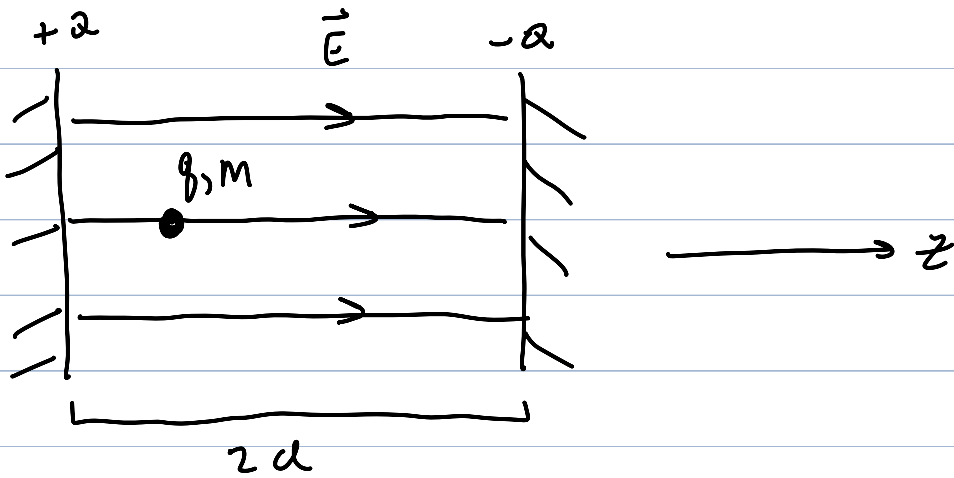
POTENTIALS 9/6/23 F.S.

→ Electrostatics isn't enough for a charged particle to be held in a stable equilibrium.

$$\vec{\nabla} \cdot \vec{E} = 0$$

<u>Detour</u>: Average force of an oscillating field.



$\phi = V_0 \cos(\Omega_T t)$

$F = m\ddot{z} = e\,E(z,t)$

$\vec{E} = -\vec{\nabla}\phi$

<u>Homogenous</u>.

Note: z is dummy variable & not necessarily axial direction.

$$F_{cap} = e\frac{\phi}{2d}$$

$$\Rightarrow \quad m\ddot{z} = \frac{V_0}{2d} e \cos(\Omega_T t)$$

$$= e E_0(z) \cos(\Omega_T t)$$

$$\int \ddot{z} = \int \frac{e}{m} E_0(z) \cos(\Omega_T t)$$

$$\int \dot{z}(t) - \cancel{\dot{z}(0)}^{\,0} = \int \frac{e}{m\Omega_T} E_0(z) \sin(\Omega_T t)$$

$$z(t) - \underbrace{z(0)}_{z_0} = -\frac{e}{m\Omega_T^2} E_0(z) \cos(\Omega_T t)$$

$$\therefore \quad \boxed{z(t) = z_0 - \frac{e}{m\Omega_T^2} E_0(z) \cos(\Omega_T t)}$$

$$\frac{1}{T}\int_0^T \cos(\Omega_T t)\, dt = 0$$

$\Rightarrow$ Average force on particle $= 0$

$\Rightarrow$ No net acceleration of e.

× Consider an in-homogenous field

→ e.g. : slightly bending the capacitor plates

$\Rightarrow \phi(z, t) \rightarrow \cos(\Omega_T t)$

• But, the spatial variation is slightly perturbed.

$$E_0(z) \simeq E_0(z_0) + \left.\frac{\partial E_0(z)}{\partial z}\right|_{z_0} (z - z_0)$$

$$z - z_0 = -\frac{e E(z_0)}{m \Omega_T^2} \cos(\Omega_T t)$$

$$E_0(z) \simeq E_0(z_0) + \frac{\partial E_0(z_0)}{\partial z_0} \left[-\frac{e E(z_0)}{m \Omega_T^2} \cos(\Omega_T t) \right]$$

Recall, $F = e E_0(z) \cos(\Omega_T t)$

$\Rightarrow$

$$F \approx \underbrace{e E_0(z_0) \cos(\Omega_T t)}_{(1)} - \underbrace{\frac{\partial E_0(z_0)}{\partial z_0} \left[\frac{e^2 E_0(z_0)}{m \Omega_T^2} \cos^2(\Omega_T t) \right]}_{(2)}$$

$$\langle F(t) \rangle = \frac{1}{T} \int_0^T F(t)\, dt$$

$$\langle (1) \rangle = 0$$

$$\langle \cos^2(\Omega_T t) \rangle = \frac{1}{2}$$

$$\Rightarrow \langle F \rangle = \frac{-e^2}{2 m \Omega_T^2} \frac{\partial E_0(z_0)}{\partial z_0} E_0(z_0)$$

$$= \frac{-e^2}{2 m \Omega_T^2} \frac{\partial}{\partial z_0} E_0^2(z_0) \frac{1}{2}$$

$$? \left(\frac{\partial E_0(z_0)}{\partial z_0} \right) E_0(z) = \frac{\partial}{\partial z_0} \left(\frac{1}{2} E_0^2(z) \right)$$

$\rightarrow$ let $\Psi_p = \dfrac{e E_0^2(z_0)}{4 m \Omega_T^2}$

$$\boxed{\therefore \langle F \rangle = -e \frac{\partial \Psi_p}{\partial z_0} \; ; \; \Psi_p = \frac{e E_0^2(z_0)}{4 m \Omega_T^2}}$$

$\Rightarrow$ Adding an inhomogenous static contribution creates an effective, time-avg. Force on the particle.

$\rightarrow$ Extending to 3D yields

$$\langle F(t) \rangle = -e\vec{E} = -e\vec{\nabla}\Psi_p \; ;$$

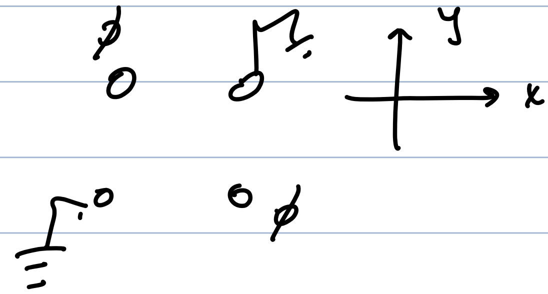
$$\Psi_p = \frac{e}{4 m \Omega_T^2} \left[E_0(x_0, y_0, z_0) \right]^2$$

<u>Note:</u>

- $E_0(z_0) = 0$ RF Null. Minimum micromotion region
- $E_{KE} = e\Psi_p(z_0) = 0 \Big|_{E_0(z_0)=0}$

* ENTER: ION TRAPPING

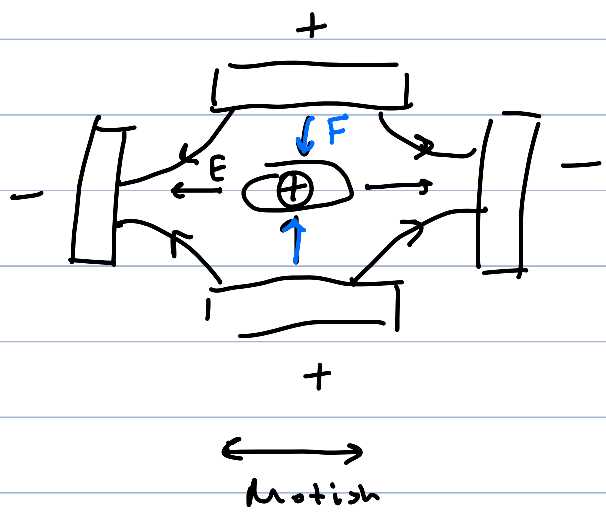
$$V = \frac{1}{2} m \left(\underbrace{\omega_x^2 X^2 + \omega_y Y^2 + \omega_z Z^2}_{\text{secular freqs.}} \right)$$

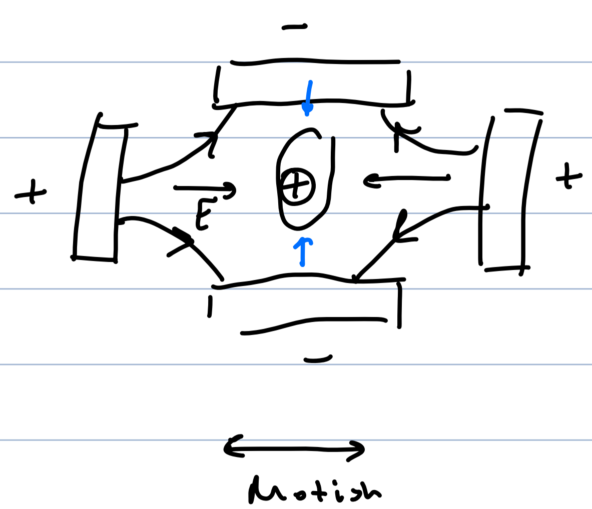


→ Quadrapole fields

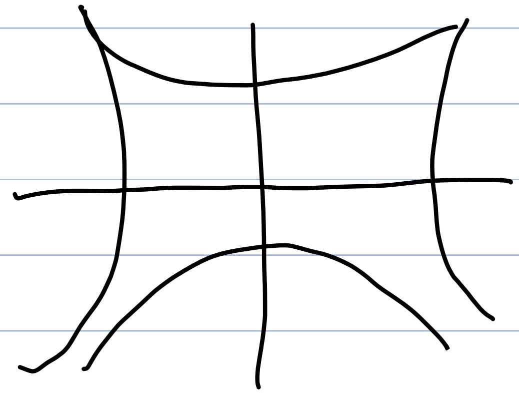
↳ RF quad. confines in the x, y (tranverse) direction.

$$\vec{F} = e\vec{E} = -e\vec{\nabla}\phi$$

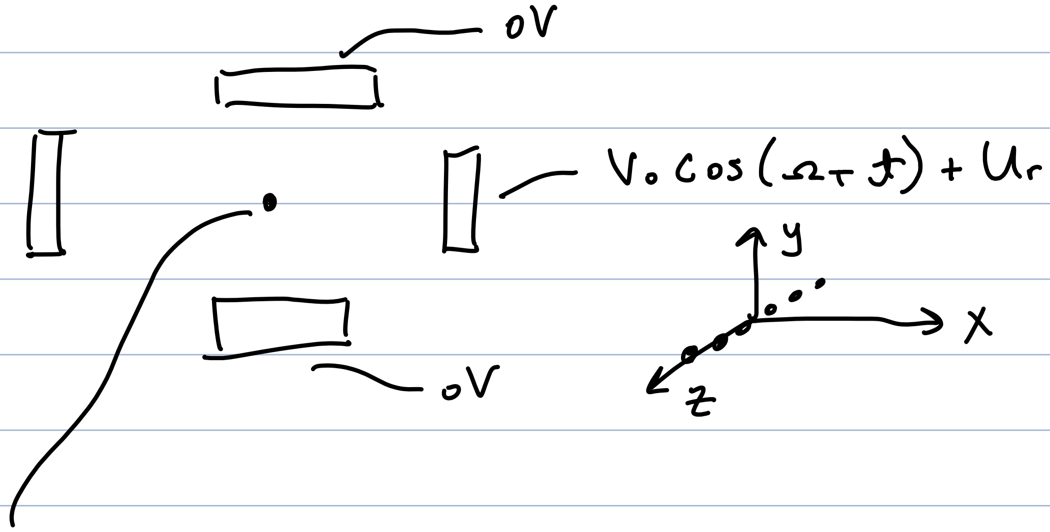




→ oscillation between confine & deconfine.



hyperbolic.



$$\phi = \frac{V_0 \cos(\Omega_T t) + U_R}{2}\left[1 + \frac{x^2 - y^2}{R^2}\right]$$

$$\Phi = \frac{\phi_0}{2r_0^2}\left(\alpha x^2 + \beta y^2 + \gamma z^2\right)$$

$$\nabla^2 \Phi^2 = 0$$

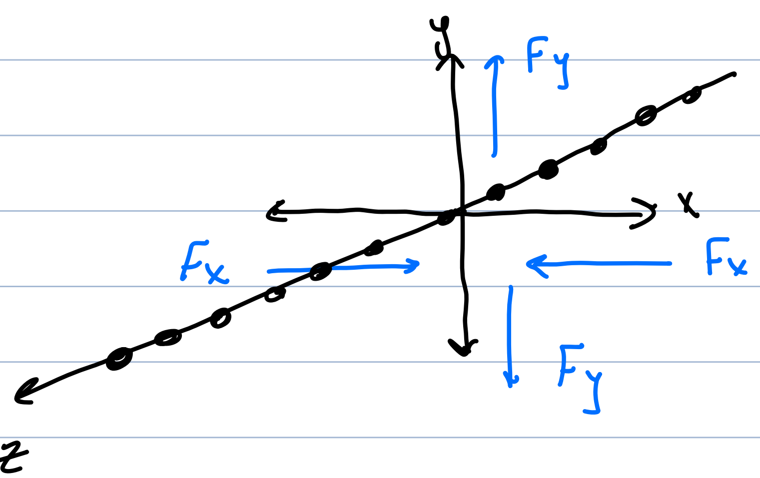
$$\Rightarrow \quad \alpha + \beta + \gamma = 0$$

i) $\gamma = 0$ & $\alpha = -\beta = +1$

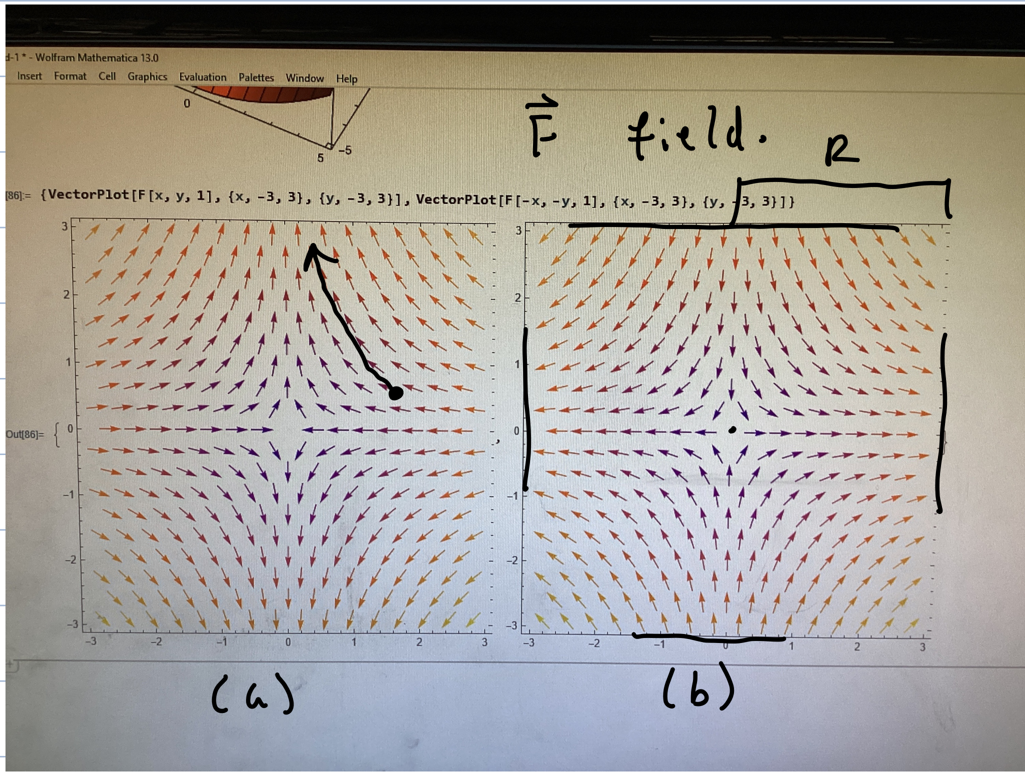
$$\Rightarrow \quad \boxed{\Phi = \frac{\phi_0}{2r_0^2}\left(x^2 - y^2\right)}$$

$$\vec{E} = -\vec{\nabla}\Phi \Rightarrow \begin{cases} E_x = -\dfrac{\phi_0}{\cancel{2}r_0^2}\, x \\ E_y = \dfrac{\phi_0}{\cancel{2}r_0^2}\, y \end{cases} \qquad E_z = 0$$

$$\Rightarrow \quad F_x = -\frac{e\,\phi_0}{\cancel{2}r_0^2}\, x \quad \& \quad F_y = \frac{e\,\phi_0}{\cancel{2}r_0^2}\, y$$



→ Analogous to 2D Harmonic oscillator



- this graph ⇒ that only static field will not confine, but rather move the particle to the electrodes

∴ we need a combination of (a) & (b) that change in time!

$$\Rightarrow \phi_0 = U_r + V_0 \cos(\Omega_T t)$$

$$\Phi = \frac{\phi_0}{2r_0^2}(x^2 - y^2)$$

$$= \frac{[U_r + V_0 \cos(\Omega_T t)]}{2r_0^2}[x^2 - y^2]$$

→ Match form in Exp. Issues paper.

$$\Phi_{RF} \simeq \frac{(V_0 \cos(\Omega_T t) + U_r)}{2}\left(1 + \frac{x^2 - y^2}{R^2}\right)$$

- this provides confinement in the x & y plane.
- this results in a "Rotating saddle".

$\Omega_T \sim 2\pi \cdot 23 \times 10^6$ Hz

$R \sim 200\ \mu m$

$$\vec{E} = -\vec{\nabla}\phi$$

$$= \left(\frac{V_0 \cos(\Omega_T t) + U_r}{R^2}\right) \langle x, -y \rangle$$

$\rightarrow U_R = 0$

$$\vec{E}(x,y) = \frac{V_0}{R^2} \langle x, -y \rangle \cos(\Omega_T t)$$

$$= -\vec{E}_0(x,y) \cos(\Omega_T t);$$

$$\vec{E}_0(x,y) = -\frac{V_0}{R^2} \langle x, -y \rangle$$

Recall: $\langle F(t) \rangle = -e\vec{E} = -e\vec{\nabla}\Psi_p$;

$$\Psi_p = \frac{e}{4m\Omega_T^2}\left[E_0(x_0, y_0, z_0)\right]^2$$

$$\Psi_p = \frac{e}{4m\Omega_T^2}\left[\frac{V_0^2}{R^4} \langle x^2, y^2 \rangle\right]$$

$$\langle \vec{F}(t) \rangle = -e \, \vec{\nabla} \Psi_p$$

$$= \frac{-e^2}{4 m \Omega_T^2} \frac{V_0^2}{R^4} \langle 2x, 2y \rangle$$

$$\langle F \rangle = \frac{-e^2 V_0^2}{2 m \Omega_T^2 R^4} \langle x, y \rangle$$

$$\begin{cases} m\ddot{x} = -\dfrac{e^2 V_0^2}{2 m \Omega_T^2 R^4} \, x \\[2ex] m\ddot{y} = -\dfrac{e^2 V_0^2}{2 m \Omega_T^2 R^4} \, y \end{cases}$$

Same functional form

$$\ddot{x} = \frac{-e^2 V_0^2}{2 m^2 \Omega_T^2 R^4} \, x$$

let $\omega_x^2 = \dfrac{e^2 V_0^2}{2 m^2 \Omega_T^2 R^4}$

$$\ddot{x}(t) = -\omega_x^2 x(t)$$

$\Rightarrow \quad x(t) = A\cos(\omega_x t);$

$$\boxed{\omega_x = \omega_y = \frac{eV_0}{\sqrt{2}\, m \Omega_T R^2}}$$

Secular Motion.

$\therefore$ Effective harmonic

Motion in the x, y

plane due to Φ

$\Omega_T = 23.8\ MHz\ (2\pi)$

$V_0 \cong 200\ V - 350\ V$

→ We also need confinement along the z direction. However, this confinement is much weaker than in x, y.

- Static in z.

$$V = \alpha x^2 + \beta y^2 + \gamma z^2$$

$$\nabla^2 V = 0 \quad \Rightarrow \quad \alpha + \beta + \gamma = 0$$

$$\alpha = \beta = -\frac{1}{2} \quad \& \quad \gamma = 1$$

$$\Rightarrow \quad V = z^2 - \frac{1}{2}(x^2 + y^2)$$

$$\Phi_s = \frac{U_0}{z_0^2}\left[\underbrace{z^2}_{\text{Conf.}} - \underbrace{\frac{1}{2}(x^2+y^2)}_{\text{Anti-conf.}}\right]$$

$$\Rightarrow \vec{E}_s = -\vec{\nabla}\phi$$

$$= -\frac{U_0}{z_0^2}\left[2z - (x+y)\right]$$

$$F_z = m\ddot{z} = e E_{s,z}$$

$$m\ddot{z} = -\frac{2U_0}{z_0^2} z$$

$$\ddot{z} = -\frac{2U_0 e}{m z_0^2} z$$

let $\omega_z^2 = \frac{2U_0 e}{m z_0^2}$

$\Rightarrow \ddot{z} = -\omega_z^2 z$

$\Rightarrow$ Harmonic motion.

$\rightarrow$ Relate $\frac{U_0}{z_0^2}$ to ω_z^2

$$\omega_z^2 = \frac{2U_0 e}{m z_0^2} \Rightarrow \frac{U_0}{z_0^2} = \frac{m}{2e}\omega_z^2$$

$$\therefore \quad \boxed{\phi_s = \frac{m}{2e}\,\omega_z^2\left(z^2 - \frac{1}{2}(x^2+y^2)\right)}$$

$$\omega_z^2 = \frac{2eU_0}{m z_0^2} \rightarrow \text{secular motion in } z$$

→ Need equations of Motion in x & y

$$\phi_{total} = \phi_{rf} + \phi_s$$

$$= \left(\frac{V_0 \cos(\Omega_T t) + U_R}{2}\right)\left(1 + \frac{x^2-y^2}{R^2}\right) + \frac{m}{2e}\,\omega_z^2\left(z^2 - \frac{1}{2}(x^2+y^2)\right)$$

$$\vec{F} = m\ddot{\vec{r}} = e\vec{E} = -e\vec{\nabla}\phi_{total}$$

$$\begin{cases} F_x = m\ddot{x} = \frac{1}{2} m\omega_z^2 x - \frac{eV_0}{R^2} x \cos(\Omega t) \\ F_y = m\ddot{y} = \frac{1}{2} m\omega_x^2 y + \frac{eV_0}{R^2} y \cos(\Omega t) \\ F_z = m\ddot{z} = -m\omega_z^2 z \end{cases}$$

or,

$$\ddot{x} = \frac{ex}{m}\left[\frac{U_0}{z_0^2} - \frac{V_0}{R^2}\cos(\Omega t)\right]$$

$$\ddot{y} = \frac{ey}{m}\left[\frac{U_0}{z_0^2} - \frac{V_0}{R^2}\cos(\Omega t)\right]$$

$$\ddot{z} = -\frac{2eU_0}{m z_0^2} z$$

→ I'm not going to solve by hand, as smarter people than I already have.

$$\frac{d^2 x}{d\xi^2} + \left[a_x + 2q_x \cos(2\xi)\right] = 0$$

$$\frac{d^2 y}{d\xi^2} + \left[a_y + 2q_y \cos(2\xi)\right] = 0$$

$\rightarrow \quad \xi = \frac{\Omega t}{2}$

offset

$$a_x = \left(\frac{4e}{m\Omega_T^2}\right)\left[\frac{U_R}{R^2} - \frac{U_0}{z_0^2}\right] \qquad q_x = \frac{2eV_0}{\Omega_T^2 m R^2}$$

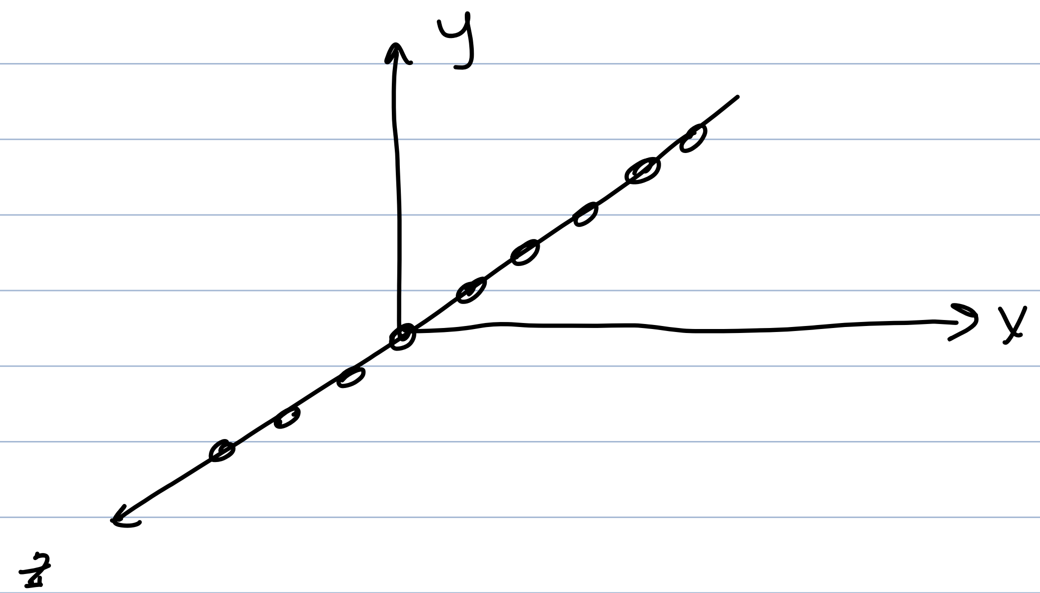
$$a_y = -\left(\frac{4e}{m\Omega_T^2}\right)\left[\frac{U_R}{R^2} + \frac{U_0}{z_0^2}\right] \; ; \quad q_y = -q_x$$

$$u_i(t) = u_{0i}\Big(\cos(\omega_i t + \varphi_i)\Big[1 + \frac{q_i}{2}\cos(\Omega_T t) + \frac{q_i^2}{32}\cos(\Omega_T t)\Big] + \beta_i \frac{q_i}{2}\sin(\omega_i t + \varphi_i)\sin(\Omega_T t)\Big)$$

$$\omega_i = \beta_i \frac{\Omega_T}{2} \quad \$ \quad \beta_i \simeq \sqrt{a_i + q_i^2/2}$$

Note: think about things in terms of confinement & secular freq.

confinement ↑ => ↑ secular freq.



$$\boxed{\omega_x = \omega_y = \frac{eV_0}{\sqrt{2}\, m \Omega_T R^2} \; ; \quad \omega_z = \sqrt{\frac{2 U_0 e}{m z_0^2}}}$$

Secular freq.'s.

→ we've already shown that

$$\ddot{z} = -w_z^2 z \quad , \quad \sqrt{\frac{2U_0 e}{m z_0^2}} = w_z ,$$

which is a H.O. in nature.

→ But, we can't say the same about transverse direction unless

$$a_i \ll \underbrace{q_i^2}_{\text{N-motion amp.}} \ll 1$$

$$\Rightarrow \quad u_i(t) = A_i \cos(w_i t + \phi_i)$$

→ which then lets us approximate $\phi_{TRANSVERSE}$ as H.O.

$$\Rightarrow e\phi_p = \frac{1}{2} m \omega_r^2 (x^2 + y^2) \; ;$$

$$\omega_r \simeq \frac{e V_0}{\sqrt{2}\, \Omega_T m R^2} = \frac{q_x \Omega_T}{2\sqrt{2}}$$

Summary

→ Φ_{RF} confines ion in transverse plane (e.g. x,y). This is achieved by creating a geometry of electrodes that is quad. pole in nature & apply an RF source to the electrodes. This creates a "local minimum" for the ion to sit in.

Another way of viewing this is alternating between confining & deconfining in the x & y direction respectively. The time average results in a harmonic pseudopotential.

→ ϕ_s provides confinement in the z (axial) direction & deconfines in the x & y (transverse) direction.

→ To give good secular frequencies, one must play a game w/ the amount of RF Voltage (V_0) & how much DC voltage (U_0) is given to the electrodes.

↳ Higher confinement ⇒ higher secular freq's.