Discussion 5B: tinyurl.com/frank-discussion 7/21/2020

1 (a) When is $P(A \cup B) = P(A) + P(B)$?

~~true when~~ iff A, B are mutually exclusive

$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{0}$ ← general

(Principal of Inclusion-Exclusion)

(b) When is $P(A \cap B) = P(A) \cdot P(B)$?

If A, B are independent, then this statement is true.

$\boxed{P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)}$

(chain rule) ↑ general

$P(B|A) = P(B)$ $P(A|B) = P(A)$

when A, B are independent

(c) Does disjoint imply independence?

$\boxed{\text{No. Disjoint} \neq \text{independence}}$

Consider rolling a dice once.

A = rolling a 1 B = rolling a 2

$P(A \cap B) \neq P(A) \cdot P(B)$ ← True iff A, B independent

$0 \neq \left(\frac{1}{6}\right) \cdot \left(\frac{1}{6}\right)$

2 (a) P(first bin empty) = ?

The outcome of every ball toss is independent of all other toss outcomes.

1st toss:

$P(\text{1st bin empty}) = \frac{n-1}{n}$ ← all other bins; ← n bins to land in

$\underbrace{\left(\frac{n-1}{n}\right)}_{\text{1st toss}} \underbrace{\left(\frac{n-1}{n}\right)}_{\text{2nd toss}} \cdots \underbrace{\left(\frac{n-1}{n}\right)}_{n\text{th toss}} = \boxed{\left(\frac{n-1}{n}\right)^n}$

(b) P(first k bins empty) =

First toss: $\frac{n-k}{n}$ ←

$\boxed{\left(\frac{n-k}{n}\right)^n}$ ← n tosses

(c) Write a union bound for $P(A)$

A = event that at least k bins are empty

$m = \binom{n}{k}$ = number of sets of k bins amongst n bins

$A = \bigcup_{i=1}^{m} A_i$ ← A_i is event that the ith set of k bins is empty

(d) Evaluate (c) union bound

"each set of k bins has equal probability of staying empty"

$P(A) = P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$

$P(A_1 \cup A_2) = P(\overbrace{A_1 = \text{True}}$ or $A_2 = \text{True})$

$P(A_1 \cup A_2 \cdots \cup A_m)$ = any set of k bins remain empty

$\left(\frac{n-k}{n}\right)^n$ ←

$P(A_i) = P\left(\begin{matrix}k \text{ bins stay} \\ \text{empty}\end{matrix}\right) = \left(\frac{n-k}{n}\right)^n$

$\boxed{P(A) \leq m\left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n}$

$\boxed{2}$ (e) $P(2^{nd}\text{ bin empty} \mid 1^{st}\text{ bin empty})$

$$= \frac{P(1^{st}\text{ bin empty AND } 2^{nd}\text{ bin empty})}{P(1^{st}\text{ bin empty})}$$

$$= \frac{\left(\frac{n-2}{n}\right)^n}{\left(\frac{n-1}{n}\right)^n} = \boxed{\left(\frac{n-2}{n-1}\right)^n}$$

(f) Are events $\overbrace{\text{"1st bin is empty"}}^{A}$, $\overbrace{\text{"first 2 bins are empty"}}^{B}$ independent?

$P(B=\text{True} \mid A=\text{False}) \neq P(B=\text{True})$ if A, B are independent

$0 \neq \left(\frac{n-2}{n}\right)^n$

$\boxed{\text{dependent}}$

(g) Are events "first bin is empty" and "second bin is empty" independent?

$$P\begin{pmatrix}1^{st}\text{ and }2^{nd}\text{ bin}\\ \text{being empty}\end{pmatrix} = P\begin{pmatrix}1^{st}\text{ bin}\\ \text{empty}\end{pmatrix} \cdot P\begin{pmatrix}2^{nd}\text{ bin}\\ \text{empty}\end{pmatrix}$$

$$\left(\frac{n-2}{n}\right)^n \neq \left(\frac{n-1}{n}\right)^n \left(\frac{n-1}{n}\right)^n$$

$\boxed{\text{dependent}}$

$\boxed{3}$ Show that:

$$P\underbrace{\begin{pmatrix}\text{at least one other TA, excluding}\\ \text{Sinho, gets same colors}\end{pmatrix}}_{T} \leq \frac{1}{2}$$

$$P\begin{pmatrix}\text{the first TA in line gets}\\ \text{2 beads of the same color}\end{pmatrix} = \left(\frac{1}{2n-1}\right)$$ only 1 matching bead

$$P\begin{pmatrix}\text{any TA in line getting}\\ \text{2 beads of the same color}\end{pmatrix} = \left(\frac{1}{2n-1}\right)$$

Define $T_1, T_2, T_3, \ldots T_n$ where T_i refers to event where i^{th} TA gets two beads of the same color.

$$P(T) = P(T_2 \cup T_3 \cup \ldots \cup T_n = \text{True}) \leq \sum_{i=2}^{n} P(T_i = \text{True})$$

$$\leq (n-1)\left(\frac{1}{2n-1}\right)$$

$$\frac{n-1}{2n-1} \leq \frac{1}{2} \quad (n \geq 2)$$

$$n-1 \leq \frac{1}{2}(2n-1)$$

$$n-1 \leq n - \frac{1}{2}$$

$$\boxed{-1 \leq -\frac{1}{2}} \leftarrow \text{True}$$

$$\frac{n-1}{2n-1} \leq \frac{n-1}{2n-2} = \frac{1}{2}$$

smaller denom

$$\boxed{P(T) \leq \frac{1}{2}}$$

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