FUNCTIONS AND RELATIONS

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1. Relations

Definition 1.1. An n-ary *relation* R on a set S is a subset:

$$R \subseteq S^n$$

We write $R(a, \ldots, z)$ whenever $(a, \ldots, z) \in R$.

Binary relations will be the main focus of this class. For these relations, it is customary to use infix notations. That is, we write aRb instead of $R(a, b)$.

Example 1.2. The less-than-or-equal-to relation $\leq$ on $\mathbb{N}$ is the subset

$$\{(0,0), (0,1), \ldots, (1,1), (1,2), \ldots\} \subseteq \mathbb{N} \times \mathbb{N}$$

Example 1.3. The divisibility relation $\mid$ on $\mathbb{Z}$ is defined by

$$a \mid b \iff \exists c.\, b = ac$$

It is the subset

$$\{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid \exists c.\, b = ac\}$$

Example 1.4. The adjacency relation in a simple graph: two vertices u are v are adjacent if they are connected by an edge. It is the subset

$$\{(u,v) \in V \times V \mid (u,v) \in E \vee (v,u) \in E\}$$

Definition 1.5 (Reflexivity)**.** A binary relation R on a set S is *reflexive* if for all elements a of S, aRa.

$$\forall a.\, aRa$$

Definition 1.6 (Symmetry)**.** A binary relation R on a set S is *symmetric* if for any two elements a, b of S, if aRb then bRa.

$$\forall a. \forall b.\, aRb \Rightarrow bRa$$

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2. Countable Sets and Uncountable Sets

Theorem 2.1. $\mathbb{N}^{\mathbb{N}}$ *is uncountable.*

Proof. Suppose that $\mathbb{N}^{\mathbb{N}}$ is countable, i.e., $\mathbb{N} \cong \mathbb{N}^{\mathbb{N}}$. A possible interpretation of this hypothesis is that every function $f : \mathbb{N} \to \mathbb{N}$ can be given a unique natural-number code. That is, there are functions

$$\begin{aligned} \mathsf{decode} &: \mathbb{N} \to \mathbb{N}^{\mathbb{N}} \\ \mathsf{encode} &: \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \end{aligned}$$

that are mutual inverses. Consider the function

$$\begin{aligned} k &: \mathbb{N} \to \mathbb{N} \\ k &: n \mapsto \mathsf{decode}(n)(n) + 1 \end{aligned}$$

Given a code n, the function k decodes n, yielding a function $\mathbb{N} \to \mathbb{N}$, then evaluates that function at n, and finally adds 1 to the result.

The function k has a unique code given by $\mathsf{encode}(k)$. Now, let's evaluate k at its own code:

$$\begin{aligned} k(\mathsf{encode}(k)) &= \mathsf{decode}(\mathsf{encode}(k))(\mathsf{encode}(k)) + 1 \\ &= k(\mathsf{encode}(k)) + 1 \end{aligned}$$

This is a contradiction. $\square$

Theorem 2.1 tells us that some functions $f : \mathbb{N} \to \mathbb{N}$ are uncomputable: there are only countably many programs that one can write, but there are uncountably many endofunctions on $\mathbb{N}$. Thus, some of those functions do not have a corresponding program that computes it.