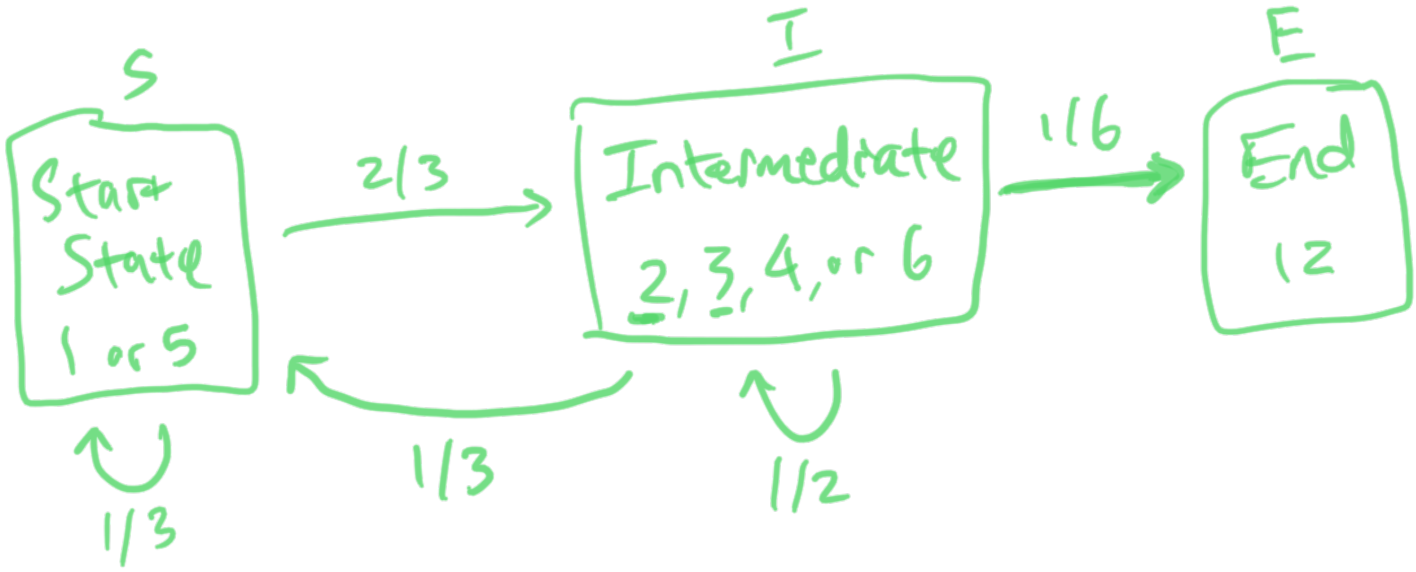
CS 70 Markov Chains Review Session

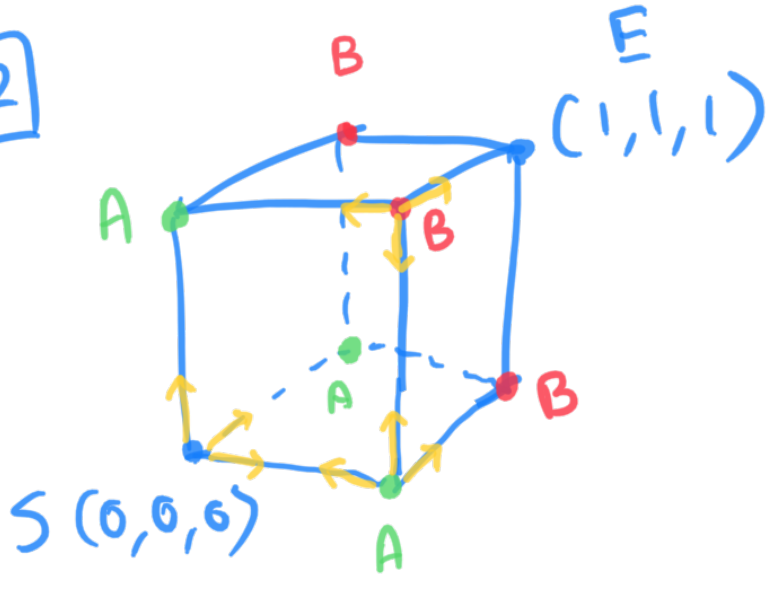
1



$$\begin{cases} \tau(S) = 1 + \frac{1}{3}\tau(S) + \frac{2}{3}\tau(I) \\ \tau(I) = 1 + \frac{1}{2}\tau(I) + \frac{1}{3}\tau(S) + \frac{1}{6}\tau(E) \\ \tau(E) = 0 \end{cases}$$

$$\boxed{\tau(S) = 10.5}$$

2



if Bug starts @ state S (0,0,0), what is the expected amt of time required to reach (1,1,1)?

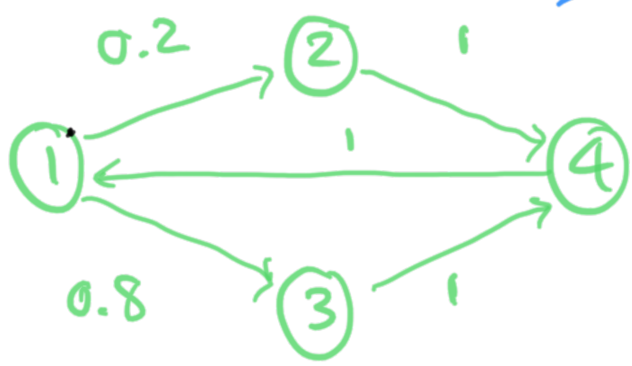
$$\begin{cases} \tau(S) = 1 + \tau(A) \\ \tau(A) = 1 + \frac{1}{3}\tau(S) + \frac{2}{3}\tau(B) \\ \tau(B) = 1 + \frac{2}{3}\tau(A) + \cancel{\frac{1}{3}\tau(E)}_{0} \\ \tau(E) = 0 \end{cases}$$

$$\tau(A) = 1 + \frac{1}{3}(1 + \tau(A)) + \frac{2}{3}\left(1 + \frac{2}{3}\tau(A)\right)$$

$$\tau(A) = 2 + \tau(A)\cdot\left(\frac{7}{9}\right)$$

$$\frac{2}{9}\tau(A) = 2 \qquad \tau(A) = 9 \qquad \boxed{\tau(S) = 10}$$

3



B, D, F, H are correct.

B: irreducible and periodic

$$D: \pi = \left[\frac{1}{3} \quad \frac{1}{15} \quad \frac{4}{15} \quad \frac{1}{3}\right] \qquad \begin{cases} \pi(1) = \pi(4) \\ \pi(2) = 0.2\pi(1) \\ \pi(3) = 0.8\pi(1) \\ \pi(4) = \pi(2) + \pi(3) \\ \pi(1) + \pi(2) + \pi(3) + \pi(4) = 1 \end{cases}$$

$\pi(i)$: long term fraction of time spent in state i

π_i = our distribution @ timestep i

F: If $\pi_0 = [1, 0, 0, 0]$, then π_n doesn't converge as $n \to \infty$

H: If $\pi_0 = [1, 0, 0, 0]$, then $\left(\frac{1}{n}\right)\sum_{m=0}^{n-1} \underline{1\{X_m = 1\}}$ converges as $n \to \infty$.

a count of the number of times I'm inside state 1

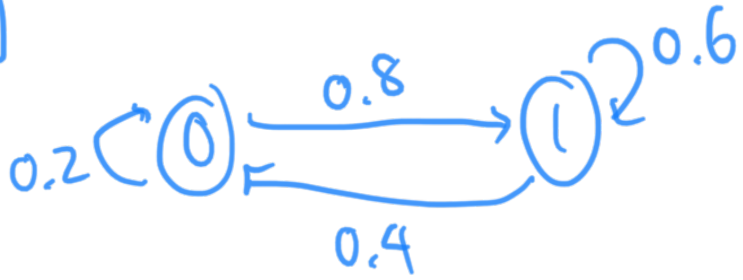
$$\mathbb{1}\{X_m = 1\}$$

$$\begin{cases} = 1 & \text{whenever } X_m = 1 \\ = 0 & \text{whenever } \underline{X_m} \neq 1 \end{cases}$$

refers to where we are at in the MC @ timestep n

quantity converges to $1/3$ as $n \to \infty$.

$\boxed{4}$



(given we start @ state 0)

(a) $P(X_2 = 0)$

$= P(X_1 = 0, X_2 = 0) + P(X_1 = 1, X_2 = 0)$

$= (0.2)(0.2) + (0.8)(0.4)$

$= \boxed{0.36}$

(b) Calculate stationary distribution:

$$\begin{cases} \pi(0) = 0.2\pi(0) + 0.4\pi(1) \\ \pi(0) + \pi(1) = 1 \end{cases}$$

$0.8\pi(0) = 0.4\pi(1)$

$\pi(1) = 2\pi(0)$

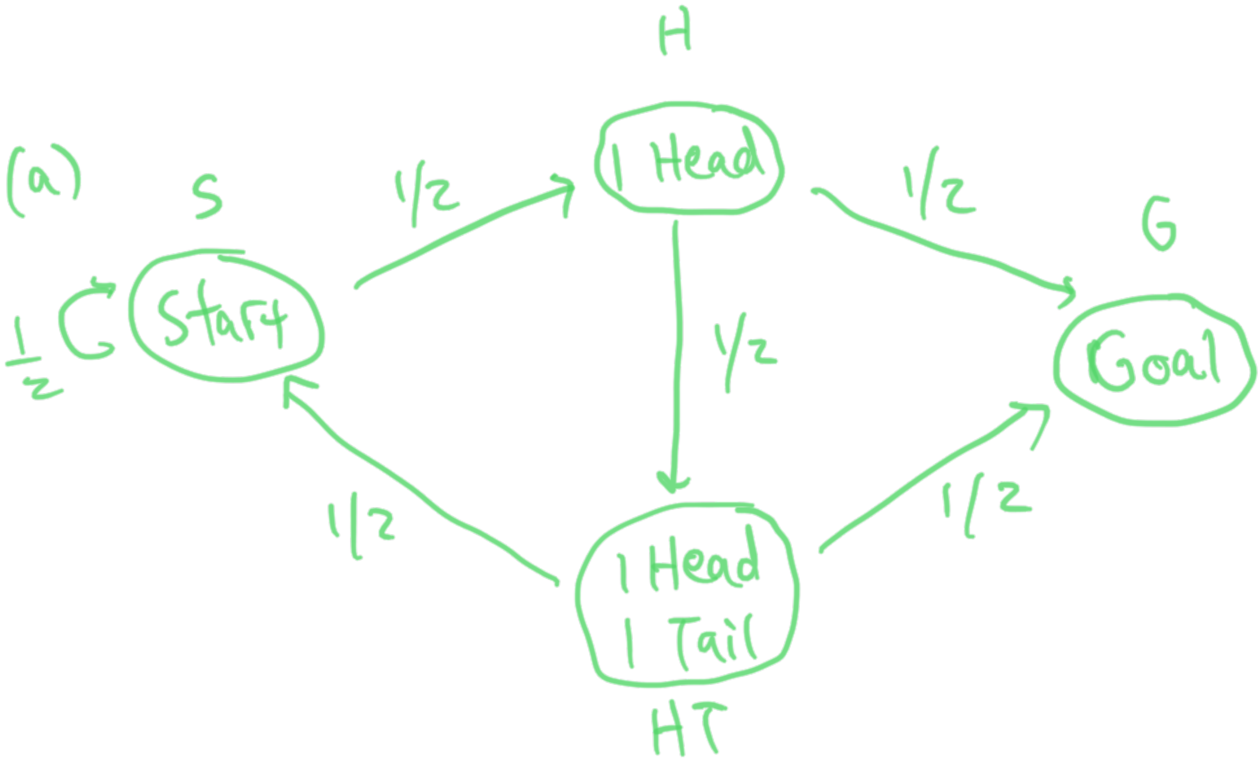
$\pi(0) = \frac{1}{3}$

$\pi(1) = \frac{2}{3}$

$\boxed{\frac{2}{3} \text{ prob.}}$

$\boxed{5}$

(a)



(b)

$$\begin{cases} \tau(S) = 1 + \frac{1}{2}\tau(S) + \frac{1}{2}\tau(H) \\ \tau(H) = 1 + \frac{1}{2}\tau(HT) + \cancel{\frac{1}{2}\overset{0}{\tau(G)}} \\ \tau(HT) = 1 + \frac{1}{2}\tau(S) + \cancel{\frac{1}{2}\underset{0}{\tau(G)}} \\ \tau(G) = 0 \end{cases}$$

$$\frac{1}{2}\tau(S) = 1 + \frac{1}{2}\tau(H)$$

$$\frac{1}{2}\tau(S) = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\tau(HT)\right)$$

$$\frac{1}{2}\tau(S) = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\left(1 + \frac{1}{2}\tau(S)\right)\right)$$

$$\frac{1}{2}\tau(S) = 1 + \frac{1}{2}\left(\frac{3}{2} + \frac{1}{4}\tau(S)\right)$$

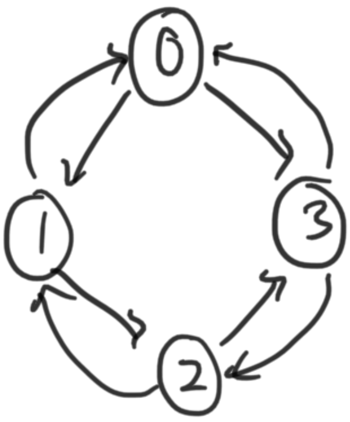
$$\frac{1}{2}\tau(S) = 1 + \frac{3}{4} + \frac{1}{8}\tau(S)$$

$$\frac{3}{8}\tau(S) = \frac{7}{4} \qquad \tau(S) = \frac{7}{\cancel{4}}\left(\frac{\overset{2}{\cancel{8}}}{3}\right)$$

$$\boxed{\tau(S) = \frac{14}{3} \text{ tosses}}$$

Clarification: Although the notes tell us that irreducible, periodic MC aren't guaranteed to converge to the stationary distribution as $n \to \infty$, this could still occur even when our initial distribution $\neq$ invariant distribution, as shown in the example below:

Initial Distribution: $\pi_0 = [0.25 \ \ 0.3 \ \ 0.25 \ \ 0.2]$



1/2 prob. along every edge drawn in this MC

$$\pi_1 = \pi_0 P$$

$$= [0.25 \ \ 0.3 \ \ 0.25 \ \ 0.2] \begin{bmatrix} 0 & .5 & 0 & .5 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ .5 & 0 & .5 & 0 \end{bmatrix}$$

$$= \underbrace{[.25 \ \ .25 \ \ .25 \ \ .25]}_{\text{our stationary distribution}}$$

Thus although this MC has period 2, it converged to the stationary distribution in one timestep without starting out at the stationary distribution.