Discussion 5D:

1

(a) $\mathbb{E}[X_i] = ?$ $\quad X_i$ = number of balls throw into bin i

Indicator RVs:

$B_1, B_2, \dots B_k$ $\quad B_i = \begin{cases} 1 & \text{if the first ball lands in bin } i \\ 0 & \text{otherwise} \end{cases}$

$$\underline{\mathbb{E}[X_i]} = \mathbb{E}[B_1 + \dots + B_k]$$

$$= k\,\mathbb{E}[B_1]$$

$$= k\left(\frac{1}{n}\right) = \boxed{\frac{k}{n}}$$

$$\mathbb{E}[B_1] = 1\left(\underline{\frac{1}{n}}\right) + \cancel{0\left(\frac{n-1}{n}\right)}$$

$$= \frac{1}{n}$$

(b) $\mathbb{E}[\text{empty bins}] = ?$

indicator RVs: $B_1, B_2, \dots \underline{B_n}$ $\leftarrow$ n bins

$B_i = \begin{cases} 1 & \text{if bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$ $\quad \underline{\mathbb{E}[B_i] = \left(\frac{n-1}{n}\right)^k (1)}$

$$\mathbb{E}[\text{empty bins}] = \mathbb{E}\left[\sum_{i=1}^{n} B_i\right] = \sum_{i=1}^{n} \mathbb{E}[B_i] = \boxed{n\left(\frac{n-1}{n}\right)^k}$$

(c) $\mathbb{E}[\text{collisions}] = ?$

Key Insight: $\begin{pmatrix}\text{number of}\\ \text{occupied}\\ \text{bins}\end{pmatrix} + \begin{pmatrix}\text{number of}\\ \text{collisions}\end{pmatrix} =$ number of balls

$$O + C = k$$

$$\mathbb{E}[O + C] = \cancel{\mathbb{E}[k]}\ k$$

$$? \longrightarrow \underline{\mathbb{E}[O]} + \underline{\mathbb{E}[C]} = k$$

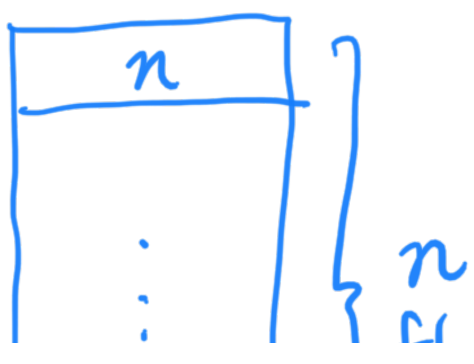
$$\underline{n - \begin{pmatrix}\text{exp. number of}\\ \text{empty}\\ \text{bins}\end{pmatrix}} + \mathbb{E}[C] = k$$

$$n - n\left(\frac{n-1}{n}\right)^k + \underline{\mathbb{E}[C]} = k$$

$$\boxed{\mathbb{E}[C] = k - n + n\left(\frac{n-1}{n}\right)^k}$$

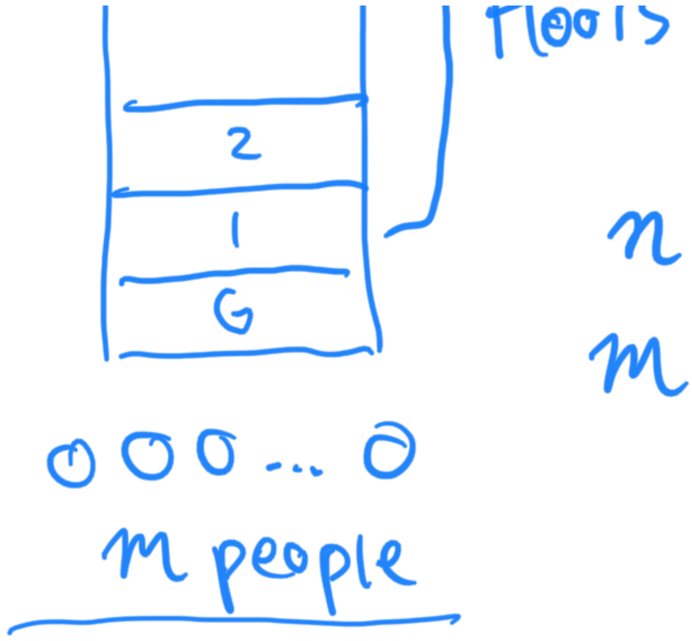
2

(a)



= balls into bins problem

where the balls are ppl,



and the bins are floors

n floors = $\underline{n \text{ bins}}$

m ppl = $\underline{m \text{ balls}}$

$$\mathbb{E}\left[\begin{matrix}\text{floors we}\\ \text{stop at}\end{matrix}\right] = \mathbb{E}\left[\begin{matrix}\text{occupied}\\ \text{bins}\end{matrix}\right] = n - \mathbb{E}\left[\begin{matrix}\text{empty}\\ \text{bins}\end{matrix}\right]$$

1(b): $\mathbb{E}\left[\begin{matrix}\text{empty}\\ \text{bins}\end{matrix}\right] = \underline{n\left(\frac{n-1}{n}\right)^k}$ $= \boxed{n - n\left(\frac{n-1}{n}\right)^m}$

k balls

n bins

(b) $\text{Var}(X) = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ <u>ONLY IF</u> X, Y are independent

$$\text{Var}\Big(\underbrace{\begin{matrix}\text{number of floors the}\\ \text{elevator doesn't stop at}\end{matrix}}_{X}\Big) \qquad \text{Var}(X) = ?$$

Define indicator RVs $X_1, X_2 \dots X_n$, where $X = X_1 + \dots + X_n$

$$X_1 = \begin{cases} 1 & \text{if don't stop @ floor 1} \\ 0 & \text{otherwise} \end{cases}$$

① Step 1: Calculate $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \mathbb{E}\left[(X_1+\dots+X_n)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] + \mathbb{E}\left[\sum_{i\neq j} X_iX_j\right]$$

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] = \sum_{i=1}^{n} 1^2\left(\frac{n-1}{n}\right)^m = n\left(\frac{n-1}{n}\right)^m$$

$$\mathbb{E}[X_iX_j] = 1\cdot \underline{P(X_i=1 \cap X_j=1)} = \left(\frac{n-2}{n}\right)^m$$

X_i	X_j	X_iX_j
0	0	0
0	1	0
1	0	0
1	1	1

Thus,

$$\boxed{\mathbb{E}[X^2] = n\left(\frac{n-1}{n}\right)^m + n(n-1)\left(\frac{n-2}{n}\right)^m}$$

② Step 2: Calculate $(\mathbb{E}[X])^2$:

$$\mathbb{E}[X] = \mathbb{E}[X_1+\dots+X_n] = n\,\mathbb{E}[X_1] = n\left[1\left(\frac{n-1}{n}\right)^m\right]$$

$$(\mathbb{E}[X])^2 = n^2\left(\frac{n-1}{n}\right)^{2m}$$

③ Step 3: Calculate $\mathrm{Var}(X)$:

$$\mathrm{Var}(X) = n\left(\frac{n-1}{n}\right)^m + n(n-1)\left(\frac{n-2}{n}\right)^m - n^2\left(\frac{n-1}{n}\right)^{2m}$$

$\boxed{3}$ 5 red
5 blue

X_1 = indicator RV if first ball is red

X_2 = if 2^{nd} ball is red

$$\text{cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$$

$$\mathbb{E}[X_1] = \overbrace{\tfrac{1}{2}(1)}^{\text{red}} + \overbrace{\cancel{\tfrac{1}{2}(0)}}^{\text{blue}} = \frac{1}{2}$$

$$\mathbb{E}[X_2] = \underbrace{\tfrac{1}{2}(1)}_{\text{red}} + \underbrace{\cancel{\tfrac{1}{2}(0)}}_{\text{blue}} = \frac{1}{2}$$

$$\mathbb{E}[X_1 X_2] = 1\big(P(\underbrace{X_1 = 1 \cap X_2 = 1}_{\text{prob. that 1st and 2nd draw are red!}})\big) = \frac{5}{10}\left(\frac{4}{9}\right) = \frac{2}{9}$$

X_1	X_2	$X_1 X_2$
0	0	0
0	1	0
1	0	0
1	1	1

$$\text{cov}(X_1, X_2) = \frac{2}{9} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{-\frac{1}{36}}$$

$$P(2^{nd} \text{ draw is red}) = P(1^{st} \text{ draw is red}) = \frac{1}{2}$$

$$P(2^{nd} \text{ draw is R} \mid 1^{st} \text{ draw is R}) = \frac{4}{9}$$