CS 70 CLT Review

Central Limit Theorem:

Suppose we take n iid observation of a random variable, and denote the outcomes $X_1, X_2, \ldots X_n$, each having common mean μ and common variance σ^2. Then:

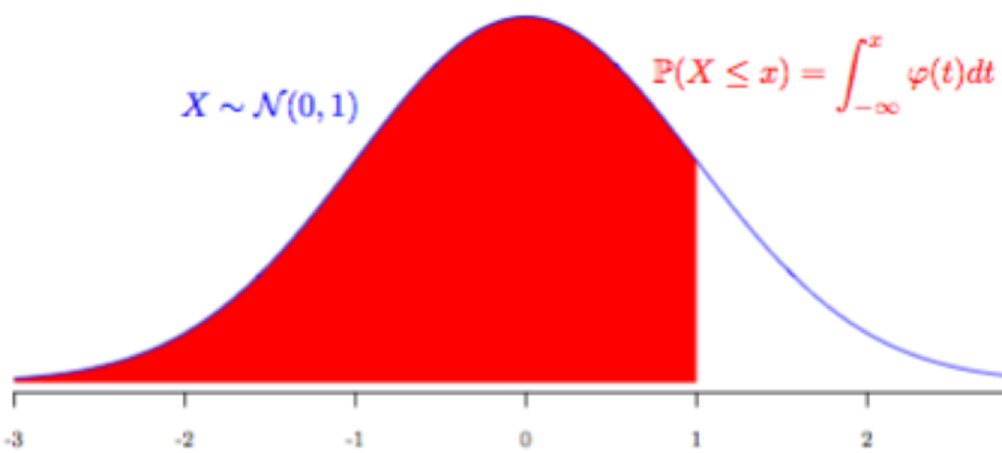
$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

sample mean estimator

divide by n in the numerator and denominator

In particular then, $P(Z_n \leq c) = \Phi(c)$ where Φ is the standard normal CDF (given inside a table).

The CDF table of standard normal distribution.



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Equivalently, given that $Z_n = \frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, we can say that

$$\hat{\mu} - \mu = \frac{\sigma}{\sqrt{n}} N(0,1)$$

$$\hat{\mu} = \mu + \frac{\sigma}{\sqrt{n}} N(0,1)$$

$$\boxed{\hat{\mu} = N\left(\mu, \frac{\sigma^2}{n}\right)}$$

$$a + bN(0,1) = N(a, b^2)$$

CLT allows us to say that our sample mean estimator is normally distributed with mean $= \mu$ and variance $= \frac{\sigma^2}{n}$.

1 Let $X_1, X_2, \ldots X_{10,000}$ denote dice roll outcomes.

(a) $$Z_n = \frac{(X_1 + \ldots + X_n) - n\mu}{\sigma\sqrt{n}} = \frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{2.8 - \mu}{\frac{4}{100}}$$

unknown since we're unsure if this dice is balanced

$n = 10^4$

$$\frac{2.8-\mu}{\frac{4}{100}} = 1.96 \qquad \frac{\sigma}{\sqrt{n}} = \frac{4}{100} = .25$$

$$2.8 - \frac{1.96\sigma}{\sqrt{n}} \le \mu \le 2.8 + \frac{1.96\sigma}{\sqrt{n}}$$

$\hat{\mu}$ distributed as $N\left(\mu, \frac{\sigma^2}{n}\right)$

so stdev is $\frac{\sigma}{\sqrt{n}}$

$$2.8 - 1.96(.25) \le \mu \le 2.8 + 1.96(.25)$$

$$\frac{1}{4}(1+4+9+16) - (2.5)^2 = \frac{15}{2} - 6.25$$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

2 Poisson(λ) $\quad \lambda \le 5 \quad \sigma = 1.96$

Normal w/ mean

$\frac{X_1+\dots+X_n}{n}$ is our sample mean estimator w/ $N(\mu, \frac{\sigma^2}{n}) = N(\lambda, \frac{\lambda}{n})$

IMPORTANT $\rightarrow$ $\boxed{1.96 \text{ stdev} = \frac{1}{2}}$

$$\frac{\frac{X_1+\dots+X_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$1.96\sqrt{\frac{\lambda}{n}} = \frac{1}{2}$$

$$\sqrt{\frac{\lambda}{n}} = \frac{1}{2(1.96)}$$

$$\frac{\lambda}{n} = \frac{1}{4(1.96)^2}$$

$$n = \lambda(4)(1.96)^2$$

$$\boxed{n = 20(1.96)^2}$$

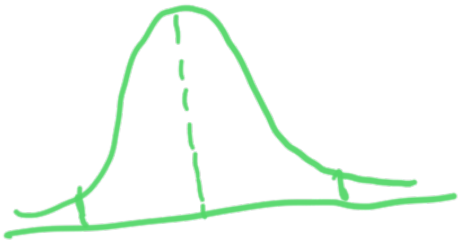
equivalent to saying:

$$\hat{\mu}-\mu = \frac{\sigma}{\sqrt{n}} N(0,1)$$

$$\hat{\mu}-\mu = N(0, \frac{\sigma^2}{n})$$

$$\hat{\mu} = \mu + N(0, \frac{\sigma^2}{n})$$

$$\hat{\mu} = N(\mu, \frac{\sigma^2}{n})$$



Collecting more and more samples to reduce CI width

$\boxed{6}$ $A_n \sim N(\frac{1}{\lambda}, \frac{1}{n\lambda^2})$

width of conf. interval on one end is $\frac{0.1}{\lambda} = \frac{1}{10\lambda}$

$\frac{0.05}{\lambda} = \frac{1}{20\lambda}$

$N(\mu, \frac{\sigma^2}{n})$

want $\frac{\sigma}{\sqrt{n}}(1.96) = \cancel{0.1}\mu$ 0.05

$$\frac{1}{\lambda\sqrt{n}}(1.96) = \frac{1}{\cancel{10}\lambda} \quad 20$$

$$2 \times 19.6 = \sqrt{n}$$

$$n = (19.6)^2(2^2)$$

$$\approx 400\,(4) = 1600$$

$\boxed{5}$ Variance: $np(1-p)$

$X_1 + \dots + X_n$

Well first, we use average estimator $A_n = \frac{\quad}{n}$ having variance $\underbrace{\frac{p(1-p)}{n}}_{\text{upperbounded by } \frac{1}{4n}}$ and mean p.

$$A_n \sim N\left(p, \frac{\sigma}{\sqrt{n}}\right)$$

$$= N\left(p, \frac{\frac{1}{2\sqrt{n}}}{\sqrt{n}}\right)$$

$$= N\left(p, \frac{1}{2n}\right)$$

$$1.96\left(\sqrt{\frac{1}{4n}}\right) \leq .01$$

$$\approx 2\left(\sqrt{\frac{1}{4n}}\right) \leq .01$$

$$\sqrt{\frac{1}{4n}} \leq .005$$

$$\frac{1}{4n} \leq \frac{1}{(200)^2}$$

$$\frac{1}{4} \leq \frac{n}{(200)^2}$$

$$n \geq \frac{(200)^2}{4} = \boxed{10{,}000}$$