2 min intro

CS 70 Discussion 1C: tinyurl.com/frank-discussion 6/23/2020

$\boxed{1}$ (a) $\forall n \in \mathbb{N}\ (n \geq 1)$, $3 \mid n^3 - n$

① Base Case ② Inductive Hypothesis

③ Show that claim holds for $n = k+1$

3 min alone
4 min break
7 min disc

① $n=1$ $n^3 - n = 1^3 - 1 = 0$ $3 \mid 0$ ✓

② $\exists k \in \mathbb{N}\ (k > 1)$ $3 \mid k^3 - k$

③ $(k+1)^3 - (k+1)$
$= k^3 + 3k^2 + 3k + 1 - (k+1)$
$= k^3 + 3k^2 + 3k - k$
$= (k^3 - k) + 3k^2 + 3k$
$= \underbrace{(k^3 - k)}_{\text{div by } 3} + \underbrace{3(k^2 + k)}$ ■ □

QED

(b) Prove that you can make exactly $n$ cents of postage with 3-cent and 7-cent stamps as long as $n \geq 12$.

① Base Case: $n = 12, 13, 14$

$n = 12 = 4(3)$ $n = 13 = 2(3) + 1(7)$

$n = 14 = 2(7)$

② $\exists k \in \mathbb{N}\ (k \geq 15)$ $\forall 12 \leq n \leq k$ that $n$ can be made using 3-cent and 7-cent coins

③ $n = k+1 - (3 \text{ cent stamp}) = k-2$ cents

We know that $(k-2)$ cents can be made using 3-cent and 7-cent stamps. So $(k+1)$ cents can be broken into a combo of 3-cent and 7-cent stamps. ■

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$\boxed{2}$ For $x \in \mathbb{R}$, $x \geq 1$, use induction to show that all entries of the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$ are $\leq xn$.

$n = 12, n = 13$ $(k = 13)$

$k+1 = 14$ $n = 14$

$(k+1) - 3 = \boxed{11}$

$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^1$ $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2x \\ 0 & 1 \end{bmatrix}$

3 min alone
3 min break
5 min disc

$\begin{bmatrix} 1 & 2x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3x \\ 0 & 1 \end{bmatrix}$

Claim: $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & nx \\ 0 & 1 \end{bmatrix}$ $(\forall n \in \mathbb{N})$ $n \geq 1$

① Base Case: $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ ✓ $(n=1)$

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② Inductive Hypothesis: $\exists k \in \mathbb{N}$ $(k > 1)$ $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}$

③ $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^{(k+1)} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (k+1)x \\ 0 & 1 \end{bmatrix}$ ■

3 Prove that every positive integer $n$ can be written in binary,
i.e. $n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \dots + c_1 \cdot 2^1 + c_0 \cdot 2^0$, for $k \in \mathbb{N}$ and $c_k \in \{0,1\}$

4 min alone
4 min break
disc

① Base case: $n = 1$

$1 = 1 \cdot 2^0$

② Inductive Hypothesis: $\exists k \in \mathbb{N}$ $(k > 1)$
$\forall n$ $\boxed{1 \le n \le k}$ satisfy the claim we're $n = k$
trying to prove.

③ $n = (k+1) \longrightarrow \boxed{n \text{ is odd}}$ $\quad$ $k+1$ is odd
$k$ is even

$\boxed{n \text{ is even}}$

$k = \underbrace{c_k \cdot 2^k + \dots + c_1 \cdot 2^1}_{\text{div by } 2} + c_0 \cdot 2^0$

$c_0 = 0$

$\cancel{c_0 = 1}$

$\boxed{\frac{n}{2} \text{ is an integer}}$ $\quad \frac{k+1}{2}$

$\boxed{k+1 = c_k \cdot 2^k + \dots + c_1 \cdot 2^1 + 1 \cdot 2^0}$

$\frac{k+1}{2} = c_k \cdot 2^k + \dots + c_1 \cdot 2^1 + c_0 \cdot 2^0$

$\boxed{k+1 = c_k \cdot 2^{k+1} + \dots + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0}$