq K$.
   - Because each $p \in I$ is an isolated point, there exists an $\epsilon_p > 0$ such that $N_{\epsilon_p}(p) \cap K = \{p\}$.
   - Let $\{N_{\epsilon_p}(p)\}_{p \in I}$ be the collection of open sets centered at each isolated point.
   - Let $\{G_\alpha\}_\alpha$ be an open cover of $K \setminus I$.
   - We have that $\{N_{\epsilon_p}(p)\}_{p \in I} \cup \{G_\alpha\}_\alpha$ is an open cover of K.
   - However, notice that it is impossible to find a finite subcover of $\{N_{\epsilon_p}(p)\}_{p \in I} \cup \{G_\alpha\}_\alpha$.
   - This is because to cover each $p \in I \subseteq K$, the subcover must contain the corresponding $N_{\epsilon_p}(p)$ which there are infinitely many.
   - Therefore, K is not compact, which is a contradiction.

   □

3. Let $\{K_\alpha\}_{\alpha \in I}$ be any collection of compact sets. Prove that $\bigcap_{\alpha \in I} K_\alpha$ is also compact.

   Proof.

   - Let $B = \{B_\beta\}_{\beta \in J}$ be an open cover of $\bigcap_{\alpha \in I} K_\alpha$. We will show that B has a finite subcover.
   - Let $K_1 \in \{K_\alpha\}_{\alpha \in I}$. Then K_1 is compact.
   - Let C be an open cover of K_1. Let $C_f = \{C_1, C_2, \ldots C_n\}$ be a finite subcover of C.
   - Notice that C_f is an open cover of $\bigcap_{\alpha \in I} K_\alpha$. This is because $\bigcap_{\alpha \in I} K_\alpha \subseteq K_1$ and C_f covers K_1.
   - We will construct a finite subcover of B from C_f. Consider $B_f = \{C_i \cap (\bigcup_{\beta \in J} B_\beta)\}_{1 \leq i \leq n}$.
   - B_f must be finite as it is constructed from iterating over C_f, which has n elements.
   - It is also a subcover of B as $C_f \cap B \subseteq B$. It must also cover $\bigcap_{\alpha \in I} K_\alpha$ because C_f covers K_1 and $K_1 \subseteq \bigcap_{\alpha \in I} K_\alpha$.
   - Therefore, $\bigcap_{\alpha \in I} K_\alpha$ is compact.

   □

Problem 3. Let (X, d_X) and (Y, d_Y) be two metric spaces. We define a metric on the product space $X \times Y$:

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Similarly, if $(X_1, d_1), ..., (X_N, d_N)$ are N metric spaces we define a metric on the product $X_1 \times X_2 \times \cdots \times X_N$ as the sum of distances in each coordinate.

1. Verify that $d_{X\times Y}$ is indeed a metric on $X \times Y$. Conclude by induction that the construction of a metric on $X_1 \times X_2 \times \cdots \times X_N$ is indeed a metric.

   Proof. **Base Case:** We will prove positivity, symmetry, and the triangle inequality for $d_{X\times Y}$.

   Positivity:

   - Let $(x_1, y_1), (x_2, y_2) \in X \times Y$.
   - Then $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$.
   - Therefore, $d_X(x_1, x_2) + d_Y(y_1, y_2) \geq 0$.
   - This implies $d_{X\times Y}((x_1, y_1), (x_2, y_2)) \geq 0$.

   Symmetry:

   - Let $(x_1, y_1), (x_2, y_2) \in X \times Y$.
   - Then $d_X(x_1, x_2) = d_X(x_2, x_1)$ and $d_Y(y_1, y_2) = d_Y(y_2, y_1)$.
   - Therefore, $d_X(x_1, x_2) + d_Y(y_1, y_2) = d_X(x_2, x_1) + d_Y(y_2, y_1)$.
   - This implies $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_{X\times Y}((x_2, y_2), (x_1, y_1))$.

   Triangle Inequality:

   - Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.
   - Then $d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3)$ and $d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3)$.
   - Therefore, $d_X(x_1, x_3) + d_Y(y_1, y_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3) + d_Y(y_1, y_2) + d_Y(y_2, y_3)$.
   - This implies $d_{X\times Y}((x_1, y_1), (x_3, y_3)) \leq d_{X\times Y}((x_1, y_1), (x_2, y_2)) + d_{X\times Y}((x_2, y_2), (x_3, y_3))$.

   Therefore, $d_{X\times Y}$ is a metric on $X \times Y$.

   Inductive Step: We assume that $d_{X_1\times X_2\times\cdots\times X_N}$ is a metric on $X_1 \times X_2 \times \cdots \times X_N$. We will prove that $d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}$ is a metric on $X_1 \times X_2 \times \cdots \times X_N \times X_{N+1}$.

   Positivity:

   - Let $(x_1, x_2, ..., x_N, x_{N+1}), (y_1, y_2, ..., y_N, y_{N+1}) \in X_1 \times X_2 \times \cdots \times X_N \times X_{N+1}$.
   - Then $d_{X_1\times X_2\times\cdots\times X_N}(x_1, x_2, ..., x_N, y_1, y_2, ..., y_N) \geq 0$ (inductive hypothesis).
   - And $d_{X_{N+1}}(x_{N+1}, y_{N+1}) \geq 0$ ($d_{X_{N+1}}$ is a metric on X_{N+1}).
   - Therefore, $d_{X_1\times X_2\times\cdots\times X_N}((x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N)) + d_{X_{N+1}}(x_{N+1}, y_{N+1}) \geq 0$.
   - This implies $d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((x_1, x_2, ..., x_N, x_{N+1}), (y_1, y_2, ..., y_N, y_{N+1})) \geq 0$.

   Symmetry:

   - Let $(x_1, x_2, ..., x_N, x_{N+1}), (y_1, y_2, ..., y_N, y_{N+1}) \in X_1 \times X_2 \times \cdots \times X_N \times X_{N+1}$.
   - Then $d_{X_1\times X_2\times\cdots\times X_N}(x_1, x_2, ..., x_N, y_1, y_2, ..., y_N) = d_{X_1\times X_2\times\cdots\times X_N}(y_1, y_2, ..., y_N, x_1, x_2, ..., x_N)$ (inductive hypothesis).
   - And $d_{X_{N+1}}(x_{N+1}, y_{N+1}) = d_{X_{N+1}}(y_{N+1}, x_{N+1})$ ($d_{X_{N+1}}$ is a metric on X_{N+1}).
   - Therefore, $d_{X_1\times X_2\times\cdots\times X_N}((x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N)) + d_{X_{N+1}}(x_{N+1}, y_{N+1})$
     $= d_{X_1\times X_2\times\cdots\times X_N}((y_1, y_2, ..., y_N), (x_1, x_2, ..., x_N)) + d_{X_{N+1}}(y_{N+1}, x_{N+1})$.

- This implies $d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((x_1,x_2,...,x_N,x_{N+1}),(y_1,y_2,...,y_N,y_{N+1}))$
$= d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((y_1,y_2,...,y_N,y_{N+1}),(x_1,x_2,...,x_N,x_{N+1}))$.

Triangle Inequality:

- Let $(x_1,x_2,...,x_N,x_{N+1}),(y_1,y_2,...,y_N,y_{N+1}),(z_1,z_2,...,z_N,z_{N+1}) \in X_1\times X_2\times\cdots\times X_N\times X_{N+1}$.
- Then $d_{X_1\times X_2\times\cdots\times X_N}((x_1,x_2,...,x_N),(z_1,z_2,...,z_N))$
$\leq d_{X_1\times X_2\times\cdots\times X_N}((x_1,x_2,...,x_N),(y_1,y_2,...,y_N)) + d_{X_1\times X_2\times\cdots\times X_N}((y_1,y_2,...,y_N),(z_1,z_2,...,z_N))$ (inductive hypothesis).
- And $d_{X_{N+1}}(x_{N+1},z_{N+1}) \leq d_{X_{N+1}}(x_{N+1},y_{N+1}) + d_{X_{N+1}}(y_{N+1},z_{N+1})$ ($d_{X_{N+1}}$ is a metric on X_{N+1}).
- Therefore, $d_{X_1\times X_2\times\cdots\times X_N}((x_1,x_2,...,x_N),(z_1,z_2,...,z_N)) + d_{X_{N+1}}(x_{N+1},z_{N+1})$
$\leq d_{X_1\times X_2\times\cdots\times X_N}((x_1,x_2,...,x_N),(y_1,y_2,...,y_N)) + d_{X_1\times X_2\times\cdots\times X_N}((y_1,y_2,...,y_N),(z_1,z_2,...,z_N))$
$+d_{X_{N+1}}(x_{N+1},y_{N+1}) + d_{X_{N+1}}(y_{N+1},z_{N+1})$.
- This implies $d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((x_1,x_2,...,x_N,x_{N+1}),(z_1,z_2,...,z_N,z_{N+1}))$
$\leq d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((x_1,x_2,...,x_N),(y_1,y_2,...,y_N,y_{N+1}))$
$+d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}((y_1,y_2,...,y_N,y_{N+1}),(z_1,z_2,...,z_N,z_{N+1}))$.

Hence, $d_{X_1\times X_2\times\cdots\times X_N\times X_{N+1}}$ is a metric on $X_1\times X_2\times\cdots\times X_N\times X_{N+1}$. By induction, we have shown that the construction of a metric on $X_1\times X_2\times\cdots\times X_N$ is indeed a metric. ☐

2. Assume $K_1 \subseteq X$ and $K_2 \subseteq Y$ are compact sets. Prove that $K_1\times K_2 \subseteq X\times Y$ is compact. Conclude that a product of compact sets in $X_1\times\cdots\times X_N$ is compact. [1]

Proof. **Base Case:** We will prove that $K_1\times K_2$ is compact.

- Let K_1 and K_2 be compact sets and $\{G_\alpha\}_{\alpha\in A}$ be an open cover of $K_1\times K_2$. We will show that it has a finite subcover.
- For each point $(a,b)\in K_1\times K_2$, we can choose some α such that $(a,b)\in G_\alpha$.
- G_α is an open set in $X\times Y$. Hence, (a,b) is contained in some open box $U_{(a,b)}\times V_{(a,b)} \subseteq G_\alpha$ where $U_{(a,b)}\subseteq K_1$ and $V_{(a,b)}\subseteq K_2$ are open sets.
- Now, let's fix a and vary b. Then every point (a,b) is contained in an open box in the product $K_1\times K_2$ and the box is itself the product of a subset of K_1 and a subset of K_2.
- Therefore, the collection of sets $\{V_{(a,b)}\}_{b\in K_2}$ is an open cover for K_2.
- Since K_2 is compact, we find a finite cover $\{V_{(a,b_j(a)}\}$ of K_2 containing finitely many open sets containing points $\{(a,b_j(a))\}$.
- Now, let $U_\alpha = \cap_j U_{(a,b_j(a))}$ where the intersection of finitely many open sets, and therefore open itself.
- Since K_1 is compact, we find a finite subcover $\{U_{\alpha_i}\}$ of K_1.
- This implies that $\{U_{\alpha_i}\times V_{(a,b_j(a))}\}$ is a finite subcover of $K_1\times K_2$.

Inductive Step: We assume that $K_1\times K_2\times\cdots\times K_N$ is compact. We will prove that $K_1\times K_2\times\cdots\times K_N\times K_{N+1}$ is compact.

- Let $\{G_\beta\}_{\beta\in B}$ be an open cover of $K_1\times\cdots\times K_N\times K_{N+1}$. We will show that it has a finite subcover.
- For any point $(x_1,\ldots,x_N,x_{N+1})\in K_1\times\cdots\times K_N\times K_{N+1}$, there exists an open set G_β containing this point. This G_β contains an open set of the form $U_1\times\cdots\times U_N\times U_{N+1}$ where each U_i is open in K_i.

[1] Actually, this holds in much greater generality — arbitrary products of compact sets are compact! (Tychonoff's Theorem)

- By the inductive hypothesis we know that $K_1 \times \cdots \times K_N$ is compact. We know that K_{N+1} is compact as well.
- By sequentially considering each K_i, we can construct a finite subcover for $K_1 \times \cdots \times K_N$, and separately, a finite subcover for K_{N+1}, similar to our base case.
- Combine these finite subcovers similar to our base case creates a finite subcover of $K_1 \times \cdots \times K_N \times K_{N+1}$.

Hence, by induction, we have shown that the product of compact sets in $X_1 \times \cdots \times X_N$ is compact.

$\square$

3. Conclude that cells $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$ are compact in $\mathbb{R}^N$ equipped with the metric as defined above.

Proof.

- Assuming $a_i < b_i$ for all $1 \leq i \leq N$, we know by the proposition in class that each $[a_i, b_i]$ is compact in $\mathbb{R}$.
- Therefore, $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$ is compact in $\mathbb{R}^N$ equipped with the metric as defined above by this problem.

$\square$

Problem 4. Let (X, d) be a metric space and let K_1 and K_2 be two compact sets in X. Assume the sets are disjoint, i.e. $K_1 \cap K_2 = \emptyset$.

1. Use the fact that

$$\forall p \in K_1,\ q \in K_2 \quad \exists r_{p,q} > 0 \quad \text{such that} \quad N_{r_{p,q}}(p) \cap N_{r_{p,q}}(q) = \emptyset,$$

to prove that there exists $r > 0$ satisfying that $\forall p \in K_1,\ q \in K_2 \quad d(p, q) > r$.

Proof.

- We first notice that the fact implies that $\forall p \in K_1,\ q \in K_2 \quad d(p, q) \geq r_{p,q}$. If this wasn't the case, then $N_{r_{p,q}}(p) \cap N_{r_{p,q}}(q) \neq \emptyset$.
- We will construct a universal r such that for all $\forall p \in K_1,\ q \in K_2 \quad d(p, q) > r$. The idea will be to use the fact that K_1 and K_2 are compact to locate a minimum from its finite subcovers $r \leq r_{p,q}$.
- We will construct open covers for K_1 and K_2. For each $p \in K_1$, consider $N_{r_{p,q}/2}(p)$. This is an open cover for K_1 because all neighborhoods are open and all $p \in K_1$ are covered. Similarly, for each $q \in K_2$, consider $N_{r_{p,q}/2}(q)$, which is an open cover for K_2.
- Because K_1 and K_2 are compact, we have finite subcovers $\{N_{r_{p_i,q_i}/2}(p_i)\}_{i=1}^{n}$ and $\{N_{r_{p_i,q_i}/2}(q_i)\}_{j=1}^{m}$ for K_1 and K_2 respectively.
- Let $r = \min\{r_{p_i,q_j}/2 : 1 \leq i \leq n, 1 \leq j \leq m\}$.
- With r chosen as above, for any $p \in K_1,\ q \in K_2,\ d(p, q) > r$. This follows because if there were any $p \in K_1,\ q \in K_2,\ d(p, q) \leq r$, their neighborhoods would overlap, contradicting our initial assumption that we can always find $r_{p,q}$ making their neighborhoods disjoint.

□

2. Is the same statement true for any two closed sets? That is, if F_1 and F_2 are closed in X with $F_1 \cap F_2 = \emptyset$, then there exists $r > 0$ for which $d(p, q) > r$ for all $p \in F_1$ and $q \in F_2$?

Proof. We will show that the statement is not true for any two closed sets by providing a counterexample.

- Let F_1 and F_2 be two closed sets in $\mathbb{R}^2$ with the standard metric.
- Let $F_1 = \{(x, 1/x) \in \mathbb{R}^2 : x > 0\}$, $F_2 = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.
- Notice that F_1 and F_2 are both closed.
- Notice that $F_1 \cap F_2 = \emptyset$. This is clear because for any point in F_1, the y-coordinate is $1/x$, which is always positive for $x > 0$. Meanwhile, the y-coordinate for any point in F_2 is always 0. Hence, there are no points in common.
- We will show that for any $r > 0$, there exists $p \in F_1$ and $q \in F_2$ such that $d(p, q) \leq r$. Pick $p = (2/r, r/2)$ in F_1 and $q = (2/r, 0)$ in F_2. The distance between p and q is given by $d(p, q) = \sqrt{((2/r) - (2/r))^2 + (r/2 - 0)^2} = \sqrt{0 + (r/2)^2} = r/2 < r$, demonstrating that there cannot be a uniform $r > 0$ such that $d(p, q) > r$ for all $p \in F_1$ and $q \in F_2$.

□

Problem 5. Let (X, d) be a metric space and let $\{K_n\}_{n\in\mathbb{N}}$ be a countable collection of non-empty compact sets satisfying $K_1 \supseteq K_2 \supseteq K_3 \supseteq ...$[2].

1. Prove that $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.[3]

   Proof.

   - Consider $G_n = K_n^c$ for each $n \in \mathbb{N}$. Notice that G_n is open, as it is the complement of a compact set which is necessarily closed.
   - Assume in contradiction that $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$. This implies that $\bigcup_{n\in\mathbb{N}} G_n = X$. In particular, this implies that $\{G_n\}_{n\in\mathbb{N}}$ is an open cover for X.
   - Since K_1 is compact, there exists a finite subcover $\{G_{n_1}, G_{n_2}, ..., G_{n_m}\}$ for K_1.
   - Define $k = \max\{n_1, n_2, ..., n_m\} + 1$. Then $K_k \subseteq K_1$ because $\{K_n\}_{n\in\mathbb{N}}$ is a nested sequence.
   - This implies that $\{G_{n_1}, G_{n_2}, ..., G_{n_m}\}$ is an open cover for K_k.
   - However, because G_n is the complement of K_n, K_k must be disjoint from the cover formed by $\{G_{n_1}, G_{n_2}, ..., G_{n_m}\}$. This is a contradiction, as K_k cannot be both disjoint from the cover but also covered by the cover.
   - Therefore, $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.

   $\square$

2. Is the same statement true for a nested sequence of closed sets? That is, is it true that given a sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty closed sets satisfying $F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$ has $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$?

   Proof. We will show that the statement is not true for a nested sequence of closed sets by providing a counterexample.

   - Consider the nested sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ where $F_n = [n, \infty)$ for each $n \in \mathbb{N}$.
   - However, $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. This is because there is no real number that is in every set F_n. For any real number $x \in \mathbb{R}$, we can always find an $n > x$ such that the real number is not in F_n.

   $\square$

Good luck!

[2] A sequence of sets satisfying this condition is called a *nested* sequence.

[3] Hint: Consider the complements $G_n = K_n^c$. This result is called Cantor's lemma and we've encountered a special case of it as part of the proof of uncountability of $(0, 1)$ in $\mathbb{R}$.