CS 70 Discussion 6C

Exponential (λ): continuous-time analog of the geometric dist.

PDF: $\lambda e^{-\lambda x}$ (for $x \geq 0$) CDF: $1 - e^{-\lambda x}$ ($x \geq 0$)

expectation: $\frac{1}{\lambda}$ variance: $\frac{1}{\lambda^2}$

complementary CDF: $e^{-\lambda x}$ gives $P(X \geq x)$

Normal(μ, σ^2): classic bell-shaped curve

PDF: $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

expectation: μ variance: σ^2

1 $X \sim \text{Exponential}(\lambda = ?)$

$X \sim \text{Exponential}(\lambda = 1/50)$

$E[X] = 50 = \frac{1}{\lambda}$

$\lambda = 1/50$

(a) $P(X < 30) = \int_0^{30} \underset{\uparrow \text{ pdf}}{\lambda e^{-\lambda x}} dx = \frac{1}{50}\int_0^{30} e^{-\frac{x}{50}} dx$

into the CDF formula for the exponential

$= \boxed{1 - e^{-3/5}} \approx 0.451$

↑ plugging into CDF

$P(X < k) = P(X \leq k)$

(b) $Y \sim \text{Exponential}(\lambda = \frac{1}{50})$

$P(Y > 30) = 1 - P(Y \leq 30) = 1 - (1 - e^{-3/5}) = \boxed{e^{-3/5}}$

≈ 0.549

(c) $X \sim \text{Exponential}(\lambda = \frac{1}{50})$

$$P(X \geq 60 \mid X > 30) = \frac{P(X \geq 60)}{P(X > 30)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1 - P(X < 60)}{e^{-3/5}}$$

$P(X \geq 60)$

$$= \int_{60}^{\infty} \lambda e^{-\lambda x} dx$$

$\lambda = \frac{1}{50}$

$$= \frac{1 - \int_0^{60} \lambda e^{-\lambda x} dx}{e^{-3/5}} = \frac{e^{-6/5}}{e^{-3/5}}$$

$$= \boxed{e^{-3/5}} \approx 0.549$$

memorylessness:

2 1. Find CDF and PDF of distribution for distance of E's and K's throw.

2. Set up integral to evaluate $P(Y > X)$

Hint: what values can X range from? how about Y?

X: distance of E's throw Y = distance of K throw

CDF of X:

$$P(X \leq k) = \frac{\pi(k^2)}{\pi(10^2)} = \left(\frac{k}{10}\right)^2$$

ratio of areas

PDF of X:

$$\frac{d}{dk} \frac{k^2}{100} = \frac{k}{50}$$

CDF of Y:

$$1-e^{-\lambda k} = 1-e^{-k/2}$$

PDF of Y:

$$\frac{1}{2}e^{-k/2}$$

$$P(X<Y) = \int_0^{10} f_X(x) P(Y>x)\, dx$$

PDF of X

multiple integral

$$= \int_0^{10}\int_x^{\infty} \boxed{f_X(x)} f_Y(y)\, dy\, dx$$

$0 \le x \le 10$

$$\int_x^{\infty} f_Y(y)\, dy = P(Y>x)$$

$$= \int_0^{10} \frac{x}{50}\left(e^{-\frac{1}{2}x}\right) dx$$

$$P(Y>x) = \int_x^{\infty} \lambda e^{-\lambda y}\, dy \quad (\lambda = 1/2)$$

$$= e^{-\frac{1}{2}x}$$

$$= \frac{1}{50}\int_0^{10} x e^{-\frac{1}{2}x}\, dx$$

$u = x \qquad dv = e^{-\frac{1}{2}x}dx$

$du = dx \qquad v = -2e^{-\frac{1}{2}x}$

$$\approx \boxed{0.0767}$$

$0 \le X \le 10$

$Y \ge x$

$\boxed{3}$ Prove that for $X \sim \text{Normal}(\mu, \sigma^2)$ that $Y = aX + b$ is distributed as $\text{Normal}(a\mu + b, a^2\sigma^2)$.

(for some constants a, b)

To argue that Y is normally distributed with parameters as specified above, we calculate the PDF and CDF of Y.

CDF of Y:

$$F_Y(x) = P[Y \le x] = P[aX + b \le x] = P\left[X \le \frac{x-b}{a}\right]$$

$$= F_X\left(\frac{x-b}{a}\right)$$

also a
normal RV having
its own CDF, F_X

PDF of Y:

$$f_Y(x) = \frac{dF_Y(x)}{dx} = \frac{d}{dx} F_X\left(\frac{x-b}{a}\right) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-\left(\frac{x-b}{a} - \mu\right)^2}{2\sigma^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma a} \exp\left\{ \frac{-\left(\frac{x-b-a\mu}{a}\right)^2}{2\sigma^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma a} \exp\left\{ \frac{-(x-b-a\mu)^2}{2a^2\sigma^2} \right\}$$

← This is the Formula for a Gaussian PDF, except mean $= b + a\mu$ and variance is $a^2\sigma^2$.

$Y \sim \text{Normal}(a\mu + b, a^2\sigma^2)$