CQF Exam 1 - Module 2 Exercise Solution

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I. QUESTION 1

Question 1. **(24 marks)** An investment universe of the following risky assets with a dependence structure (correlation) is given. Use the ready appropriate formulae from Portfolio Optimisation Lecture.

Asset	μ	σ	w
A	0.02	0.05	w_1
B	0.07	0.12	w_2
C	0.15	0.17	w_3
D	0.20	0.25	w_4

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.6 & 0.6 \\ 0.3 & 0.6 & 1 & 0.6 \\ 0.3 & 0.6 & 0.6 & 1 \end{pmatrix}$$

Consider minimum variance portfolio with a target return m.

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \qquad \text{s.t. } \boldsymbol{w}'\mathbf{1} = 1, \qquad \mu_\Pi = \boldsymbol{w}'\boldsymbol{\mu} = m$$

Q1-1

- Formulate the Lagrangian function and give its partial derivatives. No further derivation required.

Solution

The objective function is:

$$f(\omega) = \frac{1}{2}\omega'\Sigma\omega \rightarrow \underset{\omega}{\operatorname{argmin}}\, f$$

Thus, applying Lagrange including the constraint functions:

$$L(\omega, \lambda_1, \lambda_2) = \frac{1}{2}\omega'\Sigma\omega + \lambda_1(1 - \omega\bar{1}) + \lambda_2(m - \omega'\mu)$$

Thus, finding its partial derivatives (i.e., first order conditions):

i. $\frac{\partial L}{\partial \omega} = \Sigma\omega - \lambda_1\bar{1} - \lambda_2\mu = 0,$, concluding: $\omega = \Sigma^{-1}(\lambda_1\bar{1} + \lambda_2\mu)$

ii. $\frac{\partial L}{\partial \lambda_1} = 1 - \omega\bar{1} = 0$, concluding: $\bar{1}'\omega = 1$

iii. $\frac{\partial L}{\partial \lambda_2} = m - \omega'\mu = 0$, concluding: $\mu'\omega = m$

Q1-2

- Compute the allocations $\boldsymbol{w}^*$ and portfolio risk $\sigma_\Pi = \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$, for $m = 4.5\%$.
  Now, stress the correlation matrix: multiply all correlations by $\times 1.25$ and $\times 1.5$, and compute the respective optimal allocations and portfolio risk (the same $m = 4.5\%$).

Solution

First, we should find $\lambda_1 \wedge \lambda_2$ from **Q1-1** by replacing ω from (i) in the conclusions of (ii) and (ii). So:

- In (ii): $\bar{1}'(\Sigma^{-1}(\lambda_1\bar{1} + \lambda_2\mu)) = \lambda_1(\bar{1}'\Sigma^{-1}\bar{1}) + \lambda_2(\bar{1}'\Sigma^{-1}\mu) = 1$
- In (iii): $\mu'(\Sigma^{-1}(\lambda_1\bar{1} + \lambda_2\mu)) = \lambda_1(\mu'\Sigma^{-1}\bar{1}) + \lambda_2(\mu'\Sigma^{-1}\mu) = m$

Reducing the notation with the following replacements:

$$\begin{cases} A = \overline{1}' \, \Sigma^{-1} \overline{1} \\ B = \overline{1}' \, \Sigma^{-1} \mu = \mu' \, \Sigma^{-1} \overline{1} \\ C = \mu' \, \Sigma^{-1} \mu \end{cases}$$

We will get:

- $\lambda_1 A + \lambda_2 B = 1$
- $\lambda_1 B + \lambda_2 C = m$

Solving the system:

$$\lambda_1 = \frac{Bm - C}{B^2 - \mathrm{AC}} \qquad \text{and} \qquad \lambda_2 = \frac{B - Am}{B^2 - AC} \qquad \text{being: } B^2 - AC \neq 0$$

Finally, replacing $\lambda_1 \wedge \lambda_2$ back in $\omega = \Sigma^{-1}(\lambda_1 \overline{1} + \lambda_2 \mu)$, we will get the optimal weights allocation as a function of a given m, such as:

$$\omega^*{}_m = \frac{1}{B^2 - AC}\left[(Bm - C)\left(\Sigma^{-1}\overline{1}\right) + (B - Am)(\Sigma^{-1}\mu)\right]$$

With this result, we are able to test the requirements of **Q1-2.** Thus:

a) *Compute* ω^* *for* $m = 4.5\%$ *and its portfolio risk* $\sigma_\pi = \sqrt{\omega' \Sigma \, \omega}$*:*

<u>SOL</u>: Coding computations done in **`YGNACIO_CODE_E1`** file shows that:

```
- ω* :
[[0.78511066]
[0.05386419]
[0.13355472]
[0.02747042]]

- σπ :
[[0.05840091]]
```

b) *Now, stress the correlation matrix: multiply all correlations by x1.25 and x1.5, and compute the respective optimal allocations and portfolio risk (the same m = 4.5%).*

<u>SOL</u>: Coding computations done in **`YGNACIO_CODE_E1`** file shows that:

```
Results with Stressed Corr. Matrix by x1.25:
 - ω* :
[[ 0.81818944]
 [-0.00940302]
 [ 0.17896585]
 [ 0.01224773]]

 - σπ :
[[0.0607102]]

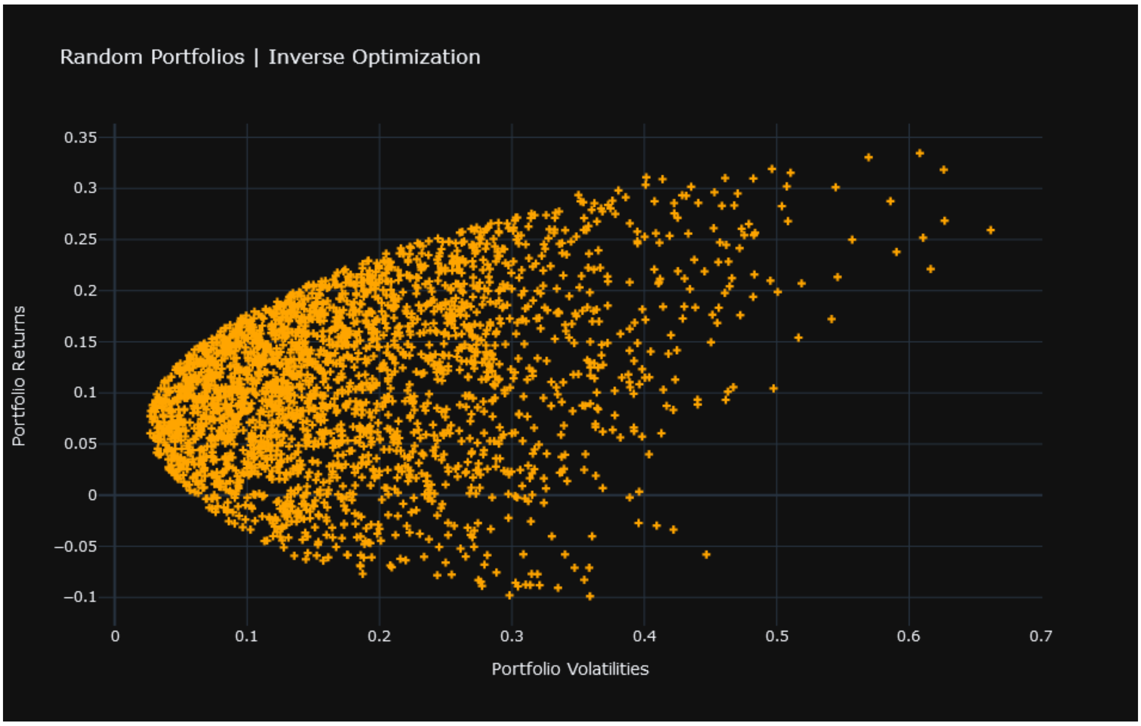
*********************************************
Results with Stressed Corr. Matrix by x1.50:
 - ω* :
[[ 0.87617647]
 [-0.14612952]
 [ 0.32570145]
 [-0.0557484 ]]

 - σπ :
[[0.06109091]]
```

Notice that there are negative values in ω*. This means we need to short some assets (asset {B} and {B, D} for x1.25 and x1.50 respectively) to achieve the min variance portfolio.

c) *Inverse optimization: generate > 2,000 random allocations sets ω'—these will not be optimal allocations. Plot the cloud of points of μ_π vertically on σ_π horizontally. Before computing of μ_π, σ_π, you can standardize to satisfy $\omega'\bar{1} = 1$.*

<u>SOL</u>: Coding computations done in **`YGNACIO_CODE_E1`** gives the following plot:



II. QUESTION 2

Question 2. (20 marks) Continue with the data from Question 1 and consider a tangency portfolio.

Q2-1

- Formulate the Lagrangian function and give its partial derivatives only.

Solution

The tangency portfolio is the portfolio fully invested in risky assets that maximizes the returns per unit of risk, i.e., that one that maximizes the Sharpe Ratio:

$$\underset{w_i}{\text{Max}}\, S_\pi = \frac{E[r_\pi]-r_f}{\sigma_\pi} \qquad s.t. \qquad \textstyle\sum_{i=1}^{n} w_i = 1$$

Where π is the portfolio of n asset with individual weights of w_i, expected return $E[r_\pi]$ and standard deviation σ_π. Notice also that the risk-free asset r_f works as a constant.

Thus, using matrices and vectors for convenience:

$$\underset{\omega}{\text{Max}}(S_\pi) = \frac{\omega'\mu - r_f}{\sqrt{\omega'\Sigma\,\omega}} \qquad s.t. \qquad \bar{1}'\omega = 1$$

In that sense, applying Lagrange, including the constraint function:

$$L(\omega, \lambda) = \frac{(\omega'\mu - r_f)}{\sqrt{\omega'\Sigma\,\omega}} + \lambda(\bar{1}'\omega - 1)$$

Finding its partial derivatives:

- $\frac{\partial L}{\partial \omega} = (\omega' \Sigma\, \omega)^{-\frac{1}{2}}\mu - \left(\frac{(\omega' \Sigma\, \omega)^{-3\frac{1}{2}}(\omega' \mu - r_f)}{2} (\Sigma\, \omega) + \frac{(\omega' \Sigma\, \omega)^{-3\frac{1}{2}}(\mu' \omega - r_f)}{2} (\Sigma\, \omega) \right) + \lambda\left(\overline{1}'\right)$

$$= (\omega' \Sigma\, \omega)^{-\frac{1}{2}}\mu - \left((\omega' \Sigma\, \omega)^{-3\frac{1}{2}} \frac{\left((\omega' \mu - r_f) + (\mu' \omega - r_f)\right)}{2} (\Sigma\, \omega) \right) + \lambda\left(\overline{1}'\right)$$

$$= (\omega' \Sigma\, \omega)^{-\frac{1}{2}}\mu - (\omega' \Sigma\, \omega)^{-3\frac{1}{2}}(\omega' \mu - r_f)(\Sigma\, \omega) + \lambda\left(\overline{1}'\right)$$

- $\frac{\partial L}{\partial \lambda} = \overline{1}'\omega - 1$

Q2-2

- For the range of tangency portfolios given by $r_f = 50bps, 100bps, 150bps, 175bps$ optimal compute allocations (ready formula) and σ_Π. Present results in a table.

Solution

The "*ready formula*" for the tangency portfolio comes from the convergence of two types of portfolios simultaneously:

- The *risky-portfolio optimized by risk-free* assets in the new efficient fronter (i.e., the Capital Market Line), and
- The *old-best portfolio of just risky-assets* in the old efficient frontier (i.e., the hyperbola).

In that sense, considering the values from **Q1-2** for A and B, the allocation vector ω for the Tangency Portfolio should satisfy:

$$\omega = \frac{\Sigma^{-1}(\mu - r\overline{1})}{B - Ar}$$

Thus, from the results in the **`YGNACIO_CODE_E1`** file and considering the portfolio information of **Q1,** the allocations for the tangency portfolios for the given r_f values are:

	rf:50bps	rf:100bps	rf:150bps	rf:175bps
asset_A	0.016835	-0.745937	-8.644854	8.103502
asset_B	-0.229367	-0.510569	-3.422571	2.751851
asset_C	0.814340	1.490249	8.489651	-6.351431
asset_D	0.398192	0.766257	4.577774	-3.503922
muPi	0.186070	0.326130	1.776525	-1.298799
sigmaPi	0.196511	0.350665	1.972392	1.473515

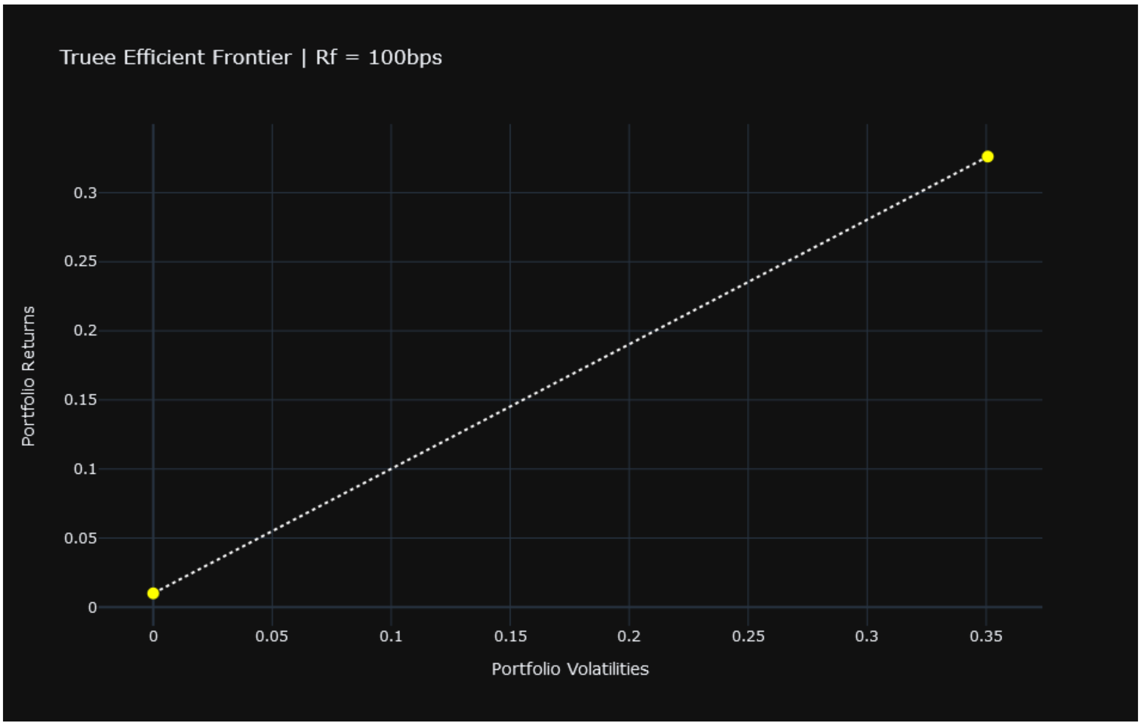
Q2-3

- Plot the true Efficient Frontier in the presence of risk-free earning asset for $r_f = 100bps, 175bps$ and specifically identify its shape.

Solution:

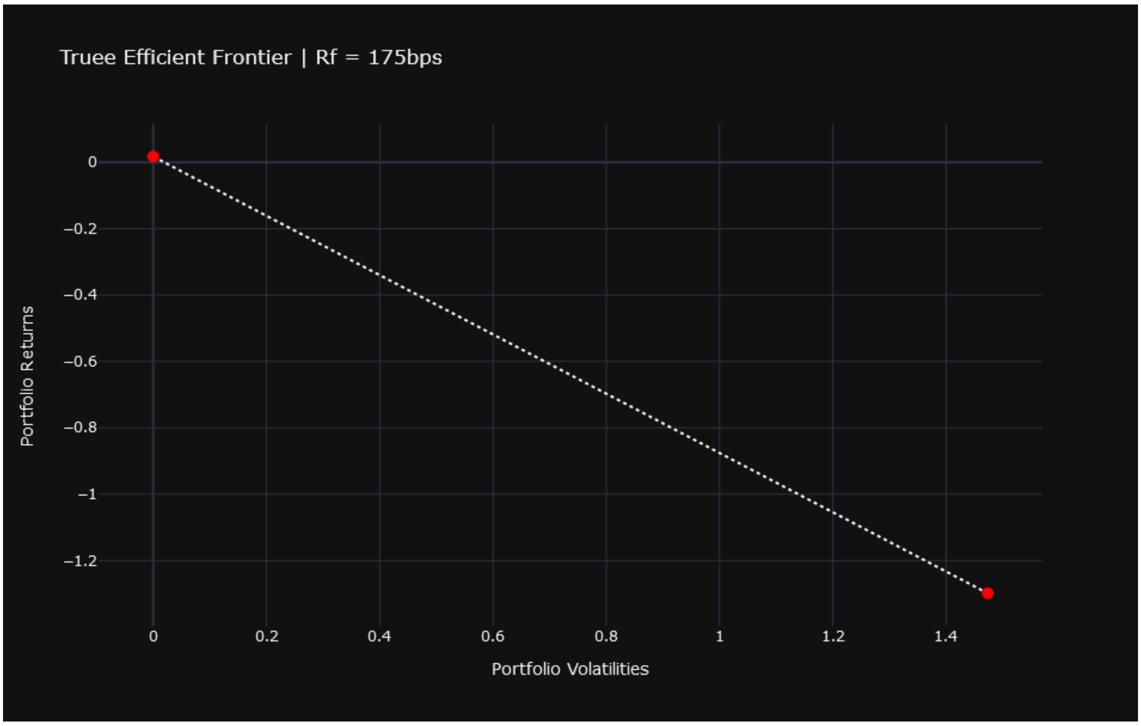
From the previous table and based on the plots computed in the **`YGNACIO_CODE_E1`** file, the True Efficient Frontiers are:

- For **rf = 100bps**:



Conclusion Q2-3 #1: The *True Efficient Frontier* in the presence of risk-free earning asset of **100bps** has a POSITIVE SLOPE.

- For **rf = 175bps**:



Conclusion Q2-3 #2: *The True Efficient frontier* in the presence of risk-free earning asset of **175bps** has a NEGATIVE SLOPE. This is due to the fact that, in the presence of a RFA earning 175bps, are more short-positions in the generated portfolio and, therefore, the general returns at the Efficient Frontier becomes negative.

III. QUESTION 3

Question 3. (42 marks) As a market risk analyst, each day you calculate VaR from the available prior data. Then, you wait ten days to compare your prediction value VaR_{t-10} to the realised return and check if the prediction about the worst loss was breached. You are given a dataset with *Closing Prices*.

Q3-1

- Implement VaR backtesting by computing 99%/10day Value at Risk using the rolling window of 21 returns to compute σ. (a) Report the percentage of VaR breaches and (b) number of consecutive breaches. (c) Provide a plot which clearly identifies breaches.

$$\text{VaR}_{10D,t} = \text{Factor} \times \sigma_t \times \sqrt{10}$$

Solution

We will assume a Normal Factor (i.e., Standard Normal Percentile) for a given confidence level C, where $0 < C < 1$. Hence, we will use the inverse Norm. Function $\Phi^{-1}(1-C)$ to estimate its value.

In that sense, for a $C = 99\%$:

$$Factor = \Phi^{-1}(1-0.99) = -2.3263478740408408$$

Thus, from the previous *Factor* value and based on the computations made in the **`YGNACIO_CODE_E1`** file, we can conclude:

(a) Percentage of VaR Breaches:

```
Breaches Report (a)
******************************
- Total breaches: 25
- Percentage of breaches: 2.053%

Important: percentage of breaches is computed only over available
datapoints. 'NaN' value-events are not considered.
```

(b) Number of consecutive Breaches:

```
Breaches Report (b)
******************************
- Number of Consecutive Breaches:  14
______________________________

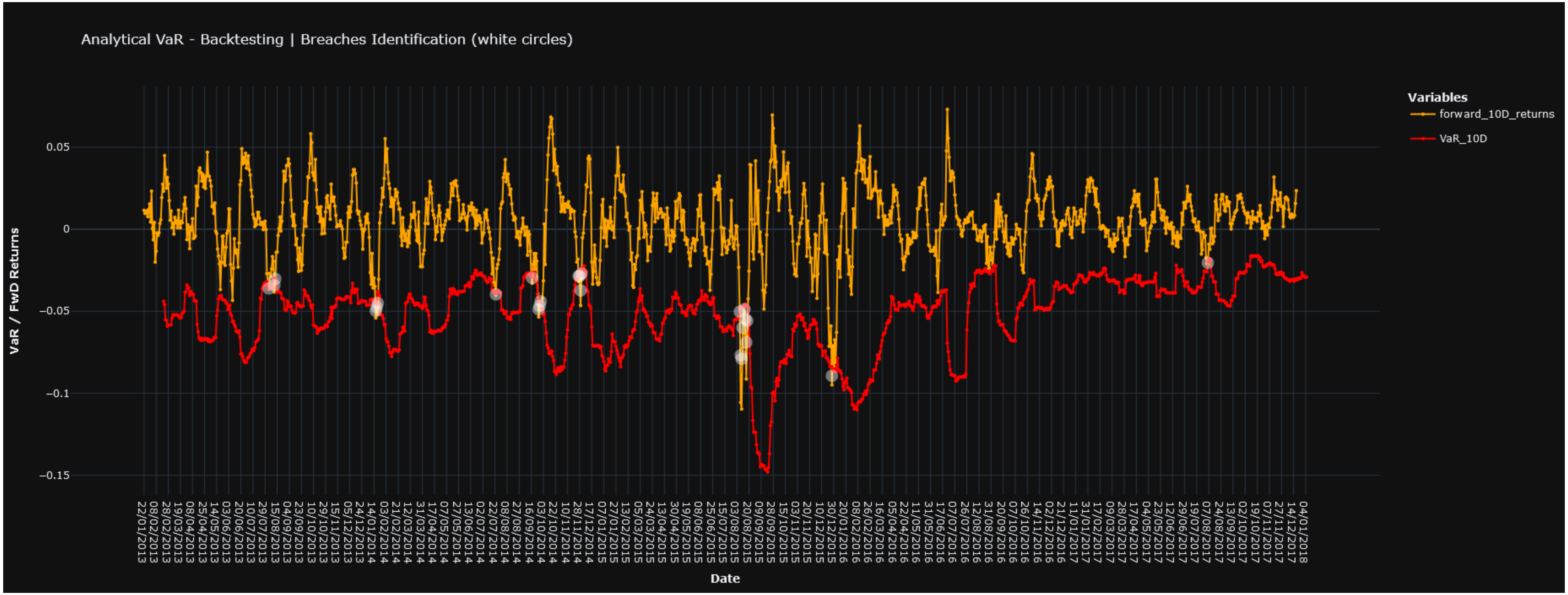
EXTRA detail about breaches:

- It's also possible to identify the number of breaches of N consecutive
breaches. So:

                           occurrences      total_consecutives
N Consecutive Breaches
1                                1                   1
2                                2                   4
3                                3                   9

Important: we are considering consecutive breaches based on the >Further
Instructions criteria<. This document says: '(...)For example, the
sequence 1, 1, 1 means two consecutive breaches occurred' (p. 3).
```

(c) Plot identifying the breaches:



Analytical VaR - Backtesting | Breaches Identification (white circles)
Variables
forward_10D_returns
VaR_10D
VaR / FwD Returns
Date
0.05
0
−0.05
−0.1
−0.15

Q3-2

- For comparison, implement backtesting using variance forecast equation below (recompute on each day). Rolling window of 21 remains for σ_t^2 (past variance) computation. The equation is known as EWMA model, and you can check how variance forecast is done in the relevant lecture.

$$\sigma^2_{t+1\,|\,t} = \lambda\,\sigma^2_{t\,|\,t-1} + (1-\lambda)\,r_t^2$$

with $\lambda = 0.72$ value set to minimise out of sample forecasting error, and r_t refers to a return. Provide the same deliverables (a), (b), and (c).

Solution

Using the same Normal Factor as **Q3-1**:

$$Factor = \Phi^{-1}(1 - 0.99) = -2.3263478740408408$$

And based on the computations made in the **`YGNACIO_CODE_E1`** file, we can conclude:

(a) Percentage of EWMA-σ VaR Breaches:

```
Breaches Report | EWMA Std for VaR | (a)
******************************
- Total breaches: 30
- Percentage of breaches: 2.465%

Important: percentage of breaches is computed only over available
datapoints. 'NaN' values-event are not considered.
```

(d) Number of consecutive EWMA-σ VaR Breaches:

```
EWMA-VaR Breaches Report (b)
******************************
- Number of Consecutive EWMA-VaR Breaches:  15
______________________________

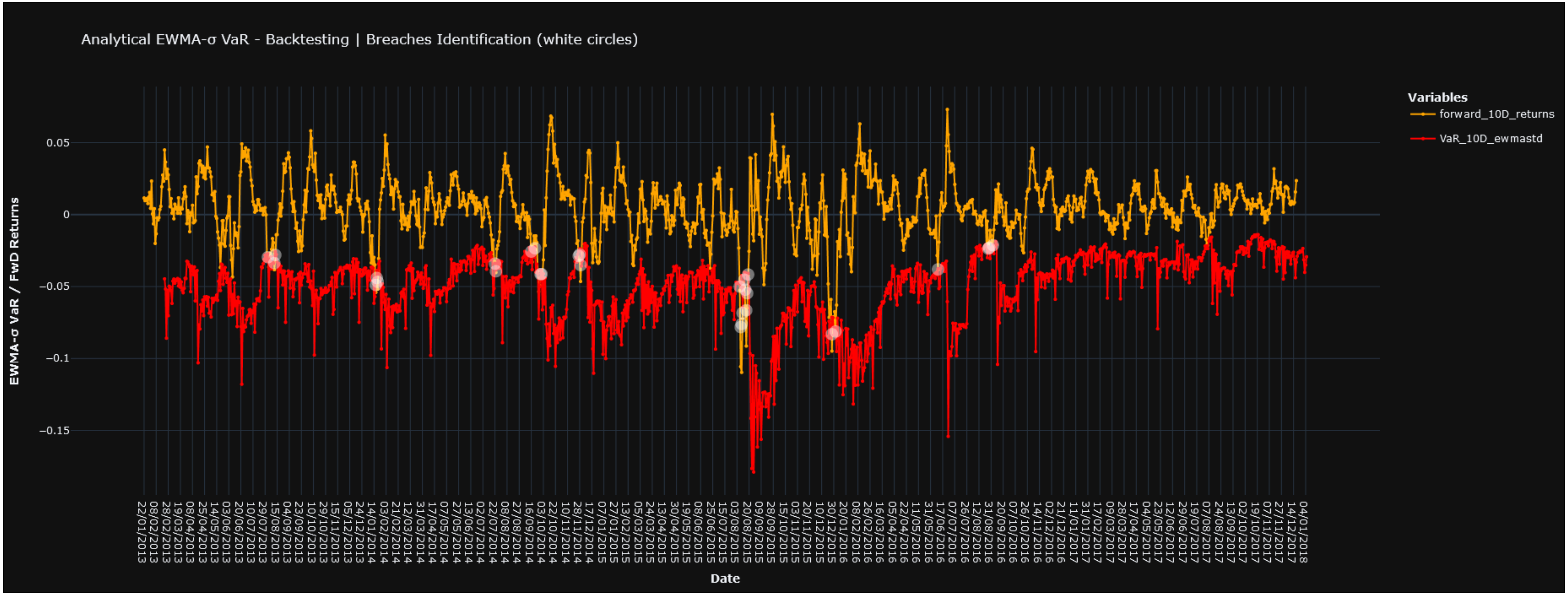
EXTRA detail about EWMA-VaR breaches:

- It's also possible to identify the number of EWMA-VaR breaches of N
consecutive breaches. So:

                                     occurrences     total_consecutives
N Consecutive EWMA-VAR Breaches
1                                         4                   4
2                                         2                   4
3                                         1                   3
4                                         1                   4

Important: we are considering consecutive breaches based on the >Further
Instructions criteria<. This document says: '(...)For example, the
sequence 1, 1, 1 means two consecutive breaches occurred' (p. 3).
```

(e) Plot identifying the EWMA-σ breaches:

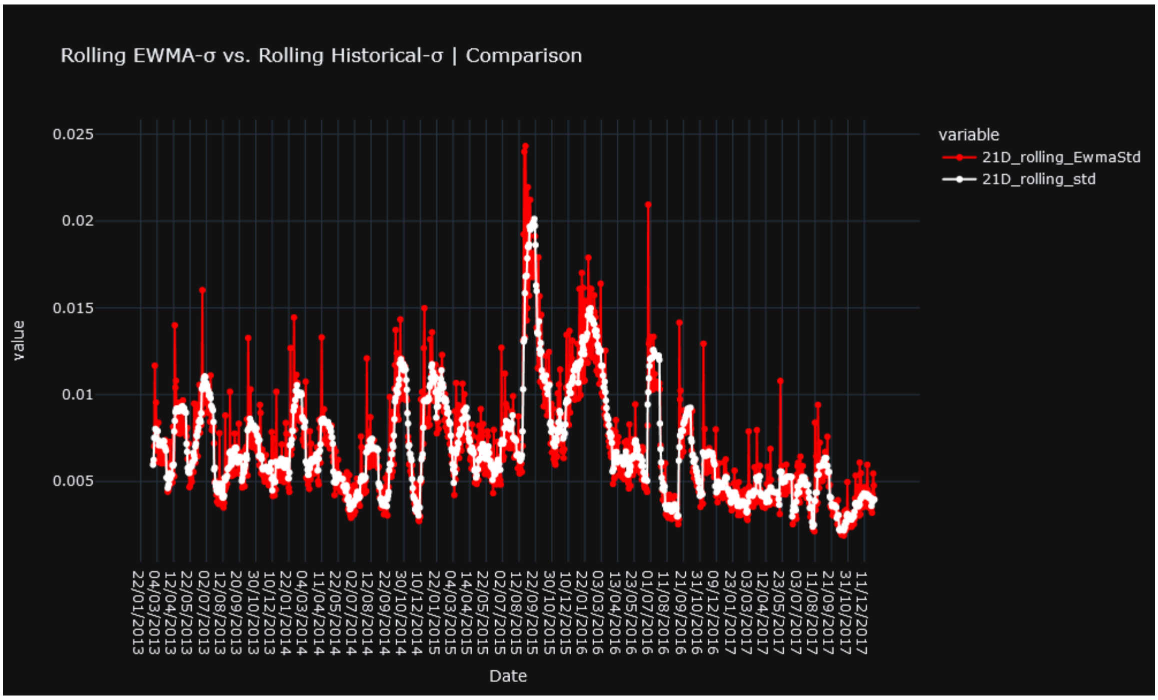

Analytical EWMA-σ VaR - Backtesting | Breaches Identification (white circles)
EWMA-σ VaR / FwD Returns
Date
Variables
forward_10D_returns
VaR_10D_ewmastd
0.05
0
−0.05
−0.1
−0.15

Conclusions – Question 3:

There are noticeable differences between the Analytical VaR99%/10D using simple rolling 21D standard deviation of log-returns and, on the other side, the same VaR99%/10D but applying an EWMA-σ forecast over this simple rolling 21D standard deviation. These key differences are:

1) EWMA-σ increases the variability of the typical standard deviation and, as a consequence, the variability of VaR.
2) Even though this increasing variability, most of the consecutive-breaches periods are still preserved in both comparisons, i.e., in the simple historical rolling-σ vs. Fwd-Returns and, also, in the rolling EWMA-σ VaR vs. Fwd-Returns.

As an extra final visualization, we can see these differences comes from the shape of the EWMA-σ against the simple rolling historical-σ, such as:



IV. Question 4

Question 4. **(14 marks)** Liquidity-Adjusted VaR (LVaR) is effectively VaR itself plus VaR of the bid/ask spread. The latter is our liquidity adjustment. It has not been introduced in Market Risk lecture, however to compute LVaR simply use the formula:

$$\begin{aligned} \text{LVaR} &= \text{VaR} + \Delta_{Liquidity} \\ &= \text{Portfolio Value} \times \left[-\mu + \text{Factor} \times \sigma + \frac{1}{2}(\mu_{Spread} + \text{Factor} \times \sigma_{Spread}) \right] \end{aligned}$$

Use the positive value of the Standard Normal Factor for the correct percentile. For the following cases, report (a) the proportion attributed to VaR and (b) the proportion attributed to liquidity adjustment.

Q4-(A)

(a) Consider a portfolio of USD 16 million composed of shares in a technology company. Daily mean and volatility of its returns are 1% and 3%, respectively. Bid-ask spread also varies with time, its daily mean and volatility are 35 bps and 150 bps. Compute 99%/1D LVaR and attributions to it,

Solution Q4-(A)

```
Solution Q4-1(A):
*************************************************************
 - (a) Proportion attributed to VaR:  0.9566469795396034 | (75.69555%)
 - (b) Proportion attributed to Liquidity Adjustment:
0.3071617448849009 | (24.30445%)
 - General Liquidity VaR:  1.2638087244245044 | (100%)
```

Q4-(B)

(b) Now consider GBP 40 million invested in UK gilts. Take the daily volatility of portfolio returns as 3% and bid-ask spread is 15 bps (no spread volatility). Compute 99%/1D LVaR and attributions. What would happen if the bid-ask spread increases to 125 bps?

Solution Q4-(B)

```
Solution Q4-1(B) | Part I:
*************************************************************
 - (a) Proportion attributed to VaR:  2.7916174488490086 | (98.93678%)
 - (b) Proportion attributed to Liquidity Adjustment:  0.03 | (1.06322%)
 - General Liquidity VaR:  2.8216174488490084 | (100%)

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Solution Q4-1(B) | Part II:
 >> Modification: increasing mu bid-ask spread to 125bps.
*************************************************************
 - (a) Proportion attributed to VaR:  2.7916174488490086 | (91.781%)
 - (b) Proportion attributed to Liquidity Adjustment:  0.25 | (8.219%)
 - General Liquidity VaR:  3.0416174488490086 | (100%)

>> Inquiry:
'What would happen if the bid-ask spread increases to 125 bps?'
>> Answer:
An increasing in the bid-ask spread will generate a proportional increase
in the Liquidity Adjustment.
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END OF EXAM – CQF Program, Jan 2022