Econ 136: Problem Set 2

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February 20, 2024

Problem 1.

(a)

$$\begin{aligned}
\mathbb{E}[Xe] &= \mathbb{E}[X(Y - X'\beta)] \\
&= \mathbb{E}[XY] - \mathbb{E}[X(X' \cdot (\mathbb{E}[X'X])^{-1} XY)] && \text{by linearity of expectation and definition of } \beta \\
&= \mathbb{E}[XY] - \mathbb{E}[XX'] (\mathbb{E}[X'X])^{-1} \mathbb{E}[XY] && \text{by linearity of expectation} \\
&= \mathbb{E}[XY] - \mathbb{E}[XY] && \text{by matrix properties} \\
&= 0
\end{aligned}$$

(b) The equation in part (a) is a set of 2 equations, one for each element of X.

$$\begin{aligned}
\mathbb{E}[1e] = \mathbb{E}[e] = 0 \\
\mathbb{E}[X_1 e] = 0
\end{aligned}$$

(c)

$$\begin{aligned}
\text{Cov}(X_1, e) &= \mathbb{E}[(X_1 - \mathbb{E}[X_1])(e - \mathbb{E}[e])] && \text{by the definition of covariance} \\
&= \mathbb{E}[X_1 e] - \mathbb{E}[\mathbb{E}[X_1]e] - \mathbb{E}[X_1\mathbb{E}[e]] + \mathbb{E}[\mathbb{E}[X_1]\mathbb{E}[e]] && \text{by the linearity of expectations} \\
&= \mathbb{E}[X_1 e] && \text{because } \mathbb{E}[e] = 0, \text{ part b} \\
&= 0 && \text{because } \mathbb{E}[X_1 e] = 0, \text{ part b}
\end{aligned}$$

(d) Let's examine the determinant of $E[XX']$.

$$\begin{aligned}
\det(E[XX']) &= \det\left(\begin{bmatrix} \mathbb{E}[1] & \mathbb{E}[X_1] \\ \mathbb{E}[X_1] & \mathbb{E}[X_1^2] \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 1 & \mu_{X_1} \\ \mu_{X_1} & \mathbb{E}[X_1^2] \end{bmatrix}\right) \\
&= 1 \cdot \mathbb{E}[X_1^2] - \mu_{X_1}^2 \\
&= Var(X_1) && \text{by the definition of variance}
\end{aligned}$$

Therefore, the determinant of $E[XX']$ being non-zero directly depends on the variance of $Var(X_1) > 0$. The variability of X_1 ensures that the components of X are not linearly dependent.

(e)

$$\begin{aligned}
\beta &= (\mathbb{E}[XX'])^{-1}\mathbb{E}[XY] && \text{by the definition of } \beta \\
&= \left(\begin{bmatrix} E[1] & E[X_1] \\ E[X_1] & E[X_1^2] \end{bmatrix}\right)^{-1} \begin{bmatrix} \mathbb{E}[Y] \\ \mathbb{E}[X_1 Y] \end{bmatrix} && \text{by opening the matrix} \\
&= \frac{1}{\det(\mathbb{E}[XX'])} \begin{bmatrix} \mathbb{E}[X_1^2] & -\mathbb{E}[X_1] \\ -\mathbb{E}[X_1] & 1 \end{bmatrix} \begin{bmatrix} \mathbb{E}[Y] \\ \mathbb{E}[X_1 Y] \end{bmatrix} && \text{by the definition matrix inverse} \\
&= \frac{1}{Var(X_1)} \begin{bmatrix} \mathbb{E}[X_1^2] & -\mathbb{E}[X_1] \\ -\mathbb{E}[X_1] & 1 \end{bmatrix} \begin{bmatrix} \mathbb{E}[Y] \\ \mathbb{E}[X_1 Y] \end{bmatrix} && \text{by part (d)} \\
&= \frac{1}{Var(X_1)} \begin{bmatrix} \mathbb{E}[X_1^2]E[Y] - \mathbb{E}[X_1]E[X_1 Y] \\ -\mathbb{E}[X_1]E[Y] + E[X_1 Y] \end{bmatrix} && \text{by matrix multiplication}
\end{aligned}$$

Therefore, β_0 and β_1 are solved as

$$\begin{aligned}
\beta_0 &= \frac{\mathbb{E}[X_1^2]E[Y] - \mathbb{E}[X_1]\mathbb{E}[X_1 Y]}{Var(X_1)} \\
&= \mathbb{E}[Y] - \frac{\mathbb{E}[X_1]Cov(X_1, Y)}{Var(X_1)} && \text{by the definition of covariance} \\
\beta_1 &= \frac{-\mathbb{E}[X_1]E[Y] + \mathbb{E}[X_1 Y]}{Var(X_1)} \\
&= \frac{Cov(X_1, Y)}{Var(X_1)} && \text{by the definition of covariance}
\end{aligned}$$

Problem 2.

$$\begin{aligned}
\mathrm{Cov}(Y, e) &= \mathrm{Cov}(X'\beta + e, e) \\
&= \mathrm{Cov}(X'\beta, e) + \mathrm{Cov}(e, e) && \text{by the properties of covariance} \\
&= \mathbb{E}[X'\beta - \mathbb{E}[X'\beta])(e - \mathbb{E}[e])] + Var[e] && \text{by the definition of covariance} \\
&= \beta\mathbb{E}[Xe] - \mathbb{E}[X'\beta]\mathbb{E}[e] + Var[e] && \text{by the linearity of expectations} \\
&= Var[e] && \mathbb{E}[Xe] = 0 \text{ and } \mathbb{E}[e] = 0 \text{ for BLP } Y \text{ given } X
\end{aligned}$$

To show that the BLP of X_1 given Y cannot be $-\frac{\beta_0}{\beta_1} - \frac{1}{\beta_1}Y$, it suffices to show that $\mathbb{E}[e \cdot g(Y)] \neq 0$ for some function $g(Y)$. Let's consider the function $g(Y) = Y - \beta_0 - \beta_1 X_1$.

$$\begin{aligned}
\mathbb{E}[e \cdot g(Y)] &= \mathbb{E}[e \cdot (Y - \beta_0 - \beta_1 X_1)] \\
&= \mathbb{E}[e^2] \\
&\neq 0 && \text{if } \mathrm{Cov}(Y, e) = Var[e] > 0
\end{aligned}$$

Alternatively, we can also consider $Cov(X_1, e)$. By problem 1, part (c), we know that $Cov(X_1, e) = 0$.

$$\begin{aligned}
Cov(X_1, e) &= Cov(-\frac{\beta_0}{\beta_1} - \frac{1}{\beta_1}Y - \frac{1}{\beta_1}e, e) \\
&= Cov(-\frac{\beta_0}{\beta_1}, e) - Cov(\frac{1}{\beta_1}Y, e) + Cov(-\frac{1}{\beta_1}e, e) \\
&= 0 - \frac{1}{\beta_1}Var(e) - \frac{1}{\beta_1}Var(e) \\
&= 0 \\
&\implies Var(e) = 0
\end{aligned}$$

This can only happen if $e = 0$, the zero vector, which implies that Y is a linear combination of X which is not the case in general. Therefore, the BLP of X_1 given Y being $-\frac{\beta_0}{\beta_1} - \frac{1}{\beta_1}Y$ is not possible.

Problem 3.

(a)

$$\begin{aligned}
Pr[Y = 1] &= E[Y] && \text{because } Y \text{ is Bernoulli} \\
&= E[E[Y|X]] && \text{by the law of iterated expectations} \\
&= E[X'\beta] && \text{because } E[e|X] = 0 \\
&= \mu_X'\beta && \text{by the linearity of expectations}
\end{aligned}$$

(b)

$$\begin{aligned}
Var[Y] &= P[Y = 1](1 - P[Y = 1]) && \text{because } Y \text{ is Bernoulli} \\
&= \mu_X'\beta(1 - \mu_X'\beta) && \text{by part (a)}
\end{aligned}$$

(c)

$$\begin{aligned} Pr[Y=1|X] &= E[Y|X] && \text{because } Y \text{ is Bernoulli} \\ &= X'\beta && \text{because } E[e|X]=0 \end{aligned}$$

(d)

$$\begin{aligned} Var[Y|X] &= P[Y=1|X](1-P[Y=1|X]) && \text{because } Y \text{ is Bernoulli} \\ &= X'\beta(1-X'\beta) && \text{by part (c)} \end{aligned}$$

(e)

$$\begin{aligned} Var[e|X] &= Var[Y-X'\beta|X] && \text{by the definition of } e \\ &= Var[Y|X]+Var[-X'\beta|X]+2Cov[Y,-X'\beta|X] && \text{by the properties of variance} \\ &= Var[Y|X]+0+0 && \text{because } -X'\beta \text{ is a constant} \\ &= X'\beta(1-X'\beta) && \text{by part (d)} \end{aligned}$$

(f) Because of part (e), we know e will be heteroskedastic because $Var[e|X]$ is a function of X.

Problem 4.

(a)

$$\begin{aligned} E[e] &= E[E[e|F,S]] && \text{by the law of iterated expectations} \\ &= E[0] && \text{because } E[e|F,S]=0 \\ &= 0 \end{aligned}$$

(b)

$$\begin{aligned} \mathbb{E}[Y|F] &= \mathbb{E}[\gamma_0+\gamma_1 F+v|F] \\ &= \gamma_0+\gamma_1\mathbb{E}[F|F]+\mathbb{E}[v|F] \\ &= \gamma_0+\gamma_1 F \end{aligned}$$

We can evaluate $\mathbb{E}[Y|F]$ by conditioning on F:

$$\begin{aligned} \mathbb{E}[Y|F=0] &= \gamma_0 \\ \mathbb{E}[Y|F=1] &= \gamma_0+\gamma_1 \end{aligned}$$

Thus, we can write

$$\begin{aligned} \gamma_0 &= \mathbb{E}[Y|F=0] \\ \gamma_1 &= \mathbb{E}[Y|F=1]-\mathbb{E}[Y|F=0] \end{aligned}$$

(c)

$$\begin{aligned} \mathbb{E}[Y|F] &= \mathbb{E}[\beta_0+\beta_1 F+\beta_2 S+e|F] \\ &= \beta_0+\beta_1\mathbb{E}[F|F]+\beta_2\mathbb{E}[S|F]+\mathbb{E}[e|F] \\ &= \beta_0+\beta_1 F+\beta_2\mathbb{E}[S|F] \end{aligned}$$

We can evaluate $\mathbb{E}[Y|F]$ by conditioning on F:

$$\begin{aligned} \mathbb{E}[Y|F=0] &= \beta_0+\beta_2\mathbb{E}[S|F=0] \\ \mathbb{E}[Y|F=1] &= \beta_0+\beta_1+\beta_2\mathbb{E}[S|F=1] \end{aligned}$$

Thus, we can write

$$\begin{aligned}\gamma_1 &= \mathbb{E}[Y|F=1] - \mathbb{E}[Y|F=0] \\ &= \beta_1 + \beta_2(\mathbb{E}[S|F=1] - \mathbb{E}[S|F=0])\end{aligned}$$

The observed wage gap between genders not controlling for schooling (γ_1) is influenced by two components: the direct effect of being a woman on wage (β_1), and the differential effect of schooling between genders (β_2 times the difference in average years of schooling between women and men).

(d) When $\mathbb{E}[S|F=1] < \mathbb{E}[S|F=0]$, the second term in the expression for γ_1 will be negative, meaning that $\gamma_1 < \beta_1$. The gender-wage gap not controlling for schooling will be smaller than the gap that does control for schooling.

(e) When $\mathbb{E}[S|F=1] > \mathbb{E}[S|F=0]$, the second term in the expression for γ_1 will be positive, meaning that $\gamma_1 > \beta_1$. The gender-wage gap not controlling for schooling will be larger than the gap that does control for schooling.