PROOF CALCULUS

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Contents

1. Proof Calculus

Logical symbols are explained by (1) how to prove it (introduction) and (2) how to use it (elimination). We use the capital Greek letter Γ to denote a list of hypotheses. A hypothesis is just a formula. The order and the number of occurrence of a given formula does not matter. For example,

$$\varphi, \psi, \chi \qquad \psi, \varphi, \chi \qquad \varphi, \varphi, \psi, \chi$$

are considered to be the set of hypotheses.

We write

$$\Gamma \vdash \varphi$$

to mean "the hypotheses in Γ *entail* φ".

1.1. Rules.

The simplest rule is the *identity rule*. It says that we can conclude φ if it is already part of the set of hypotheses.

$$\textsc{Id}\ \frac{}{\Gamma, \varphi \vdash \varphi}$$

1.1.1. *Top.*

$$\top\textsc{-Intro}\ \frac{}{\Gamma \vdash \top}$$

1.1.2. *Conjunction.*

$$\wedge\textsc{-Intro}\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \qquad \wedge\textsc{-Elim-L}\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \wedge\textsc{-Elim-R}\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

1.1.3. *Implication.*

$$\Rightarrow\textsc{-Intro}\ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi \Rightarrow \psi} \qquad \Rightarrow\textsc{-Elim}\ \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

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1.1.4. *Universal Quantification.*

$$\forall\text{-Intro}\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi}(x \notin \Gamma) \qquad \forall\text{-Elim}\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[t/x]}$$

1.1.5. *Bottom.*

$$\bot\text{-Elim}\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

1.1.6. *Disjunction.*

$$\vee\text{-Intro-L}\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \vee\text{-Intro-R}\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \vee\text{-Elim}\ \frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

1.1.7. *Existential Quantification.*

$$\exists\text{-Intro}\ \frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x.\varphi} \qquad \exists\text{-Elim}\ \frac{\Gamma \vdash \exists x.\varphi}{\Gamma \vdash \varphi[y/x]}(y \notin \Gamma)$$