CS 70 Discussion 6B

1

PDF: probability density function

CDF: cumulative density function

derivative of the CDF gives us the PDF;
integrating the PDF can give us the CDF

(a) X PDF:

$$f_X(x) = \begin{cases} cx^{-2} & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b f_X(x)\,dx = 1$$

a, b are the bounds for the domain of the RV X

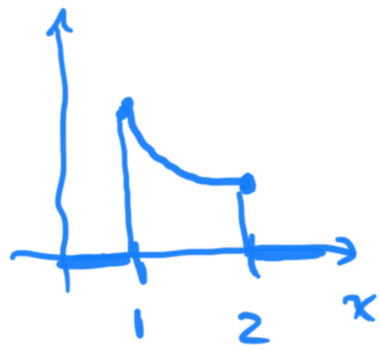
$$\int_1^2 cx^{-2}\,dx = 1$$

$$c\int_1^2 x^{-2}\,dx = 1$$

$$c\left[-x^{-1}\right]_1^2 = 1$$

$$-c\left(\frac{1}{2} - 1\right) = 1$$

$$\boxed{c = 2}$$



(b) $P\{\underset{A}{X > 1.5}\}$

$$= \int_{1.5}^2 2x^{-2}\,dx$$

$$= 2\left[-x^{-1}\right]_{1.5}^2$$

$$= -2\left[\frac{1}{2} - \frac{1}{1.5}\right] = \boxed{\frac{1}{3}}$$

Conditional PDF:

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)}$$

$$= \frac{2x^{-2}}{\frac{1}{3}} = \boxed{6x^{-2}\ \{1.5 \le x \le 2\}}$$

2 U: uniform distribution

$U[0,1]$ = a continuous RV that takes on any value in the range $[0,1]$ w/ equal probability

To calculate the CDF of a RV X, we calculate $P(X < x)$ where x is some arbitrary constant

To calculate the PDF of X, just take the derivative of the CDF.

(a) CDF of X

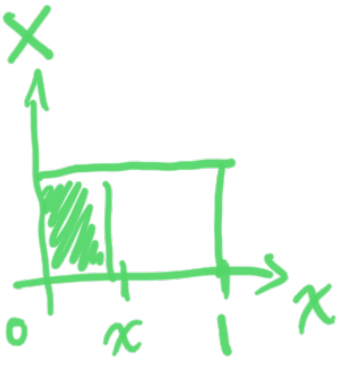
X is $\max\{X_1, X_2, \dots X_n\}$ where each X_i is $U[0,1]$

$0 \le x \le 1$

$$= P(X < x) = P(X_1, X_2, \dots X_n < x)$$

arbitrary constant

$$= \underbrace{P(X_1 < x)}_{=\frac{x}{1}} \cdot \underbrace{P(X_2 < x)}_{\frac{x}{1}} \cdot \dots \underbrace{P(X_n < x)}_{\frac{x}{1}}$$



$$= \boxed{x^n}$$

x can take on any value 0 thru 1

X = 0.1, 0.2, 0.12, 0.123

$P(X_1 < k)$

$$= \int_0^k F_{X_1}(x)\,dx = \int_0^k 1\,dx = k$$

what is the PDF? 1

In general, for a continuous uniform RV in the range $[a, b]$, it has PDF of $\frac{1}{b-a}$

(b) PDF of $\underline{X}$

CDF of X: $\underline{F_X}(x) = x^n$ PDF $= \underline{f_X}(x) = \frac{dF_X(x)}{dx} = \boxed{nx^{n-1}}$

$F_X(k) = k^n$ $f_X(k) = nk^{n-1}$

$\underline{X}$ ← RV, k ← constant; k ← arbitrary input

(c) $\mathbb{E}[X]$

$= \int_{\underline{a}}^{\underline{b}} k \cdot \underline{f_X(k)}\, dk = \int_0^1 k \cdot nk^{n-1}\, dk = n\int_0^1 k^n\, dk$

$= n\left[\frac{k^{n+1}}{n+1}\right]_0^1 = \boxed{\frac{n}{n+1}}$

(d) $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\underline{\mathbb{E}[\underline{X}^2]} = \int_a^b k^2 \cdot \underline{f_X(k)}\, dk = \int_0^1 k^2 \cdot nk^{n-1}\, dk$

↑ PDF

$= n\int_0^1 k^{n+1}\, dk$

$= n\left[\frac{k^{n+2}}{n+2}\right]_0^1 = \frac{n}{n+2}\left[k^{n+2}\right]_0^1 = \boxed{\frac{n}{n+2}}$

$\underline{(\mathbb{E}[X])^2} = \left(\frac{n}{n+1}\right)^2$

$$\boxed{\mathrm{Var}(X) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2.}$$

$\mathbb{E}[X] = \sum_{x} x \cdot P(X=x)$ $\quad$ $\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx$

PDF

$\boxed{3}$

Alice: dart has uniform prob. of landing any distance btwn [0, 1] of the center

Bob: his dart is uniformly distributed over the area of the dart board

(a) PDF of X if A throws $\quad$ $X \sim U[0,1]$

CDF: $P(X<k) = k$

PDF: $\dfrac{dF_X(k)}{dk} = \boxed{1}$

pdf is $\dfrac{1}{b-a}$ $[a,b]$ $\quad$ $[a,b] = [0,1]$

PDF: $F_X(k)$

input constant

just name of a function for RV X

(b)

CDF: $P(X<k) = \dfrac{\pi(k)^2}{\pi(1)^2} = k^2$

PDF: $\dfrac{dF_X(k)}{dk} = \boxed{2k}$

(c) Alice throws w/ prob p, Bob throws w/ prob 1-p

$P(X<k) = P(\text{Alice throws and } X<k) + P(\text{Bob throws and } X<k)$

(CDF) $= p(k) + (1-p)(k^2)$

PDF: $\frac{dF_X(k)}{dk} = \boxed{p + (1-p)(2k)}$

(d) $P[A \mid X \in [x, x+dx]]$ = prob. that A threw dart given dart is a distance of x away from center

$P[B \mid X \in [x, x+dx]]$ = same as above except for Bob

$P[A \mid \overset{X=x}{X \in [x, x+dx]}] + P[B \mid X \in [x, x+dx]] = 1$

We declare Alice to have been the thrower whenever

$P[A \mid X \in [x, x+dx]] > \frac{1}{2}$

PDF for Alice: 1

PDF for Bob: $2x$

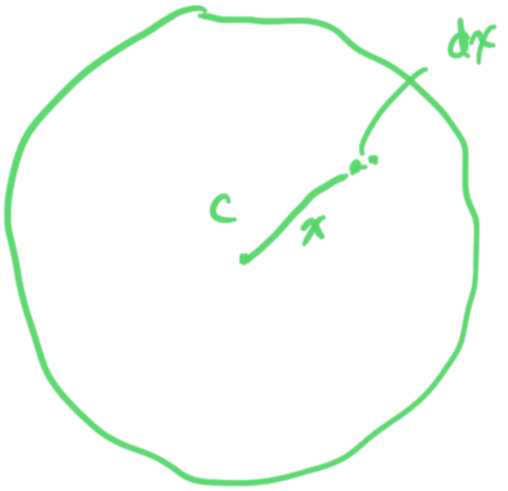
$P(X = k) \approx \underbrace{dx}_{\text{a small constant (infinitesimally)}} \cdot \underbrace{f_X(k)}_{\text{PDF}}$

$$P[A \mid X \in [x, x+dx]] = \frac{P(A \text{ and } X \in [x, x+dx])}{P(X \in [x, x+dx])}$$

$$= \frac{P(A \text{ and } X \in [x, x+dx])}{P(A \text{ and } X \in [x, x+dx]) + P(B \text{ and } X \in [x, x+dx])}$$

$$= \frac{p(\cancel{dx})(1)}{p(\cancel{dx})(1) + (1-p)(\cancel{dx})(\underline{2x})}$$

$$= \frac{p}{p + 2x(1-p)}$$ ← this is the prob. that Alice threw the dart given that $X = x$



$$\frac{p}{p+2x(1-p)} > \frac{1}{2}$$

$$p > \frac{1}{2}(p + 2x - 2xp)$$

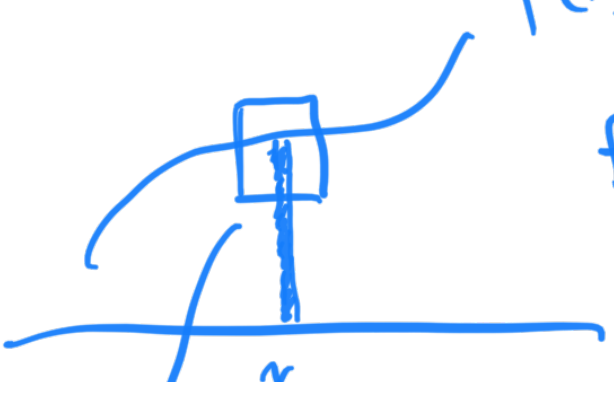
$$p > \frac{p}{2} + x - xp$$

$$\frac{p}{2} > x(1-p)$$

$$\boxed{\underline{x} < \frac{p}{2(1-p)}}$$

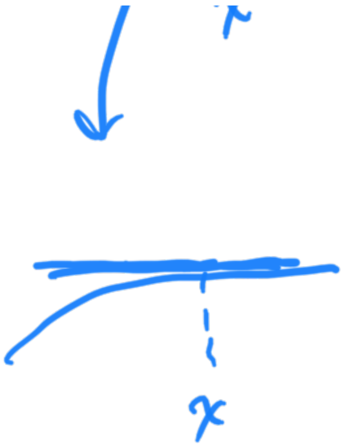
we declare A to have thrown the dart whenever $x < \frac{p}{2(1-p)}$

when $x > \frac{p}{2(1-p)}$, then Bob threw



$$f(x) \approx \underset{\sim}{dx} \cdot f'(x)$$

infinitesimally small



$$P(A|C) + P(B|C) = 1$$

not A

$$P(A|C) + P(\text{not } A|C) = 1$$

$$P(A \text{ and } C) + P(\text{not } A \text{ and } C) = P(C)$$