Analysis 1

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These are notes from a spring 2023 course at Yale University. I taught one section of the course; Taylor McAdam taught the other section. These notes should not be considered as original work; much of the material is modeled closely on the course texts, especially Rudin. I thank the course participants for pointing out many errors. Nevertheless, errors may remain.

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1. Introduction

This course has several coequal purposes:

- To develop some familiarity with the ideas and tools of analysis, which are absolutely fundamental to modern mathematics and its applications.
- To apply these tools to get a sharper understanding of a familiar set of ideas, namely the ideas of differential and integral calculus for functions of a single real variable, and to get a glimpse of some applications that go beyond this.
- More broadly — continuing the introductory sequence which begins with MATH 225/226 — to discover more of what it means to think mathematically and to be a mathematician.

You might be wondering, "what is analysis anyway?" It is a little hard to give a sharp answer, but here are some elements:

- Analysis usually concerns things which are *continuous* rather than discrete. (A remark: the real world might be continuous at the smallest scales or it might be discrete, but it is hard to deny that it looks continuous when viewed at human scales — witness the trajectories of falling objects, water waves, electric fields, the Gaussian distribution made by counting a thousand coin flips a thousand times, ...) One of the first key goals of this course will be to develop a precise understanding of what is meant by "continuous".

  You may have gotten some hint of this in earlier calculus courses, where you defined continuity in terms of the notion of *limit*, in turn defined by a procedure involving numbers δ and ϵ. We will discuss the notion of limit in this course, and put it in a much broader context, that of *metric spaces* — this includes the usual real numbers, but many other things as well.
- Here is one use of the analytic tools we will develop. One of the key uses of mathematics is to describe natural phenomena, often by writing down equations which are supposed to capture them. Often, though, just writing down the equations is not enough: we also want to *solve* the equations. This leads to the difficult question: *does a solution actually exist*? For instance, is there a solution to the equation

  $$1 - x^2 = \sin x\,? \tag{1.1}$$

  (Incidentally, what does $\sin x$ really mean, anyway?) By drawing a picture you can convince yourself that there *should* be a solution. Using the tools of analysis we will prove that there really is one.
- An important point in the above: what do we mean by saying there is a solution to (1.1)? What we mean more exactly is that there is a *real number* x satisfying this equation. But what is a real number, anyway? We will define carefully what we

mean by a real number: in fact this will be our first order of business. Only once we have made this definition can we hope to prove rigorously that this equation has a solution!

- What really is meant by the "derivative"

$$f'(x)$$

and the "integral"

$$\int_a^b f(x)\,\mathrm{d}x\,?$$

What kinds of functions can be differentiated and integrated? Why, and when, is the Fundamental Theorem of Calculus true? Using the tools of analysis we will give clear and precise answers to these questions, in the theory of the "Riemann integral." This is a precise version of the theory of Riemann sums which you might have encountered in a calculus class.

- Finally we may have time for some fancier applications. For instance: does the differential equation

$$f'(x) = x^2 f(x)^2 + 4f(x) + 1$$

have a solution $f(x)$? It's not at all obvious that it does — but it indeed does, at least for x in some domain. For another: suppose we are given a general continuous function $f : [0,1] \to \mathbb{R}$. Can we write a *polynomial* function which will approximate f well? The answer is *yes*, with an appropriate understanding of what "approximate well" means. I hope we will get to discuss these things a bit.

Let's get started!

2. Sets and functions

We start with a quick review of the language of sets and functions, which we will use extensively in this course.

2.1. **Sets.** We will not actually define what a set is. We content ourselves with the idea that a set can contain objects. For any object x and any set A, we use the notation "$x \in A$" or "x is an element of A" to mean that x is contained in A.

To begin with, the objects will be relatively simple things, such as integers, or rational numbers, or real numbers, or ordered pairs of real numbers. Later we will want to consider more interesting sets, whose elements could be e.g. functions, or sets themselves.

Example 2.1. A basic example of a set is the set $\mathbb{N}$ consisting of all natural numbers. We write it as

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

Example 2.2. Similarly we have the set $\mathbb{Z}$ of all integers. We may write it as

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

Definition 2.3. If A and B are sets, and all $x \in A$ also satisfy $x \in B$, then we say A is a *subset* of B, and write $A \subset B$.[1]

Example 2.4. Since every $x \in \mathbb{N}$ is also an element of $\mathbb{Z}$, we have $\mathbb{N} \subset \mathbb{Z}$.

We remark that if $A \subset B$ and $B \subset A$, then the elements of A are exactly the elements of B; equivalently, in this case we have $A = B$. Indeed, a useful strategy for proving that two sets A and B are equal is to first prove that $A \subset B$ and then prove that $B \subset A$.

We also define as usual:

Definition 2.5. If A and B are sets, then:

(1) The *union* $A \cup B$ is the set such that $x \in A \cup B \Leftrightarrow (x \in A \text{ or } x \in B)$.

(2) The *intersection* $A \cap B$ is the set such that $x \in A \cap B \Leftrightarrow (x \in A \text{ and } x \in B)$.

(3) The *product* $A \times B$ is the set of ordered pairs $\{(a, b) \mid a \in A, b \in B\}$.

Similarly we can define unions, intersections, and products of arbitrary collections of sets, rather than just two sets at a time.

2.2. **Functions.** We will think of functions in the traditional way:

Definition 2.6. Suppose A and B are sets. A function

$$f : A \to B$$

is a rule which, for every "input" element $a \in A$, produces an "output" element $f(a) \in B$. We say A is the *domain* of f, and B is the *codomain* of f.

Moreover, a function is nothing more than the data of the domain, codomain and the list of inputs and outputs; so, if two functions $f : A \to B$, $g : A \to B$ have $f(a) = g(a)$ for all $a \in A$, then we say $f = g$. For instance, if we define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by the formulas $f(x) = x^2$ and $g(x) = |x|^2$, then $f = g$.

The kind of functions we usually meet in one-variable calculus are ones where A and B are both subsets of $\mathbb{R}$. For example, we could just have $A = \mathbb{R}$ and $B = \mathbb{R}$, so then we are considering functions $f : \mathbb{R} \to \mathbb{R}$. Some such functions are given by a compact formula, such as $f(x) = x^2$, $f(x) = 1/(x^2 + 1)$, $f(x) = |x|, \ldots$ But not every function is given by a formula of this sort.

We quickly recall some standard terminology, hopefully familiar from MATH 225/226:

Definition 2.7. A function $f : A \to B$ is *injective* (or 1-1) if $f(x) = f(x') \implies x = x'$.

[1] Some people use the notation $\subseteq$ for this, but we will just use $\subset$. (This matches Rudin.) If we ever want to insist on a proper subset, we will use $\subsetneq$.

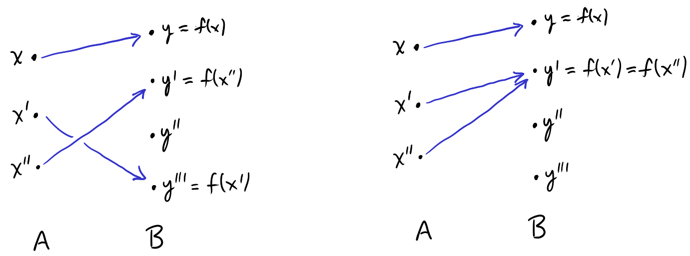

FIGURE 1. An injective function (left) and one which is not injective (right).

Example 2.8. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective, because if we take $x = 1$ and $x' = -1$, we have $f(x) = f(x')$ but $x \neq x'$.

Example 2.9. The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$ is injective. (Can you prove it?)

Definition 2.10. A function $f : A \to B$ is *surjective* (or onto) if, for any $y \in B$, there exists some $x \in A$, such that $f(x) = y$.

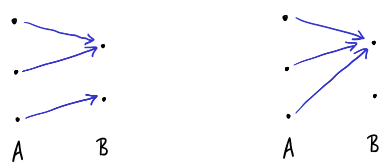

FIGURE 2. A surjective function (left) and one which is not surjective (right).

Example 2.11. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$ is not surjective, since e.g. $1 \in \mathbb{Z}$ is not $f(x)$ for any x. It is injective, since $2x = 2x'$ if and only if $x = x'$.

Example 2.12. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x + 1$ is surjective, since for all $y \in \mathbb{Z}$, $y = f(y - 1)$. It is also injective, since $x + 1 = x' + 1$ if and only if $x = x'$.

Definition 2.13. A function $f : A \to B$ is *bijective* if it is both injective and surjective.

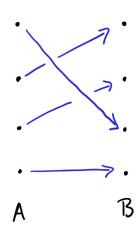

FIGURE 3. A bijective function.

Definition 2.14. Given a function $f : A \to B$ and $g : B \to C$, the *composition* $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) = g(f(x)) .$$

Definition 2.15. For any set A, the *identity function* $\mathrm{id}_A : A \to A$ is defined by

$$\mathrm{id}_A(x) = x$$

for all $x \in A$.

Here are a few important facts about functions, worth some thought if they are new to you:

Proposition 2.16. If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is also injective.

Proof. Suppose f and g are both injective, and let $h = g \circ f$. Then suppose

$$h(x) = h(x'). \tag{2.1}$$

To show that h is injective, we need to show that $x = x'$. We can rewrite (2.1) as

$$g(f(x)) = g(f(x')).$$

Using the fact that g is injective, this implies

$$f(x) = f(x').$$

Now using the fact that f is injective, we obtain

$$x = x'.$$

Thus we conclude that h is injective, as desired. □

Proposition 2.17. If $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is also surjective.

Proposition 2.18. A function $f : A \to B$ is bijective if and only if it has a two-sided inverse g, i.e. a function $g : B \to A$ obeying $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$.

When $f : A \to B$ is bijective we often use the notation $f^{-1} : B \to A$ for its two-sided inverse.

Finally we recall some notation that will be occasionally handy:

Definition 2.19. If $f : A \to B$ is a function, and $A' \subset A$, we let $f|_{A'}$ denote the *restriction* of f to A': this is the function $A' \to B$ given by

$$f|_{A'}(a) = f(a), \qquad a \in A'$$

2.3. **Images and inverse images.**

Definition 2.20. Given $f : A \to B$ and a subset $E \subset A$, we let

$$f(E) = \{f(a) \mid a \in E\} \subset B$$

and call this the *image* of E under f. The image of the whole domain A under f is also called the *range* of f.

Definition 2.21. Given $f : A \to B$ and a subset $E \subset B$, we let

$$f^{-1}(E) = \{a \mid f(a) \in E\} \subset A$$

and call this the *inverse image* of E under f.

Note that the inverse image $f^{-1}(E)$ is defined even if f is not invertible! The notation f^{-1} is a little overloaded: when f *is* invertible we use it for the inverse function as well as for the operation of taking inverse image. So $f^{-1}(y)$ for an *element* $y \in B$ only exists when f is invertible, but $f^{-1}(E)$ for a *set* $E \subset B$ always exists.

Example 2.22. Consider the map f shown in the figure.

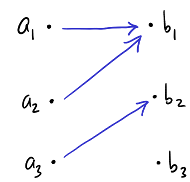

FIGURE 4. A map $f : \{a_1, a_2, a_3\} \to \{b_1, b_2, b_3\}$.

We have $f(\{a_1, a_2\}) = \{b_1\}$, $f^{-1}(\{b_1, b_3\}) = \{a_1, a_2\}$, and $f^{-1}(\{b_3\}) = \emptyset$.

Example 2.23. Consider the map $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$. Then $f(\{1,2,3\}) = \{1,4,9\}$, $f^{-1}(\{1,4,9\} = \{1,-1,2,-2,3,-3\}$, and $f^{-1}(\{-1,-2,-3,-4\}) = \emptyset$.

2.4. **Finite sets.** Let us record here a few facts about finite sets, which we will use freely.

Definition 2.24. For any $n \in \mathbb{N} \cup \{0\}$ we define the set

$$\underline{n} = \{m \in \mathbb{N} \mid m \leq n\}.$$

For example, $\underline{3} = \{1,2,3\}$, and $\underline{0}$ is the empty set $\emptyset$.

Definition 2.25. Suppose $n \in \mathbb{N} \cup \{0\}$. We say a set A is *finite of size n* if there is a bijective map

$$\phi : A \to \underline{n}.$$

If A is finite of size n, we also say A is simply *finite*. If A is not finite, then we say A is *infinite*.

Proposition 2.26.

(1) If there is an injective map $\phi : \underline{n} \to \underline{m}$, then $n \leq m$.
(2) If there is a surjective map $\phi : \underline{n} \to \underline{m}$, then $n \geq m$.
(3) If there is a bijective map $\phi : \underline{n} \to \underline{m}$, then $n = m$.

It follows that, if A is finite of size n and also finite of size m, then $m = n$. Thus, when A is finite, it has a unique size; we call that size $|A|$.

Example 2.27. The set $A = \{4, 7, 12\}$ is finite and $|A| = 3$, because there is a bijective map $\phi : A \to \underline{3}$, namely the map $\phi(4) = 1$, $\phi(7) = 2$, $\phi(12) = 3$.

Corollary 2.28.

(1) If there is an injective map $\phi : A \to B$, and B is finite, then A is finite and $|A| \leq |B|$.
(2) If there is a surjective map $\phi : A \to B$, and A is finite, then B is finite and $|A| \geq |B|$.

(3) If there is a bijective map $\phi : A \to B$, and either A or B is finite, then the other is finite and $|A| = |B|$.

Proposition 2.29. If A and B are finite, then $A \cup B$ is also finite, and $|A \cup B| \leq |A| + |B|$.

Proposition 2.30. If A and B are finite, then $A \times B$ is also finite, and $|A \times B| = |A||B|$.

3. The real numbers

One of the goals of this course is to build a firm foundation for the theory of one-variable calculus. One-variable calculus deals with real numbers. To begin with, then, we had better formulate carefully what the set $\mathbb{R}$ of real numbers is. We will do this a little bit indirectly at first.

3.1. **Fields.**

Whatever $\mathbb{R}$ is, it should be something that obeys the usual rules of arithmetic. These rules can be formalized by saying that $\mathbb{R}$ is not just a set but a *field*, defined as follows. (This will look long and painful at first, but it is really only a careful description of the rules you already know and use all the time.)

Definition 3.1. A *field* is a set F, with:

- an addition rule: for any $x, y \in F$ we have an element $x + y \in F$,
- a multiplication rule: for any $x, y \in F$ we have an element $x \cdot y \in F$ (often just written xy),
- a zero element: $0 \in F$,
- a unit element: $1 \in F$, with $1 \neq 0$,
- a negation rule: for any $x \in F$ we have an element $-x \in F$,
- an inversion rule: for any $x \in F$ with $x \neq 0$, we have an element $x^{-1} \in F$,

obeying the following rules:

(1) For all $x, y \in F$ we have $x + y = y + x$.

(2) For all $x, y, z \in F$ we have $(x + y) + z = x + (y + z)$.

(3) For all $x \in F$ we have $0 + x = x$.

(4) For all $x \in F$ we have $-x + x = 0$.

(5) For all $x, y \in F$ we have $xy = yx$.

(6) For all $x, y, z \in F$ we have $(xy)z = x(yz)$.

(7) For all $x \in F$ we have $1 \cdot x = x$.

(8) For all $x \in F$ with $x \neq 0$ we have $x \cdot x^{-1} = 1$.

(9) For all $x, y, z \in F$ we have $x(y + z) = xy + xz$.

Example 3.2. Recall that a *rational number* is a number which can be written as p/q with both p and q in $\mathbb{Z}$, and $q \neq 0$. Two rational numbers p/q and p'/q' are equal just if $pq' = p'q$. We say p/q is *in lowest terms* if the greatest common divisor of $|p|$ and $|q|$ is 1.

The rational numbers $\mathbb{Q}$ form a field. This amounts to saying that, using the usual rules for addition and multiplication of fractions, $\mathbb{Q}$ is closed under addition, multiplication, negatives, inverses, and obeys the usual laws of arithmetic.

(In this course we will take these standard facts for granted, without proving them. Still you might enjoy thinking about, e.g., how to prove that the sum of two rational numbers is a rational number.)

Example 3.3. The integers $\mathbb{Z}$ obey the usual laws of arithmetic, but they do not form a field; this is because they are not closed under taking inverses, e.g. there is no integer 2^{-1} such that $2^{-1} \cdot 2 = 1$.

Here are two more examples just for fun (we won't use them much in what follows.)

Example 3.4. If p is a prime number, the integers modulo p form a field $\mathbb{F}_p$, consisting of p elements. For example, the integers modulo 3 are $\mathbb{F}_3 = \{0, 1, 2\}$, with various funny-looking relations like $2 \cdot 2 = 1$ (remember that we work modulo 3, so 4 is the same as 1), $1 + 2 = 0$, and so on. You can check directly that they indeed obey all the rules of Definition 3.1. (By the way, this doesn't work if p is not a prime number; you might enjoy thinking about what goes wrong.)

Example 3.5. Consider $\mathbb{Q}(\sqrt{3})$, the set of all expressions of the form $a + b\sqrt{3}$, where $a, b \in \mathbb{Q}$. We can define addition and multiplication on these expressions, and it turns out that they form a field. (Can you prove it?)

From Definition 3.1 all the standard maneuvers of arithmetic can be derived. For example,

Proposition 3.6. *Suppose F is a field with $x, y, z \in F$. Then*

$$x + y = x + z \qquad \Longrightarrow \qquad y = z.$$

Proof.

$$\begin{aligned} x + y = x + z \qquad &\Longrightarrow \qquad -x + (x + y) = -x + (x + z) \\ &\Longrightarrow \qquad (-x + x) + y = (-x + x) + z \\ &\Longrightarrow \qquad 0 + y = 0 + z \\ &\Longrightarrow \qquad y = z\,. \end{aligned}$$

□

Proposition 3.7. *Suppose F is a field. Then $0 \cdot x = 0$, for all $x \in F$.*

Proof. Write

$$0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x = 0 \cdot x + 0$$

and now use Proposition 3.6 to cancel a $0 \cdot x$ from both sides, leaving

$$0 \cdot x = 0\,.$$

□

Proposition 3.8. *Suppose F is a field. Then, in F, $(-1)^2 = 1$.*

Proof. Write

$$(-1)^2 - 1 = (-1)\cdot(-1) + 1\cdot(-1) = (-1+1)\cdot(-1) = 0\cdot(-1) = 0$$

and then add 1 to both sides to get

$$(-1)^2 = 1\,.$$

□

A few more of these kinds of statements are given in Rudin.

From now on, whenever we work in a field, we will freely use the usual rules of arithmetic, and also use the convenient notation $x - y$ for $x + (-y)$, and x/y for xy^{-1}.

In almost everything you did in MATH 225/226, you only used the rules of arithmetic, i.e. the rules of a field, not anything that is specific to the field $\mathbb{R}$.[2] In this course things will be quite different: we will need to use some more properties, that distinguish $\mathbb{R}$ from all other fields. Now we'll start to identify those.

3.2. **Ordered sets.** The real numbers $\mathbb{R}$ are not just any old field: they are an *ordered* field. Now we formalize what that means. First we consider ordered sets.

Definition 3.9. An *ordered set* is a set S with a relation $<$, such that[3]

(1) For any $x, y \in S$ exactly one of the three statements $x < y$, $y < x$, $x = y$ is true.
(2) For all $x, y, z \in S$ such that $x < y$ and $y < z$, we have $x < z$.

We will also use the notation $x > y$ as a synonym for $y < x$, and use $x \geq y$ for "$x > y$ or $x = y$."

Example 3.10. $\{-7, 1, 2, 3, 8\}$ is an example of an ordered set, where we take $<$ to be the usual less-than relation on integers (so e.g. we have $-7 < 3$, $1 < 8$, etc.)

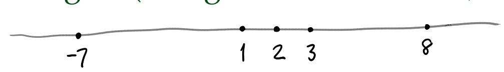

FIGURE 5. An example of an ordered set.

Example 3.11. $\mathbb{Q}$ is an example of an ordered set, where we take $<$ to be the usual less-than relation. More explicitly: when $q > 0$ and $q' > 0$, we have $p/q < p'/q'$ if and only if $pq' < p'q$ (using the usual less-than relation on $\mathbb{Z}$.)

Given an ordered set, we might ask, what is the *maximum* element of that set? This question *sometimes* has an answer, but not always.

Definition 3.12. If S is an ordered set, a *maximum* element in S is an element $y \in S$ such that for all $x \in S$ we have $x \leq y$.

Example 3.13. The ordered set in Example 3.10 has maximum element 8.

[2]The exception is in the discussion of eigenvalues, eigenvectors, diagonalizability, ... For this purpose you have to find solutions of algebraic equations, and this is guaranteed to work in some fields, not in others.

[3]We have not given a formal definition of a"relation" on a set S. One way to formalize it is to say that a relation $<$ on S is a subset P of $S \times S$, i.e. a set P of ordered pairs (x, y): then we write $x < y$ iff $(x, y) \in P$.

Example 3.14. The set of all prime numbers is an ordered set which does not have a maximum element.

Example 3.15. The set $\{x \in \mathbb{Q} \mid x \leq 1\}$ is an ordered set with maximum element 1.

Example 3.16. The set $\{x \in \mathbb{Q} \mid x < 1\}$ is an ordered set which does not have a maximum element.

Proposition 3.17. If S has a maximum element, then that maximum is unique. (In other words, if y and y' are both maximum elements in S, then $y = y'$.)

Proof. If y and y' are both maximum elements, then $y \leq y'$ and $y' \leq y$, which implies $y = y'$ as desired. □

In this case we write $\max S$ for the maximum element of S.

We have the following important fact:

Proposition 3.18. If S is a finite and nonempty ordered set, then S has a maximum element.

Hopefully this seems intuitively reasonable; just to be sure, and to work our proof-muscles a little, we provide a proof.

Proof. By induction. (Remark: almost any statement which is true for finite sets, and only true for finite sets, will ultimately boil down to induction!) For each $n \in \mathbb{N}$ we will prove: *if S is finite, with $|S| = n$, then S has a maximum element.*

First, suppose $n = 1$. Then S has just a single element, and you can check that it is indeed a maximum. Next, suppose we have proven the statement for n and we want to prove it for $n + 1$. So, suppose $|S| = n + 1$. Then choose any element $z \in S$, and let $S' = S - \{z\}$. Then $|S'| = n$, so by the inductive hypothesis, S' has a maximum element y'. We know $z \neq y'$. We consider the cases $z > y'$ and $z < y'$ separately:

- Suppose $z > y'$. Then we claim z is a maximum for S. For any $x \in S$ we have either $x \in S'$ or $x = z$. If $x \in S'$ then $x \leq y' < z$, so $x \leq z$ as desired. If $x = z$ then $x \leq z$ as desired.
- Suppose $z < y'$. Then we claim y' is a maximum for S. For any $x \in S$ we have either $x \in S'$ or $x = z$. If $x \in S'$ then $x \leq y'$ as desired. If $x = z$ then $x = z < y'$, so $x \leq y'$ as desired.

Thus, in either case S has a maximum, as desired. □

Sometimes a set does not have a maximum but still has an upper bound:

Definition 3.19. If S is an ordered set, and $A \subset S$, an *upper bound* for A is an element $y \in S$ such that, for all $x \in A$, $x \leq y$. We say A is *bounded above* if there is an upper bound for A.

If A has a maximum, then $\max A$ is an upper bound for S. A maximum is unique if it exists, but a set which is bounded above will often have many different upper bounds.

Example 3.20. The set $\{x \in \mathbb{Q} \mid x \leq 1\}$ is bounded above. More precisely, any $y \geq 1$ is an upper bound for this set (and any $y < 1$ is not). (Can you prove this?)

Example 3.21. The set $\{x \in \mathbb{Q} \mid x < 1\}$ is likewise bounded above, even though it does not have a maximum element. Any $y \geq 1$ is an upper bound for this set (and any $y < 1$ is not), just as in the previous example. (Can you prove this?)

Example 3.22. The set $\mathbb{Z} \subset \mathbb{Q}$ has no upper bound, i.e. it is not bounded above.

Among all the upper bounds for a given set A, we will be particularly interested in the smallest one, if it exists:

Definition 3.23. If S is an ordered set, and $A \subset S$, a *least upper bound* for A in S is a number $y \in S$, such that y is an upper bound for A, and any upper bound y' for A has $y' \geq y$. If y is a least upper bound for A, we also call y a *supremum* of A.

Example 3.24. The set $A = \{x \in \mathbb{Q} \mid x \leq 1\}$ has least upper bound $y = 1$. (Indeed, $y = 1$ is an upper bound for A, since all $x \in A$ have $x \leq 1$. Now suppose given some y' which is an upper bound for A; then we must have $1 \leq y'$, since $1 \in A$. This means that $y = 1$ is a least upper bound for A, as desired.)

Example 3.25. The set $\{x \in \mathbb{Q} \mid x < 1\}$ has least upper bound $y = 1$. (Can you prove it?)

Proposition 3.26. Given an ordered set S and $A \subset S$, suppose y and y' are both least upper bounds of A; then $y = y'$.

Proof. Since y and y' are both least upper bounds of A we must have $y \leq y'$ but also $y' \leq y$, from which it follows that $y = y'$. □

Thus we may speak of *the* least upper bound for A, instead of just *a* least upper bound. We also call the least upper bound the *supremum* of A, and write it $\sup A$. If A is not bounded above, then it does not have any supremum in the sense just defined; we sometimes write in this situation $\sup A = +\infty$.

Proposition 3.27. If $\max A$ exists, then $\sup A = \max A$.

In parallel we can also define *lower bound* instead of upper bound:

Definition 3.28. If S is an ordered set, and $A \subset S$, a *lower bound* for A is an element $y \in S$ such that, for all $x \in A$, $x \geq y$. We say A is *bounded below* if there is a lower bound for A.

Definition 3.29. If S is an ordered set, and $A \subset S$, a *greatest lower bound* for A in S is a number $y \in S$, such that y is a lower bound for A, and any lower bound y' for A has $y' \leq y$. If y is a greatest lower bound for A, we also call y an *infimum* of A.

Proposition 3.30. Given an ordered set S and $A \subset S$, suppose y and y' are both greatest lower bounds of A; then $y = y'$.

We write $\inf A$ for the infimum of A, if it exists.

3.3. **A hole in $\mathbb{Q}$.** Now here is a more interesting example.

Proposition 3.31. There is no $x \in \mathbb{Q}$ such that $x^2 = 2$.

Proof. Suppose $x = a/b \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$, and a/b is in lowest terms. Assume $x^2 = 2$. Then $a^2 = 2b^2$ and in particular a must be even. But then a^2 is divisible by 4, so since $a^2 = 2b^2$ it follows that b must be even. So both a and b are even, contradicting the assumption that a/b is in lowest terms. □

So $\mathbb{Q}$ contains the numbers 1, 1.4, 1.41, 1.414, 1.4142, ..., which come closer and closer to being a square root of 2, and yet $\mathbb{Q}$ does not contain an actual square root of 2. Our intuitive picture is that $\mathbb{Q}$ has a "hole" where the positive square root of 2 ought to be.

Let's formalize this a bit, as a failure of the least upper bound property. We consider the set $A \subset \mathbb{Q}$ defined by

$$A = \{x \in \mathbb{Q} \mid x^2 < 2, x > 0\}\,. \tag{3.1}$$

Then we claim:

Proposition 3.32. For $y \in \mathbb{Q}$, y is an upper bound for A if and only if $y^2 > 2$ and $y > 0$.

Proof. ($\Longleftarrow$) First suppose $y^2 > 2$ and $y > 0$. We want to show y is an upper bound for A. So consider any $x \in A$. We have $y^2 > 2 > x^2$, so $y^2 > x^2$. If $x > y$, then we would get $x^2 > xy$ and $xy > y^2$, and thus $x^2 > y^2$, which would be a contradiction. So we conclude $x \leq y$. Thus y is an upper bound for A.

($\Longrightarrow$) Now suppose y is an upper bound for A. Since $1 \in A$, we have $y \geq 1$, and $1 > 0$, so $y > 0$. We want to show $y^2 > 2$. Let's suppose the opposite: what if $y^2 \leq 2$? In that case, we will show that there is some $x \in A$ with $x > y$, contradicting the assumption that y is an upper bound. Although the strategy is simple, the process of finding such an x will take a little while to go through in detail.

First version. How do we find the desired x? We can try writing

$$x = y + \epsilon$$

for some positive $\epsilon \in \mathbb{Q}$ which we imagine to be very small. We want to make sure that $x \in A$, i.e. $x^2 < 2$. So we are trying to produce a solution ϵ to the inequalities

$$(y + \epsilon)^2 < 2, \qquad \epsilon > 0\,. \tag{3.2}$$

There are various ways to attack this.[4] Here I'll choose one which is definitely not the quickest, but illustrates some "analytic" philosophy. We first write the condition out as

$$y^2 + 2\epsilon y + \epsilon^2 < 2, \qquad \epsilon > 0\,.$$

Next note that since ϵ is very small, ϵ^2 will be absolutely tiny, and it probably does not play an essential role in the story. With that in mind, we use the same strategy we would use if the ϵ^2 were not there. Namely, we rewrite the condition as

$$2\epsilon y + \epsilon^2 < 2 - y^2, \qquad \epsilon > 0\,,$$

[4] One simple method is given in Rudin: he just picks

$$\epsilon = \frac{2 - y^2}{y + 2}\,.$$

Indeed, once you have somehow come up with this formula, it is not too hard to check that it works — the proof in Rudin is only a few lines!

then rewrite that as

$$\epsilon(2y+\epsilon) < 2 - y^2, \qquad \epsilon > 0\,,$$

then divide through (using the fact $2y + \epsilon \neq 0$) to get

$$0 < \epsilon < \frac{2-y^2}{2y+\epsilon}\,. \tag{3.3}$$

Note that all these steps are reversible: the condition (3.3) is equivalent to the desired (3.2).

We want to know that there exists a solution of (3.3). Since the numerator $2 - y^2 > 0$, and also the denominator $2y + \epsilon > 0$, things look hopeful. The tricky point is that ϵ appears on the right side as well as the left (this is an echo of the tiny ϵ^2 term we want to ignore). Here is the key: the ϵ on the right side will only cause a problem if it is *large*. But we are free to choose what ϵ is, and moreover we have been imagining that ϵ is supposed to be very small!

Let's make the smallness of ϵ explicit: suppose that $\epsilon < 1$. (We could choose any number instead of 1 here.) In that case, $2y + \epsilon < 2y + 1$, and thus $\frac{2-y^2}{2y+\epsilon} > \frac{2-y^2}{2y+1}$. Thus, instead of our original condition, we can instead look for an ϵ satisfying the stronger condition

$$0 < \epsilon < \frac{2-y^2}{2y+1}, \qquad \epsilon < 1\,,$$

or equivalently,

$$0 < \epsilon < \min\left(\frac{2-y^2}{2y+1}, 1\right)\,.$$

Now we can write down an actual $\epsilon \in \mathbb{Q}$ obeying this condition, for instance,

$$\epsilon = \frac{1}{2}\min\left(\frac{2-y^2}{2y+1}, 1\right)\,.$$

Then $x = y + \epsilon \in A$, showing that y cannot be an upper bound for A, as desired.

Second version. Here is a more streamlined version of the same proof. Let

$$\epsilon = \frac{1}{2}\min\left(\frac{2-y^2}{2y+1}, 1\right)\,.$$

Then, since $y > 0$ and $2 - y^2 > 0$, we have

$$0 < \epsilon < \min\left(\frac{2-y^2}{2y+1}, 1\right)\,.$$

Let $x = y + \epsilon$. We will show that $x \in A$. Indeed,

$$\epsilon < \frac{2-y^2}{2y+1} < \frac{2-y^2}{2y+\epsilon}\,,$$

and multiplying both sides by $2y + \epsilon$ this becomes

$$2\epsilon y + \epsilon^2 < 2 - y^2\,,$$

which rearranges to

$$(y+\epsilon)^2 < 2\,,$$

i.e. $x^2 < 2$. Also we know that $x > 0$, since both y and ϵ are positive. Thus we conclude that $x \in A$. But $x > y$, since $\epsilon > 0$. Thus y cannot be an upper bound for A. □

So now we understand the upper bounds of A completely. In particular, we can now show that there is no smallest one:

Proposition 3.33. The set $A \subset \mathbb{Q}$ given in (3.1) is bounded above in $\mathbb{Q}$, but has no least upper bound in $\mathbb{Q}$.

Proof. Suppose $y \in \mathbb{Q}$ is an upper bound for A. We will show that there is another upper bound $y' \in \mathbb{Q}$ for A which is smaller than y, so that y cannot be a least upper bound.

Since y is an upper bound for A, by Proposition 3.32 we have $y^2 > 2$ and $y > 0$. We will take $y' = y - \epsilon$, with $\epsilon \in \mathbb{Q}$, $\epsilon > 0$, but ϵ small enough that y' is still an upper bound. Again using Proposition 3.32, y' will be an upper bound if ϵ satisfies

$$(y-\epsilon)^2 > 2 \quad , \qquad y - \epsilon > 0\,.$$

Expanding this out, it becomes

$$y^2 - 2\epsilon y + \epsilon^2 > 2 \quad , \qquad y - \epsilon > 0\,. \tag{3.4}$$

Now we want to see whether there is actually a solution ϵ to (3.4). This is a problem similar to what we faced in proving Proposition 3.32: the ϵ^2 term makes it harder to see whether there is a solution. In this case, the ϵ^2 is helping us rather than hurting us — it makes the inequality easier to satisfy — so we can try just giving that ϵ^2 term up.[5] More precisely: because

$$y^2 - 2\epsilon y + \epsilon^2 > y^2 - 2\epsilon y$$

the first condition in (3.4) will be satisfied if $y^2 - 2\epsilon y > 2$. By rearranging, this is equivalent to $\epsilon < \frac{y^2-2}{2y}$. The second condition in (3.4) will be satisfied if $\epsilon < y$.

Thus, altogether, (3.4) is satisfied if $\epsilon > 0$, $\epsilon < y$ and $\epsilon < \frac{y^2-2}{2y}$. One choice which works is[6]

$$\epsilon = \frac{1}{2} \min\left(y, \frac{y^2-2}{2y}\right)\,.$$

Then $y' = y - \epsilon \in \mathbb{Q}$ is an upper bound for A with $y' < y$, as desired. □

3.4. **Ordered fields.** When a field is ordered, the ordering should obey the standard rules of arithmetic for inequalities, which we formalize below:

Definition 3.34. An *ordered field* is a field F which is also an ordered set, such that

(1) For all $x, y, z \in F$ such that $x + y < x + z$, we have $y < z$.
(2) For all $x, y \in F$ such that $x > 0$ and $y > 0$, we have $xy > 0$.

[5] This kind of maneuver — dropping a term to simplify an inequality — is very common in analysis. There is some risk involved: if we drop too much, then the inequality we are left with might not be solvable, even if the original inequality was. In this case, we are thinking of ϵ as very small, and in particular much smaller than all the other terms in our inequality; so we might imagine that dropping the ϵ^2 will not hurt us too much, and indeed this turns out to be true.

[6] In Rudin's version of this proof he chooses $\epsilon = \frac{y^2-2}{y+2}$. (Why does this work?)

Example 3.35. $\mathbb{Q}$ is an ordered field.

Example 3.36. Not every field can be ordered; for example, the field $\mathbb{F}_3$ which we introduced above cannot be made into an ordered field. (Why?)

In an ordered field, we have a notion of "positive" and "negative": $x \in F$ is positive if $x > 0$ and negative if $x < 0$.

Here are some examples of consequences of the axioms of an ordered field:

Proposition 3.37. Suppose F is an ordered field. Then, in F, $-1 < 0 < 1$.

Proof. Since $1 \neq 0$ we must have either $1 > 0$ or $0 > 1$. Suppose (for a contradiction) that $0 > 1$; then adding -1 to both sides gives $-1 > 0$, so $(-1)^2 > 0$. But $(-1)^2 = 1$ so we conclude that $1 > 0$, giving a contradiction. Thus $1 > 0$. Then adding -1 to both sides gives $0 > -1$. □

Proposition 3.38. Suppose F is an ordered field, $x > 0$ and $y < 0$. Then $xy < 0$.

Proof. Exercise. □

More generally, the axioms in Definition 3.34 imply all the standard rules of arithmetic manipulation of inequalities, and we will freely use them from now on.

3.5. **The least upper bound property.** We are getting close to being able to identify $\mathbb{R}$: it is an ordered field, but with one more key property — it has no "holes" in it, in the following sense.

Definition 3.39. An ordered set S has the *least upper bound property* if every nonempty subset $A \subset S$ which is bounded above has a least upper bound.

We have seen (in Proposition 3.33) that $\mathbb{Q}$ does not have the least upper bound property. This problem can be repaired:

Theorem 3.40. There exists an ordered field $\mathbb{R}$ with the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$, as a subfield (this means that all the arithmetic operations on this subset of $\mathbb{R}$ agree with the usual ones in $\mathbb{Q}$). Moreover, if F is any ordered field with the least upper bound property, then $F \simeq \mathbb{R}$.[7]

By *real number* we mean an element of $\mathbb{R}$.

We will not prove this theorem yet; we might prove it later in the term. There are various different methods, which lead to quite different pictures of what a real number

[7] The symbol $\simeq$ here means that F is *isomorphic to* $\mathbb{R}$, which in turn means that there is a bijective map $\phi : F \to \mathbb{R}$, and the $+, \cdot$ operations in F match with those in $\mathbb{R}$, i.e. $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$.

"is."[8] We will not commit ourselves to any particular picture; rather, everything we will use about $\mathbb{R}$ follows just from Theorem 3.40. Let us now see a few examples.

3.6. The Archimedean property and density of the rationals.

Proposition 3.41. If $x, y \in \mathbb{R}$ and $x > 0$, then there is some natural number n such that $nx > y$.

Proof. Suppose there is no such n. Then, for every $n \in \mathbb{N}$, $y \geq nx$. In that case, the set $A = \{nx \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is bounded above — indeed y is an upper bound for A. Now let $z = \sup A$ (using the least upper bound property!) Because $x > 0$, $z - x < z$; thus $z - x$ is *not* an upper bound for A, so there must exist some $n \in \mathbb{N}$ such that $nx > z - x$. But that means $(n+1)x > z$, contradicting the fact that z is an upper bound for A. □

Proposition 3.41 is sometimes called the *Archimedean property* of $\mathbb{R}$. It says roughly that there are no infinitely small or infinitely large quantities in $\mathbb{R}$: for any x, y the ratio y/x is bounded by some integer.

In our proof of Proposition 3.41 we used the least upper bound property. You might wonder if we could have gotten by without it: could it be that every ordered field has the Archimedean property? It turns out that the answer is no: there are tons of ordered fields which do not have the Archimedean property. (Google "surreal numbers" for one remarkable example.)

Here is another important fact about the real numbers: between every two reals there is a rational.

Proposition 3.42. If $x, y \in \mathbb{R}$ with $x < y$, then there exists some $z \in \mathbb{Q}$ with $x < z < y$.

Proof. The essential idea is that the distance $y - x$ is itself some real number, which (as we have just been discussing) cannot be infinitely small: by multiplying by a big enough integer, we can make that distance greater than 1, and then we should be able to find an integer inbetween.

Now let us implement this idea. By Proposition 3.41, there exists some $m \in \mathbb{N}$ such that $m(y - x) > 1$, i.e. $my - mx > 1$. Then, our intuition about real numbers would suggest that there should exist some $n \in \mathbb{Z}$ such that $mx < n < my$. If we find such an n, then dividing by m will give $x < \frac{n}{m} < y$, so taking $z = \frac{n}{m} \in \mathbb{Q}$ will complete the proof.

How to see that such an n really exists? Concretely, we could say that we "round down my to the nearest integer," i.e. we take the largest integer n such that $n < my$. The picture below suggests that this should work.

[8] If you are interested, I encourage you to look at pages 17-21 of Rudin. The approach he describes identifies real numbers with *Dedekind cuts*. By definition, a Dedekind cut is a pair (A, B) where A and B are nonempty disjoint subsets of $\mathbb{Q}$, with $\mathbb{Q} = A \cup B$, such that for any $a \in A$ and any $b \in B$ we have $a < b$, and A has no maximum element. The idea is that each Dedekind cut (A, B) should be thought of as representing some real number x, and we will have $A = \{t \in \mathbb{Q} \mid t < x\}$, $B = \{t \in \mathbb{Q} \mid t \geq x\}$. There is another nice account of this at `https://www.math.brown.edu/reschwar/INF/handout3.pdf`.

Other approaches are certainly possible — for instance, with sufficient care one could define real numbers in terms of decimal expansions, and then prove that they form a field with the least upper bound property.

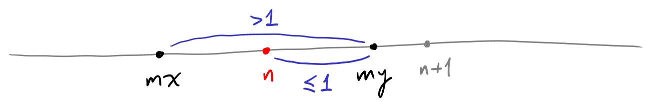

FIGURE 6. Finding an n between mx and my.

Still it is a bit tricky to complete the argument, using only the facts about $\mathbb{R}$ that we have so far. The trouble is that the set of integers $n < my$ is infinite, so we do not know that it actually has a maximum element!

To deal with this problem, we use Proposition 3.41 again, to show that there is an integer $k > my$, and another integer $k' < my$. Then we consider the set of integers n obeying $k' \leq n < my$. This set is finite (since there are only finitely many integers between k' and k), and nonempty (since it contains k'), so it has a maximum by Proposition 3.18; let n be this maximum. All that remains is to prove that $mx < n < my$ as desired.

We know already that $n < my$. On the other hand $n + 1 \geq my$ (since n is the maximum), so $n \geq my - 1$, and $my > 1 + mx$, so we conclude that $n > (1 + mx) - 1 = mx$, as needed. □

3.7. **Existence of square roots.** Now let's see one of the good features of $\mathbb{R}$.

Proposition 3.43. For every $x > 0$ there exists a unique $y \in \mathbb{R}$ such that $y^2 = x$ and $y > 0$.

Proof. Consider the set
$$A = \{t \in \mathbb{R} \mid t > 0, t^2 < x\}.$$
This set is nonempty (it contains $t = \min(\frac{1}{2}, x)$) and bounded above (by $M = \max(1, x)$). Thus it has a supremum in $\mathbb{R}$; let
$$y = \sup A.$$
We will prove that $y^2 = x$.

First suppose (for a contradiction) that $y^2 < x$. Then, we would like to show that y cannot be an upper bound for A. We will do it by producing some $t \in A$ with $t > y$. Choose ϵ so that
$$0 < \epsilon < \min\left(1, \frac{x - y^2}{2y + 1}\right),$$
and pick $t = y + \epsilon$. Then
$$(y + \epsilon)^2 - y^2 = \epsilon(2y + \epsilon) < \epsilon(2y + 1) < x - y^2$$
so
$$(y + \epsilon)^2 < x$$
as desired. This contradiction shows that we cannot have $y^2 < x$.

Next suppose (again for a contradiction) that $y^2 > x$. Then, we would like to show that y cannot be the least upper bound for A, by producing some t which is an upper bound for A and has $t < y$. The strategy is similar to the above: we write $t = y - \epsilon$, and show that for a sufficiently small $\epsilon > 0$, t is an upper bound for A. I omit the details here, since we've now done this kind of thing a few times! The conclusion is that we cannot have $y^2 > x$.

Since we have neither $y^2 > x$ nor $y^2 < x$, we conclude that $y^2 = x$, as desired.

Finally, to see the uniqueness we use the fact that the squaring function is monotone increasing for positive reals. Indeed, suppose y, y' are both positive and $y \neq y'$. If $y > y'$, then $y^2 > yy' > y'^2$, so in particular $y^2 \neq y'^2$. If $y' > y$ then similarly we get $y^2 \neq y'^2$. $\square$

Now, for $x > 0$, let $\sqrt{x}$ be the unique positive real number with $\sqrt{x}^2 = x$ (Proposition 3.43 shows that this exists).

Proposition 3.44. For $x, x' \in \mathbb{R}$ we have

$$\sqrt{x}\sqrt{x'} = \sqrt{xx'}\,.$$

Proof. Let $y = \sqrt{x}\sqrt{x'}$. First note that y is positive. Next, note that

$$y^2 = (\sqrt{x}\sqrt{x'})^2 = (\sqrt{x})^2(\sqrt{x'})^2 = xx'$$

But Proposition 3.43 shows that the only positive solution of the equation $y^2 = xx'$ is $\sqrt{xx'}$. Thus we have $y = \sqrt{xx'}$, as desired. $\square$

Here we just discussed square roots; in Rudin's book you can find a very similar discussion of n-th roots $\sqrt[n]{x}$, for any $n \in \mathbb{N}$.

3.8. **Real exponents.** In the last section we discussed n-th roots of positive real numbers x. Once we have these defined, we can also define rational powers of x. First, there is an important technical fact: if $p/q = p'/q'$ with $q, q' > 0$, then we have (exercise!)

$$(\sqrt[q]{x})^p = (\sqrt[q']{x})^{p'}\,.$$

With this in mind we define:

Definition 3.45. If $\beta \in \mathbb{Q}$ and $x \in \mathbb{R}$, $x > 0$, represent β as p/q with $q > 0$, and let

$$x^\beta = (\sqrt[q]{x})^p\,.$$

More generally we can define *real* powers of x:

Definition 3.46. Supose $x \in \mathbb{R}$, $x > 1$, and $\alpha \in \mathbb{R}$. Then

$$x^\alpha = \sup\{x^\beta \mid \beta \in \mathbb{Q}, \beta \leq \alpha\} \in \mathbb{R}\,.$$

Suppose $x \in \mathbb{R}$, $x < 1$, and $\alpha \in \mathbb{R}$. Then

$$x^\alpha = \inf\{x^\beta \mid \beta \in \mathbb{Q}, \beta \leq \alpha\} \in \mathbb{R}\,.$$

With these definitions we have all the standard properties of exponentiation: in particular $x^\alpha x^{\alpha'} = x^{\alpha+\alpha'}$.

4. Countable and uncountable

4.1. **Countability.** Earlier we discussed the notion of finite and infinite sets.

We may say that two sets A, B have the same *cardinality* if there is a bijective map $A \to B$.[9] Then two finite sets A, B have the same cardinality if and only if $|A| = |B|$. One might wonder whether all infinite sets have the same cardinality — in other words, are all infinities the same size?

[9] Informally, read "cardinality" as "size."

As it turns out, the answer is no. It will take us a little while to see why. We start by considering a particular kind of infinite set:

Definition 4.1. We say S is *countable* if there is a bijective map $f : \mathbb{N} \to S$.

Proposition 4.2. If S is finite, then S is not countable.

Proof. We leave this as an exercise (hint: do it by induction on $|S|$). □

Informally, S is countable if we can organize its elements into an infinite list: indeed, we could write

$$S = \{f(1), f(2), f(3), \dots\}$$

or, using the notation $x_n = f(n)$, we could also write

$$S = \{x_1, x_2, x_3, \dots\}$$

Example 4.3. $\mathbb{N}$ is countable. Indeed we have a bijection $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = n.$$

Example 4.4. $\mathbb{Z}$ is countable. Indeed we can list its elements:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

More formally, we can write a bijection $f : \mathbb{N} \to \mathbb{Z}$ by $f(1) = 0$, $f(2) = 1$, $f(3) = -1, \dots,$ or

$$f(n) = \begin{cases} -\frac{1}{2}(n-1) & \text{if } n \text{ is odd} \\ \frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$$

Proposition 4.5. If A is countable, and there is a bijection $\phi : A \to B$, then B is also countable.

Proof. Exercise. □

Proposition 4.6. If S is countable, and A is an infinite subset of S, then A is also countable.

Proof. The intuition here is: we write the list of elements of S as

$$S = \{x_1, x_2, x_3, \dots\}$$

To list the elements of A, we run along this list of elements of S, starting from the beginning; whenever we encounter an element of A, we add it to our new list. We will eventually hit every element of A in this way, so the resulting list will consist exactly of the elements of A, as desired.

Just to be complete (and to allay any fears we are sneaking something under the rug), let's give a more formal account of this construction. We define a sequence of numbers $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}, \dots$ iteratively:

m_1 is the smallest number with $x_{m_1} \in A$,
m_2 is the smallest number with $x_{m_2} \in A$ and $m_2 > m_1$,
$\dots,$
m_n is the smallest number with $x_{m_n} \in A$ and $m_n > m_{n-1}$,

...

(At each stage, you might ask: why does such a number m_n exist? We are defining m_n as the minimum element of a certain set, so we should be careful to check that this set actually has a minimum element! Fortunately, *every* nonempty subset of $\mathbb{N}$ has a minimum element. This is sometimes expressed as the statement that $\mathbb{N}$ is *well-ordered*. We'll consider this as part of our background knowledge about $\mathbb{N}$.)

Then we define $g : \mathbb{N} \to A$ by

$$g(n) = x_{m_n}.$$

It only remains to prove that this g is a bijection:

- To prove g is injective, we use $n \neq n' \implies m_n \neq m_{n'} \implies x_{m_n} \neq x_{m_{n'}}$.
- To prove g is surjective, consider some $x_N \in A$. Let n be the smallest index such that $m_n \geq N$. We will show that also $m_n \leq N$. Indeed, this follows from the definition of m_n, and the facts that $x_N \in A$ and $m_{n-1} < N$. So we conclude that $m_n = N$. Then $g(n) = x_{m_n} = x_N$, as desired.

□

Example 4.7. The set of all prime numbers is an infinite subset of $\mathbb{N}$, so it is countable.

Definition 4.8. If S is a set, a *sequence* in S is any map $\phi : \mathbb{N} \to S$. Giving such a map is equivalent to giving the list

$$\phi(1), \phi(2), \phi(3), \ldots$$

We often write that list instead as

$$x_1, x_2, x_3, \ldots$$

and use the notation $(x_n)_{n=1}^{\infty}$, or just (x_n), for the sequence.

Example 4.9. The function $\phi(n) = -n^2$ gives a sequence in $\mathbb{Z}$, which we also write $x_n = -n^2$, or

$$(x_n) = -1, -4, -9, \ldots$$

Example 4.10. A sequence is allowed to repeat elements: e.g. the sequence $x_n = (-1)^n$, or

$$(x_n) = -1, 1, -1, \ldots$$

Note that according to this definition a sequence is always an infinite list, not a finite one; e.g. the list $1, 3, 9, 27, 81$ is not a sequence by this definition.

Definition 4.11. We say S is *at most countable* if it is either finite or countable.

Corollary 4.12. Given a sequence (x_n) in any set S, the set $A = \{x \in S \mid x = x_n \text{ for some } n\}$ is at most countable.

Proof. The intuition is: we want to build a list of all elements of A. For this purpose we just go down the sequence

$$x_1, x_2, x_3, \ldots$$

and every time we meet an element we have not seen previously, we add it to the list of elements in A.

More formally, we can say this as follows. We consider the set

$$B = \{n \in \mathbb{N} \mid x_n \neq x_m \text{ for all } m < n\}.$$

This is a subset of $\mathbb{N}$, so it is at most countable by Proposition 4.6. There is a map $\phi : B \to A$, taking $\phi(n) = x_n$. This map is injective, by definition of B. To see it is surjective, consider any $x \in A$, and let $C = \{n \in \mathbb{N} \mid x_n = x\}$. C is a nonempty set of natural numbers, so it has a smallest element m. Then $m \in B$ and $\phi(m) = x$. So $\phi : B \to A$ is bijective; thus, since B is at most countable, A is also at most countable. $\square$

Example 4.13. Suppose A and B are countable; then $A \cup B$ is countable. Indeed, if $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ then we can consider the sequence

$$a_1, b_1, a_2, b_2, a_3, \dots$$

$A \cup B$ is the set of elements of this sequence, and thus it is at most countable; but it contains A which is infinite, so it cannot be finite; this it is countable.

You can similarly convince yourself that e.g. the union of two countable sets is countable, or the union of three countable sets, or generally the union of any finite number of countable sets.

Now what if we take the union of *infinitely* many sets?

Proposition 4.14. If each A_n is at most countable, then $\bigcup_{n=1}^{\infty} A_n$ is at most countable.

Proof. It is easiest to think of the construction via a picture. We arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in a grid, as shown below.

$$\begin{array}{l}
A_1 = \{a_{11} \to a_{12},\ a_{13} \to a_{14}, \dots\} \\
A_2 = \{a_{21},\ a_{22},\ a_{23},\ a_{24}, \dots\} \\
A_3 = \{a_{31},\ a_{32},\ a_{33},\ a_{34}, \dots\} \\
A_4 = \{a_{41},\ a_{42},\ a_{43},\ a_{44}, \dots\} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

(If some A_n is finite, then we just repeat the last element infinitely many times on the n-th row.) Now following the arrows we make a sequence which contains all of these elements:

$$a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, \dots$$

The set $\bigcup_{n=1}^{\infty} A_n$ is the set of elements which occur in this sequence, so it is at most countable. $\square$

Proposition 4.15. If S is countable, then the set of all n-tuples

$$S^n = \{(x_1, \dots, x_n) \mid x_i \in S \text{ for each } i\}$$

is also countable.

Proof. We just give the proof for $n = 2$ here. (The general case is similar by induction, or see Rudin.) So we assume S is countable and we want to show S^2 is countable. For each fixed $x \in S$ we let $S_x = \{(x,y) \mid y \in S\}$. Then

$$S^2 = \bigcup_{x \in S} S_x$$

which is a union of a countable number of countable sets, so it's countable by Proposition 4.14. □

Corollary 4.16. $\mathbb{Q}$ is countable.

Proof. $\mathbb{Q} \subset \{(p,q) \mid p,q \in \mathbb{Z}\}$, so it is a subset of a countable set; moreover we know $\mathbb{Q}$ is infinite (e.g. because it contains $\mathbb{N}$); so $\mathbb{Q}$ is countable. □

Corollary 4.17. $\mathbb{Q}^n$ is countable.

4.2. **The uncountability of $\mathbb{R}$.** We have seen that lots of infinite sets are countable, and you might at this point believe that every infinite set is countable.

Definition 4.18. We say S is *uncountable* if it is neither finite or countable.

Now here is a surprising fact:

Theorem 4.19 (Cantor). The interval $(0,1) \subset \mathbb{R}$ is uncountable.

Proof. Suppose given any infinite sequence of elements in $(0,1)$,

$$x_1, x_2, x_3, \ldots$$

We want to show that this list cannot contain *all* the elements of $(0,1)$, by constructing an element $y \in (0,1)$ which has $y \neq x_n$ for all n.

First, let $I_1 = [a_1, b_1]$ be any closed interval contained in $(0,1)$, with $b_1 > a_1$, and $x_1 \notin I_1$. Then, let $I_2 = [a_2, b_2]$ be any closed interval, with $b_2 > a_2$, such that $I_2 \subset I_1$ and $x_2 \notin I_2$. Continuing in this way, iteratively construct a nested sequence of intervals $I_n = [a_n, b_n]$ such that each $I_{n+1} \subset I_n$ and $x_n \notin I_n$.[10]

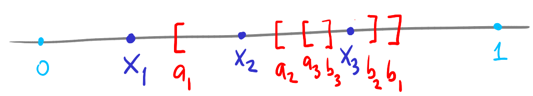

FIGURE 7. Building a nested sequence of intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subset I_n$ and $x_n \notin I_n$.

We claim that there exists some $y \in (0,1)$, such that $y \in I_n$ for all n. Indeed, we can let

$$y = \sup A, \qquad A = \{a_n \mid n \in \mathbb{N}\}.$$

[10]How do we know such I_n exists? To remove all doubt we could give an algorithm for constructing I_n from the data of I_{n-1} and x_n. Here is one algorithm that would work: if $x_n \notin I_{n-1}$ then we just take $I_n = I_{n-1}$; if $x_n = a_{n-1}$ then we take $a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$, $b_n = b_{n-1}$; if $x_n \in (a_{n-1}, b_{n-1}]$ then we take $a_n = a_{n-1}$, $b_n = \frac{1}{2}(a_{n-1} + x_n)$.

This y has $y \geq a_n$ for all n (since it is an upper bound for A) but also $y \leq b_n$ for all n (since every b_n is also an upper bound for A). Thus $y \in I_n$ for all n. But since $x_n \notin I_n$ this means that $y \neq x_n$ for all n. □

So we have shown that the interval $(0, 1)$ is not countable: any effort to list all its elements is doomed to miss one. Since $(0, 1) \subset \mathbb{R}$ it follows that $\mathbb{R}$ is not countable either.

Incidentally, in the process of this proof, we have *en passant* proven the following:

Proposition 4.20. If $I_1, I_2, \ldots$ is a sequence of closed intervals in $\mathbb{R}$, with $I_{n+1} \subset I_n$, then the intersection $\bigcap_{n=1}^{\infty} I_n$ is not empty.

4.3. **Cantor's diagonal argument.** What we gave above is roughly Cantor's original proof of the uncountability of $(0, 1)$, published in 1874. The statement was tremendously controversial at the time. Perhaps *because* it was controversial, Cantor gave several other proofs over the following decades. His most famous proof goes via the "diagonal argument," as follows.

Cantor used the representation of real numbers as infinite decimals.[11] Suppose that we have a list of real numbers in the interval $(0, 1)$, and we imagine writing out all the entries as infinite decimals: for instance:

$$\begin{aligned} x_1 &= 0.649514576... \\ x_2 &= 0.871768678... \\ x_3 &= 0.999888777... \\ x_4 &= 0.717657810... \\ x_5 &= 0.662901691... \end{aligned}$$

We highlight the k-th digit of each number x_k:

$$\begin{aligned} x_1 &= 0.\mathbf{6}49514576... \\ x_2 &= 0.8\mathbf{7}1768678... \\ x_3 &= 0.99\mathbf{9}888777... \\ x_4 &= 0.717\mathbf{6}57810... \\ x_5 &= 0.6629\mathbf{0}1691... \end{aligned}$$

We now write out a new number y as follows: at the k-th decimal place we write 6, unless the k-th digit of x_k is 6, in which case we write 7. So for the list above, we would write

$$y = 0.76676...$$

Now y is a real number in the interval $(0, 1)$. By the way we constructed y, we see that $y \neq x_1$ (they differ in the first decimal place), $y \neq x_2$ (they differ in the second decimal place), and similarly, $y \neq x_n$ for all n. Thus our list cannot contain the number y!

(You might wonder: why can't we just add y to the list? We can — but once we do that, we can apply the same procedure to the *new* list, and produce a new y' which is not on that list. What the diagonal argument shows is that, for *any* list of real numbers we might write down, there is always some number that is missing.)

[11] We haven't developed this approach, but never mind that; we already gave a proof that $\mathbb{R}$ is uncountable in our framework; we're just looking for more intuition, now.

This proof was informal, just because we used the idea that real numbers are decimals, without carefully developing it. But the same diagonal argument can be used rigorously to prove that various other sets are uncountable. For example:

Example 4.21. Let $S = \{0, 1\}$, a set with two elements. The set of all sequences in S is uncountable. You can prove this using a diagonal argument. (Try it!)

5. Metric spaces

In analysis we are often concerned with the issue of how *close* two things are, where "things" could be real numbers, but could also be something more abstract, like n-tuples of real numbers, or even infinite sequences, or functions, n-tuples of functions, ...

Even though all these situations are different, they have important common features. So we are going to develop some general technology for dealing with all these situations at once.

5.1. The space $\mathbb{R}^n$ and its distance function.

Recall that $\mathbb{R}^n$ is the space of n-tuples in $\mathbb{R}$. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. We think of an n-tuple either as a vector (arrow) or as a point, depending on the context.

Definition 5.1. The *inner product* or *dot product* of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \,.$$

Note this is *not* the multiplication law of a field (except when $n = 1$): it maps $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ to $x \cdot y \in \mathbb{R}$, not to an element of $\mathbb{R}^n$.

Definition 5.2. The *norm* or *length* of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \in \quad \mathbb{R}.$$

When $n = 1$, this reduces to $\|x\| = \sqrt{x_1^2} = |x|$ (using the absolute value function $|\cdot|$ on $\mathbb{R}$.)

5.2. The Cauchy-Schwarz inequality and the triangle inequality.

If you are given a vector $x \in \mathbb{R}^n$, its first component x_1 obeys the elementary inequality $x_1 \leq \|x\|$. To prove this inequality one might consider the other components:

$$0 \leq x_2^2 + \cdots + x_n^2 = \|x\|^2 - x_1^2$$

and thus

$$x_1^2 \leq \|x\|^2 \,.$$

Then taking square roots gives

$$|x_1| \leq \|x\|$$

and since $x_1 \leq |x_1|$ this gives

$$x_1 \leq \|x\|$$

as desired.

There is a useful generalization of this, where instead of taking the x_1 component we take the component along a general combination of directions, encoded in another vector $y \in \mathbb{R}^n$:

Proposition 5.3 (Cauchy-Schwarz inequality). If $x, y \in \mathbb{R}^n$, we have

$$x \cdot y \leq \|x\| \|y\| .$$

Proof. If $\|y\| = 0$, then $y = 0$, so $x \cdot y = 0$, and the inequality holds.

Otherwise, we consider the component of x orthogonal to y, and write out the fact that its length is nonnegative:

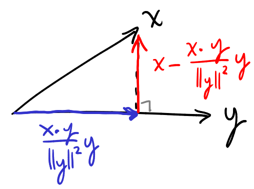

FIGURE 8. Vectors x, y in $\mathbb{R}^2$. The component of x orthogonal to y is indicated in red.

$$\begin{aligned} 0 &\leq \left\| x - \frac{x \cdot y}{\|y\|^2} y \right\|^2 \\ &= \|x\|^2 - 2\frac{(x \cdot y)^2}{\|y\|^2} + \frac{(x \cdot y)^2}{\|y\|^2} \\ &= \frac{1}{\|y\|^2} \left(\|x\|^2 \|y\|^2 - (x \cdot y)^2 \right) \end{aligned}$$

so

$$(x \cdot y)^2 \leq \|x\|^2 \|y\|^2 .$$

Taking square roots, we get

$$|x \cdot y| \leq \|x\| \|y\|$$

and since $x \cdot y \leq |x \cdot y|$ this gives the desired

$$x \cdot y \leq \|x\| \|y\| .$$

□

One of the key uses of Cauchy-Schwarz is to prove the *triangle inequality*. This expresses a fact which is probably familiar to you from plane geometry: if a triangle has side lengths a, b, c then $a \leq b + c$.

Proposition 5.4 (Triangle inequality). If $x, y \in \mathbb{R}^n$, we have

$$\|x + y\| \leq \|x\| + \|y\| .$$

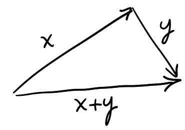

FIGURE 9. Vectors appearing in the triangle inequality.

Proof. Using Proposition 5.3 we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2(x \cdot y) \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

which gives the desired inequality. □

There are some other forms of the triangle inequality which are occasionally useful: in particular, relabeling $x \to x - y$ and rearranging gives

$$\|x - y\| \geq \|x\| - \|y\|,$$

and then taking $y \to -y$ we also have

$$\|x + y\| \geq \|x\| - \|y\|.$$

5.3. **Metric spaces.** Now we are ready to formulate our general notion of a "set with a notion of distance."

Definition 5.5. A *metric space* is a set X with a function $d : X^2 \to \mathbb{R}$, such that

(1) For all $p, q \in X$, $d(p,q) > 0$ if $p \neq q$,
(2) For all $p \in X$, $d(p,p) = 0$,
(3) For all $p, q \in X$, $d(p,q) = d(q,p)$,
(4) For all $p, q, r \in X$, $d(p,q) \leq d(p,r) + d(r,q)$.

We often refer to the elements of a metric space as "points."

Here are some examples of metric spaces. In each example, you should think about how to prove that all the axioms in Definition 5.5 are indeed satisfied.

Example 5.6. $\mathbb{R}^n$ is a metric space, with the distance function $d(x,y) = \|x - y\|$. (From now on, whenever we refer to $\mathbb{R}^n$ as a metric space without specifying a distance function, we mean to use this one.) For instance, when $n = 3$ we have

$$d((1,2,4),(5,1,4)) = \|(-4,1,0)\| = \sqrt{17}.$$

Example 5.7. $\mathbb{R}^n$ is a metric space, with the "taxicab distance" $d(x,y) = \sum_{i=1}^n |x_i - y_i|$.

Example 5.8. Any set X can be made into a metric space, with the distance function

$$d(p,q) = \begin{cases} 0 & \text{if } p = q, \\ 1 & \text{if } p \neq q. \end{cases}$$

Example 5.9. Pick any prime number p. For any $n \in \mathbb{Z}$ let $\|n\|_p = p^{-r}$, where r is the number of times p appears in the prime factorization of $|n|$ (and $\|0\|_p = 0$). Then $\mathbb{Z}$ is a metric space, with the distance function

$$d(m,n) = \|m - n\|_p.$$

(We won't use it much in this course, but this one is surprisingly useful in number theory! It is called the *p-adic distance*.)

5.4. **Open sets.** Now we introduce some terminology which will be very useful in our study of continuity.

Definition 5.10. Suppose X is a metric space. Then:

(1) For $p \in X$ and $\epsilon > 0$, the *ϵ-neighborhood* of p is the set
$$N_\epsilon(p) = \{q \in X \mid d(p,q) < \epsilon\} \subset X .$$
A *neighborhood* of p is the ϵ-neighborhood of p for some ϵ.

FIGURE 10. The ϵ-neighborhood of p. We draw it as the interior of a circle; this is literally the correct picture if $X = \mathbb{R}^2$, otherwise it is just a sort of cartoon which is meant to help your intuition.

(2) If $E \subset X$, $p \in E$ is an *interior point* of E if E contains a neighborhood of p.

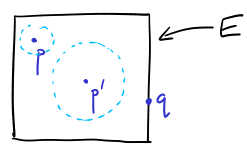

FIGURE 11. A subset $E \subset \mathbb{R}^2$. The points p,p' are interior points of E. The point q is a point of E which is not an interior point.

(3) $E \subset X$ is *open* if all points of E are interior points.

Proposition 5.11. Suppose X is a metric space. For any $p \in X$, every neighborhood of p is open.

Proof. Suppose given some $\epsilon > 0$. We want to show that $N_\epsilon(p)$ is open. Fix $q \in N_\epsilon(p)$; we need to show q is an interior point, i.e. we need to find some h such that $N_h(q) \subset N_\epsilon(p)$.

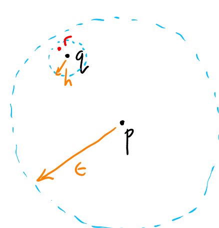

FIGURE 12. The neighborhood $N_\epsilon(p)$, containing q and the neighborhood $N_h(q)$, with $r \in N_h(q)$.

We choose any h with

$$0 < h \le \epsilon - d(p,q)\,.$$

(Note $\epsilon - d(p,q) > 0$ since $q \in N_\epsilon(p)$, so such an h indeed exists; we could even take $h = \epsilon - d(p,q)$ for instance.) Then, for any $r \in N_h(q)$, the triangle inequality gives

$$\begin{aligned} d(r,p) &\le d(r,q) + d(q,p) \\ &< h + d(q,p) \\ &\le \epsilon - d(p,q) + d(q,p) \\ &= \epsilon \end{aligned}$$

so $r \in N_\epsilon(p)$; thus $N_h(q) \subset N_\epsilon(p)$ as desired. □

Example 5.12. The interval $(0,1) \subset \mathbb{R}$ is open, because it is $N_{\frac{1}{2}}(\frac{1}{2})$. More generally, any $(a,b) \subset \mathbb{R}$ is open, because it is $N_{\frac{b-a}{2}}(\frac{a+b}{2})$.

Let's see how to make more examples of open sets. Suppose U_1 and U_2 are both open sets (in X). Then you can probably convince yourself that $U_1 \cup U_2$ is also open. Indeed, *any* union of open sets — even infinitely many — is still open. For intersections, things are more delicate. The intersection of two open sets is still open, and similarly the intersection of finitely many open sets is open. We codify this as a proposition:

Proposition 5.13.

(1) If we have a (finite or infinite) collection of open sets $G_\alpha \subset X$, the union $\bigcup_\alpha G_\alpha$ is open.

(2) If we have a finite collection of open sets $G_1, \ldots, G_n \subset X$, the intersection $\bigcap_{i=1}^n G_i$ is open.

Proof.

(1) Say $x \in \bigcup_\alpha G_\alpha$. Then there exists some particular α such that $x \in G_\alpha$. Since G_α is open, it follows that G_α contains some neighborhood N of x. This $N \subset \bigcup_\alpha G_\alpha$, so x is an interior point of $\bigcup_\alpha G_\alpha$, as desired.

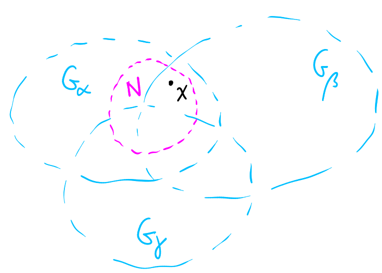

FIGURE 13. $x \in G_\alpha \subset \bigcup_\alpha G_\alpha$ has a neighborhood $N \subset G_\alpha \subset \bigcup_\alpha G_\alpha$, so x is an interior point of $\bigcup_\alpha G_\alpha$. We can do the same for any $x \in \bigcup_\alpha G_\alpha$, so the union $\bigcup_\alpha G_\alpha$ is open.

(2) Say $x \in \bigcap_{i=1}^n G_i$. Then for each i there exists some ϵ_i-neighborhood N_i of x, with $N_i \subset G_i$. Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then $N_\epsilon(x) \subset G_i$ for all i, and thus $N_\epsilon(x) \subset \bigcap_{i=1}^n G_i$. Thus x is an interior point of $\bigcap_{i=1}^n G_i$, as desired.

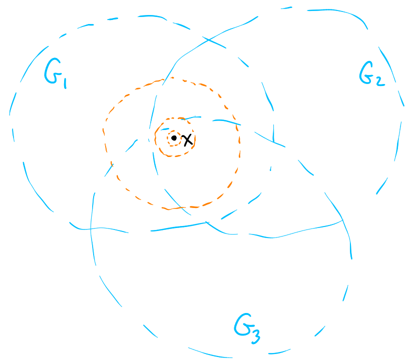

FIGURE 14. $x \in \bigcap_{i=1}^3 G_i$ has neighborhoods $N_{\epsilon_i}(x) \subset G_i$ for each i. The smallest one, $N_{\epsilon_3}(x)$, is contained in all the others, and thus in $\bigcap_{i=1}^3 G_i$; thus x is an interior point of $\bigcap_{i=1}^3 G_i$. We can do similarly for any $x \in \bigcap_{i=1}^3 G_i$, so $\bigcap_{i=1}^3 G_i$ is open.

□

But the intersection of infinitely many open sets need not be open, as the following example shows:

Example 5.14. Say $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$. Then the intersection $\bigcap_{n=1}^\infty U_n = \{0\}$ which is not open in $\mathbb{R}$.

Here is one important comment regarding the notion of "open set": when we say "G is open" it really involves not just the set G but also the metric space X that G sits inside. So if we are careful we would always say "G is open in X," not just "G is open." In practice, though, we are often working within a fixed metric space X, and then we omit mentioning X explicitly.

Example 5.15. For any metric space X, $\emptyset$ is open in X.

Example 5.16. For any metric space X, X is open in X. So, for example, the real line $\mathbb{R}$ is open in $\mathbb{R}$.

Example 5.17. Suppose we consider the metric space $X = \mathbb{R}^2$. The line $\{(x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ is *not* open in X. (Why?) Thus we cannot always say "the real line is open" — it depends on which ambient metric space we consider.

So, for $E \subset Y \subset X$, "E is open in Y" is not generally the same as "E is open in X." Nevertheless, there *is* a relation between "open in Y" and "open in X," given in the next proposition:

Proposition 5.18. If $E \subset Y \subset X$, then E is open in Y if and only if $E = Y \cap G$ for some G which is open in X.

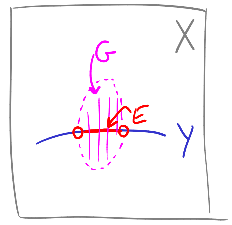

FIGURE 15. The subset $E \subset Y$ shown above is open in Y, but not in X. According to Proposition 5.18, we must have $E = G \cap Y$ for some $G \subset X$ which is open in X; one such G is shown.

Proof. For the purpose of this proof we use N^Y to indicate neighborhoods in Y, and N^X for neighborhoods in X.

($\implies$) Since E is open in Y, for each $x \in E$, there is some $\epsilon_x > 0$ such that $N^Y_{\epsilon_x}(x) \subset E$. Let $G = \bigcup_{x \in X} N^X_{\epsilon_x}(x)$. This is a union of open sets in X, so it is open in X; and $E = Y \cap G$ as desired (why?)

($\impliedby$) Suppose $E = Y \cap G$ and fix some $x \in E$. Then $x \in G$, and G is open in X, so there is some $\epsilon > 0$ such that $N^X_\epsilon(x) \subset G$. Then $N^Y_\epsilon(x) \subset E$, so E is open in Y as desired. $\square$

5.5. **Limit points and closed sets.**

Definition 5.19. Suppose X is a metric space. Then:

(1) Suppose $E \subset X$, and $p \in X$. We say p is a *limit point* of E if every neighborhood of p contains some $q \in E$ with $q \neq p$.

(2) $E \subset X$ is *closed* if E contains all of its limit points.

Example 5.20. If $E = (a, b) \subset \mathbb{R}$, then any $x \in [a, b]$ is a limit point of E, and any other x is not. (Prove it!) Thus E is not closed, since it does not contain the limit points a and b.

Example 5.21. If $E = [a, b] \subset \mathbb{R}$, then any $x \in [a, b]$ is a limit point of E, and any other x is not. (Prove it!) Thus E is closed, since it contains all of its limit points.

Example 5.22. If $E = \mathbb{Q} \subset \mathbb{R}$, then any $x \in \mathbb{R}$ is a limit point of E. (Prove it!) Thus E is not closed.

Example 5.23. If $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then $x = 0$ is the only limit point of E. Thus E is not closed, since it does not contain its limit point.

Note from this example that points in E need not be limit points of E.

Proposition 5.24. If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

Proof. We prove the contrapositive: so we assume that there is some $\epsilon > 0$ such that $N_\epsilon(p)$ contains only finitely many points of E, say $q_1, \ldots, q_n$, and we want to show p cannot be a limit point of E.

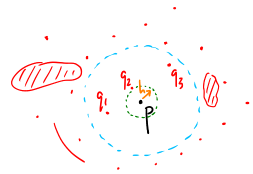

FIGURE 16. The neighborhood $N_\epsilon(p)$ containing only finitely many points of E, and the sub-neighborhood $N_h(p)$ containing none.

For this, let

$$h = \min\{d(p, q_i) \mid 1 \le i \le n\}.$$

The neighborhood $N_h(p)$ does not contain any point of E. Thus p cannot be a limit point. $\square$

Corollary 5.25. If E is finite, then E has no limit points.

Corollary 5.26. If E is finite, then E is closed.

Definition 5.27. If X is a metric space and $E \subset X$, we write

$$E^c = X \setminus E = \{x \in X \mid x \notin E\}.$$

Proposition 5.28. If X is a metric space, $E \subset X$ is open if and only if $E^c \subset X$ is closed.

Proof. ($\Longrightarrow$) Assume E is open. Let x be a limit point of E^c; then any neighborhood of x contains a point of E^c, and thus x cannot be an interior point of E. But every point of E is an interior point of E, since E is open. Thus $x \notin E$, i.e. $x \in E^c$. Thus E^c is closed as desired.

($\Longleftarrow$) Assume E^c is closed, and suppose $x \in E$. Then $x \notin E^c$, and thus (since E^c is closed) x is not a limit point of E^c. Thus there is some neighborhood N of x which does not contain any point of E^c, i.e. N is contained in E. Thus x is an interior point of E, and E is open as desired. $\square$

Corollary 5.29. If X is a metric space, $E \subset X$ is closed if and only if $E^c \subset X$ is open.

We note that "a set is not a door": it need not be either open or closed, and it can be both open and closed.

Example 5.30. The set $E = (a, b] \subset \mathbb{R}$ is neither open nor closed.

Example 5.31. For any metric space X, the set $E = \emptyset$ is both open and closed, as is the set $E = X$.

Sets which are both open and closed are sometimes called "clopen."

In Proposition 5.13 we have proved some facts about unions and intersections of open sets. By taking complements we get analogous statements about closed sets — but with the role of union and intersection reversed, because $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Corollary 5.32.

(1) If we have a (finite or infinite) collection of closed sets $F_\alpha \subset X$, the intersection $\bigcap_\alpha F_\alpha$ is closed.
(2) If we have a finite collection of closed sets $F_1, \ldots, F_n \subset X$, the union $\bigcup_{i=1}^n F_i$ is closed.

Again the finiteness is essential:

Example 5.33. The closed sets $F_n = [\frac{1}{n}, 1]$ have union $\bigcup_{n=1}^\infty F_n = (0, 1]$ which is not closed.

Example 5.34. A point is a closed set, so in fact *any* set is a union of closed sets.

Definition 5.35. If $E \subset X$, the *closure* of E is

$$\overline{E} = \{x \mid x \in E \text{ or } x \text{ is a limit point of } E\}.$$

Equivalently, a point $x \in \overline{E}$ just if every neighborhood of x meets E.

Proposition 5.36. For any $E \subset X$, $\overline{E}$ is closed.

Proof. We prove that $\overline{E}^c$ is open.

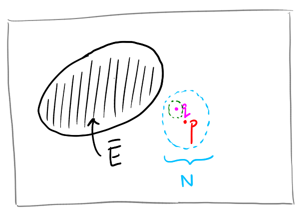

FIGURE 17. A point $p \in \overline{E}^c$, with a neighborhood N disjoint from E, and $q \in N$, with a neighborhood in N.

So, suppose $p \in \overline{E}^c$. Then p has a neighborhood N which is disjoint from E. Now consider any $q \in N$; then $q \notin E$, and we now show q cannot be a limit point of E either. Indeed, since N is open, N contains a neighborhood of q, which cannot contain any point of E. So $q \notin \overline{E}$, and thus $N \subset \overline{E}^c$, so p is an interior point of $\overline{E}^c$. This shows $\overline{E}^c$ is open as desired. $\square$

Example 5.37. If $X = \mathbb{R}$ and $E = (0, 1]$, then $\overline{E} = [0, 1]$.

Example 5.38. If $X = \mathbb{R}$ and $E = \mathbb{Q}$, then $\overline{E} = \mathbb{R}$.

Here are a few handy facts, whose proofs we leave as exercises:

Corollary 5.39. $E = \overline{E}$ if and only if E is closed in X.

Corollary 5.40. If $E \subset F$ and F is closed in X, then $\overline{E} \subset F$.

Proposition 5.41. If $E \subset \mathbb{R}$ is bounded above, then $\sup E \in \overline{E}$.

5.6. **Bounded sets.** We have discussed the notion of bounded above or bounded below for ordered sets. For subsets of a metric space, we also have a notion of boundedness:

Definition 5.42. If X is a metric space, we say a set $E \subset X$ is *bounded* if there exists some $M \in \mathbb{R}$ such that, for all $p, q \in E$, $d(p,q) < M$. We say E is *unbounded* if it is not bounded.

These two notions of boundedness are closely related:

Example 5.43. A subset $E \subset \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below. (Prove it!)

Example 5.44. If X is any metric space, every finite subset $E \subset X$ is bounded. (Why?)

Example 5.45. The subset $\mathbb{Z}^3 \subset \mathbb{R}^3$ is unbounded. (Why?)

Rudin's definition of boundedness is slightly different, but the two definitions are equivalent as long as X is not empty, as we now show.

Proposition 5.46. If X is not empty, then $E \subset X$ is bounded if and only if there exists some $M \in \mathbb{R}$ and some $p \in X$ such that $E \subset N_M(p)$.

Proof. ($\Longrightarrow$) Suppose X is not empty, and $E \subset X$ is bounded. Then there is some $M \in \mathbb{R}$ such that $d(q,r) < M$ for all $q, r \in E$. If E is empty, then choose any $p \in X$; then $E \subset N_1(p)$ as desired. If E is not empty, choose any point $p \in E$; then $d(p,q) < M$ for any $q \in E$, i.e. $E \subset N_M(p)$ as desired.

($\Longleftarrow$) Suppose there exists some $M \in \mathbb{R}$ and $p \in X$ such that $E \subset N_M(p)$. Then for any $q, r \in E$ we have $d(q,r) \leq d(q,p) + d(p,r) < M + M = 2M$. □

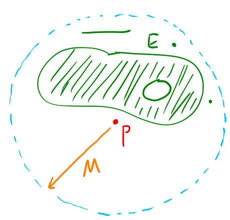

FIGURE 18. The set E is bounded, since it is contained entirely in $N_M(p)$.

5.7. **Connected sets.**

Definition 5.47. Suppose X is a metric space, and $E \subset X$. We say E is *disconnected* if we can write $E = A \cup B$, with A and B both nonempty subsets of X, and $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$. We say E is *connected* if it is not disconnected.

Proposition 5.48. Suppose $E \subset \mathbb{R}$. E is connected if and only if, for every $x, y \in E$ with $x < y$, we have $[x, y] \subset E$.

Proof. ($\Longrightarrow$) Suppose E is connected, $x, y \in E$, and $x < z < y$. Now, for a contradiction, suppose $z \notin E$. Then define

$$A = \{t \in E \mid t < z\}, \qquad B = \{t \in E \mid t > z\}.$$

Then $x \in A$ and $y \in B$, so both are nonempty. We have $A \cap B = \emptyset$, so to show $\overline{A} \cap B = \emptyset$, it suffices to show that no $t \in B$ is a limit point of A. But indeed, for any $t \in B$, taking $\epsilon = t - z$, $N_\epsilon(t)$ does not meet A; thus t is not a limit point of A. Similarly we can show $A \cap \overline{B} = \emptyset$. Thus E is disconnected.

($\Longleftarrow$) Suppose E is not connected, so $E = A \cup B$ with A, B nonempty, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$. Pick $x \in A$, $y \in B$. We may assume $x < y$ (otherwise swap the roles of A and B).

Let $z = \sup\{a \in A \mid a < y\}$. We have $z \in \overline{A}$ (using Proposition 5.41), so $z \notin B$. If $z \notin A$ then we have $x < z < y$, $x, y \in E$ but $z \notin E$, so we are done. Thus we only have to deal with the case $z \in A$. In that case $z \notin \overline{B}$, so there exists some w with $z < w < y$ such that $w \notin B$. This $w \notin A$ also, since $z = \sup A$. Thus $z < w < y$, with $z, y \in E$ but $w \notin E$, so we are done. □

Example 5.49. From Proposition 5.48 it follows that $\mathbb{Q} \subset \mathbb{R}$ is not connected. This might be confusing, so let's see it explicitly; we consider any irrational number α; then the two sets $A = \{x \in \mathbb{Q} \mid x < \alpha\}$ and $B = \{x \in \mathbb{Q} \mid x > \alpha\}$ separate $\mathbb{Q}$. (Crucially, we do have $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, although $\overline{A} \cap \overline{B} = \{\alpha\}$.)

There is a slightly more abstract characterization of connectedness, which is sometimes convenient because it avoids making any reference to the ambient metric space X:

Proposition 5.50. $E \subset X$ is disconnected if and only if we can write $E = A \cup B$ where A and B are both nonempty, both open in E, and disjoint.

Proof. Given $E = A \cup B$ with A, B disjoint, we need to show that "A is open in E" is equivalent to "$A \cap \overline{B} = \emptyset$". In other words, given $A \subset E$, we want to show "A is open in E" equivalent to "A does not contain any limit points of $E \setminus A$." But indeed these are equivalent, since both are equivalent to "$E \setminus A$ is closed in E." □

5.8. **Compact sets.** Now we come to the notion of *compact set*, whose importance will take a few weeks to fully reveal itself.

(One often hears that "compact set" is a natural generalization of "finite set" in a metric space. This is true, but I remember that when I first read it, it seemed to me like a totally useless statement. So I don't expect you to think much of this at first, but hold tight.)

Definition 5.51. Suppose X is a metric space, and $E \subset X$. An *open cover* of E is a collection of open sets G_α, such that $E \subset \bigcup_\alpha G_\alpha$. We call the cover *finite* if there are only finitely many G_α, otherwise *infinite*.

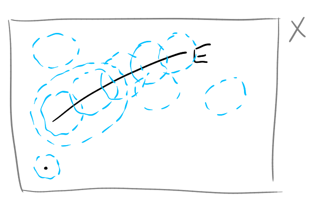

FIGURE 19. A set $E \subset X$ with an open cover by sets $G_1, \ldots, G_{10}$.

Example 5.52. If $X = \mathbb{R}$ and $E = [0,1]$, take $G_1 = (-2, \frac{1}{2})$, $G_2 = (\frac{1}{4}, 3)$, $G_3 = (\frac{3}{4}, 2)$, $G_4 = (5,6)$; then $\{G_1, G_2, G_3, G_4\}$ is a finite open cover of E.

Example 5.53. If $X = \mathbb{R}$ and $E = (0, \infty)$, take $G_n = (n-1, n+\frac{1}{10})$ for any $n \in \mathbb{N}$; then $\{G_n \mid n \in \mathbb{N}\}$ is an infinite open cover of E.

Definition 5.54. Suppose X is a metric space, and $E \subset X$, with an open cover $\{G_\alpha\}$. Then a *subcover* of this open cover is a subset of $\{G_\alpha\}$ which is still an open cover of E.

Example 5.55. If $X = \mathbb{R}$ and $E = (0,1]$, take $G_x = (x - \frac{1}{10}, x + \frac{1}{10})$ for any $x \in \mathbb{R}$. Then $\{G_x \mid x \in \mathbb{R}\}$ is an infinite (even uncountable) open cover of E. $\{G_0, G_{\frac{1}{10}}, \ldots, G_1\}$ is a finite subcover of this open cover.

Example 5.56. If $X = \mathbb{R}$ and $E = (0,1]$, take $G_n = (\frac{1}{n}, 2)$. Then $\{G_n \mid n \in \mathbb{N}\}$ is an infinite open cover of E which has no finite subcover.

Now we are ready for the key definition:

Definition 5.57. Suppose X is a metric space and $K \subset X$. We say K is *compact* if every open cover of K has a finite subcover.

Example 5.58. If $X = \mathbb{R}$ and $E = (0,1]$, then E is not compact (see Example 5.56.)

Example 5.59. If X is any metric space and E is any finite set, then E is compact. (Prove it!)

Example 5.60. The set $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ is compact. (Prove it!)

Example 5.61. The set $\mathbb{Z} \subset \mathbb{R}$ is not compact. (Prove it!)

As I wrote above, compact sets have many of the nice properties that finite sets have. Here are two examples.

Proposition 5.62. If X is a metric space, and $K \subset X$ is compact, then K is closed in X.

Proof. We will prove K^c is open. So, fix some $p \in K^c$. Then, for any $q \in K$, let $V_q = N_{d(p,q)/2}(p)$ and $W_q = N_{d(p,q)/2}(q)$.

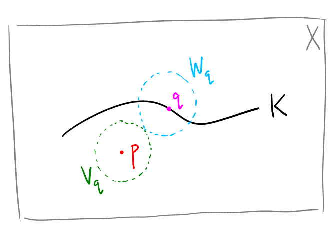

FIGURE 20. For fixed $p \in K^c$, and $q \in K$, we define neighborhoods V_q and W_q. As q varies, the W_q make up an open cover of K, while the V_q vary in size.

The sets $\{W_q \mid q \in K\}$ form an open cover of K, which therefore has a finite subcover $\{W_{q_1}, \ldots, W_{q_n}\}$. The intersection $V_{q_1} \cap \cdots \cap V_{q_n}$ is a neighborhood of p which does not intersect any of the W_{q_i} and thus does not intersect K. Thus p is interior in K^c, as desired. $\square$

Proposition 5.63. If X is a metric space, and $K \subset X$ is compact, then K is bounded.

Proof. Consider the collection of all neighborhoods $N_\epsilon(q)$, for all $\epsilon > 0$ and all $q \in K$. These give an open cover of K, which must have a finite subcover since K is compact; thus K is covered by finitely many neighborhoods $N_{\epsilon_1}(q_1), \ldots, N_{\epsilon_n}(q_n)$. Then let $M = \max\{d(q_i, q_j) + \epsilon_i + \epsilon_j \mid 1 \leq i, j \leq n\}$. Using the triangle inequality we can show that any two points in K are separated by a distance less than M, so K is bounded, as desired (exercise!) $\square$

Much like the definition of "open," the definition of "compact" for a set E in principle depends on which metric space X we see E as a subset of. But actually, unlike for the notion of "open", for "compact" the ambient metric space turns out not to matter:

Proposition 5.64. If $K \subset Y \subset X$, then K is compact in X if and only if K is compact in Y.

Proof. We use Proposition 5.18 to relate open sets in X to open sets in Y.

($\Longleftarrow$) Suppose $\{G_\alpha\}$ is an open cover of K in X. We want to find a finite subcover. Let $H_\alpha = G_\alpha \cap Y$. Then $\{H_\alpha\}$ is an open cover of K in Y. Since K is compact in Y, it follows that there is a finite subcover $\{H_1, \ldots, H_n\}$ of K. The corresponding sets $\{G_1, \ldots, G_n\}$ then give the desired finite subcover of K in X.

($\Longrightarrow$) Suppose $\{H_\alpha\}$ is an open cover of K in Y. We want to find a finite subcover. For each α, there exists some open G_α in X such that $H_\alpha = G_\alpha \cap Y$. Then $\{G_\alpha\}$ is an open cover of K in X. Since K is compact in X, it follows that there is a finite subcover $\{G_1, \ldots, G_n\}$ of K. The corresponding sets $\{H_1, \ldots, H_n\}$ then give the desired finite subcover of K in Y. $\square$

With this in mind, when K is compact we sometimes just take the ambient metric space to be K itself — this allows us to simplify our thinking a bit.

Here is another good property of compact sets, which will be essential for us (we will use it to prove that various sorts of sequences have limits):

Proposition 5.65. If K is compact, $E \subset K$, and E is infinite, then E has a limit point in K.

Proof. Suppose E does not have a limit point in K. Then, for every $q \in K$, there exists a neighborhood V_q of q such that $V_q \cap E \subset \{q\}$. The collection $\{V_q\}$ gives an open covering of K, which cannot have a finite subcover (since each V_q contains at most one point of E, and E is infinite.) $\square$

So far we have been discussing compact sets in abstract metric spaces, and the only examples we have given are finite sets. But the notion of compactness would not be very useful if we couldn't find other examples. Now we specialize to $X = \mathbb{R}$, where we can say very concretely what the compact sets are. Here is the most fundamental example:

Theorem 5.66. Any closed interval $I = [a, b] \subset \mathbb{R}$ is compact.

Proof. We work by contradiction. Let $I_0 = I$. Suppose there is an open cover $\{G_\alpha\}$ of I_0 with no finite subcover. Say $c = \frac{1}{2}(a+b)$. Then $\{G_\alpha\}$ is also an open cover of both $[a,c]$ and $[c,b]$, and at least one of these two must have no finite subcover (otherwise combining the finite subcovers would give a finite subcover of I). Let I_1 be one of these two intervals, such that the cover $\{G_\alpha\}$ of I_1 has no finite subcover. Now bisect I_1, and continue in the same way. We get an infinite chain of closed intervals

$$I_0 \supset I_1 \supset I_2 \supset \cdots,$$

such that no I_n can be covered by any finite subcover of $\{G_\alpha\}$, and the interval I_n has length $\frac{b-a}{2^n}$.

We proved in Proposition 4.20 that given such an infinite chain, there is at least one x such that $x \in I_n$ for all n. Here comes the trouble: this x must be in some G_α, since the G_α cover I; then this G_α (being open) must contain some $N_\epsilon(x)$. Now, there exists some n such that $2^n > \frac{b-a}{\epsilon}$, or in other words $\frac{b-a}{2^n} < \epsilon$; for this n, the neighborhood $N_\epsilon(x)$ contains I_n (exercise!) Thus $I_n \subset G_\alpha$, contradicting the fact that the cover $\{G_\alpha\}$ of I_n has no finite subcover. □

So, for example, we obtain the following corollary:

Corollary 5.67. Any infinite subset of $[a,b]$ has a limit point.

One can similarly prove:

Theorem 5.68. Any closed cell $I = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_k,b_k] \subset \mathbb{R}^k$ is compact.

Proof. The proof uses the same ideas we used for $\mathbb{R}$ above (see Rudin). □

Finally we collect a few miscellaneous facts about compact sets, for later convenience:

Proposition 5.69. Suppose K is compact and $F \subset K$ closed. Then F is compact.

Proof. Suppose $\{V_\alpha\}$ is an open cover of F. Then adding F^c to this collection gives an open cover of K. Since K is compact, this open cover of K has a finite subcover. Removing F^c from the subcover if it is included, we obtain the desired finite subcover of the cover $\{V_\alpha\}$ of F. □

Corollary 5.70. If F is closed and K is compact, then $F \cap K$ is compact.

Proposition 5.71. Suppose X is a metric space and $\{K_\alpha\}$ is a nonempty collection of compact subsets of X. If all finite intersections $K_1 \cap \cdots \cap K_n$ are nonempty, then $\bigcap_\alpha K_\alpha$ is also nonempty.

Proof. Let K be one of the sets K_α. Suppose that $\bigcap_\alpha K_\alpha$ is empty. Then the sets K_α^c give an open cover of K. This open cover must have a finite subcover, since K is compact. This means that some finite union $K_1^c \cup \cdots \cup K_n^c$ contains K, so the finite intersection $K \cap K_1 \cap \cdots \cap K_n$ is empty. □

Corollary 5.72. If $K_1 \supset K_2 \supset K_3 \supset \cdots$ with all K_n compact and nonempty, then $\bigcap_{n=1}^\infty K_n$ is also nonempty.

Now we can say precisely what the compact subsets of $\mathbb{R}^k$ are:

Proposition 5.73. For $E \subset \mathbb{R}^k$ the following are equivalent:

(1) E is closed and bounded.

(2) E is compact.

(3) Every infinite subset of E has a limit point in E.

Proof. (1) $\implies$ (2): If E is closed and bounded, then E is a closed subset of some cell. But the cell is compact by Theorem 5.68, so E is compact by Proposition 5.69.

(2) $\implies$ (3): this follows from Proposition 5.65.

(3) $\implies$ (1): we prove the contrapositive, i.e. suppose that E is either not closed or not bounded, and show that there is some infinite subset of E without a limit point in E.

First, suppose E is not closed. Then there is some $x \notin E$ which is a limit point of E. Since x is a limit point of E, there exists a sequence (x_n) in E such that $d(x_n, x) < \frac{1}{n}$. Then the set $\{x_n \mid n \in \mathbb{N}\}$ is infinite, and has x as its only limit point in X (exercise!); thus it has no limit point in E.

Second, suppose E is not bounded. Then fixing any $x \in E$, there is some sequence (x_n) in E such that $d(x_n, x) > n$. The set $\{x_n \mid n \in \mathbb{N}\}$ has no limit point in X (exercise!) $\square$

In a general metric space X, though, Proposition 5.73 would not be true; we would still have (2) $\Leftrightarrow$ (3) and (2) $\implies$ (1), but not necessarily (1) $\implies$ (2).

We should remark that compact sets can be more interesting than you first think. Here is a pretty example:

Example 5.74. Consider a set defined as follows. Let

$$C_0 = [0, 1].$$

We delete the open middle third, leaving

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Next we delete the open middle thirds of each of these intervals, leaving

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Then we delete the open middle third of each of these intervals, and so on: to get C_n we remove the open middle third of each interval in C_{n-1}. Each C_n is a union of finitely many closed intervals, thus closed. Finally we define the "middle-thirds Cantor set"

$$C = \bigcap_{n=1}^{\infty} C_n .$$

Since C is an intersection of closed sets it is also closed; and C is also bounded; so C is compact. C is a rather peculiar set. C does not contain any interval, but nevertheless, every point of C is a limit point, and C is uncountable. See Rudin for some discussion of these facts.

6. Convergence

In practical situations, we often have some mathematical problem, where we do not see how to just write down the solution in one step: for instance

- the solution of an algebraic equation,
- the solution of a differential equation,
- the value of an integral,
- the limiting weights for a neural network after a large number of training steps,
- ...

We often approach this as follows. First, we find some metric space X where the solution to our problem should live. It might be $X = \mathbb{R}$, or $X = \mathbb{R}^k$, or $X = \{\text{functions } (a, b) \to \mathbb{R}\}$, or something else. Then, instead of writing down the desired solution $p \in X$ in one step, we first build an *approximate* solution $p_1 \in X$, then we apply some procedure to improve it to a better approximate solution $p_2 \in X$, and iteratively $p_3, p_4, p_5, \dots$ Then, we hope to find the desired p as the *limit* of this sequence.

6.1. Convergent sequences.

Definition 6.1. Suppose X is a metric space and (p_n) a sequence in X. We say (p_n) *converges to* $q \in X$ if, for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad d(p_n, q) < \epsilon\,.$$

We also write "(p_n) converges to q" as "$p_n \to q$", or "$\lim_{n\to\infty} p_n = q$".

The picture one should have is: if $p_n \to q$, then no matter how small a neighborhood we draw around q, the terms (p_n) of the sequence will eventually be trapped in that neighborhood.

Example 6.2. The sequence $x_n = \frac{1}{n}$ in $\mathbb{R}$ converges to 0. (Indeed, for any $\epsilon > 0$, if $N > 1/\epsilon$, then $1/N < \epsilon$, and thus $n \geq N \implies |x_n - 0| < \epsilon$ as needed.)

Example 6.3. The sequence $x_n = 3$ in $\mathbb{R}$ converges to 3. (Indeed, for any $\epsilon > 0$, we can take $N = 1$, since $n \geq 1 \implies |x_n - 3| < \epsilon$ as needed.)

Example 6.4. The sequence

$$x_n = \begin{cases} 0 & \text{for } n \text{ prime,} \\ \frac{1}{n} & \text{for } n \text{ composite} \end{cases}$$

converges to 0. (Prove it!)

A given sequence need not have a limit, but if it does have one, the limit is unique:

Proposition 6.5. If $p_n \to q$ and $p_n \to q'$, then $q = q'$.

Proof. Suppose $p_n \to q$ and $p_n \to q'$. Fix any $\epsilon > 0$. There exists N such that

$$n \geq N \implies d(p_n, q) < \frac{1}{2}\epsilon\,.$$

Also, there exists N' such that

$$n \geq N' \implies d(p_n, q') < \frac{1}{2}\epsilon\,.$$

Then combining these, taking $n = \max(N, N')$ we have

$$d(q, q') \leq d(q, p_n) + d(p_n, q') < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon\,.$$

But $\epsilon > 0$ was arbitrary, so we conclude that $d(q, q') = 0$, i.e. $q = q'$. □

Definition 6.6. If (p_n) converges to some q, we say (p_n) is *convergent*. If (p_n) does not converge to any $q \in X$, we say (p_n) *diverges* or is *divergent*.

There are various different ways of diverging, illustrated by the next few examples:

Example 6.7. The sequence $x_n = n$ in $X = \mathbb{R}$ diverges.

Example 6.8. The sequence $x_n = \frac{1}{n}$ in the metric space $X = (0, 1)$ diverges. (It has a limit in $\mathbb{R}$, but not in X.)

Example 6.9. The sequence $x_n = (-1)^n + \frac{1}{n}$ in $X = \mathbb{R}$ diverges.

We will want to be able to prove that various different kinds of sequences converge. The first two kinds of divergence we listed above — "diverging to infinity" or "diverging to a hole" — could be avoided by requiring X to be compact. This still leaves the third kind, "diverging by oscillating" as in Example 6.9.

Now we note that, even though the sequence in Example 6.9 diverges, its divergence is a mild sort: if we look only at the odd-numbered terms, for instance,

$$x_1, x_3, x_5, \ldots$$

and think of those as a sequence itself, say

$$x'_1 = x_1, \quad x'_2 = x_3, \quad x'_3 = x_5, \quad \ldots, \quad x'_n = x_{2n+1},$$

then the new sequence x'_n is convergent. We say that the odd-numbered terms make up a *convergent subsequence* of the original sequence. Let's formalize this behavior:

Definition 6.10. If (p_n) is a sequence in X, then a *subsequence* of (p_n) is a sequence (p'_n) where each $p'_n = p_{k_n}$, and $k_1 < k_2 < \cdots$

Proposition 6.11. If X is compact, and (p_n) is a sequence in X, then there is a convergent subsequence (p'_n) of (p_n).

Proof. Consider the set

$$E = \{p_n \mid n \in \mathbb{N}\}\,.$$

First suppose E is finite. In this case, the sequence (p_n) must repeat some value p infinitely many times. This gives a subsequence of the form

$$p, p, p, \ldots$$

and this is convergent to p.

On the other hand, suppose E is infinite. In this case, since X is compact, E must have a limit point p in X (by Proposition 5.65). Then, for every $n \in \mathbb{N}$, there exists some k_n such that $d(p_{k_n}, p) < \frac{1}{n}$; and we can choose the k_n iteratively so that

$$k_1 < k_2 < k_3 < \cdots$$

Then the subsequence $p'_n = p_{k_n}$ has $p'_n \to p$ (exercise!) ☐

We record for future convenience another handy fact:

Proposition 6.12. If $(p_n) \to p$, then any subsequence of (p_n) also converges to p.

6.2. **Cauchy sequences.** How can we detect when a sequence is convergent? For instance, we might consider the sequence of partial sums of

$$x_n = \sum_{k=1}^{n} \frac{1}{2^k k^2},$$

i.e.

$$(x_n) = \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{16}, \quad \frac{1}{2} + \frac{1}{16} + \frac{1}{72}, \quad \frac{1}{2} + \frac{1}{16} + \frac{1}{72} + \frac{1}{256}, \quad \ldots \tag{6.1}$$

With the tools we have so far, it seems hard to see whether this sequence has a limit or not. We can see that the terms are getting closer *to each other* — the gaps between terms are getting small — but we cannot easily see what the limit is, if any. Now we want to show that the limit exists anyway.

Definition 6.13. Suppose X is a metric space and (p_n) a sequence in X. We say (p_n) is *Cauchy* if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Longrightarrow \quad d(p_n, p_m) < \epsilon\,.$$

The notion of Cauchy sequence is similar to that of convergent sequence, but not quite the same on its face. In a convergent sequence the p_n are getting arbitrarily close *to the limit*, while in a Cauchy sequence, the terms p_n are getting arbitrarily close *to each other*.

Proposition 6.14. If (p_n) is convergent, then (p_n) is Cauchy.

Proof. Suppose $p_n \to y$. Fix $\epsilon > 0$. Then there exists some N such that

$$n \geq N \quad \Longrightarrow \quad d(p_n, q) < \frac{\epsilon}{2}$$

Then, for this N, we have

$$n, m \geq N \quad \Longrightarrow \quad d(p_n, p_m) \leq d(p_n, q) + d(p_m, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so (p_n) is Cauchy as desired. ☐

It would be nice if every Cauchy sequence in a metric space X were convergent — this would allow us to show, e.g., that our example (6.1) is actually convergent. It doesn't work if X is any old metric space, as the next few examples show.

Example 6.15. If $X = \mathbb{R} \setminus \{0\}$, the sequence $x_n = \frac{1}{n}$ is Cauchy, but not convergent in X. (Informally it is "trying" to converge to 0, but 0 is not an element of X.)

Example 6.16. If $X = \mathbb{Q}$, the sequence

$$1, 1.4, 1.41, 1.414, \ldots$$

is a Cauchy sequence in $\mathbb{Q}$ (why?), but not convergent in $\mathbb{Q}$. (It is "trying" to converge to $\sqrt{2}$.)

But it becomes true if we consider metric spaces which are sufficiently nice. For instance:

Proposition 6.17. Suppose X is a compact metric space, and (p_n) is a Cauchy sequence in X. Then (p_n) is convergent in X.

Proof. Since X is compact, there is at least a *subsequence* of (p_n) which converges, to some limit $p \in X$. We want to show that in fact the whole sequence (p_n) converges to p. The idea is that if all the terms of the sequence are getting close to one another, and also some of the terms are getting close to p, then in fact all of the terms must be getting close to p.

So, suppose given some $\epsilon > 0$. Because (p_n) is Cauchy, there exists some M such that $n, m \geq M \implies d(p_n, p_m) < \frac{\epsilon}{2}$. Moreover, because of the convergent subsequence, there is some $m \geq M$ such that $d(p_m, p) < \frac{\epsilon}{2}$.

Then, for any $n \geq M$, we have

$$d(p_n, p) \leq d(p_n, p_m) + d(p_m, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and thus $p_n \in N_\epsilon(p)$. This shows that $p_n \to p$ as desired. □

Corollary 6.18. If (x_n) is a Cauchy sequence in $\mathbb{R}^k$, then (x_n) is convergent in $\mathbb{R}^k$.

Proof. Since (x_n) is a Cauchy sequence, it follows that all the points x_n lie in some cell $K \subset \mathbb{R}^k$ (exercise!) Then use Proposition 6.17, applied to the metric space K. □

The property that Cauchy sequences converge is important enough that we give it a name:

Definition 6.19. The metric space X is *complete* if every Cauchy sequence in X converges in X.

Corollary 6.20.

(1) If X is compact, then X is complete.

(2) $\mathbb{R}^k$ is complete.

So, one way to show that the sequence (6.1) is convergent would be to show that it is Cauchy.

6.3. **Monotonic sequences of real numbers.** There is also a somewhat simpler way: we notice that (6.1) has another important property — the terms only go up, never down. Then we might expect that the only way this sequence could possibly diverge would be if its terms go to $+\infty$, i.e. if it is not bounded above. This is indeed true as we now show.

Definition 6.21. A sequence (p_n) in a metric space X is *bounded* if $E = \{p_n \mid n \in \mathbb{N}\}$ is bounded.

Proposition 6.22. Every convergent sequence is bounded.

Proof. Exercise. □

Definition 6.23. A sequence (x_n) in $\mathbb{R}$ is *monotonically increasing* if for all $n \in \mathbb{N}$ we have $x_{n+1} \geq x_n$. It is *monotonically decreasing* if for all $n \in \mathbb{N}$ we have $x_{n+1} \leq x_n$. It is *monotonic* in either case.

Theorem 6.24. If (x_n) is a monotonic sequence in $\mathbb{R}$, then (x_n) converges if and only if it is bounded.

Proof. ($\Longrightarrow$) This direction is Proposition 6.22.

($\Longleftarrow$) Suppose (x_n) is monotonically increasing and bounded. Then let

$$x = \sup\{x_n \mid n \in \mathbb{N}\};$$

we'll show that $x_n \to x$. Fix any $\epsilon > 0$. There is some $N \in \mathbb{N}$ such that $x_N > x - \epsilon$. But then, for all $n \geq N$, we have $x_n > x - \epsilon$, and also $x_n \leq x$, so $|x_n - x| < \epsilon$. Thus $x_n \to x$ as desired.

The proof is similar if (x_n) is monotonically decreasing. (Or, we could consider the sequence $(-x_n)$ instead, which is monotonically increasing, and apply the above to conclude $-x_n \to x$, then $x_n \to -x$.) □

Example 6.25. Consider the sequence

$$(x_n) = 1.001, 1.0010001, 1.0010001000001, \ldots$$

where the number of zeroes we insert at each stage is a prime number. This sequence is convergent, since it is monotonically increasing and bounded (by 1.1 say).

6.4. **Lim sup and inf.** In this section we introduce some convenient language — it's not strictly necessary but sometimes helps to simplify proofs.

When we have two sequences (x_n) and (y_n), such that $x_n \leq y_n$ for all n, it would be tempting to conclude that $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$. Unfortunately, we can't draw this conclusion, unless we already know that the two limits exist. The notion of $\limsup$ and $\liminf$ are a way of working around this problem and similar ones — they always exist, and by studying them we can sometimes reach conclusions about the actual convergence or divergence of a sequence.

Definition 6.26. The *extended real numbers* $\mathbb{R}_{\text{ext}} = \mathbb{R} \cup \{\infty, -\infty\}$. We extend the usual ordering from $\mathbb{R}$ to $\mathbb{R}_{\text{ext}}$, by adding the rules that $-\infty < x$ for all $x \in \mathbb{R}$, $x < \infty$ for all $x \in \mathbb{R}$, and $-\infty < \infty$. Then $\mathbb{R}_{\text{ext}}$ is not a field (there is no way to extend the addition and multiplication laws to $\mathbb{R}_{\text{ext}}$ while obeying all the field axioms), but it is an ordered set.

Proposition 6.27. Any subset of $\mathbb{R}_{\text{ext}}$ has a supremum and an infimum.

Proof. We just prove that a supremum exists; the proof that an infimum exists is parallel.

If $\infty \in A$, then ∞ is a maximum element for A, thus a supremum. So, assume $\infty \notin A$.

Next, if $A \cap \mathbb{R}$ is empty, then either $A = \emptyset$ or $A = \{-\infty\}$, and in either case $-\infty$ is a supremum. So, assume $A \cap \mathbb{R}$ is nonempty.

If $A \cap \mathbb{R}$ is bounded above in $\mathbb{R}$, then it has a supremum, and this is also a supremum for A. If $A \cap \mathbb{R}$ is not bounded above in $\mathbb{R}$, then ∞ is a supremum for A. ☐

Definition 6.28. Suppose (x_n) is a sequence in $\mathbb{R}$. We say (x_n) *converges to* ∞ if, for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \implies \quad x_n > M\,.$$

We also write this as $x_n \to \infty$.

Example 6.29. The sequence $x_n = n^3$ converges to ∞. (Prove it!)

This language is a bit tricky: we simultaneously say that the sequence $x_n = n^3$ *diverges* (in $\mathbb{R}$) and also that it *converges to* ∞ (in $\mathbb{R}_{\text{ext}}$).

Definition 6.30. Suppose (x_n) is a sequence in $\mathbb{R}$. We say (x_n) *converges to* $-\infty$ if, for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \implies \quad x_n < -M\,.$$

We also write this as $x_n \to -\infty$.

Example 6.31. The sequence $x_n = -n^3$ converges to $-\infty$. (Prove it!)

Now, for *any* sequence in $\mathbb{R}$, we define:

Definition 6.32. Suppose (x_n) is a sequence in $\mathbb{R}$. Then we define

$$\lim_{n\to\infty} \sup x_n = \sup\{x \in \mathbb{R}_{\text{ext}} \mid \text{ there is a subsequence of } (x_n) \text{ which converges to } x\}$$

and

$$\lim_{n\to\infty} \inf x_n = \inf\{x \in \mathbb{R}_{\text{ext}} \mid \text{ there is a subsequence of } (x_n) \text{ which converges to } x\}\,.$$

Example 6.33. If $x_n = (-1)^n$, then $\lim_{n\to\infty} \sup x_n = 1$ and $\lim_{n\to\infty} \inf x_n = -1$.

Example 6.34. Suppose (x_n) is a sequence enumerating all the rationals (we know such a sequence exists since the rationals are countable). Then $\limsup x_n = \infty$ and $\limsup x_n = -\infty$. (Indeed, in this case every real number is the limit of some subsequence of (x_n)!)

Here are some basic facts about the lim sup and lim inf.

Proposition 6.35. $\liminf x_n \leq \limsup x_n$.

Proof. Exercise. ☐

Proposition 6.36.

(1) If for some $\alpha \in \mathbb{R}$ we have $\limsup x_n < \alpha$, then there is some $N \in \mathbb{N}$ such that $n \geq N \implies x_n < \alpha$.

(2) If for some $\alpha \in \mathbb{R}$ we have $\liminf x_n > \alpha$, then there is some $N \in \mathbb{N}$ such that $n \geq N \implies x_n > \alpha$.

Proof. We prove (1). Suppose there is no such N. Then there must be a subsequence of (x_n) with terms all $\geq \alpha$. Moreover this subsequence lies in $[\alpha, M]$ for some M (otherwise we could find a further subsequence which converges to ∞, which is impossible since $\limsup x_n$ is finite). Then using Proposition 6.11 we can pass to a further subsequence which converges to some limit in $[\alpha, M]$. But that contradicts $\limsup x_n < \alpha$.

(2) is similar. □

Because of Proposition 6.36, one way to think about lim sup and lim inf is that they describe the smallest range to which the sequence is "essentially confined" — for any $\epsilon > 0$, all but finitely many terms x_n fall into the range $(\liminf x_n - \epsilon, \limsup x_n + \epsilon)$.

Proposition 6.37. Suppose $x \in \mathbb{R}$. The sequence (x_n) converges to x if and only if

$$\liminf x_n = \limsup x_n = x\,.$$

Proof. ($\implies$) Suppose $x_n \to x$. Then every subsequence of (x_n) also converges to x. It follows that $\liminf x_n = \limsup x_n = x$ as desired.

($\impliedby$) This direction is more interesting. Assume that $\liminf x_n = \limsup x_n = x$. This means that every convergent subsequence of (x_n) converges to x. We want to conclude that the whole sequence $x_n \to x$.

So, suppose given some $\epsilon > 0$. Using Proposition 6.36, we know that there is some $N \in \mathbb{N}$ such that $n \geq N \implies x_n < x + \epsilon$. Again using Proposition 6.36, we know that there is some $N' \in \mathbb{N}$ such that $n \geq N' \implies x_n > x - \epsilon$. Then for all $n \geq \max(N, N')$ we have $|x_n - x| < \epsilon$; thus $x_n \to x$ as desired. □

Proposition 6.38. If for some $N \in \mathbb{N}$ we have

$$n \geq N \implies x_n \leq y_n,$$

then

$$\limsup x_n \leq \limsup y_n\,, \qquad \liminf x_n \leq \liminf y_n\,.$$

Proof. Let $X = \limsup x_n$ and $Y = \limsup y_n$. We will prove $X \leq Y$. (The proof for lim inf is parallel.)

Fix some $\epsilon > 0$.

There is some subsequence of (x_n) which converges to a limit exceeding $X - \epsilon/2$. It follows that, for any $N' \in \mathbb{N}$, there exists some $n > N'$ such that $x_n > X - \epsilon$. On the other hand, using Proposition 6.36 we know that for some N', $n > N' \implies y_n < Y + \epsilon$.

Combining these, we have for $n > \max(N, N')$

$$X - \epsilon < x_n \leq y_n < Y + \epsilon\,.$$

Since this holds for every $\epsilon > 0$, we conclude that $X \leq Y$, as desired. □

Here is a corollary, sometimes called the "Squeeze Theorem." (You don't really *need* the technology of lim sup and lim inf to prove this, but it's a handy illustration.)

Corollary 6.39. If there is some N such that $a_n \leq x_n \leq b_n$ for all $n \geq N$, and $a_n \to L$, $b_n \to L$, then $x_n \to L$.

Proof. We have $L \leq \liminf x_n \leq L$ and $L \leq \limsup x_n \leq L$, so $\liminf x_n = \limsup x_n = L$, so $x_n \to L$. □

6.5. **Useful facts about sequences in $\mathbb{R}$.** We often want to do arithmetic operations on sequences. For example:

Proposition 6.40. Suppose we have sequences (x_n), (y_n) in $\mathbb{R}$ with $x_n \to x$ and $y_n \to y$. Then

(1) $x_n + y_n \to x + y$,

(2) $cx_n \to cx$,

(3) $x_n y_n \to xy$,

(4) If all $x_n \neq 0$ and $x \neq 0$, then $\frac{1}{x_n} \to \frac{1}{x}$.

Proof. We just prove part (3), leaving the rest as exercises. Also we assume $x, y \neq 0$, leaving the cases $x = 0$, $y = 0$ as exercises. Then, start with

$$|x_n y_n - xy| = |x_n(y_n - y) + (x_n - x)y| \leq |x_n||y_n - y| + |y||x_n - x|$$

Now, fix some $\epsilon > 0$. We want to show that, for sufficiently large n, each of the two terms on the right is smaller than $\frac{1}{2}\epsilon$.

Using the fact $x_n \to x$ and $x \neq 0$, we can show that for large enough n we will have $|x_n| < 2|x|$. Then, since $y_n \to y$, for large enough n we have

$$|y_n - y| < \frac{\epsilon}{4|x|} < \frac{\epsilon}{2|x_n|}.$$

Also, since $x_n \to x$, for large enough n we have

$$|x_n - x| < \frac{\epsilon}{2|y|}.$$

Combining these, we get

$$|x_n y_n - xy| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

as desired. ☐

Here are some simple examples of limits which occur often:

Proposition 6.41.

(1) If $p \in \mathbb{R}$ and $p > 0$, then $\frac{1}{n^p} \to 0$.

(2) If $x \in \mathbb{R}$ and $x > 0$, then $x^{1/n} \to 1$.

(3) If $x \in \mathbb{R}$ and $|x| < 1$, $x^n \to 0$.

(4) $n^{1/n} \to 1$.

Proof. See Rudin for these. The main tool is the binomial expansion, e.g. for (2), in case $x > 1$, we set $y = x^{1/n} - 1$ and use $x = (1 + y)^n = 1 + ny + \cdots \geq 1 + ny$, so $0 < ny \leq x - 1$. ☐

7. Series in $\mathbb{R}$

7.1. **Definition and first examples.**

Definition 7.1. For a sequence (x_n) in $\mathbb{R}$,

(1) $\sum_{n=p}^{q} x_n$ means $x_p + x_{p+1} + \cdots + x_q$.
(2) The *partial sums* of (x_n) are $s_k = \sum_{n=1}^{k} x_n$.
(3) If there exists $s \in \mathbb{R}$ such that $s_n \to s$, then we write $\sum_{n=1}^{\infty} x_n = s$, and we say that this infinite sum *converges to s*. Otherwise, we say that the sum *diverges*.

You have probably seen various examples of convergent and divergent sums before. Two famous examples which we recall right away:

Example 7.2. The sum
$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$
converges to 1. Indeed,
$$s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \to 1$$

Example 7.3. The sum
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
diverges. Indeed, if $n = 2^m - 1$ then we have
$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} \\ &> \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^m} + \cdots + \frac{1}{2^m}\right) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= \frac{m}{2} \end{aligned}$$
Thus, for any $m \in \mathbb{N}$, there is some n for which $s_n > \frac{m}{2}$; it follows that (s_n) is not bounded, which means it cannot converge.

7.2. Tests for convergence.

When does an infinite sum converge?

Proposition 7.4. For a sequence (x_n) in $\mathbb{R}$, $\sum_{n=1}^{\infty} x_n$ converges if and only if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$$m \geq n \geq N \implies \left| \sum_{k=n}^{m} x_n \right| < \epsilon \, .$$

Proof. The sequence of partial sums (s_n) is convergent if and only if it is a Cauchy sequence; and $s_{m+1} - s_n = \sum_{k=n}^{m} x_k$. This gives the desired statement. □

Corollary 7.5. If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to 0$.

We emphasize that $x_n \to 0$ is not a *sufficient* condition for $\sum_{n=1}^{\infty} x_n$ to converge, only a necessary one. The necessary and sufficient condition is the one given in Proposition 7.4.

Proposition 7.6. For a sequence (x_n) in $\mathbb{R}$, with all $x_n \geq 0$, $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums is bounded.

Proof. The sequence of partial sums is monotone increasing, so (using Theorem 6.24) it converges if and only if it is bounded. ☐

Proposition 7.7 (Comparison test for convergence). If there exists some N such that $n \geq N \implies |x_n| \leq c_n$, and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. For $m \geq n \geq N$, we have

$$\left| \sum_{k=n}^{m} x_k \right| \leq \sum_{k=n}^{m} |x_k| \leq \sum_{k=n}^{m} c_k \,.$$

Now, since $\sum_{k=1}^{\infty} c_k$ is convergent, there exists some N' for which $m \geq n \geq N'$ implies $\sum_{k=n}^{m} c_k < \epsilon$. Thus, for $m \geq n \geq \max(N, N')$ we get

$$\left| \sum_{k=n}^{m} x_k \right| < \epsilon$$

and thus, by Proposition 7.4, $\sum_{n=1}^{\infty} x_k$ converges, as desired. ☐

Corollary 7.8 (Comparison test for divergence). If there exists some N such that $n \geq N \implies x_n \geq d_n \geq 0$, and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. If $\sum_{n=1}^{\infty} x_n$ converged, then by Proposition 7.7, $\sum_{n=1}^{\infty} d_n$ would also converge. ☐

Here are some standard and important examples.

Proposition 7.9 (Convergence and divergence of geometric series).

(1) If $x \in (-1, 1)$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.
(2) If $|x| \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Proof. For $x \in [0, 1)$ we have

$$s_{n-1} = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} \,,$$

and $x^n \to 0$ (Proposition 6.41), so $\frac{1-x^n}{1-x} \to \frac{1}{1-x}$ using Proposition 6.40. For $|x| \geq 1$ we do not have $x^n \to 0$, and thus the sum must diverge. ☐

Proposition 7.10 (Convergence and divergence of p-series). The sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and diverges for $p \leq 1$.

Proof.

- For $p = 1$ we have already checked (in Example 7.3) that the sum diverges.
- For $p < 1$, we have $\frac{1}{n^p} > \frac{1}{n}$, so the sum diverges by Corollary 7.8.

- For $p > 1$, we group the terms into groups of size 2^k, and estimate each group by

$$\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n^p} \leq 2^k \cdot \frac{1}{2^{kp}} = \frac{1}{2^{k(p-1)}}.$$

Thus, with $x = \frac{1}{2^{p-1}} < 1$, the partial sums are bounded above by

$$\sum_{k=0}^{m} x^k < \frac{1}{1-x}.$$

Since a monotonic bounded sequence converges (Theorem 6.24), this shows that the sum converges as desired.

□

Here are some tests which are very useful in practice:

Proposition 7.11 (Root test). Suppose (x_n) is a series in $\mathbb{R}$, with all $x_n \geq 0$. Let

$$\alpha = \limsup \sqrt[n]{x_n}.$$

(1) If $\alpha < 1$, then $\sum_{n=1}^{\infty} x_n$ converges.
(2) If $\alpha > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. If $\alpha < 1$, then choose some β with $\alpha < \beta < 1$; for sufficiently large n we must have $\sqrt[n]{x_n} < \beta$ (using Proposition 6.36), thus $x_n < \beta^n$, and then by the comparison test $\sum x_n$ converges. If $\alpha > 1$, then the x_n do not converge to 0 (exercise), so the sum must diverge. □

If $\alpha = 1$, the root test does not give us any information: the sum could converge or diverge.

Proposition 7.12 (Ratio test). Suppose (x_n) is a series in $\mathbb{R}$, with all $x_n \geq 0$. Then,

(1) If $\limsup \frac{x_{n+1}}{x_n} < 1$, then $\sum_{n=1}^{\infty} x_n$ converges.
(2) If $\frac{x_{n+1}}{x_n} \geq 1$ for all $n \geq N$, then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. If $\limsup \frac{x_{n+1}}{x_n} < 1$, then for some $\beta < 1$ and N we have $n \geq N \implies \frac{x_{n+1}}{x_n} < \beta$, and thus $x_n < \beta^{n-N} x_N$, so $\sum_{n=1}^{\infty} x_n$ converges by comparison to the geometric series $\sum_{n=1}^{\infty} (\beta^{-N} x_N) \beta^n$.

If $\frac{x_{n+1}}{x_n} \geq 1$ for $n \geq N$, then we cannot have $x_n \to 0$ (exercise), so $\sum_{n=1}^{\infty} x_n$ cannot converge. □

Corollary 7.13. If $\liminf \frac{x_{n+1}}{x_n} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

We emphasize that in Corollary 7.13 we need to use lim inf, not lim sup; it is a nice exercise to construct a sequence (x_n) for which $\sum x_n$ converges but $\limsup \frac{x_{n+1}}{x_n} = \infty$.

Here is one more useful criterion for convergence:

Proposition 7.14. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n + b_n$ also converges, and

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\,.$$

Proof. Exercise. □

7.3. **Absolute and conditional convergence.** So far we mostly discussed series with only positive terms. These are simpler since (as we explained) the partial sums form a monotone sequence, and for those the convergence just comes down to boundedness. For series which are not necessarily positive, there are in a sense two different ways to converge. The stronger way is to converge *absolutely*:

Definition 7.15. If (x_n) is a sequence in $\mathbb{R}$, we say $\sum_{n=1}^{\infty} x_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |x_n|$ converges.

Proposition 7.16. If $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it converges.

Proof. We use the criterion in Proposition 7.4. We have the bound

$$\left|\sum_{k=m}^{n} x_k\right| \leq \sum_{k=m}^{n} |x_k|\,.$$

If the sum converges absolutely, suppose given $\epsilon > 0$; for some N and $m, n \geq N$ we have

$$\sum_{k=m}^{n} |x_k| < \epsilon$$

and thus

$$\left|\sum_{k=m}^{n} x_k\right| < \epsilon$$

as needed. □

Example 7.17. The sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely, because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2^n}\right| = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges. (What does $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converge to?)

The other option is to converge *conditionally*:

Definition 7.18. If (x_n) is a sequence in $\mathbb{R}$, we say $\sum_{n=1}^{\infty} x_n$ *converges conditionally* if it converges but does not converge absolutely.

Conditional convergence is a delicate thing: it means that the sum converges only by virtue of some cancellations between the positive and negative terms.

Example 7.19. Suppose

$$x_n = 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots$$

Then the partial sums of $\sum x_n$ are

$$s_n = 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots$$

which converge to 0 (exercise!) Thus $\sum_{n=1}^{\infty} x_n$ converges to 0. On the other hand, $\sum_{n=1}^{\infty}|x_n|$ does not converge, much as in Example 7.3. So $\sum_{n=1}^{\infty} x_n$ converges *conditionally* to 0.

When we deal with conditionally convergent series we must be rather careful. For instance, the sum of a conditionally convergent series can be affected by reordering the terms!

Definition 7.20. A *rearrangement* of a sequence (x_n) is a sequence (y_n) where $y_n = x_{\phi(n)}$ for some bijection $\phi : \mathbb{N} \to \mathbb{N}$.

Example 7.21. Consider again the sequence in Example 7.19. Now rearrange it by taking two positive terms before each negative one:

$$x'_n = 1, \frac{1}{2}, -1, \frac{1}{3}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{5}, \frac{1}{6}, -\frac{1}{3}, \dots$$

In this case all the partial sums are at least $\frac{1}{2}$:

$$s'_n = 1, \frac{3}{2}, \frac{1}{2}, \frac{5}{6}, \frac{13}{12}, \frac{7}{12}, \dots$$

Thus, if $\sum_{n=1}^{\infty} x'_n$ converges, it must converge to a value $\geq \frac{1}{2}$. In particular, it cannot converge to 0, so the rearrangement has changed the value of the sum.

(In fact, $\sum_{n=1}^{\infty} x'_n$ does converge, to the value $\log 2$; we will prove this later, I hope. By other rearrangements we could actually get *any* value for the limit, including $\pm\infty$.)

Rudin gives a slightly different but similar example.

In contrast, absolutely convergent series can be treated liberally. For example,

Proposition 7.22. If $\sum x_n$ is absolutely convergent, and (x'_n) is any rearrangement of (x_n), then

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x'_n .$$

Proof. The idea is that no matter how we rearrange, we cannot delay the big terms forever: if we go far enough down the rearranged sequence we will capture them all. Let us formalize this. Suppose $x'_n = x_{\phi(n)}$. Fix $\epsilon > 0$. Then by Proposition 7.4 there exists N such that, for all $m, n \geq N$, $\sum_{k=m}^{n}|x_k| < \frac{\epsilon}{2}$. Choose some N' large enough that $\{1, \dots, N\} \subset \{\phi(1), \dots, \phi(N')\}$. Then for $n > N'$ the differences of partial sums $|s'_n - s_n|$ are sums of finitely many x_k with $k \geq N$, and thus

$$n > N' \quad \Longrightarrow \quad |s'_n - s_n| < \frac{\epsilon}{2} .$$

But the s_n converge to some α, and thus for some N''

$$n > N'' \quad \Longrightarrow \quad |s_n - \alpha| < \frac{\epsilon}{2} .$$

Combining these, for $n > \max(N', N'')$ we get $|s'_n - \alpha| < \epsilon$, so $s'_n \to \alpha$, as desired. ☐

Any reasonable-looking manipulation of absolutely convergent series can usually be justified. For instance: suppose we have two absolutely convergent series, and we want to determine the product

$$(x_1 + x_2 + x_3 + \cdots)(y_1 + y_2 + y_3 + \cdots)$$

This product involves an infinite number of terms indexed by $\mathbb{N}^2$ rather than $\mathbb{N}$, and it is not immediately obvious how to sum them. Here is one thing we could try: organize the products as

$$x_1y_1 + (x_2y_1 + x_1y_2) + (x_3y_1 + x_2y_2 + x_1y_3) + \cdots$$

which we think of as the sum of a new series,

$$z_2 + z_3 + \cdots$$

where

$$z_n = \sum_{k=1}^{n-1} x_k y_{n-k}\,.$$

The series (z_n) so defined is called the "Cauchy product" of the original two series.[12] The following proposition says that this manipulation gives the right answer:

Proposition 7.23. Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are both absolutely convergent. Define $z_n = \sum_{k=1}^{n-1} x_k y_{n-k}$. Then $\sum_{n=1}^{\infty} z_n$ is also absolutely convergent, and

$$\sum_{n=1}^{\infty} z_n = \left(\sum_{n=1}^{\infty} x_n\right)\left(\sum_{n=1}^{\infty} y_n\right)\,.$$

Proof. Let X_n, Y_n, Z_n be the corresponding partial sums.

First we consider the case where all of the series are positive. Then

$$Z_n = x_1y_1 + (x_2y_1 + x_1y_2) + \cdots + (x_{n-1}y_1 + \cdots + x_1y_{n-1})$$

while

$$X_nY_n = (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)$$

It follows that $X_{\lfloor \frac{n}{2} \rfloor} Y_{\lfloor \frac{n}{2} \rfloor} \leq Z_n \leq X_n Y_n$. We assumed $X_n \to X$, $Y_n \to Y$; then likewise $X_{\lfloor \frac{n}{2} \rfloor} \to X$, $Y_{\lfloor \frac{n}{2} \rfloor} \to Y$; thus by the squeeze theorem $\lim_{n\to\infty} Z_n = XY$.

Now we consider the more general case where the series are not necessarily positive. Define $\tilde{z}_n = \sum_{k=1}^{n} |x_k y_{n-k}|$. Then the preceding part shows that $\sum_{n=1}^{\infty} \tilde{z}_n$ converges. We have $|z_n| \leq \tilde{z}_n$, so it follows that $\sum_{n=1}^{\infty} z_n$ converges absolutely. It only remains to show that $\sum_{n=1}^{\infty} z_n$ converges to XY. For this, note $X_nY_n - Z_n$ is a sum of terms $x_k y_l$, and using the previous part, the sum of the $|x_k y_l|$ goes to 0 as $n \to \infty$. It follows that $X_nY_n - Z_n \to 0$, so $Z_n \to XY$ as desired. $\square$

See Rudin for the proof of something slightly stronger: we actually only need one of the two sequences to be absolutely convergent. On the other hand, the statement is not true if both sequences are conditionally convergent; indeed Rudin gives an example of two conditionally convergent series whose Cauchy product is not even convergent.

[12] With our conventions, $z_1 = 0$; if we had considered sequences starting with x_0 instead of x_1 then we would have $z_0 \neq 0$ in general; you can do things either way, and different choices are convenient at different times.

7.4. **Summation by parts and the alternating series test.** Frequently we meet series which arise as products of simpler parts, like $\sum x_n y_n$. To study such series it is often convenient to use a maneuver which is a discrete analog of the familiar "integration by parts" formula $\int_a^b f(x)G(x)\mathrm{d}x = -\int_a^b F(x)g(x)\mathrm{d}x + F(a)G(a) - F(b)G(b)$:

Proposition 7.24 (Summation by parts). Suppose given two sequences (x_n) and (Y_n). Let $X_n = \sum_{k=1}^n x_k$ (partial sum), and let $y_n = Y_{n+1} - Y_n$ (discrete derivative). Then

$$\sum_{n=p}^{q} x_n Y_n = -\sum_{n=p}^{q-1} X_n y_n + X_q Y_q - X_{p-1} Y_p \,.$$

Proof. We rewrite

$$\sum_{n=p}^{q} x_n Y_n = \sum_{n=p}^{q} (X_n - X_{n-1}) Y_n = \sum_{n=p}^{q} X_n Y_n - \sum_{n=p-1}^{q-1} X_n Y_{n+1} \,,$$

and then combine the two sums to get

$$\sum_{n=p}^{q-1} X_n (Y_n - Y_{n+1}) + X_q Y_q - X_{p-1} Y_p$$

as desired. □

Here is an application:

Proposition 7.25. Suppose (x_n) and (Y_n) are sequences in $\mathbb{R}$, such that:

- The partial sums X_n of $\sum x_n$ form a bounded sequence,
- (Y_n) is monotonically decreasing,
- $Y_n \to 0$.

Then $\sum_{n=1}^{\infty} x_n Y_n$ is convergent.

(We chose the unsymmetrical notation x_n, Y_n with the proof in mind.)

Proof. We'll use summation by parts. So, let $y_n = Y_{n+1} - Y_n$. By Proposition 7.24 the partial sums of $x_n Y_n$ are

$$-\sum_{k=1}^{n-1} X_k y_k + X_n Y_n \,.$$

The last term $X_n Y_n \to 0$ since (X_n) is bounded and $Y_n \to 0$. Thus it is sufficient to show that $\sum_{k=1}^{\infty} X_k y_k$ is convergent. Indeed this sum is absolutely convergent, since

$$\sum_{k=1}^{n} |X_k||y_k| \leq M \sum_{k=1}^{n} |y_k| = M \sum_{k=1}^{n} (Y_k - Y_{k+1}) = M(Y_1 - Y_{n+1}) \leq MY_1,$$

and thus it is convergent. □

Corollary 7.26 (Alternating series test). Suppose (Y_n) is monotonically decreasing and $Y_n \to 0$. Then $\sum_{n=1}^{\infty} (-1)^n Y_n$ is convergent.

Example 7.27. The alternating-sign version of the harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

is (conditionally) convergent. (We will compute its value later.)

7.5. **The number e.** Now we are in a position to define "Euler's constant" e.

Definition 7.28. We define $e \in \mathbb{R}$ by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(Note that, by the ratio test, this sum is indeed convergent.)

Numerically one can compute $e \approx 2.71828$. (It takes 10 terms to get this level of precision: the partial sum $\sum_{n=0}^{9} \frac{1}{n!} > 2.71828$.)

Proposition 7.29. We have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. (From Rudin, with some details added.) Let $t_n = \left(1 + \frac{1}{n}\right)^n$ and $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. We multiply out t_n directly using the binomial theorem:

$$t_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(\prod_{j=0}^{k-1} (n-j)\right) \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right)$$

or, written out,

$$t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)$$

We'll use this to get bounds in both directions. First,

$$t_n \le 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} = s_n$$

and thus

$$\limsup t_n \le \limsup s_n = e.$$

But also, if $n \ge m$ we can truncate the sum to m terms, getting

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

Thus, for any fixed m, we get the bound

$$\liminf t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m.$$

Since this is true for every m we get (exercise!)

$$\liminf t_n \ge \limsup s_m = e.$$

So we have shown

$$e \le \liminf t_n \le \limsup t_n \le e,$$

from which it follows that $\lim_{n\to\infty} t_n = e$ as desired. □

8. Limits and continuity

Throughout this section let X, Y be metric spaces.

Now we are in a position to start discussing *functions*

$$f : X \to Y$$

or more generally

$$f : E \to Y$$

where $E \subset X$. What does it mean for such a function to be continuous at a point p of its domain?

8.1. Defining limits of functions.

The usual definition of continuity involves *limits* $\lim_{x\to p} f(x)$. So let us start by considering the notion of limit.

A preliminary comment: $\lim_{x\to p} f(x)$ does not make much sense unless it is actually possible for x to approach p. For example, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $f(x) = \sqrt{x}$, we cannot meaningfully ask about $\lim_{x\to -1} f(x)$. We formalize this by saying that the point p has to be a limit point for the domain of f.

Now here is the (famous, dreaded?) "ϵ-δ" definition of limit. It is the same one you may have seen before — except that if you saw it before you probably saw it for $X = Y = \mathbb{R}$, and now we formulate it using arbitrary metric spaces.

Definition 8.1. Suppose $E \subset X$, p is a limit point of E, and we have a function $f : E \to Y$. Then we say

$$\lim_{x\to p} f(x) = q$$

if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < d_X(x,p) < \delta \quad \Longrightarrow \quad d_Y(f(x),q) < \epsilon . \tag{8.1}$$

We could equivalently phrase the condition (8.1) as

$$f(N_\delta^X(p) \setminus \{p\}) \subset N_\epsilon^Y(q) .$$

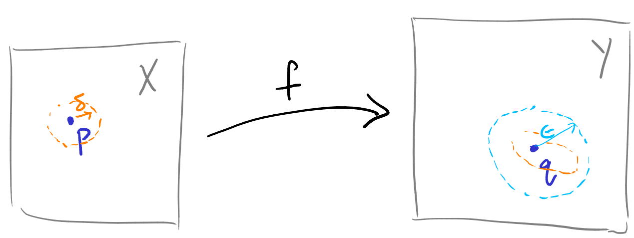

Figure 21. The condition $\lim_{x\to p} f(x) = q$ implies that, for any neighborhood N of q (blue), there must exist a deleted neighborhood N' of p (orange) which is mapped into N.

It is sometimes helpful to think of δ and ϵ as "tolerances," as in a manufacturing process — then the definition says that, if we want to hit q within an output tolerance ϵ, we can do so by choosing x to be within an input tolerance δ of p.

Example 8.2. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 5x$, then we have $\lim_{x\to 2} f(x) = 10$. More generally $\lim_{x\to a} f(x) = 5a$. (Prove it! Hint: take $\delta = \epsilon/5$.)

Example 8.3. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then we have $\lim_{x\to 2} f(x) = 4$. More generally $\lim_{x\to a} f(x) = a^2$. (This one is a little harder to prove; we'll prove it as a consequence of something more general below. We can also prove it directly. For instance, for $a \neq 0$, we have $|x^2 - a^2| = |x+a||x-a|$. Now suppose that $|x - a| < \delta$, and also suppose that $\delta \leq |a|$, so that $|x + a| \leq 2|a|$. Then we will have $|x^2 - a^2| = |x+a||x-a| < 2|a|\delta$, so if we assume $\delta \leq \frac{\epsilon}{2|a|}$ we will have $|x^2 - a^2| < \epsilon$. Altogether, then, we assumed both $\delta \leq |a|$ and $\delta \leq \frac{\epsilon}{2|a|}$, and concluded that $|x - a| < \delta \implies |x^2 - a^2| < \epsilon$. Thus we can take $\delta = \min(\frac{\epsilon}{2|a|}, |a|)$. Note that here, unlike the previous example, δ has to depend on a! To complete the proof we should also deal with the case $a = 0$; in that case we can just take $\delta = \sqrt{\epsilon}$.)

Example 8.4. If $f : X \to X$ is the identity map, then $\lim_{x\to p} f(x) = p$. (Prove it! Hint: take $\delta = \epsilon$.)

Example 8.5. If $f : (0,1) \to \mathbb{R}$ is $f(x) = 1/x$, then $\lim_{x\to 1} f(x) = 1$ (we'll prove it later), while $\lim_{x\to 0} f(x)$ does not exist. (Also $\lim_{x\to -1} f(x)$ does not exist, but for a different reason: -1 is not a limit point of the domain $(0,1)$.)

Example 8.6. If $f : \mathbb{R} \to \mathbb{R}$ is

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0 \end{cases}$$

then $\lim_{x\to 0} f(x) = 0$. (Prove it!)

Example 8.7. If $f : \mathbb{R} \to \mathbb{R}$ is

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

then $\lim_{x\to 0} f(x)$ does not exist.

Example 8.8. If $f : \mathbb{R} \to \mathbb{R}$ is

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ x & \text{if } x \in \mathbb{Q} \end{cases}$$

then:

(1) $\lim_{x\to 0} f(x) = 0$,
(2) $\lim_{x\to \alpha} f(x)$ does not exist if $\alpha \neq 0$.

The notion of limit of a function seems similar to the notion of limit of a sequence, and you might wonder if there is a relation. Sure enough, there is, as follows: if $\lim_{x\to p} f(x) = q$, then if we consider a sequence of points (p_n) approaching p, the values $f(p_n)$ approach the limit q. Moreover, the converse is also true, as long as we consider *every* possible sequence (p_n) approaching p. There is one tricky point: the definition of $\lim_{x\to p} f(x)$

involves the value of the function at points x very near p, but not the value at p itself; so for both directions of the implication, we should only consider sequences where all $p_n \neq p$.

Now we formulate this precisely:

Proposition 8.9. Suppose $E \subset X$, p is a limit point of E, and $f : E \to Y$. Then, the following are equivalent:

(1) $\lim_{x\to p} f(x) = q$.

(2) For all sequences (p_n) with $p_n \to p$, and $p_n \neq p$ for all n, $f(p_n) \to q$.

Proof. This really amounts just to writing out the definitions.

(1) $\implies$ (2): Fix any sequence (p_n), $p_n \to p$ with $p_n \neq p$ for all n. Also fix $\epsilon > 0$. By definition, there is some $\delta > 0$ such that $0 < d(x,p) < \delta \implies d(f(x),q) < \epsilon$. Now, since $p_n \to p$ and all $p_n \neq p$, there exists N such that $n \geq N \implies 0 < d(p_n,p) < \delta$. Combining these, $n \geq N \implies d(f(p_n),q) < \epsilon$. This says that $f(p_n) \to q$ as desired.

(2) $\implies$ (1): We prove the contrapositive. So suppose $\lim_{x\to p} f(x) \neq q$. That would mean there is some $\epsilon > 0$ such that, for every $\delta > 0$, there is an x with $d(x,p) < \delta$ but $d(f(x),q) > \epsilon$. Now we can use this to build a sequence p_n for which $f(x_n)$ doesn't converge to q, as follows: picking $\delta = \frac{1}{n}$, we get a p_n with $d(p_n,p) < \frac{1}{n}$ but $d(f(p_n),q) > \epsilon$; then the sequence $p_n \to p$, but $f(p_n)$ does not converge to q (exercise!) □

In practice Proposition 8.9 gives a very convenient intuition for what a limit is. Moreover, because of Proposition 8.9, we can leverage facts we've already proved about limits of sequences, to get parallel facts about limits of functions. For example, limits of functions are unique if they exist:

Corollary 8.10. If $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} f(x) = q'$, then $q = q'$.

For another example, arithmetic of limits works as you'd expect:

Corollary 8.11. Say $E \subset X$ and $f, g : E \to \mathbb{R}$, with $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$. Then,

(1) $\lim_{x\to p}(f+g)(x) = A+B$.

(2) $\lim_{x\to p}(fg)(x) = AB$.

(3) If $B \neq 0$ then $\lim_{x\to p}(f/g)(x) = A/B$.

8.2. **Defining continuity.** Now we can say what it means for a function to be continuous.

Definition 8.12. Suppose $E \subset X$, $p \in E$, and $f : E \to Y$. Then we say f is *continuous at p* if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that, for $x \in E$,

$$d_X(x,p) < \delta \quad \implies \quad d_Y(f(x), f(p)) < \epsilon \,. \tag{8.2}$$

We say f is *continuous* (on E) if f is continuous at every $p \in E$.

We could equivalently write the condition (8.2) as

$$f(N_\delta^X(p)) \subset N_\epsilon^Y(f(p)) \,.$$

Example 8.13. The identity map $f : X \to X$ is continuous. (For any $p \in X$ we can take $\delta = \epsilon$.)

Example 8.14. The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ is continuous. To prove this: fix any $p \in X$ and take $\delta = \epsilon/2$. Then, suppose $d_{\mathbb{R}}(p, x) < \delta$. We want to show that then $d_{\mathbb{R}}(f(p), f(x)) < \epsilon$. Writing this out, it says $|(2p+3) - (2x-3)| < \epsilon$, i.e. $|2p - 2x| < \epsilon$. But by assumption $|x - p| = d_{\mathbb{R}}(p, x) < \delta = \epsilon/2$, so $|2x - 2p| < \epsilon$, as desired.

Example 8.15. Suppose X is any metric space, and $q \in X$. Then the function $f : X \to \mathbb{R}$ given by $f(p) = d(p, q)$ is continuous. (For any $p \in X$ we can take $\delta = \epsilon$. It's a good exercise to prove it!)

Note that continuity is automatic at points $p \in E$ which are not limit points (sometimes called *isolated points* of E):

Proposition 8.16. Suppose $E \subset X$ and $p \in E$. If p is not a limit point of E, then every function $f : E \to Y$ is continuous at p.

Proof. If p is not a limit point, then there exists some $\delta > 0$ such that $d(x, p) < \delta \implies x = p$. Then for this δ we have $d(x, p) < \delta \implies d(f(x), f(p)) = 0 < \epsilon$, no matter what ϵ is. □

On the other hand, if p is a limit point of E, then the definition of "continuity at p" is equivalent to the one you might have learned in your calculus classes: f is continuous at p just if its value at p agrees with its limit as $x \to p$.

Proposition 8.17. Suppose $E \subset X$, $p \in E$, and $f : E \to Y$. If p is a limit point of E, then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof. The definition of "f is continuous at p" says that for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, p) < \delta \implies d(f(x), f(p)) < \epsilon$, and this precisely matches the definition of "$\lim_{x \to p} f(x) = f(p)$," when p is a limit point of E. □

Here is another simple equivalence which is sometimes convenient:

Proposition 8.18. $f : E \to Y$ is continuous at $p \in E$ if and only if, for all sequences (p_n) in E with $p_n \to p$, $\lim_{n \to \infty} f(p_n) = f(p)$.

Proof. Exercise. □

How to prove that interesting functions are continuous? One very important fact is that the composition of continuous functions is again continuous:

Proposition 8.19. Suppose X, Y, Z are metric spaces and $E \subset X$. Suppose $f : E \to Y$ and $g : f(E) \to Z$. Then define $h : E \to Z$ by $h = g \circ f$. If f is continuous at $p \in E$, and g is continuous at $f(p) \in f(E)$, then h is continuous at p.

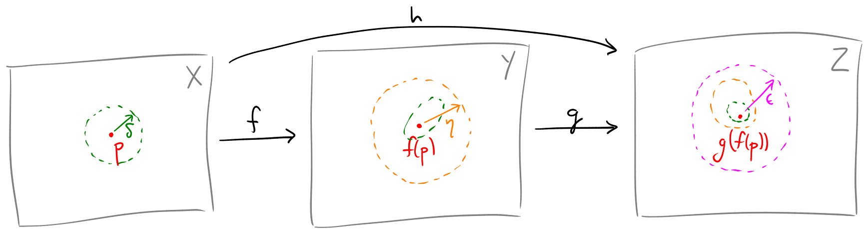

FIGURE 22. Various neighborhoods appearing in the proof that the composition of continuous functions is continuous.

Proof. Fix some $\epsilon > 0$. Then by continuity of g at $f(p)$ there exists $\eta > 0$ such that

$$d(y, f(p)) < \eta \implies d(g(y), g(f(p))) < \epsilon,$$

and by continuity of f at p there exists $\delta > 0$ such that

$$d(x, p) < \delta \implies d(f(x), f(p)) < \eta.$$

Combine these, setting $y = f(x)$, to get

$$d(x, p) < \delta \implies d(g(f(x)), g(f(p))) < \epsilon$$

i.e.

$$d(x, p) < \delta \implies d(h(x), h(p)) < \epsilon$$

as desired. $\square$

Arithmetic operations on continuous functions also give continuous functions:

Proposition 8.20. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are both continuous at p, then $f + g$, fg are also continuous at p. If $g(p) \neq 0$, then f/g is also continuous at p.

Proof. Exercise (use Corollary 8.11). $\square$

Using this fact we can easily prove continuity in some familiar examples:

Example 8.21. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous.

Example 8.22. If P is any polynomial with real coefficients, then the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = P(x)$ is continuous.

8.3. **Topology and continuity.** Here is a reformulation of the definition of continuity which you have probably not seen before. It turns out to be extremely handy in proofs, as we will see momentarily.

Recall that, whether or not $f : X \to Y$ is invertible, we can always talk about $f^{-1}(A) \subset X$ for any subset $A \subset Y$ (see Definition 2.21).

Proposition 8.23. $f : X \to Y$ is continuous if and only if, for all open $U \subset Y$, $f^{-1}(U) \subset X$ is open.

Proof. ($\Longrightarrow$) Suppose f is continuous and $U \subset Y$ open. We want to show that $f^{-1}(U)$ is also open. So pick some $p \in f^{-1}(U)$. Then $f(p) \in U$, and since U is open, there is some $\epsilon > 0$ such that $N_\epsilon(f(p)) \subset U$. Since f is continuous at p, there is some $\delta > 0$ such that $f(N_\delta(p)) \subset N_\epsilon(f(p)) \subset U$, so $N_\delta(p) \subset f^{-1}(U)$. Thus p is an interior point in $f^{-1}(U)$. Since $p \in f^{-1}(U)$ was arbitrary it follows that $f^{-1}(U)$ is open.

($\Longleftarrow$) Now consider some $p \in X$. We want to show that f is continuous at p, i.e., for any $\epsilon > 0$ we want to show that there is $\delta > 0$ such that $f(N_\delta(p)) \subset N_\epsilon(f(p))$. Now, let $U = N_\epsilon(f(p)) \subset Y$. U is open, so $f^{-1}(U)$ must also be open. Since $p \in f^{-1}(U)$, this means p is an interior point of $f^{-1}(U)$, i.e. there is some $\delta > 0$ such that $N_\delta(p) \subset f^{-1}(U)$, as desired. □

Example 8.24. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 2x$. Let $U = (a, b) \subset \mathbb{R}$. Then $f^{-1}(U) = (\frac{a}{2}, \frac{b}{2})$ which is open, as expected since f is continuous.

Example 8.25. Consider the "Heaviside step function" $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } x \geq 0\,. \end{cases}$$

Let $U = (\frac{1}{2}, \frac{3}{2}) \subset \mathbb{R}$. Then $f^{-1}(U) = [0, \infty)$ which is not open. It follows that f is not continuous.

Corollary 8.26. $f : X \to Y$ is continuous if and only if, for all closed $G \subset Y$, $f^{-1}(G) \subset X$ is closed.

One intuitive consequence of Corollary 8.26 is that, given any continuous $f : X \to Y$ and any target value $y \in Y$, the set $\{x \mid f(x) = y\} \subset X$ is closed. Said otherwise, the solution-set of an equation is closed, so long as the equation is defined by a continuous function. Similarly for sets defined by closed inequalities, e.g. $\{x \mid a \leq f(x) \leq b\} \subset X$ is closed if $f : X \to \mathbb{R}$ is continuous.

Example 8.27. Suppose X is a metric space, $q \in X$, and $L \in \mathbb{R}$. Then $\{p \in X \mid d(p, q) < L\}$ is open, and $\{p \in X \mid d(p, q) \leq L\}$ is closed.

Here is a first indication of how useful Proposition 8.23 is in practice.

Proposition 8.28. Suppose $f : X \to Y$ is continuous and X is compact. Then $f(X) \subset Y$ is compact.

Proof. Suppose $\{U_\alpha\}$ is an open cover of $f(X)$. Then by Proposition 8.23 $\{f^{-1}(U_\alpha)\}$ is an open cover of X. Since X is compact this cover must have a finite subcover, i.e.

$$X \subset f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)\,.$$

But $f(f^{-1}(S)) \subset S$, so it follows that

$$f(X) \subset f(f^{-1}(U_1)) \cup \cdots \cup f(f^{-1}(U_n)) \subset U_1 \cup \cdots \cup U_n\,,$$

i.e. $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{U_\alpha\}$. Thus $f(X)$ is compact. □

Corollary 8.29. Suppose $f : X \to Y$ is continuous and X is compact. Then $f(X) \subset Y$ is closed and bounded.

Note that we really need the compactness of X for this: it's not true that the continuous image of every bounded set is bounded (for instance, consider $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$), and also not true that the continuous image of every closed set is closed (for instance, consider $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = 1/n$.)

Here is a useful corollary. Given a function $f : X \to \mathbb{R}$ we are often interested in finding the global maximum of f. An important subtlety is that a global maximum doesn't necessarily exist — e.g. consider $f : [0,1) \to \mathbb{R}$ given by $f(x) = x$, or $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$. But if we restrict to continuous functions with compact domains, then a global maximum always exists:

Corollary 8.30. Suppose $f : X \to \mathbb{R}$ is continuous, and X is nonempty and compact. Then there is some $p \in X$ such that, for all $x \in X$, $f(p) \geq f(x)$.

Proof. Since $f(X)$ is compact, it would be enough to show that every compact subset $K \subset \mathbb{R}$ has a maximum element. To see this, note that K is bounded, so $\sup K$ exists; moreover $\sup K \in \overline{K}$ (otherwise there would be a smaller upper bound); so since K is closed we must have $\sup K \in K$, which means $\sup K$ is a maximum element of K as desired. □

(Of course, the global maximum need not be unique.)

Here is another appealing corollary. It would be tempting to think that the inverse of a continuous bijection is always continuous. This isn't quite true: for instance, let $Y \subset \mathbb{R}^2$ be the unit circle, and consider the map $f : [0,2\pi) \to Y$ given by $f(t) = (\cos t, \sin t)$. Then f is a continuous bijection, but $f^{-1} : Y \to [0,2\pi)$ is not continuous; indeed $\lim_{x\to(1,0)} f^{-1}(x)$ does not even exist. However, this problem does not occur if the domain is compact:

Corollary 8.31. Suppose $f : X \to Y$ is continuous and bijective, and X is compact. Then $f^{-1} : Y \to X$ is continuous.

Proof. Suppose $U \subset X$ is open; we want to show that $(f^{-1})^{-1}(U) \subset Y$ is also open. We have $(f^{-1})^{-1}(U) = \{y \in Y \mid f^{-1}(y) \in U\} = f(U)$, so we just need to show $f(U)$ is open. For this note that U^c a closed subset of the compact space X, and thus U^c is compact; thus $f(U^c)$ is also compact, and in particular is is closed; thus $f(U^c)^c = f(U)$ is open as desired. □

Example 8.32. Suppose $f : [0,L] \to [0,L^2]$ is $f(x) = x^2$. Then $f^{-1} : [0,L^2] \to [0,L]$ is the square-root function $f^{-1}(x) = \sqrt{x}$. Thus, by Corollary 8.31, the square-root function is continuous on the interval $[0,L^2]$.[13]

[13] With more work, you can use this fact to show that the square root is continuous on the full $[0,\infty)$. To formulate this precisely, we should carefully distinguish the square-root functions with different domains. Any $y \in [0,\infty)$ is contained in an open set $U \subset [0,\infty)$ which is also contained in some $[0,L^2]$. We know that the square root function $\sqrt{}|_{[0,L^2]}$ is continuous. Thus $\sqrt{}|_U$ is also continuous (the restriction of a continuous function is always continuous). From this it follows that the full $\sqrt{}|_{[0,\infty)}$ is also continuous at y.

A continuous bijection $f : X \to Y$ such that $f^{-1} : Y \to X$ is also continuous is sometimes called a *homeomorphism*. When f is a homeomorphism, $U \subset X$ is open if and only if $f(U) \subset Y$ is open. If a homeomorphism $f : X \to Y$ exists, one says that X and Y are *homeomorphic*. Loosely speaking, this means that X and Y are topologically "the same," even though they need not be precisely the same metric spaces.

One more useful topological property:

Proposition 8.33. Suppose $f : X \to Y$ is continuous and $E \subset X$ is connected. Then $f(E) \subset Y$ is also connected.

Proof. We show the contrapositive. So, suppose $f(E)$ is disconnected, i.e. $f(E) = A \cup B$ with A, B nonempty and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. We want to show that E is disconnected too.

Then let $A' = f^{-1}(A) \cap E$ and $B' = f^{-1}(B) \cap E$. These have $f(A') = A$ and $f(B') = B$ (why?) Also $E = A' \cup B'$, and both A', B' are nonempty.

All that remains is to show that $\overline{A'} \cap B' = \emptyset$ and $A' \cap \overline{B'} = \emptyset$. For this, we use a result which you have proven in a problem set: $f(\overline{A'}) \subset \overline{f(A')}$. Since $f(A') = A$ this means $f(\overline{A'}) \subset \overline{A}$; combining this with $f(B') = B'$ we get $f(\overline{A'}) \cap f(B') \subset \overline{A} \cap B = \emptyset$. This implies $\overline{A'} \cap B' = \emptyset$ as desired. The proof for $A' \cap \overline{B'} = \emptyset$ is parallel, just exchanging the roles of A and B above.

(Incidentally here is an alternative "slicker" proof: use Proposition 5.50. Without loss of generality we can assume $E = X$, and $f(E) = Y$. Then, suppose Y is disconnected; then $Y = A \cup B$ with A, B nonempty, open, disjoint. Then $X = f^{-1}(A) \cup f^{-1}(B)$, and $f^{-1}(A)$, $f^{-1}(B)$ are likewise nonempty, open, disjoint, so X is disconnected.) $\square$

Corollary 8.34 (Intermediate value theorem). Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If $f(a) < y < f(b)$, then there exists some $x \in [a, b]$ such that $f(x) = y$.

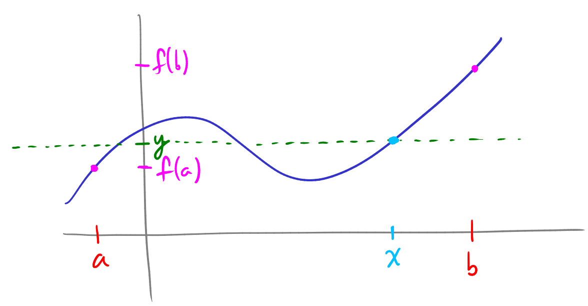

FIGURE 23. The intermediate value theorem: because $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < y < f(b)$, there exists at least one $x \in [a, b]$ with $f(x) = y$. (In this case there are actually three such x.)

Proof. By Proposition 5.48 we know that $[a, b]$ is connected. Thus $f([a, b])$ is also connected. Then use Proposition 5.48 again: since this connected set contains $f(a)$ and $f(b)$ it must also contain every value between them. $\square$

8.4. **Uniform continuity.** There is a refinement of continuity which is sometimes important when we want to study properties that are global — using the values of a function

f all over a space X, instead of in a small region of X — as we will do when we discuss integration, for instance.

The definition of "f is continuous at p" requires that for every ϵ there exists a δ with some property. We might loosely say that f is "more continuous" at p if the δ can be taken bigger, and "less continuous" at p if δ has to be taken smaller. Sometimes it's useful to know that f is in some sense equally continuous at every p:

Definition 8.35. If $f : X \to Y$, we say f is *uniformly continuous* (on X) if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(p, p') < \delta \quad \Longrightarrow \quad d_Y(f(p), f(p')) < \epsilon \,. \tag{8.3}$$

Proposition 8.36. If f is uniformly continuous on X, then f is continuous on X.

The key difference between Definition 8.35 and Definition 8.12 is that in Definition 8.35 we require that the *same* δ works for *every* p, while in Definition 8.12 we first fix a p and only then find the δ. Said otherwise: in Definition 8.12, δ is allowed to be a function of p as well as of ϵ; in Definition 8.35, δ is allowed to be a function of ϵ, but not of p.

Example 8.37. For any $c \in \mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = cx$ is uniformly continuous. Indeed, for any given $\epsilon > 0$, we may choose any δ with $c\delta \leq \epsilon$; more concretely, if $c \neq 0$, the choice $\delta = \frac{\epsilon}{c}$ will do the job, and if $c = 0$, any δ will do. (Prove it!)

Example 8.38. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous, but not uniformly continuous. To prove this, suppose we pick $\epsilon = 1$, and consider any $\delta > 0$. Then, consider $x = \frac{1}{\delta}$. We have

$$\left| f\left(x + \frac{\delta}{2}\right) - f(x) \right| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} = 1 + \frac{\delta^2}{4} > \epsilon \,.$$

So the condition (8.3) is not satisfied with this δ.

Example 8.39. The function $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous, but not uniformly continuous. (Prove it!)

In these two counterexamples the domain is not compact. Fortunately, when the domain is compact there is no difference between continuity and uniform continuity:

Proposition 8.40. If X is compact, and $f : X \to Y$ is continuous on X, then f is uniformly continuous on X.

Proof. Fix $\epsilon > 0$. Since f is continuous, for every $p \in X$ there exists some $\phi(p)$ such that

$$f(N_{\phi(p)}(p)) \subset N_{\frac{1}{2}\epsilon}(f(p)) \,. \tag{8.4}$$

Now we consider the neighborhoods

$$J(p) = N_{\frac{1}{2}\phi(p)}(p).$$

The collection $\{J(p) \mid p \in X\}$ gives an open cover of X. Since X is compact, this cover has a finite subcover $\{J(p_1), \ldots, J(p_n)\}$. We let

$$\delta = \frac{1}{2}\min\{\phi(p_1), \ldots, \phi(p_n)\}.$$

It remains to show that (8.3) indeed holds with this δ. So, suppose $p, p' \in X$ and $d(p, p') < \delta$. Then there is some p_m such that $d(p, p_m) < \frac{1}{2}\phi(p_m)$. Using the triangle inequality we also have $d(p', p_m) < \frac{1}{2}\phi(p_m) + \delta < \phi(p_m)$.

The conclusion of the above paragraph is that both p and p' are in $N_{\phi(p_m)}(p_m)$. Using this and (8.4) we can complete the proof:

$$d(f(p), f(p')) < d(f(p), f(p_m)) + d(f(p_m), f(p')) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

as desired. $\square$

9. The derivative in one variable

9.1. **Definition and basic properties.** We are finally ready to define the derivative of a real-valued function in one real variable.

Definition 9.1. Suppose $a, b \in \mathbb{R}_{\text{ext}}$, with $b > a$, and $f : [a, b] \to \mathbb{R}$. Then for any $x \in [a, b]$ consider

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

If this limit exists, we call it $f'(x)$, and say f is *differentiable at* x. If f is differentiable at all $x \in [a, b]$ we just say f is *differentiable*.

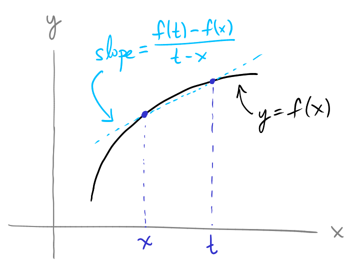

FIGURE 24. The function f depicted here is differentiable at x; as $t \to x$ the slope of the dashed light blue line approaches $f'(x)$.

Note that the function $g(t) = \frac{f(t)-f(x)}{t-x}$ which appears in Definition 9.1 is defined on the smaller domain $[a, b] \setminus \{x\}$. But this is OK: x is still a limit point of this domain, so $\lim_{t \to x} g(t)$ makes sense even though $g(x)$ is not defined.

Definition 9.2. Let

$$E = \{x \in [a, b] \mid f \text{ is differentiable at } x\}.$$

The function $f' : E \to \mathbb{R}$ is called the *derivative* of f.

Here are the most basic examples.

Example 9.3. If $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = cx$, then f is differentiable at every $x \in \mathbb{R}$, and

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{ct - cx}{t - x} = \lim_{t\to x} c = c.$$

Example 9.4. If $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = |x|$, then f is differentiable at all $x \neq 0$, and

$$f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

But f is not differentiable at $x = 0$. (Prove it!)

Example 9.5. If $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = c$, then f is differentiable at every $x \in \mathbb{R}$, and

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{c - c}{t - x} = \lim_{t\to x} 0 = 0.$$

Proposition 9.6. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable at x. Then f is continuous at x.

Proof.

$$\lim_{t\to x} f(t) - f(x) = \left(\lim_{t\to x} \frac{f(t) - f(x)}{t - x}\right)\left(\lim_{t\to x} t - x\right) = f'(x)0 = 0\,.$$

Here, to "split up" the limit $\lim_{t\to x} f(t) - f(x)$, we use the fact that we already know the two limits on the right side exist (because f is differentiable): otherwise such a splitting would not be legitimate! □

Proposition 9.7. Suppose $f, g : [a, b] \to \mathbb{R}$ are both differentiable at x. Then:

(1) $f + g$ is differentiable at x, and $(f + g)'(x) = f'(x) + g'(x)$.

(2) fg is differentiable at x, and $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Proof. For (1) we use the definition of f' directly:

$$\begin{aligned}
\lim_{t\to x} \frac{(f + g)(t) - (f + g)(x)}{t - x} &= \lim_{t\to x} \frac{f(t) + g(t) - f(x) - g(x)}{t - x} \\
&= \lim_{t\to x} \frac{f(t) - f(x)}{t - x} + \lim_{t\to x} \frac{g(t) - g(x)}{t - x} \\
&= f'(x) + g'(x)
\end{aligned}$$

as desired.

For (2):

$$\begin{aligned}
\lim_{t\to x} \frac{(fg)(t) - (fg)(x)}{t - x} &= \lim_{t\to x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\
&= \lim_{t\to x} \frac{f(t)g(t) - f(x)g(t)}{t - x} + \lim_{t\to x} \frac{f(x)g(t) - f(x)g(x)}{t - x} \\
&= \left(\lim_{t\to x} g(t)\right) \lim_{t\to x} \frac{f(t) - f(x)}{t - x} + (f(x)) \lim_{t\to x} \frac{g(t) - g(x)}{t - x} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

as needed. (To get the first term on the last line we used the fact that g is continuous, by Proposition 9.6.) ☐

Proposition 9.8. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$, then f is differentiable, and $f'(x) = nx^{n-1}$.

Proof. By induction. For $n = 0$ this just says that when $f(x) = 1$ we have $f'(x) = 0$. Now suppose we know the statement for $n-1$ and want to prove it for n. We compute:

$$(x^n)' = (xx^{n-1})' = x'x^{n-1} + x(x^{n-1})' = x^{n-1} + x(n-1)x^{n-2} = nx^{n-1}$$

as desired. ☐

Proposition 9.9. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 1/x$, then f is differentiable, and $f'(x) = -1/x^2$.

Proof. If we already knew that f is differentiable, then we could prove this using the product rule. But we don't know that, so we'll work more directly:

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{\frac{1}{t} - \frac{1}{x}}{t - x} = \lim_{t \to x} \frac{x - t}{tx(t - x)} = \lim_{t \to x} -\frac{1}{tx} = -\frac{1}{x^2}\,.$$

☐

Another way of thinking about the derivative is that it provides an affine-linear approximation to f, of the form $f(t) \approx a + bt$:

$$f(t) \approx f(x) + f'(x)(t - x)\,. \tag{9.1}$$

The precise meaning of (9.1) is nothing more than the definition of the derivative: the difference between the two sides goes to 0 when $t \to x$, even if we divide it by $t - x$. In particular, the meaning of (9.1) is *only* a statement about the limit as $t \to x$; it doesn't say anything about what happens at any specific value of t.

The intuition (9.1) is often useful, both for applications and for proofs; for example:

Proposition 9.10 (Chain rule). Suppose:

- $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable at x.
- $g : I \to \mathbb{R}$ where I is an interval containing $f([a, b])$, and g is differentiable at $f(x)$.
- $h = g \circ f$.

Then h is differentiable at x and $h'(x) = g'(f(x))f'(x)$.

Proof. Here is the idea. When t is close to x, $f(t)$ is also close to $f(x)$, and then the differentiability of g should say that

$$g(f(t)) - g(f(x)) \approx g'(f(x))(f(t) - f(x))\,.$$

In turn the differentiability of f says that

$$f(t) - f(x) \approx f'(x)(t - x)\,.$$

Combining these two would give

$$h(t) - h(x) \approx g'(f(x))f'(x)(t - x)$$

which is what we want.

To formalize this idea we need to keep track of the errors in our approximations. Define

$$a(t) = g(f(t)) - g(f(x)) - g'(f(x))(f(t) - f(x))$$

and

$$b(t) = f(t) - f(x) - f'(x)(t - x)$$

We want to say that these errors are small as $t \to x$. More precisely: by the definition of the derivative we have

$$\lim_{t \to x} \frac{b(t)}{t - x} = 0\,,$$

and also we have

$$\begin{aligned}
\lim_{t \to x} \frac{a(t)}{t-x} &= \lim_{t \to x} \frac{g(f(t)) - g(f(x)) - g'(f(x))(f(t) - f(x))}{t - x} \\
&= \left(\lim_{t \to x} \frac{g(f(t)) - g(f(x)) - g'(f(x))(f(t) - f(x))}{f(t) - f(x)} \right) \left(\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \right) \\
&= \left(\lim_{s \to y} \frac{g(s) - g(y) - g'(s)(y - s)}{s - y} \right) f'(x) \\
&= 0 \cdot f'(x) \\
&= 0\,,
\end{aligned}$$

where in the third line we set $y = f(x)$, $s = f(t)$ and used the continuity of f at x to transfer the limit from $t \to x$ to $s \to y$.[14] Then we have

$$\lim_{t \to x} \frac{g(f(t)) - g(f(x)) - g'(f(x))f'(x)(t - x)}{t - x} = \lim_{t \to x} \frac{a(t) + g'(f(x))b(t)}{t - x} = 0$$

which shows $(g \circ f)'(x)$ is $g'(f(x))f'(x)$, as desired. □

Corollary 9.11. If f is differentiable at x, and h is defined by $h(x) = \frac{1}{f(x)}$, then $h'(x) = -\frac{f'(x)}{f(x)^2}$.

Proof. Note that $h = g \circ f$ where $g(x) = 1/x$. Then use the chain rule. □

Similarly you could derive the quotient rule from the rules we have already written.

Now there are various things you might wonder. For instance, if f is differentiable, is f' always continuous? Here is a series of famous examples, which give some warning about what kind of strange things can happen:

Example 9.12. The function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at $x = 0$, thus not differentiable at $x = 0$.

[14] To spell this out more: we want to evaluate a limit of the form $\lim_{t \to x} k(f(t))$. To show this limit is L, it's sufficient to consider any sequence $t_n \to x$ with all $t_n \neq x$, and show $k(f(t_n)) \to L$. Let $s_n = f(t_n)$. We have $s_n \to y$ since f is continuous at x. Thus, if we know $\lim_{s \to y} k(s) = L$, we can conclude that $k(s_n) \to L$, as needed.

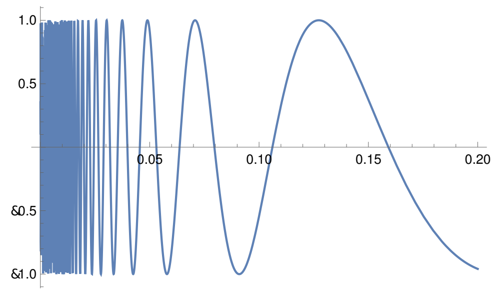

FIGURE 25. The function $f(x) = \sin(\frac{1}{x})$.

Example 9.13. The function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$, but still not differentiable at $x = 0$, because $\lim_{t\to 0} \frac{t\sin(\frac{1}{t})-0}{t}$ does not exist.

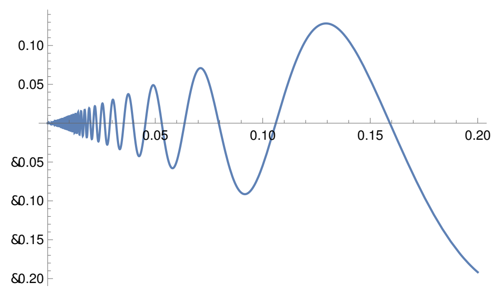

FIGURE 26. The function $f(x) = x\sin(\frac{1}{x})$.

Example 9.14. The function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$ and has $f'(0) = 0$, but $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ for $x \neq 0$, from which it follows that f' is not continuous at $x = 0$.

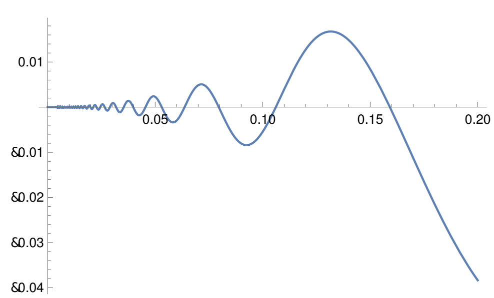

FIGURE 27. The function $f(x) = x^2\sin(\frac{1}{x})$.

9.2. Extrema.

Definition 9.15. We say $f : X \to \mathbb{R}$ has a *maximum* or *global maximum* at $p \in X$ if, for all $q \in X$, $f(p) \geq f(q)$.

Definition 9.16. We say $f : X \to \mathbb{R}$ has a *local maximum* at $p \in X$ if there exists $\delta > 0$ such that

$$d(p, q) < \delta \quad \Longrightarrow \quad f(q) \leq f(p) .$$

Said otherwise, f has a local maximum at p just if, for some $\delta > 0$, $f|_{N_\delta(p)}$ has a global maximum at p. Every global maximum is also a local maximum, but a local maximum need not be a global maximum.

We similarly define local and global minima. We use "extremum" as shorthand for "maximum or minimum."

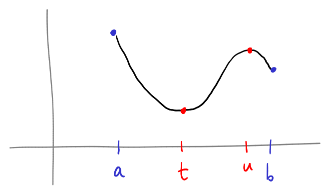

FIGURE 28. The function $f : [a, b] \to \mathbb{R}$ has a global maximum at a and a global minimum at t. It has local maxima at a, u and local minima at t, b.

Now, one of the fundamental applications of the derivative is the fact that one can use it to detect extrema:

Theorem 9.17. If $f : [a, b] \to \mathbb{R}$ has a local extremum at $x \in (a, b)$, and f is differentiable at x, then $f'(x) = 0$.

Proof. We prove it in the case of a local maximum. Fix δ such that $f|_{(x-\delta, x+\delta)}$ has a global maximum at x. Then, for $t \in (x - \delta, x)$ we have $\frac{f(t)-f(x)}{t-x} \geq 0$. It follows that $f'(x) \geq 0$ (why?) On the other hand, for $t \in (x, x + \delta)$ we have $\frac{f(t)-f(x)}{t-x} \leq 0$, and it follows that $f'(x) \leq 0$. Combining these we get $f'(x) = 0$. □

As you probably remember from calculus, though, the converse of Theorem 9.17 is not true: if $f'(x) = 0$, x need not be even a local extremum of f (consider $f(x) = x^3$ at $x = 0$ for a counterexample.)

9.3. The mean value theorem.

More generally, suppose I know something about f'; how can I transfer it to know something about f? For example, suppose $f'(x) \geq 0$ for all x; how can we conclude that f is monotonically increasing? One fundamental tool in this direction is:

Theorem 9.18 (Mean value theorem). Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $t \in (a, b)$ such that

$$f'(t) = \frac{f(b) - f(a)}{b - a} .$$

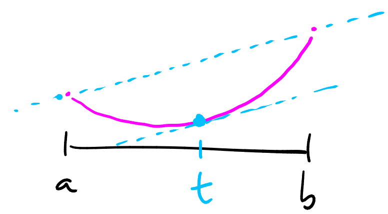

FIGURE 29. An illustration of the mean value theorem.

Proof. First consider the special case where $f(b) = f(a)$. In that case we are looking for $t \in (a, b)$ with $f'(t) = 0$. If f is constant, then any t will do. Otherwise, either the global maximum of f or the global minimum of f must be attained at some $t \in (a, b)$, and then we have $f'(t) = 0$ by Theorem 9.17.

Now consider the general case, where $f(b)$ need not equal $f(a)$. We adjust f by a linear function to reduce to the previous case: let $\tilde{f}(t) = f(t) - \frac{f(b)-f(a)}{b-a} t$. Then $\tilde{f}(a) = \tilde{f}(b)$, so by the previous part there is some t with $\tilde{f}'(t) = 0$, and this means

$$f'(t) = \frac{f(b) - f(a)}{b - a}$$

as desired. □

Here is a simple application.

Corollary 9.19. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then:

(1) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(2) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(3) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

Proof. We prove (1): if $a < x_1 < x_2 < b$, then by Theorem 9.18 there exists some $x \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1) f'(x) \geq 0$, thus $f(x_2) \geq f(x_1)$. It follows that f is monotonically increasing on (a, b). Using the continuity of f we can extend this to show that f is monotonically increasing on $[a, b]$ (exercise).

The proofs of (2), (3) are similar. □

9.4. **The Taylor polynomial.** As we have discussed, knowing f' allows us to make a local model for the function f, representing it approximately as an affine-linear function, i.e. a polynomial of degree at most 1.

If we know something about the higher-order derivatives we can often do better.

Definition 9.20. For $f : [a, b] \to \mathbb{R}$ let $f^{(0)} = f$ and define $f^{(n)}$ iteratively by letting $f^{(n)}$ be the derivative of $f^{(n-1)}$.

So $f^{(1)} = f'$, $f^{(2)} = f''$ and so on. (Note that each successive derivative is defined on a possibly smaller domain, and that in order for (say) f'' to be defined at x, f' must be defined on some closed interval containing x.)

Now we make successive estimates

$$f(t) \approx f(a),$$

$$f(t) \approx f(a) + f'(a)(t-a),$$

$$f(t) \approx f(a) + f'(a)(t-a) + \frac{f''(a)}{2}(t-a)^2,$$

$$f(t) \approx f(a) + f'(a)(t-a) + \frac{f''(a)}{2}(t-a)^2 + \frac{f'''(a)}{3!}(t-a)^3,$$

and so on. At each step, the polynomial $P_n(t;a)$ on the right side was determined by the condition that $f(t) - P_n(t;a)$ and its first $n-1$ derivatives are all zero at $t = a$; in other words, as far as the first $n-1$ derivatives at a are concerned, $f(t)$ and $P_n(t;a)$ agree. Continuing this pattern we define:

Definition 9.21. If f is n times differentiable at a, the *n-th Taylor polynomial of f at a* is

$$P_n(t;a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k .$$

Then, we expect that

$$f(t) \approx P_n(t;a)$$

up to some error. Directly from the definitions we can see that the error is small as $t \to a$ in an appropriate sense:

$$\lim_{t \to a} \frac{f(t) - P_n(t;a)}{(t-a)^n} = 0 .$$

Often, though, we are interested in a different kind of question: we hold t fixed, and want to know how big the error is. To make a sharp estimate we give a higher-order analogue of the mean value theorem:

Theorem 9.22. Suppose $f : [a,b] \to \mathbb{R}$, and $f^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $t \in (a,b)$ such that

$$f(b) = P_{n-1}(b;a) + \frac{f^{(n)}(t)}{n!}(b-a)^n .$$

(When $n = 1$ this is the mean value theorem, Theorem 9.18.)

Proof. First we consider the special case where $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and also $f(b) = 0$. Then $P_{n-1}(b;a) = 0$ and we want to prove there is some t such that

$$f^{(n)}(t) = 0 .$$

First, since $f(a) = f(b) = 0$, there is some $t_1 \in (a,b)$ with $f'(t_1) = 0$. Next, since $f'(a) = f'(t_1) = 0$, there is some $t_2 \in (a,t_1)$ with $f^{(2)}(t_2) = 0$. Continuing inductively we get $t_k \in (a, t_{k-1})$ with $f^{(k)}(t_k) = 0$. The desired t is $t = t_n$.

Now, for the general case, we shift f by a polynomial of degree $\leq n$ in order to return to the special case. Namely, we consider

$$\tilde{f}(t) = f(t) - P_{n-1}(t;a) - M(t-a)^n .$$

For any M, this has $\tilde{f}^{(k)}(a) = 0$ for $0 \leq k \leq n-1$ (why?) We choose M to arrange that $\tilde{f}(b) = 0$, i.e.

$$M = \frac{f(b) - P_{n-1}(b;a)}{(b-a)^n}\,. \tag{9.2}$$

Then the above proof shows there is some t with $\tilde{f}^{(n)}(t) = 0$, i.e.

$$0 = f^{(n)}(t) - Mn!\,,$$

which unwinds to the desired equation after substituting (9.2). □

If we can control $f^{(n)}$ somehow, then Theorem 9.22 can be used to get good information about $f(b)$. For instance, if $|f^{(n)}|$ is bounded by some M on the interval $[a,b]$, we can conclude our polynomial approximation of $f(b)$ by $P_{n-1}(b;a)$ is good, in the sense that

$$|f(b) - P_{n-1}(b;a)| \leq \frac{M}{n!}|b-a|^n.$$

Example 9.23. Let's use Theorem 9.22 to get a rigorous estimate of $f(x) = \sqrt{1+x}$ for $x > 0$, using a Taylor polynomial at $a = 0$. We have $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$. For $x \in [0,b]$ we estimate $f'''(x) \leq \frac{3}{8}$.

The Taylor polynomial

$$P_2(b;0) = f(0) + f'(0)b + \frac{f''(0)}{2}b^2 = 1 + \frac{1}{2}b - \frac{1}{8}b^2\,,$$

and our error bound is, for $b > 0$,

$$|f(b) - P_2(b;0)| = \frac{|f'''(t)|}{6} \leq \frac{1}{16}b^3\,.$$

We can test this at e.g. $b = \frac{1}{10}$: then our estimate and error bound are

$$P_2(b;0) = 1 + 0.05 - 0.00125 = 1.0487500, \quad \frac{1}{16}b^3 = 0.0000625\,.$$

As a check, the actual value and the error are

$$f(b) = \sqrt{1.1} \approx 1.0488088, \quad |f(b) - P_2(b;0)| \approx 0.0000588,$$

just within the bound.

One particularly nice thing that could happen is for the polynomial approximations $P_n(b;a)$ to *converge* to $f(b)$ as $n \to \infty$. If this happens, then we can write

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(b-a)^n = f(b)\,.$$

Many functions f which "occur in nature" have this property: for instance, this does happen for the approximations to $\sqrt{1+b}$ we were computing above, so long as $|b| < 1$. (You could prove this by showing that the error bound goes to 0 as $n \to \infty$.) Oddly enough, this property becomes easier to understand when one thinks of the variable as complex instead of real; see MATH 310 for more about why, and about the many marvelous consequences.

But the news is not always so good. First, the approximations $P_n(b;a)$ might not converge to anything — e.g. this would happen if we take $|b| \geq 1$ in the example above. Maybe more surprising, even if they do converge, they might converge to a limit which is not equal to $f(b)$!

Example 9.24. A famous counterexample: we consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0 \end{cases}$$

One can show that all $f^{(n)}$ exist everywhere, including $x = 0$, and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ — despite the fact that $f(x)$ is not constant in any neighborhood of 0!

Since all the derivatives vanish, the Taylor polynomials $P_n(b;0) = 0$ for all n. Thus the $P_n(b;0)$ converge to 0 as $n \to \infty$, for any b. On the other hand, for $b > 0$ we have $f(b) \neq 0$. So this is an example where the sequence $(P_n(b;a))$ converges, but *not* to the value $f(b)$.

9.5. **L'Hospital's rule.** We briefly discuss one simple version of L'Hospital's rule, sometimes useful for evaluating limits of the formal shape "$\frac{0}{0}$."

Theorem 9.25. Suppose $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are differentiable, with $g'(x) \neq 0$, $f(x) = 0$, $g(x) = 0$. Then

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)} .$$

Proof. The idea is that $f(t) \approx (t-x)f'(x)$ and $g(t) \approx (t-x)g'(x)$. We factorize

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \left(\lim_{t \to x} \frac{f(t)}{t-x} \right) \left(\lim_{t \to x} \frac{t-x}{g(t)} \right) = f'(x) g'(x)^{-1}$$

as desired. □

See Rudin for a fancier version, with fewer assumptions: the functions don't have to be differentiable at x, only the limit $\lim_{t \to x} \frac{f'(t)}{g'(t)}$ has to exist. Once one has this version one can also extend it to apply to limits of the form "$\frac{\infty}{\infty}$."

10. The Riemann integral

Now we are ready to discuss integration: given a function $f : [a,b] \to \mathbb{R}$ we want to make sense of

$$\int_a^b f(x) \, dx . \tag{10.1}$$

As you know from your 1-variable calculus course, (10.1) is meant to represent a kind of "weighted average" of $f(x)$, with the weighting given by lengths of intervals. There are various approaches to defining the integral. For our purposes in this course we will use a relatively simple one, the *Riemann integral*, which is close to how the integral was probably motivated in your calculus course. In later courses (particularly MATH 305) you can learn about more sophisticated notions of integration, which are more powerful in some respects, and can be extended to more complicated situations.

10.1. **Basic definitions.**

Definition 10.1. A *partition* P of $[a, b]$ is a set $\{x_0, \dots, x_n\}$ with $a = x_0 \le x_1 \le \dots \le x_n = b$. Given such a partition let $\Delta x_i = x_i - x_{i-1}$.

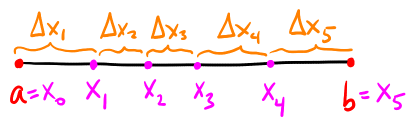

FIGURE 30. A partition of $[a, b]$ into 5 subintervals.

Thus a partition divides the interval $[a, b] = [x_0, x_n]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-1}, x_n]$, with widths $\Delta x_1, \dots, \Delta x_n$.

Definition 10.2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function, and $P = \{x_0, \dots, x_n\}$ a partition of $[a, b]$. Then let

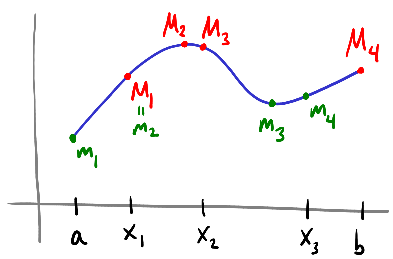

FIGURE 31. A partition of $[a, b]$ into 4 subintervals and the values of M_i, m_i for these 4 intervals.

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

define the *upper sum* and *lower sum*

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and define the *upper Riemann integral* and *lower Riemann integral*

$$\overline{\int_a^b} f(x)\, dx = \inf\{U(P, f) \mid P \text{ a partition of } [a, b]\},$$
$$\underline{\int_a^b} f(x)\, dx = \sup\{L(P, f) \mid P \text{ a partition of } [a, b]\}.$$

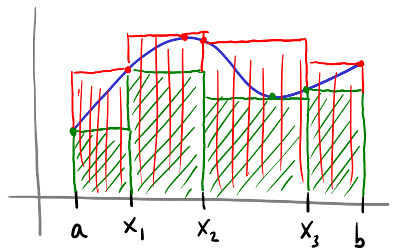

FIGURE 32. The function f and partition P are as in the figure above. The upper sum $U(P, f)$ is the sum of the red shaded areas; the lower sum $L(P, f)$ is the sum of the green ones. Evidently $L(P, f) < U(P, f)$ in this example.

The rough expectation is that, as the partition P gets finer, the lower sums will increase, approaching the lower integral from below; likewise and the upper sums will decrease, approaching the upper integral from above. We will see momentarily that this is indeed the correct picture.

Note that for each partition P we have $L(P, f) \leq U(P, f)$ (since each $m_i \leq M_i$). We would like to say more: that *all* lower sums are lower than *all* upper sums. For this we use the following:

Definition 10.3. If P, P' are partitions of $[a, b]$, we say P' is a *refinement* of P if $P \subset P'$. Given two partitions P_1, P_2 of $[a, b]$ a *common refinement* is any partition containing $P_1 \cup P_2$.

Every pair of partitions P_1, P_2 has a common refinement, namely $P_1 \cup P_2$.

Proposition 10.4. If P' is a refinement of P, then $U(P', f) \leq U(P, f)$ and $L(P', f) \geq L(P, f)$.

Proof. We just prove the statement for the upper sums. By induction, it's sufficient to deal with the case where P' includes only one more point than P, say $x_i < x_* < x_{i+1}$.

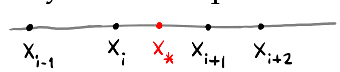

FIGURE 33. Refining a partition by adding one extra point.

Let $\widehat{M}$ be the supremum of f on $[x_i, x_{i+1}]$, $\widehat{M}_1$ the supremum of f on $[x_i, x_*]$, and $\widehat{M}_2$ the supremum of f on $[x_*, x_{i+1}]$. We have $\widehat{M}_1 \leq \widehat{M}$, $\widehat{M}_2 \leq \widehat{M}$. Let U_0 be the sum of contributions from the other intervals,

$$U_0 = \sum_{j=1, j\neq i+1}^{n} M_j \Delta x_j \,.$$

Then we have

$$\begin{aligned} U(P', f) &= U_0 + \widehat{M}_1(x_* - x_i) + \widehat{M}_2(x_{i+1} - x_*) \\ &\leq U_0 + \widehat{M}(x_{i+1} - x_1) \\ &= U(P, f) \end{aligned}$$

as desired. The proof for lower sums is parallel. □

Corollary 10.5. For any partitions P and P' of $[a, b]$ we have $L(P, f) \leq U(P', f)$.

Proof. Take a common refinement P'' of P and P'. Then we have

$$L(P,f) \le L(P'',f) \le U(P'',f) \le U(P',f)$$

as desired. ☐

Corollary 10.6. For any $f : [a,b] \to \mathbb{R}$ bounded,

$$\underline{\int_a^b} f(x)\,\mathrm{d}x \le \overline{\int_a^b} f(x)\,\mathrm{d}x\,.$$

Definition 10.7. We say $f : [a,b] \to \mathbb{R}$ is *Riemann integrable* on $[a,b]$ if f is bounded and $\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$. In that case we define the *Riemann integral* (or just *integral*) by

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x\,.$$

We can see at once that constant functions are integrable, and their integral is what we expect:

Example 10.8. Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = c\,.$$

For any partition P of $[a,b]$, all lower sums $L(P,f) = c(b-a)$ and likewise all upper sums $U(P,f) = c(b-a)$. Thus we have

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = c(b-a) = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

and so this function is integrable, with

$$\int_a^b f(x)\,\mathrm{d}x = c(b-a)\,.$$

We can also see that not every function is integrable:

Example 10.9. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

For any partition P of $[0,1]$, all lower sums $L(P,f) = 0$ while all upper sums $U(P,f) = 1$. (Prove it!) Thus we have

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = 0 \ne 1 = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

and so this function is not integrable.

(You might feel that this function ought to be integrable: after all, it is 0 at every point except for a countable set, so morally it should have $\int_0^1 f(x)\mathrm{d}x = 0$. The definition of Riemann integral does not match with this intuition. But there is a different notion of integral, the Lebesgue integral, for which this function actually is integrable, and sure enough it has integral 0. The Lebesgue integral is discussed in MATH 305.)

10.2. **Some conditions for integrability.**

Proposition 10.10. For any $f : [a, b] \to \mathbb{R}$ bounded, f is Riemann integrable if and only if, for all $\epsilon > 0$, there exists a partition P with $U(P, f) - L(P, f) < \epsilon$.

Proof. We use $\overline{I}$, $\underline{I}$ for the upper and lower integrals.

($\Longleftarrow$) Fix $\epsilon > 0$ and the corresponding P. Then we have

$$0 \le \overline{I} - \underline{I} \le U(P, f) - L(P, f) < \epsilon\,.$$

Since $\epsilon > 0$ was arbitrary it follows that

$$\overline{I} - \underline{I} = 0.$$

($\Longrightarrow$) By definition of the upper and lower integrals, there exists P' with $U(P', f) < \overline{I} + \frac{1}{2}\epsilon$ and P'' with $L(P'', f) > \underline{I} - \frac{1}{2}\epsilon$. If $\overline{I} = \underline{I}$, then this implies $U(P', f) - L(P'', f) < \epsilon$. Let P be a common refinement of P' and P''; then $U(P, f) - L(P, f) < \epsilon$. □

Now we can use this criterion to start proving that some familiar functions are integrable.

Theorem 10.11. If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$.

Proof. Suppose $\epsilon > 0$. Let $\eta = \frac{\epsilon}{b-a}$. Since f is continuous on the compact interval $[a, b]$, we know that f is bounded, and also by Proposition 8.40 that f is uniformly continuous on $[a, b]$. Thus there exists $\delta > 0$ such that for $x, t \in [a, b]$

$$|x - t| < \delta \quad\Longrightarrow\quad |f(x) - f(t)| < \eta\,.$$

Now let P be any partition of $[a, b]$ such that the widths $\Delta x_i < \delta$ for all i. Then we have $M_i - m_i \le \eta$, and thus

$$U(P, f) - L(P, f) < \eta(b - a) = \epsilon$$

Using Proposition 10.10 we conclude that f is Riemann integrable on $[a, b]$. □

More generally:

Theorem 10.12. If $f : [a, b] \to \mathbb{R}$ is bounded, and discontinuous only at finitely many points of $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof. The idea is that we can choose a partition which traps the discontinuities in very short intervals, and thus make the contributions from those intervals very small.[15] Then we use uniform continuity to make the rest of the contributions small, as we did above.

Precisely: let $M = \sup\{|f(t)| \mid t \in [a, b]\}$. Take open intervals I_n containing the discontinuities of f, with total length less than $\frac{\epsilon}{4M}$. Then f is uniformly continuous on the compact set $K = [a, b] \setminus (I_1 \cup \cdots \cup I_n)$. Choose δ so that for $t, x \in K$, $|t - x| < \delta \implies$

[15] As usual, by "very" we really mean "arbitrarily."

$|f(t) - f(x)| < \frac{\epsilon}{2(b-a)}$. Now take a partition P which includes the endpoints of the I_n and has all $\Delta x_i < \delta$. Then

$$U(P, f) - L(P, f) < 2M\frac{\epsilon}{4M} + (b-a)\frac{\epsilon}{2(b-a)} = \epsilon\,.$$

□

Using the same device of "trapping the bad points in a small interval" one can prove the following:

Proposition 10.13. If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, and $g : [a, b] \to \mathbb{R}$ has $g(x) = f(x)$ for all x except one, then g is integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$.

Inductively, this means that changing an integrable function f at any finite number of points will not change the integral. (On the other hand, changing f at countably many points can destroy the integrability of f.)

Proof. Exercise. □

There are many more integrable functions! For example, one can show that a version of Theorem 10.12 is still true even if f is allowed to have countably many discontinuities. Moreover, even some functions with uncountably many discontinuities are allowed:

Example 10.14. Let $C \subset [0, 1]$ be the Cantor set from Example 5.74, and define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin C, \\ 1 & \text{if } x \in C \end{cases}$$

Then f is integrable on $[0, 1]$. (Prove it!)

Here is another nice class of integrable functions:

Proposition 10.15. If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is Riemann integrable on $[a, b]$.

(Note this is true even if f has infinitely many discontinuities.)

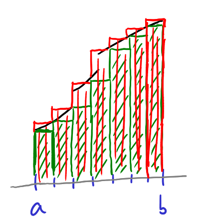

FIGURE 34. Lower and upper sums for a monotone function, with a uniform partition. Note that the terms in the lower sum (except the first) match with the terms in the upper sum (except the last), just by sliding over the rectangles by one unit. Thus the difference between the upper and lower sums just comes from the first and last rectangles.

Proof. We use a uniform partition P, where all subintervals have the same width $\Delta x_i = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, and

$$U(P,f) - L(P,f) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \frac{b-a}{n}(f(b) - f(a))$$

which we can make arbitrarily small by choosing large enough n. $\square$

Here is a technical point you might have wondered about: what if you chose some other sampling points, such as "left endpoints" or "right endpoints" as you may have done in your calculus class, or midpoints, or random sampling points? The following proposition shows that, if f is integrable and we choose a sufficiently fine partition, we will get within ϵ of the integral no matter which sampling points we use:

Proposition 10.16. Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable, and P is a partition of $[a,b]$ into n subintervals, such that $U(P,f) - L(P,f) < \epsilon$. Suppose given $x_i^* \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, n$. Then

$$\left| \sum_{i=1}^{n} f(x_i^*)\Delta x_i - \int_a^b f(x)\,\mathrm{d}x \right| < \epsilon\,.$$

Proof. This follows from the facts

$$L(P,f) \le \sum_{i=1}^{n} f(x_i^*)\Delta x_i \le U(P,f)$$

and

$$L(P,f) \le \int_a^b f(x)\,\mathrm{d}x \le U(P,f)\,.$$

$\square$

10.3. **Basic properties.** Now let's formulate some of the basic properties of the Riemann integral:

Proposition 10.17.

(1) If f_1 and f_2 are Riemann integrable on $[a,b]$, then $f_1 + f_2$ is also Riemann integrable on $[a,b]$, and

$$\int_a^b f_1(x)\,\mathrm{d}x + \int_a^b f_2(x)\,\mathrm{d}x = \int_a^b (f_1 + f_2)(x)\,\mathrm{d}x\,.$$

(2) If f is Riemann integrable on $[a,b]$, and $c \in \mathbb{R}$, then cf is also Riemann integrable on $[a,b]$, and

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x\,.$$

(3) If f_1 and f_2 are Riemann integrable on $[a,b]$, and $f_1(x) \le f_2(x)$ for all $x \in [a,b]$, then

$$\int_a^b f_1(x)\,\mathrm{d}x \le \int_a^b f_2(x)\,\mathrm{d}x\,.$$

(4) If f is Riemann integrable on $[a,b]$ and $a<c<b$, then $f(x)$ is Riemann integrable on $[a,c]$ and $[c,b]$, and

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x\,. \tag{10.2}$$

(5) If f is Riemann integrable on $[a,b]$, and $|f(x)| \le M$ for all $x \in [a,b]$, then

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le M|b-a|\,.$$

Proof. We'll prove just part (4), leaving the rest as exercises. For this part, pick a partition P of $[a,b]$ with $U(P,f)-L(P,f)<\epsilon$. We may assume that P contains c (if not, we refine it to include c, which still preserves the property $U(P,f)-L(P,f)<\epsilon$.) Divide P into partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively. Then

$$U(P_1,f)-L(P_1,f) \le U(P,f)-L(P,f) < \epsilon$$

and similarly

$$U(P_2,f)-L(P_2,f) \le U(P,f)-L(P,f) < \epsilon$$

Thus f is integrable on $[a,c]$ and $[c,b]$. Next, to actually determine the integral:

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \ge L(P_1,f)+L(P_2,f) = L(P,f) > \int_a^b f(x)\,\mathrm{d}x - \epsilon$$

and this holds for any $\epsilon>0$, thus

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \ge \int_a^b f(x)\,\mathrm{d}x$$

Similarly considering upper sums we show

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x$$

and combining these gives the desired equality. □

10.4. **Other limits of integration.** So far we have defined $\int_a^b f(x)\,\mathrm{d}x$ only in case $b>a$. It is occasionally useful to define it more generally. In particular, to formulate Theorem 10.22 we will find it useful to define

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

independent of the function f. With this definition, (10.2) continues to be true even when $a \le c \le b$ (check it!)

It is also sometimes convenient to define the integral even for $a>b$. Then the conventional rule is to declare that by definition

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x\,.$$

With this definition, (10.2) continues to be true for arbitrary a,b,c, as long as f is integrable on all the relevant intervals (check it!) Using this at intermediate steps will allow us to somewhat shorten the proof of Theorem 10.22.

10.5. **Integrability of transformed functions.** It is nice to know e.g. that if f, g are integrable then fg is integrable. We prove something a bit more general, which will have other useful spinoffs:

Proposition 10.18. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and f maps $[a, b] \to [m, M] \subset \mathbb{R}$. Suppose given a continuous function $\phi : [m, M] \to \mathbb{R}$, and let $h : [a, b] \to \mathbb{R}$ be $h(x) = \phi(f(x))$. Then h is Riemann integrable on $[a, b]$.

Proof. Here is the idea. We introduce some convenient terminology: given a function f on an interval, say the "max-min range" of f is the difference between the sup and inf of the values of f on the interval. In $U(P, f) - L(P, f)$ the contribution from each interval is the product of the max-min range of f on that interval and the width of the interval.

We consider some partition P where $U(P, f) - L(P, f)$ is small. Since ϕ is continuous on a compact set it is uniformly continuous; this means that on subintervals where the max-min range of f is small, the max-min range of h will also be small. There may be a few subintervals where f varies a lot, but then those intervals must have small width since $U(P, f) - L(P, f)$ is small. So on each interval we can control *either* the max-min range of h *or* the width of the interval; thus we expect to be able to control $U(P, h) - L(P, h)$.

Now we describe the details. Fix $\epsilon > 0$. We'll pick some ϵ' small (to be specified later). Since $[m, M]$ is compact, ϕ is uniformly continuous there. Thus we can find $\delta > 0$ such that $|s - t| \leq \delta \implies |\phi(s) - \phi(t)| \leq \epsilon'$.

We pick a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \delta \cdot \epsilon'$. Let A be the set of subintervals where the max-min range of f is less than δ; let B be the rest. Now we need to estimate the contribution to $U(P, h) - L(P, h)$ from each of these sets:

- On each subinterval in A, the max-min range of f is less than δ, so the max-min range of h is at most ϵ'. The sum of the widths of these subintervals is at most the total width of the interval, $b - a$. Thus the total contribution of these subintervals to $U(P, h) - L(P, h)$ is at most $\epsilon'(b - a)$.
- On each subinterval in B, the max-min range of h is at most $2K$, where K is an overall bound for ϕ. The good news is that these subintervals are very short: their total length is less than ϵ' (this follows from the fact that $U(P, f) - L(P, f) < \delta \cdot \epsilon'$.) So the total contribution of these subintervals to $U(P, h) - L(P, h)$ is less than $2K\epsilon'$.

Altogether then we have

$$U(P, f) - L(P, f) < \epsilon'(b - a) + 2K\epsilon'$$

So, if we pick $\epsilon' = \frac{\epsilon}{b-a+2K}$ we get the desired

$$U(P, f) - L(P, f) < \epsilon\,.$$

□

Corollary 10.19. If f and g are Riemann integrable on $[a, b]$, then fg is also Riemann integrable on $[a, b]$.

Proof. Since $\phi(x) = x^2$ is integrable we know that f^2, g^2, $(f + g)^2$ are integrable. Then use $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$. □

Corollary 10.20. If f is integrable on $[a,b]$ then $|f|$ is also integrable on $[a,b]$, and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. For the integrability we just use the fact that $\phi(x) = |x|$ is continuous. For the inequality, note that for some $c = \pm 1$ we have

$$\left|\int f(x)\mathrm{d}x\right| = c\int f(x)\,\mathrm{d}x = \int cf(x)\,\mathrm{d}x \le \int |f(x)|\mathrm{d}x\,.$$

□

10.6. **Change of variables.** Now we consider a situation that comes up very often in practice. How would you evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x\,\mathrm{d}x$? The standard device is to make a "substitution": one writes $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$, and then converts the integral to

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos x\,\mathrm{d}x = \int_0^1 u^3\,\mathrm{d}u\,.$$

How should we think about this process? It involves recognizing the integrand as a product of two pieces:

- One part of the integrand, $\sin^3 x$, involves the function $g : [0, \frac{\pi}{2}] \to \mathbb{R}$ given by
$$g(x) = \sin^3 x\,.$$
We have two different domains, $[A,B] = [0, \frac{\pi}{2}]$ and $[a,b] = [0,1]$, and we relate them by the map $\varphi : [A,B] \to [a,b]$ given by
$$\varphi(x) = \sin x\,.$$
Note that φ is a nice map: it is differentiable and strictly increasing (so in particular it is bijective). Then, we recognize $g : [A,B] \to \mathbb{R}$ as built from $f : [a,b] \to \mathbb{R}$,
$$g = f \circ \varphi, \qquad f(u) = u^3\,.$$
- The other part of the integrand, $\cos x$, we identify as $\varphi'(x)$. (Or, if it helps to remember, include the $\mathrm{d}x$ here: $\cos x\,\mathrm{d}x = \varphi'(x)\,\mathrm{d}x$.)

Rewritten in this language, the substitution rule we are using is

$$\int_A^B g(x)\,\varphi'(x)\,\mathrm{d}x = \int_a^b f(u)\,\mathrm{d}u\,.$$

Let's formulate this as a theorem:

Theorem 10.21 (Change of variables formula for Riemann integrals). Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. Suppose $\varphi : [A,B] \to [a,b]$ is differentiable and strictly increasing, and φ' is Riemann integrable on $[A,B]$. Then, define $g : [A,B] \to \mathbb{R}$ by $g = f \circ \varphi$; $g\varphi'$ is Riemann integrable on $[A,B]$, and

$$\int_A^B g(x)\,\varphi'(x)\,\mathrm{d}x = \int_a^b f(u)\,\mathrm{d}u\,.$$

Before giving a detailed proof we had better explain the idea. We consider some partition $P = \{A, x_1, x_2, \ldots, B\}$ of the domain $[A, B]$. Using φ this maps to a partition $\widetilde{P} = \{a, u_1, u_2, \ldots, b\}$ of $[a, b]$ where each $u_i = \varphi(x_i)$.

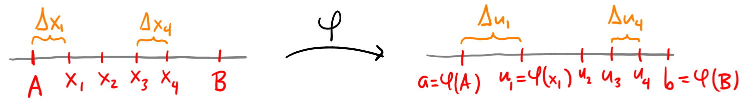

FIGURE 35. The change-of-variables map φ maps a partition P of $[A, B]$ to a partition $\widetilde{P}$ of $[a, b]$.

The integral on the right side is about

$$\int_a^b f(u)\,\mathrm{d}u \approx \sum_{i=1}^{n} \Delta u_i f(u_i^s)$$

where u_i^s are some sampling points chosen in the respective intervals of the partition $\widetilde{P}$. On the left side we have a similar formula

$$\int_A^B g(x)\varphi'(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} \Delta x_i f(\varphi(x_i^s))\varphi'(x_i^s)\,.$$

Now, suppose we choose the sampling points $u_i^s = \varphi(x_i^s)$, and also take the partition fine; then $\Delta u_i \approx \Delta x_i\, \varphi'(x_i^s)$ (why?) and so the two integrals are approximately equal. What remains is to turn this from a rough approximation into a proof, by keeping more careful track of the errors.

Proof. Fix $\epsilon > 0$ and choose a partition P of $[A, B]$ with

$$U(P, \varphi') - L(P, \varphi') < \epsilon\,. \tag{10.3}$$

As above, we get a corresponding partition $\widetilde{P}$ of $[a, b]$. We want to compare the upper sums $U(P, g\varphi')$ and $U(\widetilde{P}, f)$.

Using the mean value theorem Theorem 9.18, the lengths of the subintervals of the two partitions are related by

$$\Delta u_i = \Delta x_i\, \varphi'(x_i^*) \tag{10.4}$$

for some $x_i^* \in [x_{i-1}, x_i]$. When we compute $U(P, g\varphi')$ we need to consider the values of φ' at all points x_i^s in the interval, not only at x_i^*. Using (10.3), for any choice of sampling points x_i^s, we have good control over the differences $\varphi'(x_i^s) - \varphi(x_i^*)$:

$$\sum_{i=1}^{n} |\varphi'(x_i^s) - \varphi'(x_i^*)|\Delta x_i < \epsilon$$

Multiplying in $g(x_i^s)$ at each i this gives

$$\sum_{i=1}^{n} |g(x_i^s)\varphi'(x_i^s) - g(x_i^s)\varphi'(x_i^*)|\Delta x_i < M\epsilon$$

where M denotes a bound for $|g|$ on $[a, b]$. Now, let $u_i^s = \varphi(x_i^s)$; then $f(u_i^s) = g(x_i^s)$, and using (10.4) this becomes

$$\sum_{i=1}^{n} |g(x_i^s)\varphi'(x_i^s)\Delta x_i - f(u_i^s)\Delta u_i| < M\epsilon\,.$$

In particular, we have the one-sided bound

$$\sum_{i=1}^{n} g(x_i^s)\varphi'(x_i^s)\Delta x_i < \sum_{i=1}^{n} f(u_i^s)\Delta u_i + M\epsilon\,.$$

The sum on the right side is bounded above by $U(\widetilde{P}, f)$, whatever the u_i^s are. Then, by taking the sup over all choices of the sampling points x_i^s, we can replace the left side by $U(P, g\varphi')$. Thus

$$U(P, g\varphi') \le U(\widetilde{P}, f) + M\epsilon\,.$$

Now we compare this to the integrals. On the left, by definition, the upper sum is at least the upper integral. The inequality holds for any refinement of P, so we can take the inf over all refinements, to replace the upper sum on the right side by the upper integral. Thus we get

$$\overline{\int_A^B} g(x)\varphi'(x)\,\mathrm{d}x \le \overline{\int_a^b} f(u)\,\mathrm{d}u + M\epsilon$$

and since this holds for any ϵ we conclude

$$\overline{\int_A^B} g(x)\varphi'(x)\,\mathrm{d}x \le \overline{\int_a^b} f(u)\,\mathrm{d}u\,.$$

Doing similarly for the lower integrals we have

$$\underline{\int_a^b} f(u)\,\mathrm{d}u \le \underline{\int_A^B} g(x)\varphi'(x)\,\mathrm{d}x \le \overline{\int_A^B} g(x)\varphi'(x)\,\mathrm{d}x \le \overline{\int_a^b} f(u)\,\mathrm{d}u$$

and using the integrability of f (which we didn't use up to this point!) completes the proof. □

One point in the statement of Theorem 10.21 might seem strange: shouldn't φ' automatically be integrable, since it is the derivative of something? But we have to watch out: there are examples of functions $\varphi : [a, b] \to \mathbb{R}$ which are everywhere differentiable but have φ' unbounded; probably there are even examples of this where φ is strictly increasing.[16]

10.7. **The fundamental theorem of calculus.** Finally we can discuss the fundamental theorem of calculus, which gives the essential link between differentiation and integration.

The essential idea is: given $f : [a, b] \to \mathbb{R}$ we build a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t\,. \tag{10.5}$$

Then we consider changing x a bit:

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t \approx hf(x)\,,$$

so roughly we would like to say that

$$F'(x) = f(x)\,.$$

[16] An example seems to be discussed at `https://math.stackexchange.com/questions/2791574/bounded-derivative-of-increasing-function`.

In this sense we can say that the integral of f (considered as a function of the upper endpoint, as in (10.5)) is an antiderivative of f.

Now we want to prove this statement. As usual, we have to take some care, particularly when the functions f we consider are not continuous. Here is one version that works, assuming only that f is integrable:

Theorem 10.22 (Fundamental theorem of calculus). Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t\,.$$

Then F is continuous at all $x \in [a,b]$, and if f is continuous at x, then F is differentiable at x and

$$F'(x) = f(x)\,.$$

Proof. First we show F is continuous. For this, note for $x, y \in [a,b]$ we have

$$F(y) - F(x) = \int_a^y f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t = \int_x^y f(t)\,\mathrm{d}t,$$

and so

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,\mathrm{d}t\right| \leq M|y-x|\,,$$

where M is any bound for f (f is bounded since it is integrable). Now, for any $\epsilon > 0$, take $\delta = \frac{\epsilon}{M}$; then $|x-y| < \delta \implies |F(x) - F(y)| < \epsilon$, so F is indeed continuous (actually uniformly continuous) on $[a,b]$.

Next, suppose f is continuous at x. We will show $F'(x) = f(x)$. For this, note that for $t \neq x$

$$\frac{F(t) - F(x)}{t-x} - f(x) = \frac{1}{t-x}\int_x^t \left(f(u) - f(x)\right)\,\mathrm{d}u\,.$$

We want to show that the limit as $t \to x$ vanishes. Fix $\epsilon > 0$. Since f is continuous at x, there exists some δ such that $|u-x| < \delta \implies |f(u) - f(x)| < \epsilon$. Then for $0 < |t-x| < \delta$ we have

$$\left|\frac{F(t) - F(x)}{t-x} - f(x)\right| \leq \frac{1}{t-x}\int_x^t |f(u) - f(x)|\,\mathrm{d}u \leq \epsilon\,.$$

Then by the definition of limit we have

$$\lim_{t \to x}\left(\frac{F(t) - F(x)}{t-x} - f(x)\right) = 0\,,$$

i.e.

$$F'(x) - f(x) = 0$$

as desired. ∎

Example 10.23. We consider $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } x \geq 0\,. \end{cases}$$

Then the integral is

$$F(x) = \int_{-1}^{x} f(t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } x < 0\,, \\ x & \text{if } x \geq 0\,, \end{cases}$$

which is indeed continuous on $[-1,1]$ and differentiable away from $x = 0$ with $F'(x) = f(x)$, as Theorem 10.22 predicts.

Now we can deduce a corollary, which is often used in practice as a way of computing integrals.

Corollary 10.24 (Fundamental theorem of calculus, second version). Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If we also have $F : [a,b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)\,.$$

Proof. Consider the function $G(x) = \int_a^x f(t)\,\mathrm{d}t$. This function has $G'(x) = f(x) = F'(x)$. It follows that $G(x) - F(x)$ is constant, so $G(x) = F(x) + c$ for some constant $c \in \mathbb{R}$. Thus

$$G(b) - G(a) = F(b) - F(a),$$

which is the desired result. □

(See Rudin for a slightly stronger version, where f only has to be Riemann integrable, not continuous.)

A function $F(x)$ such that $F'(x) = f(x)$ is called an *antiderivative* of f. (It's also sometimes called an *indefinite integral* of f, but I think this terminology is better avoided; using this language too much will tend to get you confused about what the definition of "integral" is.)

Example 10.25. Suppose we want to compute $\int_1^4 x^3\,\mathrm{d}x$. We observe that the function $f(x) = x^3$ has antiderivative $F(x) = \frac{1}{4}x^4$. Then it follows from Corollary 10.24 that

$$\int_a^4 x^3\,\mathrm{d}x = F(4) - F(1) = \frac{1}{4}4^4 - \frac{1}{4}1^4 = \frac{255}{4}\,.$$

Corollary 10.26 (Integration by parts). Suppose $F, G : [a,b] \to \mathbb{R}$ are both differentiable on $[a,b]$, and $F' = f, G' = g$ both continuous on $[a,b]$. Then

$$\int_a^b F(x)g(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,\mathrm{d}x\,.$$

Proof. Apply Corollary 10.24 to $H(x) = F(x)G(x)$. □

(Again Rudin has a version where f, g are only assumed integrable.)

11. Contractions and fractals

This section is not for examination; it is just meant to give the flavor of an interesting application of the ideas we have been developing in this course.

Here is the main tool:

Theorem 11.1 (Contraction mapping theorem). Suppose X is a complete metric space and $E \subset X$ is a nonempty closed subset. Suppose $\phi : E \to E$ is a *contraction* mapping, which means there is some $c \in (0,1)$ such that for all $x, y \in X$

$$d(\phi(x), \phi(y)) \leq cd(x,y) .$$

Then, there exists a unique $x_* \in X$ such that

$$\phi(x_*) = x_* .$$

Proof. The idea: start with any $x_0 \in E$, then build a sequence (x_n) in E by $x_n = \phi(x_{n-1})$. Prove that this sequence is Cauchy (if $d(x_0, x_1) = L$ then $d(x_n, x_{n+1}) \leq c^n L$, from which you show that $m, n \geq N \implies d(x_m, x_n) \leq \frac{c^N L}{1-c}$, using comparison to a geometric series.) Then since X is complete we have $x_n \to x_* \in X$, and since $E \subset X$ is closed, the limit $x_* \in E$. Show $\phi(x_*) = x_*$ by showing $x_n \to \phi(x_*)$ and using the uniqueness of limits. Finally, if $x \neq x_*$ we have $d(\phi(x), x_*) \leq cd(x, x_*)$ and $d(x, x_*) \neq 0$; thus in particular $d(\phi(x), x_*) \neq d(x, x_*)$, which implies $\phi(x) \neq x$. □

This theorem gets used for many practical purposes. One of the main ones is *solving equations*: if you have an equation you want to solve and you can rewrite it in the form $\phi(x) = x$, where ϕ is a contraction mapping acting on a closed subset in a complete metric space, then you can use Theorem 11.1 to guarantee that your equation has a unique solution.[17] But here let's discuss something fun that you can do with it, that will use a little more of the technology of abstract metric spaces.

We fix some closed $X \subset \mathbb{R}^n$ and let $\mathcal{S}(X)$ be the set of all nonempty compact subsets of X. We are going to put a metric on $\mathcal{S}(X)$.

Definition 11.2. For any $K \in \mathcal{S}(X)$ and $r > 0$, define the *r-expansion* of K by

$$E_r(K) = \bigcup_{x \in K} N_r(x) \quad \subset \quad \mathbb{R}^n .$$

Then, for $K, K' \in \mathcal{S}(X)$ define

$$d(K, K') = \inf \{ r \in \mathbb{R} \mid r > 0, K \subset E_r(K'), K' \subset E_r(K) \} \in \mathbb{R} .$$

(This d is called the *Hausdorff distance*.)

Proposition 11.3. The distance function d makes $\mathcal{S}(X)$ into a metric space.

It is worth thinking a little about what the Cauchy sequences in $\mathcal{S}(X)$ look like.

Proposition 11.4. $\mathcal{S}(X)$ is complete.

Now, suppose we have a continuous map $\phi : X \to X$. Then we can consider applying this map as well to compact subsets of X to get other compact subsets, so we get an

[17] See https://kconrad.math.uconn.edu/blurbs/analysis/contraction.pdf and https://kconrad.math.uconn.edu/blurbs/analysis/contraction2.pdf for some very clear notes with examples, including Newton's method for solving equations, and one of the basic existence-uniqueness theorems for solutions of ordinary differential equations (Picard's theorem), as well as the fun application we discuss here. In MATH 302, the contraction theorem will be used to prove the extremely important "inverse function theorem" for differentiable maps $f : \mathbb{R}^n \to \mathbb{R}^k$.

induced map $\Phi : \mathcal{S}(X) \to \mathcal{S}(X)$ by

$$\Phi(K) = \phi(K) .$$

Proposition 11.5. If $\phi : X \to X$ is a contraction mapping, then $\Phi : \mathcal{S}(X) \to \mathcal{S}(X)$ is also a contraction mapping.

Applying the contraction mapping theorem to Φ won't give anything too interesting. But now we consider a little variant, as follows. Suppose we have a bunch of continuous maps $\phi_1, \ldots \phi_n$ each mapping $X \to X$. Then we can define an induced map $\Phi : \mathcal{S}(X) \to \mathcal{S}(X)$ again by

$$\Phi(K) = \phi_1(K) \cup \cdots \cup \phi_n(K)$$

and again we have:

Proposition 11.6. If $\phi_i : X \to X$ is a contraction mapping for each $i = 1, 2, \ldots, n$, then $\Phi : \mathcal{S}(X) \to \mathcal{S}(X)$ is also a contraction mapping.

Now we can apply Theorem 11.1 to Φ. The conclusion is:

Corollary 11.7. Suppose $X \subset \mathbb{R}^n$ is nonempty and closed. Suppose $\phi_1, \ldots, \phi_n$ are contraction maps $X \to X$. Then there exists a unique compact subset $K \subset X$ such that

$$K = \phi_1(K) \cup \cdots \cup \phi_n(K) .$$

Example 11.8. Suppose $X = \mathbb{R}$, and

$$\phi_0(x) = \frac{x}{3}, \quad \phi_1(x) = 1 - \frac{1-x}{3} .$$

Both ϕ_0 and ϕ_1 are indeed contractions (their unique fixed points are 0 and 1 respectively). Thus Corollary 11.7 says there is a unique compact $K \subset \mathbb{R}$ with

$$K = \phi_0(K) \cup \phi_1(K) .$$

In fact we have seen this K already: it is the Cantor set from Example 5.74.

Example 11.9. Suppose $X = \mathbb{R}^2$. For any $p \in \mathbb{R}^2$ let $\phi_p(x)$ be the midpoint of the segment from x to p. Then ϕ_p is a contraction (its unique fixed point is p). Now choose 3 distinct points $p_1, p_2, p_3 \in \mathbb{R}^2$. Then Corollary 11.7 says there is a unique compact $K \subset \mathbb{R}^2$ with

$$K = \phi_{p_1}(K) \cup \phi_{p_2}(K) \cup \phi_{p_3}(K) .$$

This K is called a *Sierpinski triangle*.[18]

[18] Image by Beojan Stanislaus, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=8862246.

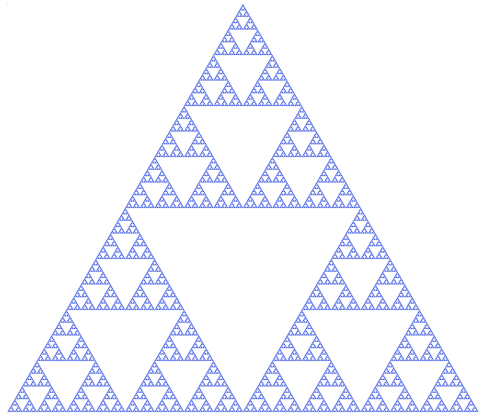

FIGURE 36. A Sierpinski triangle, as in Example 11.9. The points p_1, p_2, p_3 are the vertices of the big triangle.

Note that the proof of Theorem 11.1 also tells us how to make these fractals: we can start with *any* nonempty compact subset $K_0 \subset X$, iterate the map Φ to define $K_n = \Phi(K_{n-1})$, then take $\lim_{n\to\infty} K_n$ in the metric of $\mathcal{S}(X)$.

For instance, the usual construction of the Sierpinski triangle would arise if we take K_0 to be a solid triangle; then each application of Φ deletes some finite number of subtriangles. In this case each iteration makes the set smaller, i.e. we get a sequence $K_0 \supset K_1 \supset K_2 \supset \cdots$, and the limit K is the infinite intersection of the K_n. The usual construction of the Cantor set arises in a similar way. But alternatively, we could go in the opposite direction, taking K_0 to be a finite set which just consits of the fixed points of the ϕ_i: then we get a sequence $K_0 \subset K_1 \subset K_2 \subset \cdots$, where each K_n is finite. In this case the limit K is not just the union of the K_n, rather it is the closure of the union.

Now you can see many possible generalizations, e.g. using larger numbers of maps, higher dimensions, nonlinear maps instead of linear ones. With a little more care, you can also develop a version where instead of $X \subset \mathbb{R}^n$ you use a more general complete metric space.

12. THINGS WE SHOULD HAVE DONE

There are various important topics in one-variable real analysis which we haven't had time to discuss. I list a couple of them here just so you are aware. This material is not for examination.

12.1. **Power series.** The biggest gap is probably that we didn't systematically discuss the theory of functions defined by *power series*: these are functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \tag{12.1}$$

where the coefficients $a_n \in \mathbb{R}$. Such a function is defined only when the sum converges; this happens when x lies in the *interval of convergence* of the series. This interval may be of the form $(c-r, c+r)$ or $[c-r, c+r)$ or $(c-r, c+r]$ or $[c-r, c+r]$, for some $r \geq 0$, or it may be $(-\infty, \infty)$.

Power series give a good way of defining the important functions e^x, $\sin x$, $\cos x$, $\log x$, ... which we didn't discuss at all, and also proving their basic properties. To get a feeling for the flavor of this, we can consider the sum

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

As you show on midterm 2, this sum converges for all x — i.e. its interval of convergence is $(-\infty, \infty)$. Moreover, it agrees with e^x for all $x \in \mathbb{Q}$. With a little more work one can show it agrees with e^x for all $x \in \mathbb{R}$.

Now, one of the most important properties of $F(x) = e^x$ is the formula $F'(x) = F(x)$. How to prove it? If we are allowed to "differentiate term by term" in the sum, then since the derivative of $\frac{x^n}{n!}$ is $\frac{x^{n-1}}{(n-1)!}$, we would indeed get $F'(x) = F(x)$. So there is a question: does this kind of termwise differentiation applied to infinite series really produce correct answers? The answer is *yes*, at least for x in the interior of the interval of convergence; see e.g. Rudin for the proof of this important fact. This technique also allows you to prove e.g. that the derivative of $F(x) = \sin x$ is $F'(x) = \cos x$.

One can ask (and answer) similar questions about termwise integration.

12.2. **Uniform convergence.** Another (related) point we did not discuss is the following. Suppose we have a sequence of functions $f_n : [a,b] \to \mathbb{R}$, each integrable. Moreover suppose that there is another function $f(x)$ such that, for all $x \in [a,b]$,

$$f_n(x) \to f(x)\,. \tag{12.2}$$

Then we might ask various questions. For instance, if all f_n are continuous, does f have to be continuous? If all f_n are differentiable, does f have to be differentiable, and does $f_n'(x)$ converge to $f'(x)$? Or, suppose all f_n are integrable, and f is also integrable; then is it true that "the integral of the limit equals the limit of the integrals,"

$$\int_a^b f_n(x)\,\mathrm{d}x \;\to\; \int_a^b f(x)\,\mathrm{d}x\,?$$

The answer to this kind of question turns out to be *no* in general. For instance, consider the functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{if } x \geq \frac{1}{n}, \\ 0 & \text{if } x = 0. \end{cases}$$

In this case we have $\int_0^1 f_n(x)\,\mathrm{d}x = 1$ for all n, so $\int_0^1 f_n(x)\,\mathrm{d}x \to 1$; but $f_n(x) \to 0$ for all x (why?), and $\int_0^1 0\,\mathrm{d}x = 0$. So in this case the integral of the limit is 0 while the limit of the integrals is 1.

The point is that the kind of convergence given by (12.2) is not strong enough for what we need. There is a stronger notion, where we require not only convergence at every point x, but we require that the speed of convergence is uniform across the whole domain. This is similar to the notion of uniform continuity which we defined in Definition 8.35, and it gets a similar name: it is called *uniform convergence*. For sequences of functions which converge uniformly, the answers to the questions above are *yes*. Moreover, the partial sums of a power series (12.1) do indeed converge uniformly, everywhere except at the endpoints of the interval of convergence. So this uniform convergence is a very useful tool in practice. Again, see e.g. Rudin for the definitions and theorems.