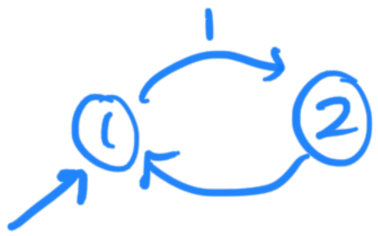
CS 70 Discussion 7D

$d(i) = \gcd\{n > 0 \mid P^n(i,i) = P[X_n = i \mid X_0 = i] \geq 0\}$

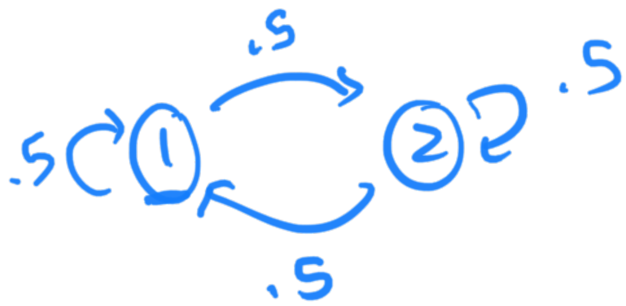
$i \in X$ if $d(i) = 1$ $\forall i \in X$, then MC is aperiodic.

- given that I start at state i inside a MC, how long does it take me to return to state i?



period: 2

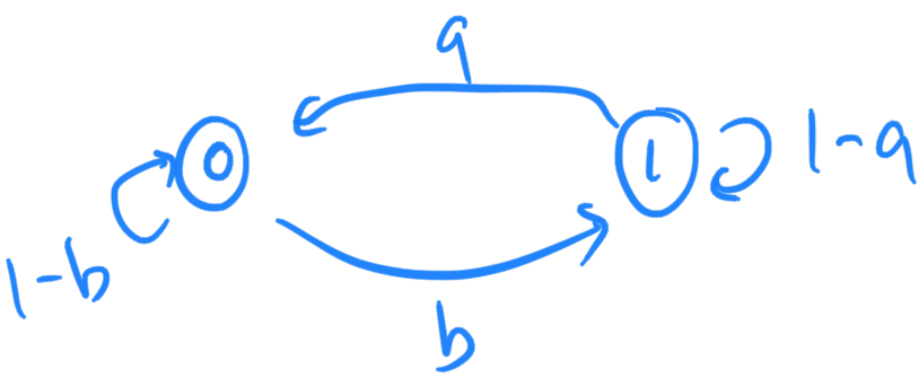
$\gcd(2, 4, 6, \ldots) = 2$



$\pi(1) = .5\pi(1) + .5\pi(2)$

a balance equation

$\boxed{1}$

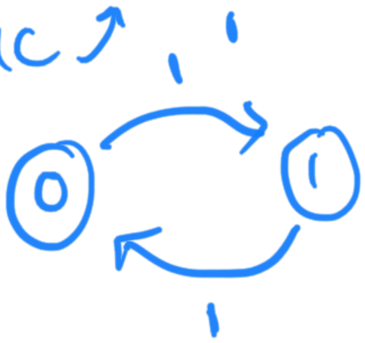


(a) $0 < a, b \leq 1$ the MC is irreducible

$\boxed{a \text{ or } b = 0}$ the MC is reducible

$b = 0,\ a = 1$

(b) MC

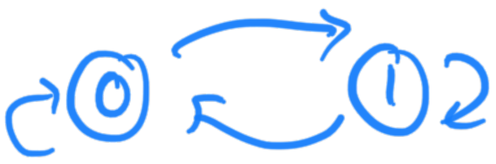


$d(0): 2, 4, 6, \ldots$

$d(1) = 2, 4, 6, \ldots$

This MC has period 2 and thus is periodic.

(c)



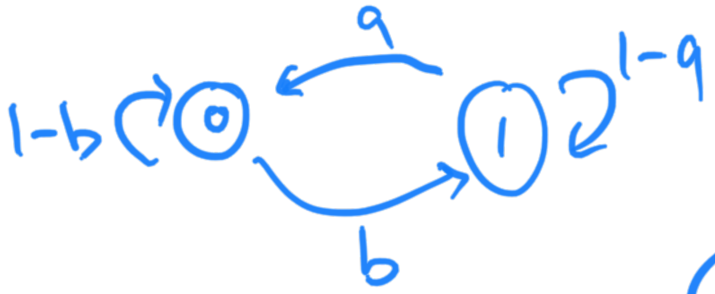
You have a nonzero probability of traversing a self loop. So therefore I can return to any state I start at in just 1 step. Therefore, MC is aperiodic.

(d)

$$P = \begin{array}{cc} 0\to 0 & 0\to 1 \\ \left[\begin{array}{cc} 1-b & b \\ a & 1-a \end{array}\right] & \\ 1\to 0 & 1\to 1 \end{array}$$

$$\boxed{\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}}$$

(e)



$$\begin{cases} \pi(0) + \pi(1) = 1 \\ \pi(0) = (1-b)\pi(0) + a\pi(1) \end{cases}$$

$$b\pi(0) = a\pi(1)$$

$$\pi(0) = \frac{a}{b}\pi(1)$$

$$\pi(1)\left(\frac{a+b}{b}\right) = 1$$

$$\boxed{\pi(1) = \frac{b}{a+b} \quad \pi(0) = \frac{a}{a+b}}$$

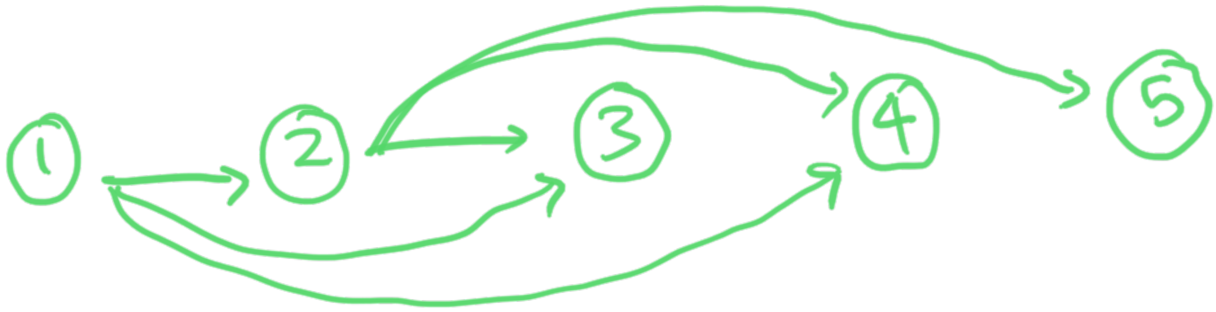
$$\underline{\pi(1) = \frac{b}{a+b}}$$

$$\pi(0) + \pi(1) = 1$$

$$\pi(0) = 1 - \pi(1)$$

$$= \frac{a+b}{a+b} - \frac{b}{a+b} = \frac{a}{a+b}$$

2



Each arrow drawn in the above MC has $\frac{1}{3}$ prob. of occurring.

Let $\alpha(i)$ denote the probability of reaching the target (state 3) before overshooting, given that we start at state i.

$\alpha(4) = 0 \qquad \alpha(5) = 0 \qquad \alpha(3) = 1$

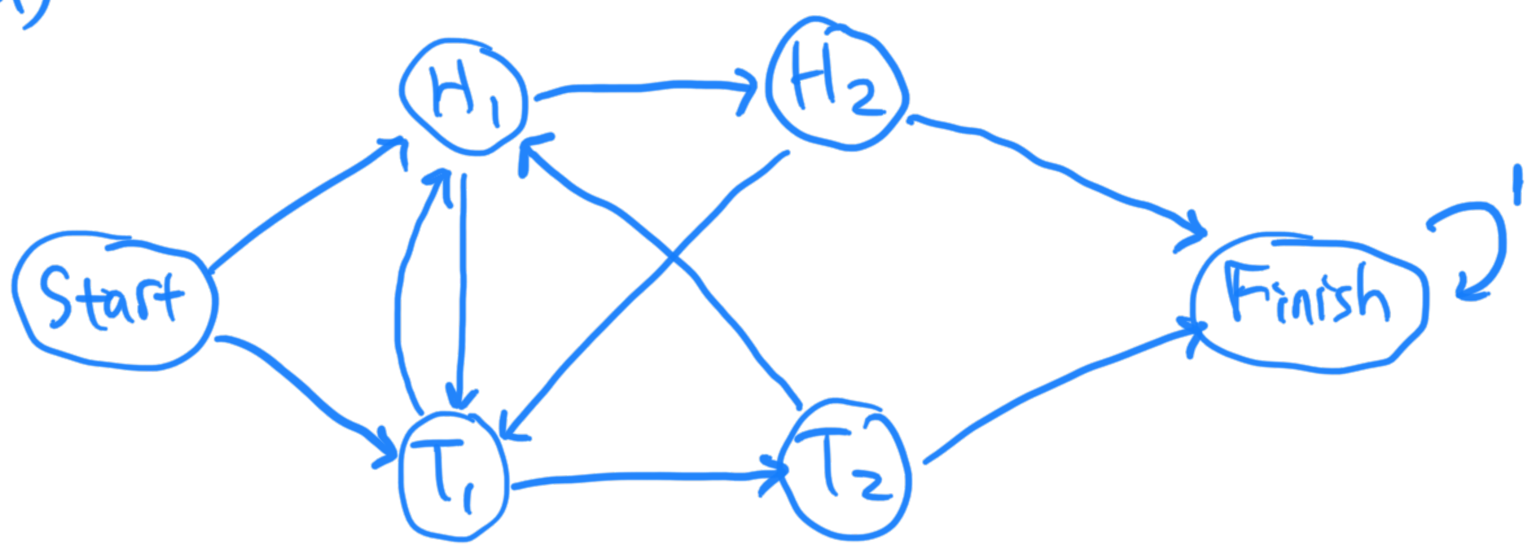
$$\underline{\alpha(2)} = \underline{\tfrac{1}{3}}\,\cancel{\alpha(3)}^{1} + \underline{\tfrac{1}{3}}\,\cancel{\alpha(4)}^{0} + \underline{\tfrac{1}{3}}\,\cancel{\alpha(5)}^{0} = \tfrac{1}{3}$$

$$\underline{\alpha(1)} = \tfrac{1}{3}\,\cancel{\alpha(2)}^{1/3} + \tfrac{1}{3}\,\cancel{\alpha(3)}^{1} + \tfrac{1}{3}\,\cancel{\alpha(4)}^{0} = \boxed{\tfrac{4}{9}}$$

$$\begin{cases} \alpha(2) = \frac{1}{3}\left(\alpha(3) + \alpha(4) + \alpha(5)\right) \\ \alpha(1) = \frac{1}{3}\left(\alpha(2) + \alpha(3) + \alpha(4)\right) \end{cases}$$

3 Hint: Construct a MC with states based upon how many consecutive heads or tails you've seen so far.

(a)



Every edge in the above MC has 0.5 prob. of occurring

(b) H1 is where we start

Let $\beta(i)$ denote the expected number of flips to reach the end state, given that we start @ state i.

$$\begin{cases} \beta(\text{Finish}) = 0 \\ \beta(H_1) = 1 + \frac{1}{2}\beta(H_2) + \frac{1}{2}\beta(T_1) \\ \beta(H_2) = 1 + \frac{1}{2}\beta(\text{Finish}) + \frac{1}{2}\beta(T_1) \\ \beta(T_1) = 1 + \frac{1}{2}\beta(T_2) + \frac{1}{2}\beta(H_1) \\ \beta(T_2) = 1 + \frac{1}{2}\beta(\text{Finish}) + \frac{1}{2}\beta(H_1) \end{cases}$$

$\boxed{\beta(H_1) = 6}$

$\beta(H_2) = 4$

$\beta(T_1) = 6$

$\beta(T_2) = 4$

(c) $\beta(\text{Start}) = 1 + \frac{1}{2}\beta(H_1) + \frac{1}{2}\beta(T_1)$

$$= 1 + \frac{1}{2}(6+6) = \boxed{7}$$

Aperiodic MC without a self loop:

If a MC has a self-loop, immediately aperiodic

If a MC lacks a self loop, could be periodic or aperiodic.

Periodic MC without a self loop: