Fix a monoidal category V.

Def. A **V-category** $\mathcal{C}$ (a category enriched in V) consists of the following data.

- A class of objects $ob(\mathcal{C})$.
- For each pair of objects $X, Y \in \mathcal{C}$, an object $\mathcal{C}(X,Y) \in V$, called the **hom object**.
- For each triple of objects $X, Y, Z \in \mathcal{C}$, a morphism

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \xrightarrow{m_{x,y,z}} \mathcal{C}(X,Z)$$

in V called the **composition**.

- For each object X in $\mathcal{C}$, a morphism

$$1 \xrightarrow{i_X} \mathcal{C}(X,X)$$

called the **identity** on X.

These data are subject to the following conditions.

- $$\begin{tikzcd}
(\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y)) \otimes \mathcal{C}(W,X) \arrow[r, "\alpha"] \arrow[dd, "m \otimes id"'] & \mathcal{C}(Y,Z) \otimes (\mathcal{C}(X,Y) \otimes \mathcal{C}(W,X)) \arrow[d, "id \otimes m"] \\
 & \mathcal{C}(Y,Z) \otimes \mathcal{C}(W,Y) \arrow[d, "m"] \\
\mathcal{C}(X,Z) \otimes \mathcal{C}(W,X) \arrow[r, "m"'] & \mathcal{C}(W,Z)
\end{tikzcd}$$

- $$\begin{tikzcd}
\mathcal{C}(X,Y) \otimes 1 \arrow[r, "\lambda"] \arrow[d, "id \times i"'] & \mathcal{C}(X,Y) \arrow[d, equal] & 1 \otimes \mathcal{C}(X,Y) \arrow[l, "\rho"'] \arrow[d, "i \times id"] \\
\mathcal{C}(X,Y) \otimes \mathcal{C}(X,X) \arrow[r, "m"'] & \mathcal{C}(X,Y) & \mathcal{C}(Y,Y) \otimes \mathcal{C}(X,Y) \arrow[l, "m"]
\end{tikzcd}$$

Def. Let $\mathcal{C}$ and $\mathcal{D}$ be V-categories. A **V-functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.

- An assignment on objects $ob(\mathcal{C}) \to ob(\mathcal{D})$, $X \mapsto FX$
- For each $\mathcal{C}(X,Y)$, a morphism

$$\mathcal{C}(X,Y) \xrightarrow{F_{XY}} \mathcal{D}(FX,FY)$$

in V.

These data are subject to the following conditions.

- $$\begin{tikzcd}
\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \arrow[r, "F \otimes F"] \arrow[d, "m"'] & \mathcal{D}(FY,FZ) \otimes \mathcal{D}(FX,FY) \arrow[d, "m"] \\
\mathcal{C}(X,Y) \arrow[r, "F"'] & \mathcal{D}(FX,FY)
\end{tikzcd}$$

- $$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "i_X"'] & 1 \arrow[d, "i_{FX}"] \\
\mathcal{C}(X,X) \arrow[r, "F"'] & \mathcal{D}(FX,FX)
\end{tikzcd}$$

The composite and identity V-functors are defined in the evident way.

Def. Let $F, G : \mathcal{C} \to \mathcal{D}$ be V-functors. A **V-natural transformation** $\alpha : F \Rightarrow G$ consists of the following data.

- For each $X \in \mathcal{C}$, a morphism

$$1 \xrightarrow{\alpha_X} \mathcal{D}(FX,GX)$$

called the **component**.

This data is subject to the following condition.

- $$\begin{tikzcd}
 & \mathcal{C}(X,Y) \arrow[dl, "\rho^{-1}"'] \arrow[dr, "\lambda^{-1}"] & \\
\mathcal{C}(X,Y) \otimes 1 \arrow[dd, "G \otimes \alpha_X"'] & & 1 \otimes \mathcal{C}(X,Y) \arrow[d, "\alpha_Y \otimes F"] \\
 & & \mathcal{D}(FY,GY) \otimes \mathcal{D}(FX,FY) \arrow[d, "m"] \\
\mathcal{D}(GX,GY) \otimes \mathcal{D}(FX,GX) \arrow[rr, "m"'] & & \mathcal{D}(FX,GY)
\end{tikzcd}$$

The identity V-natural transformation is defined in the obvious way.

Def. Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be V-natural transformations. The **vertical composition** $\beta\alpha : F \Rightarrow H$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\alpha \Downarrow"] \arrow[r, phantom, shift right=4, "\beta \Downarrow"] \arrow[r, phantom, shift left=8, "\Downarrow \beta\alpha" near end] & \mathcal{D}
\end{tikzcd}$$

is given by the following composite for each $X \in \mathcal{C}$.

ρ^{-1} also works since $\rho_1 = \lambda_1$

$$\begin{tikzcd}
1 \arrow[r, "\beta\alpha_X"] \arrow[d, "\lambda^{-1}"', "\cong"] & \mathcal{D}(FX,HX) \\
1 \otimes 1 \arrow[r, "\beta_X \otimes \alpha_X"'] & \mathcal{D}(GX,HX) \otimes \mathcal{D}(FX,GX) \arrow[u, "m"']
\end{tikzcd}$$

Def. Given the following V-natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathcal{E}
\end{tikzcd}$$

The **horizontal composition** $\beta * \alpha$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F'F"] \arrow[r, bend right, "G'G"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & \mathcal{E}
\end{tikzcd}$$

is given by the following composite for all $X \in \mathcal{C}$.

$$\begin{tikzcd}
1 \arrow[r, "(\beta * \alpha)_X"] \arrow[d, "\lambda^{-1}"', "\cong"] & \mathcal{E}(F'FX, G'GX) \\
1 \otimes 1 \arrow[d, "\beta_{GX} \otimes \alpha_X"'] & \\
\mathcal{E}(F'GX, G'GX) \otimes \mathcal{D}(FX,GX) \arrow[r, "id \otimes F'"'] & \mathcal{E}(F'GX, G'GX) \otimes \mathcal{E}(F'FX, F'GX) \arrow[uu, "m"']
\end{tikzcd}$$

Def. A **V-adjunction** $L \dashv R$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"]
\end{tikzcd}$$

consists of the following data.

- A V-natural transformation $\eta : 1_{\mathcal{C}} \to RL$ called the **unit**.
- A V-natural transformation $\varepsilon : LR \to 1_{\mathcal{D}}$ called the **counit**.

These data are subject to the following conditions.

$$\begin{tikzcd}
 & RLR \arrow[dr, "1_R * \varepsilon"] & \\
R \arrow[ur, "\eta * 1_R"] \arrow[rr, "1_R"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & LRL \arrow[dr, "\varepsilon * 1_L"] & \\
L \arrow[ur, "1_L * \eta"] \arrow[rr, "1_L"'] & & L
\end{tikzcd}$$