A monoidal category consists of the following data.

- A category $\mathcal{C}$
- A functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ called the monoidal product.
- An object $\mathbb{1} \in \mathcal{C}$ called the monoidal unit.
- A natural isomorphism called the associativity isomorphism.

$$(X\otimes Y)\otimes Z \xrightarrow[\cong]{\alpha_{X,Y,Z}} X\otimes(Y\otimes Z)$$

- Natural isomorphisms called the left unit (resp., right unit) isomorphism.

$$\mathbb{1}\otimes X \xrightarrow[\cong]{\lambda_X} X \qquad\qquad X\otimes\mathbb{1} \xrightarrow[\cong]{\rho_X} X$$

These data are subject to the following conditions

- $$\begin{tikzcd} (X\otimes\mathbb{1})\otimes Y \arrow[r, "\alpha_{X,\mathbb{1},Y}"] \arrow[d, "\rho_X\otimes Y"'] & X\otimes(\mathbb{1}\otimes Y) \arrow[d, "X\otimes\lambda_Y"] \\ X\otimes Y \arrow[r, equal] & X\otimes Y \end{tikzcd}$$ commutes

  and $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ (this equality can be derived).

- $$\begin{tikzcd} & (W\otimes X)\otimes(Y\otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\ ((W\otimes X)\otimes Y)\otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y}\otimes Z"'] & & W\otimes(X\otimes(Y\otimes Z)) \\ (W\otimes(X\otimes Y))\otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & W\otimes((X\otimes Y)\otimes Z) \arrow[u, "W\otimes\alpha_{X,Y,Z}"'] \end{tikzcd}$$ Commutes.

- A strict monoidal category is a monoidal category in which α, λ, ρ are identities.

Def. Given a monoidal category $(\mathcal{C},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ the reverse monoidal category consists of the following data.

$\mathcal{C}' := \mathcal{C}$, $\quad \otimes' := C\times C \xrightarrow{\tau} C\times C \xrightarrow{\otimes} C \quad \mathbb{1} := \mathbb{1}$

$\alpha_{X,Y,Z} := \alpha^{-1}_{Z,Y,X} \quad \lambda' := \rho \quad \rho' := \lambda$

Def. Given a monoidal category $(\mathcal{C},\otimes,\mathbb{1},\alpha,\lambda,\rho)$, the opposite monoidal category consists of the following data.

$\mathcal{C}' := \mathcal{C}^{op} \quad \otimes' := C^{op}\times C^{op} \cong (C\times C)^{op} \xrightarrow{\otimes^{op}} C^{op}$

$\mathbb{1}' := \mathbb{1} \quad \alpha' := \alpha^{-1} \quad \lambda' := \lambda^{-1} \quad \rho' := \rho^{-1}$

Def. A monoid in a monoidal category $\mathcal{C}$ consists of the following data.

- An object $X \in \mathcal{C}$.
- $\mu : X\otimes X \to X$ a morphism called the multiplication
- $1 : \mathbb{1} \to X$ a morphism called the unit.

These data are required to satisfy the following conditions.

$$\begin{tikzcd} (X\otimes X)\otimes X \arrow[r, "\alpha"] \arrow[dd, "\mu\otimes X"'] & X\otimes(X\otimes X) \arrow[d, "X\otimes\mu"] \\ & X\otimes X \arrow[d, "\mu"] \\ X\otimes X \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}\otimes X \arrow[r, "1\otimes X"] \arrow[d, "\lambda"', "\cong"] & X\otimes X \arrow[d, "\mu"] & X\otimes\mathbb{1} \arrow[l, "X\otimes 1"'] \arrow[d, "\rho", "\cong"'] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

A morphism of monoids $f : (X,\mu_X,1_X) \to (Y,\mu_Y,1_Y)$ is a morphism $f : X \to Y$ such that

$$\begin{tikzcd} X\otimes X \arrow[r, "f\otimes f"] \arrow[d, "\mu_X"'] & Y\otimes Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "1_X"] \arrow[d, equal] & X \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

The category of monoids in a monoidal category $\mathcal{C}$ is denoted $\mathrm{Mon}(\mathcal{C})$.

Def. For monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a monoidal functor consists of the following data.

- A functor $F : \mathcal{C} \to \mathcal{D}$
- A natural transformation

$$FX\otimes FY \xrightarrow{F_2} F(X\otimes Y)$$

- A morphism

$$\mathbb{1}_D \xrightarrow{F_0} F\mathbb{1}_C$$

These data are subject to the following conditions.

- $$\begin{tikzcd} (FX\otimes FY)\otimes FZ \arrow[r, "\alpha_D"] \arrow[d, "F_2\otimes FZ"'] & FX\otimes(FY\otimes FZ) \arrow[d, "FX\otimes F_2"] \\ F(X\otimes Y)\otimes FZ \arrow[d, "F_2"'] & FX\otimes F(Y\otimes Z) \arrow[d, "F_2"] \\ F((X\otimes Y)\otimes Z) \arrow[r, "F\alpha_C"'] & F(X\otimes(Y\otimes Z)) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1}_D\otimes FX \arrow[r, "\lambda_D"] \arrow[d, "F_0\otimes FX"'] & FX \\ F\mathbb{1}_C\otimes FX \arrow[r, "F_2"'] & F(\mathbb{1}_C\otimes X) \arrow[u, "F\lambda_C"'] \end{tikzcd} \qquad \begin{tikzcd} FX\otimes\mathbb{1}_D \arrow[r, "\rho_D"] \arrow[d, "FX\otimes F_0"'] & FX \\ FX\otimes F\mathbb{1}_C \arrow[r, "F_2"'] & F(X\otimes\mathbb{1}_C) \arrow[u, "F\rho_C"'] \end{tikzcd}$$

A strong monoidal functor is a monoidal functor in which F_2 and F_0 are isomorphisms. A strict monoidal functor is a monoidal functor in which F_2 and F_0 are identities.

Thm. (Mac Lane's coherence theorem). For each monoidal category $\mathcal{C}$, there is a strict monoidal category $\mathcal{C}_{st}$ and an adjoint equivalence

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{st} \arrow[l, bend left, "R"] \end{tikzcd}$$

with both L and R strong monoidal functors and $RL = 1_{\mathcal{C}}$.

Def. A symmetric monoidal category is a monoidal category $\mathcal{C}$ equipped with an additional natural isomorphism ζ

$$X\otimes Y \xrightarrow[\cong]{\zeta_{X,Y}} Y\otimes X$$

This data is subject to the following conditions.

- $$\begin{tikzcd} X\otimes Y \arrow[r, "\zeta_{X,Y}"] \arrow[dr, equal] & Y\otimes X \arrow[d, "\zeta_{Y,X}"] \\ & X\otimes Y \end{tikzcd}$$

- $$\begin{tikzcd} X\otimes\mathbb{1} \arrow[r, "\zeta_{X,\mathbb{1}}"] \arrow[d, "\rho"'] & \mathbb{1}\times X \arrow[d, "\lambda"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}\otimes X \arrow[r, "\zeta_{\mathbb{1},X}"] \arrow[d, "\lambda"'] & X\otimes\mathbb{1} \arrow[d, "\rho"] \\ X \arrow[r, equal] & X \end{tikzcd}$$
  (this can be derived)

- $$\begin{tikzcd} & X\otimes(Z\otimes Y) \arrow[r, "X\otimes\zeta_{Z,Y}"] & X\otimes(Y\otimes Z) \arrow[dr, "\alpha^{-1}"] & \\ (X\otimes Z)\otimes Y \arrow[ur, "\alpha_{X,Z,Y}"] \arrow[dr, "\zeta_{X\otimes Z,Y}"'] & & & (X\otimes Y)\otimes Z \\ & Y\otimes(X\otimes Z) \arrow[r, "\alpha^{-1}"'] & (Y\otimes X)\otimes Z \arrow[ur, "\zeta_{Y,X}\otimes Z"'] & \end{tikzcd}$$

  A symmetric monoidal category is strict if its underlying monoidal category is strict.

Def. A commutative monoid in a symmetric monoidal category is a monoid $(X,\mu,1)$ such that

$$\begin{tikzcd} X\otimes X \arrow[r, "\zeta_{X,X}"] \arrow[d, "\mu"'] & X\otimes X \arrow[d, "\mu"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

Def. A symmetric monoidal functor is a monoidal functor such that

$$\begin{tikzcd} FX\otimes FY \arrow[r, "\zeta_{FX,FY}", "\cong"'] \arrow[d, "F_2"'] & FY\otimes FX \arrow[d, "F_2"] \\ F(X\otimes Y) \arrow[r, "\cong", "F\zeta_{X,Y}"'] & F(Y\otimes X) \end{tikzcd}$$

Def. A symmetric monoidal category is closed if the functor

$$-\otimes X : \mathcal{C} \to \mathcal{C}$$

admits a right adjoint $[X,-]$ called the internal hom.

Def. A braided monoidal category is a monoidal category equipped with a braiding ζ.

$$X\otimes Y \xrightarrow[\cong]{\zeta_{X,Y}} Y\otimes X$$

subject to the following conditions.

- $$\begin{tikzcd} X\otimes\mathbb{1} \arrow[r, "\zeta_{X,\mathbb{1}}"] \arrow[d, "\rho"'] & \mathbb{1}\otimes X \arrow[d, "\lambda"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

- $$\begin{tikzcd} & (Y\otimes X)\otimes Z \arrow[r, "\alpha"] & Y\otimes(X\otimes Z) \arrow[dr, "Y\otimes\zeta_{X,Z}"] & \\ (X\otimes Y)\otimes Z \arrow[ur, "\zeta_{X,Y}\otimes Z"] \arrow[dr, "\alpha"'] & & & Y\otimes(Z\otimes X) \\ & X\otimes(Y\otimes Z) \arrow[r, "\zeta_{X,Y\otimes Z}"'] & (Y\otimes Z)\otimes X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

- $$\begin{tikzcd} & X\otimes(Z\otimes Y) \arrow[r, "\alpha^{-1}"] & (X\otimes Z)\otimes Y \arrow[dr, "\zeta_{X,Z}\otimes Y"] & \\ X\otimes(Y\otimes Z) \arrow[ur, "X\otimes\zeta_{Y,Z}"] \arrow[dr, "\alpha^{-1}"'] & & & (Z\otimes X)\otimes Y \\ & (X\otimes Y)\otimes Z \arrow[r, "\zeta_{X\otimes Y,Z}"'] & Z\otimes(X\otimes Y) \arrow[ur, "\alpha^{-1}"'] & \end{tikzcd}$$