CS 70 Discussion 6D

$\boxed{1}$ $f_{XY}(X=x, Y=y) = \underline{C}xy \quad 0 \le x \le 1$

$\uparrow ?$ $\quad 0 \le y \le 2$

(a) $\int_0^1 \int_0^2 \underline{f_{XY}}\, dy\, dx = 1$ PDF is prob. per unit area

$= \int_0^1 \int_0^2 \underline{Cxy}\, \underline{dy}\, dx = 1$

$\boxed{f_{XY}(X=x, Y=y) = xy}$

$= \int_0^1 Cx \underbrace{\int_0^2 y\, dy}\, dx = 1$

$\int_0^1 Cx \left[\frac{1}{2}y^2\right]_0^2 dx = 1$

$\int_0^1 \underline{C}x(\underline{2})\, dx = 1$

$2C \int_0^1 x\, dx = 1$

$2C \left[\frac{1}{2}x^2\right]_0^1 = 1$

$2C\left[\frac{1}{2} - 0\right] = 1$

$\boxed{C = 1}$

(b) $f_X(x)$ = marginal dist. of X

$f_X(x) = \int_0^2 \underline{f_{XY}(x,y)}\, dy = \int_0^2 xy\, dy = x\int_0^2 y\, dy$

joint PDF

$= x\left[\frac{1}{2}y^2\right]_0^2 = x(2) = \boxed{2x}$

$(0 \leq x \leq 1)$

$$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx = \int_0^1 xy\,dx = y\int_0^1 x\,dx$$

$$= y\left[\tfrac{1}{2}x^2\right]_0^1 = \boxed{\frac{y}{2}} \leftarrow$$

$(0 \leq y \leq 2)$

(c) $f_{Y|X}(Y=y \mid X=x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{xy}{2x} = \boxed{\dfrac{y}{2}}$

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$(0 \leq y \leq 2)$

(d) X, Y independent?

$\boxed{\text{Yes, X, Y are indep.}}$

$$f_{XY}(x,y) \stackrel{?}{=} f_X(x) f_Y(y)$$

$$xy \stackrel{?}{=} (2x)\left(\tfrac{y}{2}\right)$$

$$xy = xy$$

$$f_{Y|X}(y) = f_Y(y)$$

$$\frac{y}{2} = \frac{y}{2}$$

PDF of Y: $\lambda_Y e^{-\lambda_Y y}$

PDF of X: $\lambda_X e^{-\lambda_X x}$

$\boxed{3}$ complementary CDF of an exponential RV:

$$P(X \geq k) = e^{-\lambda k} = \int_k^\infty \lambda e^{-\lambda x}\,dx$$

$X \sim \text{Exp}(\lambda_X)$ $\quad U = \min\{X, Y\}$

$Y \sim Exp(\lambda_Y)$ $\quad V = \max\{X, Y\}$ $\quad W = V - U$

(a) $P(U > t, X \le Y)$ for $t \ge 0$

$U = X$

$$= P(X > t, X \le Y) = \int_t^\infty \lambda_X e^{-\lambda_X x} \underbrace{\int_x^\infty \lambda_Y e^{-\lambda_Y y}\, dy}\, dx$$

$$= \int_t^\infty \lambda_X e^{-\lambda_X x} \underbrace{e^{-\lambda_Y x}}\, dx$$

$Y \ge X$

$U = X$

$V = Y$

$W = Y - X$

$$= \lambda_X \int_t^\infty e^{-x(\lambda_1 + \lambda_2)}\, dx$$

$$= \frac{\lambda_X}{\lambda_X + \lambda_Y} \underbrace{\int_t^\infty (\lambda_X + \lambda_Y) e^{-x(\lambda_X + \lambda_Y)}\, dx}_{\text{PDF of } Exp(\lambda_X + \lambda_Y)}$$

$$= \boxed{\frac{\lambda_X}{\lambda_X + \lambda_Y} e^{-t(\lambda_X + \lambda_Y)}}$$

(b) $P(X \le Y)$

$= P(X > 0, X \le Y)$

$t = 0$

(a): $P(X > t, X \le Y) =$

$$\frac{\lambda_X}{\lambda_X + \lambda_Y} \cancel{e^{-0(\lambda_X + \lambda_Y)}} = \boxed{\frac{\lambda_X}{\lambda_X + \lambda_Y}}$$

(c) $P(W > t \mid X \le Y)$

$$= \frac{P(W > t \cap X \le Y)}{P(X \le Y)}$$

$W = \underline{V} - \underline{U}$ $\leftarrow \min\{X, Y\}$

$\uparrow$

$\max\{X, Y\}$

① Evaluate $P(W>t \cap X \le Y)$ $\quad V=Y$, and $U=X$

$$=P(\underbrace{Y-X>t}) = P(\underbrace{Y>X+t})$$

$$=\int_0^\infty \lambda_X e^{-\lambda_X x} \underbrace{\int_{x+t}^\infty \lambda_Y e^{-\lambda_Y y}\, dy}\, dx$$

$$=\int_0^\infty \lambda_X e^{-\lambda_X x}\, e^{-\lambda_Y (x+t)}\, dx$$

$$=\int_0^\infty \lambda_X e^{-\lambda_X x}\, e^{-\lambda_Y x}\, e^{-\lambda_Y t}\, dx$$

$$=\lambda_X e^{-\lambda_Y t}\int_0^\infty e^{-x(\lambda_X+\lambda_Y)}\, dx$$

$$=\frac{\lambda_X e^{-\lambda_Y t}}{\lambda_X+\lambda_Y}\underbrace{\int_0^\infty (\lambda_X+\lambda_Y) e^{-x(\lambda_X+\lambda_Y)}\, dx}_{\text{PDF of Exp}(\lambda_X+\lambda_Y) \; =1}$$

$$=\frac{\lambda_X e^{-\lambda_Y t}}{\lambda_X+\lambda_Y}$$

② $P(X \le Y) = \dfrac{\lambda_X}{\lambda_X+\lambda_Y}$

Answer: $$\frac{\dfrac{\cancel{\lambda_X} e^{-\lambda_Y t}}{\cancel{\lambda_X+\lambda_Y}}}{\dfrac{\cancel{\lambda_X}}{\cancel{\lambda_X+\lambda_Y}}} = \boxed{e^{-\lambda_Y t}}$$

$W=Y-X$

(d) $P(W>t)$: either $W \ge t$ and $X \le Y$, or

$W \geq t$ and $Y \leq X$ $W = X - Y$

$$P(W > t) = P(X \leq Y)P(W \geq t \mid X \leq Y) +$$
$$P(Y \leq X)P(W > t \mid Y \leq X)$$

$P(W > t \mid X \leq Y) = e^{-\lambda_Y t}$ $\quad P(X \leq Y) = \dfrac{\lambda_X}{\lambda_X + \lambda_Y}$

$P(W > t \mid Y \leq X) = e^{-\lambda_X t}$ $\quad P(Y \leq X) = \dfrac{\lambda_Y}{\lambda_X + \lambda_Y}$

$$= \boxed{\left(\frac{\lambda_X}{\lambda_X + \lambda_Y}\right)\left(e^{-\lambda_Y t}\right) + \left(\frac{\lambda_Y}{\lambda_X + \lambda_Y}\right)\left(e^{-\lambda_X t}\right)}$$

(e) $P(U > u, W > w)$ For $w > u > 0$

$$= \underbrace{P(U > u, W > w, Y \geq X)}_{U = X \quad W = Y - X} + \underbrace{P(U > u, W > w, X \geq Y)}_{?}$$

$Y - X > w$

$Y > X + w$

$$= P(X > u, \underbrace{Y - X > w}_{Y > X + w})$$

$$= \int_u^\infty \lambda_X e^{-\lambda_X x} \int_{x+w}^\infty \lambda_Y e^{-\lambda_Y y}\, dy\, dx$$

$$= \int_u^\infty \lambda_X e^{-\lambda_X x}\, e^{-\lambda_Y (x + w)}\, dx$$

$$= \int_u^\infty \lambda_X e^{-x(\lambda_X + \lambda_Y)}\, e^{-\lambda_Y w}\, dx$$

$$e^{-\lambda_Y w} \int^\infty e^{-x(\lambda_X + \lambda_Y)}\, dx$$

$$= \lambda_X e \int_u e \, dx$$

$$= \frac{\lambda_X e^{-\lambda_Y w}}{\lambda_X + \lambda_Y} \int_u^\infty \underbrace{(\lambda_X + \lambda_Y) e^{-x(\lambda_X + \lambda_Y)}}_{\text{PDF of } \mathrm{Exp}(\lambda_X + \lambda_Y)} dx$$

$$= \frac{\lambda_X e^{-\lambda_Y w}}{\lambda_X + \lambda_Y} e^{-u(\lambda_X + \lambda_Y)} + \frac{\lambda_Y e^{-\lambda_X w}}{\lambda_X + \lambda_Y} e^{-u(\lambda_X + \lambda_Y)}$$

$$\boxed{\frac{e^{-u(\lambda_X + \lambda_Y)}}{\lambda_X + \lambda_Y} \left(\lambda_X e^{-\lambda_Y w} + \lambda_Y e^{-\lambda_X w} \right)}$$

$$P(U > u, W > w) = \underbrace{P(U > u)} \cdot \underbrace{P(W > w)}$$