Module 1 Assignment Jan 2022

Instructions: This is a non-assessed assignment; there are a number of exercises to help consolidate some of the key material covered in module 1. The aim should be to work through the numerous problems in this sheet to help measure understanding and performance. Complete solutions will follow. Throughout this sheet $W,\ W(t), W_t$ all refer to a standard Brownian motion. All queries should be directed to Riaz Ahmad: riaz.ahmad@fitchlearning.com

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W} dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(t), t)$ where $dW(t)$ is an increment of a Brownian motion.
If $W(0) = 0$ evaluate

$$\int_0^t \tau^2 \sin W \, dW(\tau).$$

b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS = \mu S dt + \sigma S dW.$$

Obtain a SDE $df(S, t)$ for each of the following functions

i $f(S, t) = \alpha^t + \beta t S^n$ $\qquad \alpha,\ \beta$ are constants

ii $f(S, t) = \log tS + \cos tS$

2. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\ dr_t &= \gamma (m - r_t) dt + c dW_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}\left[dW_t^{(1)} dW_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

3. An equity price S evolves according to Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are constants. We know that an explicit solution is

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}$$

where S_0 is S_t at time $t = 0$.
By working through all the integration steps, deduce that the expected value of S_t at time $t > 0$, given S_0, is

$$\mathbb{E}\left[S_t | S_0 \right] = S_0 e^{\mu t}.$$

You are required to present all your integration steps to obtain the expectation.

4. Consider the diffusion process for the state variable u which evolves according to the Ornstein-Uhlenbeck process

$$du = -\theta u dt + \sigma dW.$$

Both θ and σ are constants. Obtain the **steady state** probability distribution $p_\infty(u)$, which is given by

$$\sqrt{\frac{\theta}{\sigma^2 \pi}} \exp\left(-\frac{\theta}{\sigma^2} u^2 \right).$$

By looking at p_∞, write down the mean and standard deviation for this distribution.

5. Consider the spot rate r, which evolves according to the popular form

$$dr = u(r)\,dt + \nu r^{\beta} dW \ , \tag{1}$$

where ν and β are constants.
Suppose such a model has a **steady state transition probability density function** $p_{\infty}(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies that the drift structure of (1) is given by

$$u(r) = \nu^2 \beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}(\log p_{\infty}).$$

a. Let X_t, Y_t be two one-dimensional stochastic processes, where

$$\begin{aligned} dX_t &= a(t, X_t)\,dt + b(t, X_t)\,dW_t^{(1)}, \\ dY_t &= c(t, Y_t)\,dt + d(t, Y_t)\,dW_t^{(2)}. \end{aligned} \tag{2}$$

The Wiener processes are correlated such that $\mathbb{E}\left[W_t^{(1)} W_t^{(2)}\right] = \rho t$.

i. Derive the Itô rule for products $X_t Y_t$.

ii. Hence deduce the following *integration by parts formula*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s.$$

iii. Derive the Itô rule for quotients X_t / Y_t.

b. Consider a function $V(t, X_t, Y_t)$, where X_t, Y_t are defined by (2). Using Itô and suitable integration over $[0, T]$ obtain an expression for $V(T, X_T, Y_T)$.

c. Suppose the pair of stochastic differential equations (2) are to be simulated as discrete processes. Outline a scheme for doing this. Your discussion should include a method for correlating $W_t^{(1)}$ and $W_t^{(2)}$. Parte final. M1L5. Pag. 40 - 46. (*) Guardar como problema modelo.

6. In this question $t \geq 0$.

a. For which values of k is the process

$$Y_t = W_t^4 - 6tW_t^2 + kt^2,$$

a martingale?

b. Show that $X_t = \cosh(\theta W_t)\, e^{-\theta^2 t/2};\ \theta \in \mathbb{R}$, is a martingale.

7. Consider the following model, where the risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	$a^2 S$
ω_2	S	aS	S
ω_3	S	$a^{-1} S$	S
ω_4	S	$a^{-1} S$	$a^{-2} S$

NB
For this question you are pricing a European call where the strike is equal to the initial asset level S over the two periods.

S is the initial asset value at $t = 0$ and $a > 1$ is a constant.

a. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price, where X is the payoff function.

b. Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.