# MODULE 1 - Assignment

1)

a)

$$\int_0^t \frac{\delta F}{\delta W} \cancel{dW_r} = \int_0^t r^2 \cdot \sin(W)\, \cancel{dw_r}$$

- $\frac{\delta F}{\delta W} = t^2 \cdot \sin(W)$

$$\int \cancel{\delta} F = \int t^2 \cdot \sin(w)\, \delta w$$

- $F = -t^2 \cos(W)$

- $\frac{\delta^2 F}{\delta W^2} = r^2 \cdot \cos(w)$
- $\frac{\delta F}{\delta t} = 2t \sin(w)$

Itô Integral:

$$\int_0^t \frac{\delta F}{\delta W} dw_r = -t^2 \cos(w_t) - (-\cancel{(0)^2} \cdot w_0)^{2} - \int_0^t \left(2r \cdot \sin(w) + \frac{1}{2} \cdot r^2 \cos(w)\right) dr$$

$$\therefore \int_0^t r^2 \sin(w_r)\, dw_r = -t^2 \cos(w_t) - \int_0^t \left(2r \sin(w) + \frac{1}{2} r^2 \cdot \cos(w)\right) dr$$

b)

i. Itô IV:

- $f(s,t) = \alpha^t + \beta t s^n$

$\frac{\delta f}{\delta s} = \beta t n \cdot s^{n-1}$ $\quad$ $\frac{\delta^2 f}{\delta s^2} = \beta t n(n-1) s^{n-2}$ $\quad$ $\frac{\delta f}{\delta t} = t \cdot \alpha^{t-1} + \beta s^n$

$$\Rightarrow df = \left[\left(t \cdot \alpha^{t-1} + \beta \cdot s^n\right) + \upsilon \cancel{s}\left(\beta t n \cdot \cancel{s^{n-1}}^{\,n}\right) + \frac{1}{2} \cdot 6^2 \cancel{s^2}\left(\beta \cdot t \cdot n(n-1) \cancel{s^{n-2}}^{\,n}\right)\right] dt + 6\cancel{s}\left[\beta t n \cancel{s^{n-1}}^{\,n}\right] ds$$

$$\therefore df = \left(t \cdot \alpha^{t-1} + \beta \cdot s^n + \upsilon \cdot \beta \cdot t \cdot n \cdot s^n + \frac{1}{2}\left(6^2 \cdot \beta \cdot t \cdot n \cdot (n-1) \cdot s^n\right) dt + \left(6 \cdot \beta \cdot t \cdot n \cdot s^n\right) ds\right.$$

ii. $\underline{\text{Itô IV}}$:

- $f = \log(ts) + \cos(ts)$ ... ⚠ I'm assuming the whole term "t.s" belongs to the log and cosine functions!

- $\frac{\delta f}{\delta s} = \frac{1}{s} - \sin(ts)t$     - $\frac{\delta^2 f}{\delta s^2} = -\frac{1}{s^2} - \cos(ts)t^2$

- $\frac{\delta f}{\delta t} = \frac{1}{t} - \sin(ts)s$

$$\Rightarrow df = \left[\left(\frac{1}{t} - \sin(ts)s\right) + \nu.s\left(\frac{1}{s} - \sin(ts)t\right) + \frac{1}{2}\sigma^2.s^2\left(-\frac{1}{s^2} - \cos(ts)t^2\right)\right]dt + \sigma.s\left[\frac{1}{s} - \sin(ts)t\right]ds$$

$$\therefore df = \left(\frac{1}{t} - \sin(ts)s + \nu - \sin(ts)t.\nu.s - \sigma^2/2 - \frac{\cos(ts)}{2}(t\sigma.s)^2\right)dt + \left(\sigma - \sin(ts)t.\sigma.s\right)ds$$

2) - Deducing a general form for "$\underline{\text{Itô VI}}$":

* Let: $dS_1 = a_1(t,S_1)dt + b_1(t,S_1)dW_t^{(1)} \wedge dS_2 = a_2(t,S_2)dt + b_2(t,S_2)dW_t^{(2)}$

- From $\underline{\text{TSE}}$ of $V(t,S_1,S_2)$ for $V(t+\Delta t, S_1+\Delta S_1, S_2+\Delta S_2) = V'$:

$$V' = V + \frac{\delta V}{\delta t}\Delta t + \frac{\delta V}{\delta S_1}\Delta S_1 + \frac{\delta V}{\delta S_2}\Delta S_2 + \frac{1}{2}\left(\frac{\delta^2 V}{\delta^2 t}\Delta t^2 + \frac{\delta^2 V}{\delta^2 S_1}\Delta S_1^2 + \frac{\delta V^2}{\delta S_2^2}\Delta S_2^2 + 2.\frac{\delta V^2}{\delta S_1 \delta S_2}\Delta S_1 \Delta S_2 + \ldots\right.$$

- As long as $V' - V = dV$ and $\Delta t = dt$, $\Delta S_1 = dS_1 \wedge \Delta S_2 = dS_2$, and $\frac{\delta^2 V}{\delta^2 t}\Delta t^2 \to 0$:

$$\Rightarrow dV = \frac{\delta V}{\delta t}dt + \frac{\delta V}{\delta S_1}dS_1 + \frac{\delta V}{\delta S_2}dS_2 + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S_1}dS_1^2 + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S_2}dS_2^2 + \frac{\delta^2 V}{\delta S_1 \delta S_2}dS_1 dS_2$$

- Replacing eq., simplifying and applying $dS_{(n)}^2 = b_{(n)}^2 dt \wedge dS_1 dS_2 = \rho b_1.b_2 dt$.

$$\circledast\ dV = \left(\frac{\delta V}{\delta t} + a_1\frac{\delta V}{\delta S_1} + a_2\frac{\delta V}{\delta S_2} + \frac{1}{2}b_1^2\frac{\delta^2 V}{\delta^2 S_1} + \frac{1}{2}b_2^2\frac{\delta^2 V}{\delta^2 S_2} + \rho b_1.b_2\frac{\delta^2 V}{\delta S_1 \delta S_2}\right)dt + b_1\frac{\delta V}{\delta S_1}dW_t^{(1)} + b_2\frac{\delta V}{\delta S_2}dW_t^{(2)}$$

$\Rightarrow$ Approximate general form for Itô $\underline{\text{VI}}$!

• Using ⊛ for $dS_t$ & $dr_t$ in $V(t,s,r)$:

$$dV = \left(\frac{\delta V}{\delta t} + (\nu s_1)\frac{\delta V}{\delta s} + (\gamma(m-r))\frac{\delta V}{\delta r} + \frac{1}{2}(\sigma s)^2\frac{\delta^2 V}{\delta^2 s} + \frac{1}{2}(cd)^2\frac{\delta^2 V}{\delta r^2} + \rho(\sigma s)\frac{\delta^2 V}{\delta s \delta r}\right)dt$$
$$(\sigma s\, \delta V/\delta s)\, dW_t^{(1)}$$
$$-(cd\, \delta V/\delta r)\, dW_t^{(2)}$$

• Applying Integration over $(T,0)$:

$$\int_0^T dV = \int_0^T \left(\frac{\delta V}{\delta t} + (\nu s)\frac{\delta V}{\delta s} + (\gamma(m-r))\frac{\delta V}{\delta r} + \frac{1}{2}(\sigma s)^2\frac{\delta^2 V}{\delta^2 s} + \frac{1}{2}(cd)^2\frac{\delta^2 V}{\delta r^2} + \rho(\sigma s)(cd)\frac{\delta^2 V}{\delta s \delta r}\right)dt$$
$$+ \int_0^t (\sigma s\, \delta V/\delta s)\, dW_t^{(1)}$$
$$+ \int_0^t (cd\, \delta V/\delta r)\, dW_t^{(2)}$$

$$V\Big|_0^T = \int_0^T (\ldots)dt + \int_0^T (\sigma s\, \delta V/\delta s)\, dW_t^{(1)} + \int_0^T \left(cd\, \frac{\delta V}{\delta r}\right) dW_t^{(2)}$$

• Assuming $V(0, S_0, r_0) = v$:

$$\therefore V_{(T,S_T,r_T)} = v + \int_0^T \left(\frac{\delta V}{\delta t} + (\nu s)\frac{\delta V}{\delta s} + (\gamma(m-r))\frac{\delta V}{\delta r} + \frac{1}{2}(\sigma s)^2\frac{\delta^2 V}{\delta^2 s} + \frac{1}{2}(cd)^2\frac{\delta^2 V}{\delta r^2} + \rho(\sigma s)(cd)\frac{\delta^2 V}{\delta s \delta r}\right)dt$$
$$+ \int_0^T (\sigma s\, \delta V/\delta s)\, dW_t^{(1)} + \int_0^T \left(cd\, \frac{\delta V}{\delta r}\right) dW_t^{(2)}$$

3) * Rearranging:

$$\Rightarrow dS_t = S_t(\nu\, dt + \sigma W_t)$$
$$\frac{1}{S_t} dS_t = \nu\, dt + \sigma W_t$$

* Integrating over $(t,0)$:

$$\Rightarrow \int_0^t \frac{1}{S_t} dS_t = \int_0^t \nu\, dt + \int_0^t \sigma W_t$$
$$\ln(s)\Big|_0^t = \nu(t-0) + \int_0^t \sigma W_t$$
$$\Rightarrow \ln\left(\frac{S_t}{S_0}\right) = \nu t + \sigma\int_0^t dW_t$$

$$\Rightarrow \ln\left(\frac{S_t}{S_0}\right) = \nu t + \sigma\int_0^t dW_t$$
$$e^{\ln(S_t/S_0)} = e^{\nu t} \cdot e^{\sigma\int_0^t dW_t}$$
$$S_t = S_0\, e^{\nu t} \cdot e^{\sigma\int_0^t dW_t}$$

* Finding $\mathbb{E}[S_t | S_0]$, where $S_0$ is known.

$$\mathbb{E}[S_t|S_0] = S_0\, \mathbb{E}[e^{\nu t} \cdot e^{\sigma\int_0^t dW_t}]$$
$$= S_0\, \mathbb{E}[e^{\nu t}] \cdot \mathbb{E}[e^{\sigma \cancel{\int_0^t dW_t}^{\,0}}] \quad \text{... by Martingala}$$
$$\mathbb{E}[S_t|S_0] = S_0\, \mathbb{E}[e^{\nu t}] \cdot (1)$$
$$\therefore \mathbb{E}[S_t|S_0] = S_0 e^{\nu t}$$

4) * Knowing the SSD distribution satisfies: $\frac{1}{2}\frac{d^2}{dy'^2}(B^2 P_\infty) = \frac{d}{dy'}(A P_\infty)$, where $y' = u'$

$$\Rightarrow \frac{1}{2}(\sigma^2)\frac{d^2 P_\infty}{du'^2} - \left(-\theta \frac{d}{du'}(u P_\infty)\right) = 0$$

* Replacing $u' = u \wedge P_\infty = P$ for simplicity:

$$\Rightarrow \frac{1}{2}(\sigma^2)\frac{d^2P}{du^2} + \theta \frac{d}{du}(uP) = 0$$

$$\frac{1}{2}(\sigma^2)\frac{d^2P}{du^2} = -\theta \frac{d}{du}(uP)$$

$$\frac{1}{2}(\sigma^2)\int \frac{d^2P}{du^2} = -\theta \int \frac{d}{du}(uP)$$

$$\frac{1}{2}(\sigma^2)\,dP/du = -\theta(uP) + \cancel{k}, \quad k = 0 \begin{cases} u \to 0 \\ P \to 0 \\ dP/du \to 0 \end{cases}$$

$$\Rightarrow \frac{1}{P}dP = -2\theta/\sigma^2 \cdot (u)\,du$$

$$\int \frac{1}{P}dP = -2\theta/\sigma^2 \int u\,du$$

$$\ln(P) = -\cancel{2}\theta/\sigma^2 \, \cancel{\tfrac{1}{2}} u^2$$

$$\ln(P) = -\theta/\sigma^2\, u^2$$

$$\cancel{e}^{\cancel{\ln}(P)} = e^{-\theta/\sigma^2 u^2}$$

$$\Rightarrow P = A \cdot e^{-\theta/\sigma^2 (u)^2}$$

* Knowing that $\int_{\mathbb{R}} P = 1$, thus:

$$\int_{\mathbb{R}} P = \int_{\mathbb{R}} e^{-\frac{\theta}{\sigma^2}(u)^2} du = 1 \quad \begin{cases} \text{say } x = \sqrt{\theta}/\sigma \cdot u \Rightarrow dx = x'\,du \\ x' = \sqrt{\theta}/\sigma \qquad \frac{\sigma}{\sqrt{\theta}}dx = du \end{cases}$$

* Replacing:

$\Rightarrow$ Being $(x)^2 = \frac{\theta}{\sigma^2}(u)^2$

$$k \int e^{-(x)^2} \frac{\sigma}{\sqrt{\theta}} dx = 1$$

$$k \frac{\sigma}{\sqrt{\theta}} \underbrace{\int e^{-x^2} dx}_{\sqrt{\pi}} = 1$$

$$k \cdot \frac{\sigma}{\sqrt{\theta}} \cdot \sqrt{\pi} = 1$$

$$\Rightarrow k = \sqrt{\frac{\theta}{\pi}} \cdot \frac{1}{\sigma} = \sqrt{\frac{\theta}{\sigma^2 \pi}}$$

* Finally:

$$\therefore P_\infty(u') = \sqrt{\frac{\theta}{\sigma^2 \pi}}\, e^{\left(-\frac{\theta}{\sigma^2} u'^2\right)}$$

* Comparing to $\mathbb{N}(\mu, \sigma^2)$:

$$\sqrt{\frac{\theta}{\pi}} \cdot \cancel{\frac{1}{\sigma}}\, e^{\left(-\frac{\theta}{\sigma^2} u'^2\right)} = \cancel{\frac{1}{\sigma}} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}}$$

$\Rightarrow \theta = 1/2$ to fits the equality, so:

$$\cancel{e}^{-\cancel{\frac{1}{2}}\left(\frac{1}{\sigma^2}\cdot u'^2\right)} = \cancel{e}^{-\cancel{\frac{1}{2}}\left(\frac{1}{\sigma^2}\cdot (x-\mu)^2\right)}$$

$$\cancel{1/\sigma^2} u'^2 = \cancel{1/\sigma^2}(x-\mu)^2 \rightsquigarrow \underline{u'^2 = (x-\mu)^2} \begin{cases} \text{"}\mu\text{" should} \\ \text{be 0!} \end{cases}$$

* Thus, mean and std for $P_\infty(u')$ becomes:

$$\therefore \mu = 0 \wedge \text{std} = \sigma$$

5) * Knowing that SSD-FKE is " $\underbrace{\frac{1}{2}\frac{d^2}{dy'^2}(B^2 P_\infty) - d/dy'(A P_\infty) = 0}$ ", thus:

$$\Rightarrow A = U_r \wedge B = V.r^{\beta}$$

* Solving to find $U_r$:

$$\Rightarrow \frac{1}{2}V^2.\, d^2/dr^2\, (r^\beta)^2.p - d/dr\,(Up) = 0$$

$$\frac{1}{2}V^2\, d^2/dr^2 (r^{2\beta})\, p = d/dr\,(Up)$$

$$\frac{1}{2}V^2 \cancel{\int} \cancel{d^2/dr^2}\, (r^{2\beta})\, p = \cancel{\int} \cancel{d/dr}\,(Up)$$

$$\frac{1}{2}V^2 d/dr\, \underline{(r^{2\beta}).p} = Up$$

$$\frac{1}{2}V^2\left(d/dr\,(r^{2\beta}).p + r^{2\beta}.\,(d/dr\, p)\right) = Up$$

$$\frac{1}{2}V^2\left(2\beta.r^{2\beta-1}.p + r^{2\beta}.\, d/dr \cdot p\right) = Up$$

$$\Rightarrow V^2.\beta r^{2\beta-1}.p + \frac{1}{2}V^2 r^{2\beta}\, d/dr\, p = Up$$

* Simplifying 'p':

$$\Rightarrow V^2\beta r^{2\beta-1} + \frac{1}{2}V^2 r^{2\beta}\left(\frac{d}{dr}.p\right)\underline{\left(\frac{1}{p}\right)} = U$$

$$V^2\beta r^{2\beta-1} + \frac{1}{2}V^2 r^{2\beta}\left(\frac{d}{dr}p\right)\left(\frac{d}{dp}\ln(p)\right) = U$$

$$V^2\beta r^{2\beta} + \frac{1}{2}V^2 r^{2\beta}\left(d/dr\, \ln(p)\right)\cancel{\left(\frac{d}{dp}p\right)}^{1} = U$$

* Finally, for $U_r$:

$$\therefore V^2\beta r^{2\beta} + \frac{1}{2}V^2 r^{2\beta}\left(d/dr\, \ln(P_\infty)\right) = U_r$$

a) i. Assuming $F = XY$, using TSE for $F$:

$$dF = \underbrace{\delta F/\delta X}\, dx + \underbrace{\delta F/\delta Y}\, dy + \frac{1}{2}\left(\underline{\delta^2 F/\delta X^2}\, dx + \underline{\delta^2 F/\delta Y^2}\, dy + 2\underline{\delta^2 F/\delta X\delta Y}\, dxdy + \ldots\right)$$

$$= Y dx + X dy + \frac{1}{2}\left(\cancel{0dx} + \cancel{0dy} + 2(1)dxdy\right)$$

$$\therefore dF = Ydx + Xdy + dxdy \quad \ldots \text{ Itô Product Rule!}$$

ii. Integrating in Itô Product Rule over $(t, 0)$:

$$\Rightarrow \int_0^t \cancel{d}F = \int_0^t Ydx + \int_0^t Xdy + \int_0^t dxdy$$

$$F\Big|_0^t = \int_0^t Ydx + \int_0^t Xdy + \int_0^t dxdy$$

$$(F_t - F_0) = \int_0^t Ydx + \int_0^t Xdy + \int_0^t dxdy$$

$$\Rightarrow X_t Y_t - X_0 Y_0 = \int_0^t Ydx + \int_0^t Xdy + \int_0^t dxdy$$

* Thus:

$$\therefore X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s = \int_0^t X_s dY_s$$

iii. Using TSE for $F$, such as $F = X/Y$:

$$dF = \frac{1}{Y}dx - \frac{X}{Y^2}dy + \frac{1}{2}\left(\cancel{0dx^2} - 2.\frac{1}{Y^2}dxdy + \frac{2X}{Y^3}dy^2\right)$$

$$\Rightarrow dF = \frac{1}{Y}dx - \frac{X}{Y^2}dy - \frac{1}{Y^2}dxdy + \frac{X}{Y^3}dy^2$$

* Simplificando:

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dXdY}{Y} + \frac{dY^2}{Y^2}\right) \Rightarrow \therefore d\left(\frac{X}{Y}\right) = \left(\frac{Ydx - Xdy - dxdy}{Y^2} + \frac{X}{Y^3}dy^2\right)$$

b) Using general form for "Itô VI" deduced in question 2 (*):

(*) $$dV = \left(\delta V/\delta t + a_1 \delta V/\delta S_1 + a_2 \delta V/\delta S_2 + \frac{1}{2}b_1^2 \delta^2 V/\delta^2 S_1 + \frac{1}{2}b_2^2 \delta^2 V/\delta^2 S_2 + \rho b_1 . b_2 \delta^2 V/\delta S_1 \delta S_2\right)dt + b_1 \delta V/\delta S_1 dW_t^{(1)} + b_2 \delta V/\delta S_2 dW_t^{(2)}$$

* Replacing: $a_1 = a(t, X_t) \wedge a_2 = c(t, Y_t)$

$b_1 = b(t, X_t) \wedge b_2 = d(t, Y_t)$

* And integrating for $V(t, X_t, Y_t)$ over $(T, 0)$, assuming $V(0, X_0, Y_0) = v$:

$$\therefore V_{(T, S_T, r_T)} = v + \int_0^T \left(dV/\delta t + (a)\delta V/\delta X + (c)\delta V/\delta Y + \frac{1}{2}(b)^2 \delta^2 V/\delta^2 X + \frac{1}{2}(d)^2 \delta^2 V/\delta Y^2 + \rho.(b.d)\delta^2 V/\delta X \delta Y\right)dt + \int_0^T \left(b . dV/\delta X\right)dW_t^{(1)} + \int_0^T \left(d\, \delta V/\delta Y\right)dW_t^{(2)}$$

c) * Transforming $dX_t \wedge dY_t$ to discrete time, considering $\mathbb{E}[W_t^{(1)} W_t^{(2)}] = \rho t$.

$\therefore dX_t \Rightarrow X_{i+1} - X_i = X_i(a\delta t + b.(\phi_x \sqrt{\delta t}))$

$\therefore dY_t \Rightarrow Y_{i+1} - Y_i = Y_i(c\delta t + d(\phi_Y \sqrt{\delta t}))$ , $\phi \sim N(0,1)$

* Deducing $\mathbb{E}[\phi_x \sqrt{\delta t} . \phi_Y \sqrt{\delta t}] = \mathbb{E}[\phi_x \phi_Y \delta t] = \rho \delta t = \rho$ ... (1)

* Thus, in order to choose $\phi_x \wedge \phi_Y$ to satisfy (1):

$\phi_x = \varepsilon_1 \wedge \phi_y = \alpha \varepsilon_1 + \beta \varepsilon_2$, where $\alpha, \beta \in \mathbb{R}$.

* Solving to find $\alpha \wedge \beta$, knowing that $\phi_x \wedge \phi_y \sim \underline{N(0,1)}$ (thus, $\varepsilon_1 \wedge \varepsilon_2$ too):

$$\Rightarrow \mathbb{E}[\phi_x \phi_y] = \mathbb{E}[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)] = \alpha \underbrace{\cancel{\mathbb{E}[\varepsilon_1^2]}}_{1} + \beta \underbrace{\cancel{\mathbb{E}[\varepsilon_1]}}_{0} . \mathbb{E}[\varepsilon_2]$$

$$\therefore \mathbb{E}[\phi_x \phi_y] = \alpha = \rho$$

$$\Rightarrow \mathbb{E}[\phi_y^2] = \mathbb{E}[(\alpha\varepsilon_1 + \beta\varepsilon_2)^2] = \mathbb{E}[(\alpha\varepsilon_1)^2 + 2\alpha\varepsilon_1\beta\varepsilon_2 + (\beta\varepsilon_2)^2]$$

$$= \alpha^2 \cancel{\mathbb{E}[\varepsilon_1^2]}^{1} + 2\alpha\beta \cancel{\mathbb{E}[\varepsilon_1]}^{0} . \mathbb{E}[\varepsilon_2] + \beta^2 \mathbb{E}[\varepsilon_2^2]$$

$$= \alpha^2 + \beta^2 \cancel{\mathbb{E}[\varepsilon_2^2]}^{1}$$

$$\therefore \mathbb{E}[\phi_y^2] = \alpha^2 + \beta^2 = 1$$

$$\beta = \sqrt{1-\alpha^2}$$

6) a) * Using "$\underline{\text{Itô II}}$":

$\cdot \ \delta y/\delta t = -6w^2 + 2kt \quad \cdot \ \delta y/\delta w = 4w^3 - 12tw \quad \cdot \ \delta^2 y/\delta w^2 = 12w^2 - 2t$

$$\Rightarrow dy = \left((2kt - \cancel{6w^2}) + \tfrac{1}{\cancel{2}}(\cancel{12}^{6}\cancel{w^2} - \cancel{12}^{6}t)\right)dt + (4w^3 - 12tw)\,dW_t$$

$$dy = (2kt - 6t)\,dt + (4w^3 - 12tw)\,dW_t$$

* For be a Martingala: $2kt - 6t = 0$, $\therefore k = 3$

b) * Using "$\underline{\text{Itô II}}$":

$\cdot \ \delta x/\delta t = -\cosh(\theta W_t)\frac{\theta^2}{2}e^{-\theta^2 t/2} \quad \cdot \ \delta x/\delta w = e^{-\theta^2 t/2} . \theta . \sinh(\theta W_t) \quad \cdot \ \delta^2 x/\delta w^2 = e^{-\theta^2 t/2} . \theta^2 . \cosh(\theta W_t)$

$$\Rightarrow dX = \left[\cancel{\left(-\cosh(\theta W_t)\frac{\theta^2}{2}e^{-\theta^2 t/2}\right)} + \frac{1}{2}\cancel{\left(e^{-\theta^2 t/2} . \theta^2 . \cosh(\theta W_t)\right)}\right]dt + \left(e^{-\theta^2 t/2} \theta . \sinh(\theta W_t)\right)dW_t$$

$\Rightarrow dx = (0)\,dt + \left(e^{-\theta^2 t/2} \theta . \sinh(\theta W_t)\right)dW_t$, So as long as drifft $= 0$,

$\therefore X_t :=$ is a Martingale.