GRAPHS

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Contents

1. DEGREES

Definition 1.1. Let $G = (V, E)$ be a simple (no self loops) undirected graph. A *neighbor* of a given vertex $v \in V$ is a vertex $u \in V$ such that $\{ u, v \} \in E$.

Definition 1.2. Let $G = (V, E)$ be a simple undirected graph. The *degree* of a given vertex $v \in V$, denoted as $\deg(v)$, is the number of neighbors of v.

Lemma 1.3 (Handshake Lemma). *Let $G = (V, E)$ be a simple undirected graph.*

$$2|E| = \sum_{v \in V} \deg(v)$$

Proof. **(1·1)** Proceed by induction on the number of vertices. In the base case, there is no vertex, so there is no edges either. In the induction step, there are $k + 1$ vertices. By removing an arbitrary vertex u from this graph, we are left with a graph with k vertices. By the induction hypothesis,

$$2|E'| = \sum_{v \in V \setminus \{ u \}} \deg(v)$$

where E' is the set of remaining edges after removing u.

(1·2) Note that $2|E'| + 2\deg(u) = 2|E|$ because we had to remove exactly $\deg(u)$ edges when we removed u. And $2\deg(u) + \sum_{v \in V \setminus \{ u \}} \deg(v) = \sum_{v \in V} \deg(v)$.

□

Definition 1.4. A simple undirected graph G is *complete* if every vertex is adjacent to every other vertex.

Corollary 1.5. *The complete graph K_n with n vertices has $\frac{n(n-1)}{2}$ edges.*

Proof. Since every vertex is adjacent to every other vertex, the degree of each vertex is $n-1$. By Lemma 1.3,

$$2|E| = \sum_{v\in V} \deg(v) = \sum_{v\in V}(n-1) = n(n-1)$$

□

2. GRAPH HOMOMORPHISMS

(2·1) The word *homomorphism* comes from Greek: *homos* means "same" and *morphe* means "shape." A homomorphism is a structure-preserving mapping.

(2·2) A graph is a set V equipped with a structure (a binary relation $E \subseteq V \times V$).

Definition 2.1. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *graph homomorphism* $\varphi : G \longrightarrow G'$ consists of an underlying function $|\varphi| : V \longrightarrow V'$ that preserves the relation E, i.e., for all $u, v \in V$, if $(u, v) \in E$ then $(|\varphi|(u), |\varphi|(v)) \in E'$.

Lemma 2.2. *Let $G = (V, E)$ be a graph. The identity function $\mathrm{id}_V : V \longrightarrow V$ is a graph homomorphism, denoted as $\mathrm{id}_G : G \longrightarrow G$.*

Proof. **(2·3)** Let $u, v \in V$ be any vertices of G. Suppose that $(u, v) \in E$. Then clearly $(\mathrm{id}_V(u), \mathrm{id}_V(v)) = (u, v) \in E$.

□

(2·4) A graph homomorphism is not just a function on vertex sets. The graph structures are important. For example, given a set $V = \{ A, B, C \}$, any subset $E \subseteq V \times V$ is a graph structure on V. Two obvious graphs are the indiscrete graph $I = (V, \varnothing)$ and the discrete graph $D = (V, V \times V)$. The identity function on V is a graph homomorphism $I \longrightarrow D$, but it is not a graph homomorphism $D \longrightarrow I$.

Lemma 2.3. *Let $G = (V, E)$, $G' = (V', E')$, and $G'' = (V'', E'')$ be graphs and $\varphi : G \longrightarrow G'$ and $\varphi' : G' \longrightarrow G''$ be graph homomorphisms. The composite $|\varphi'| \circ |\varphi| : V \longrightarrow V''$ is a graph homomorphism, denoted as $\varphi' \circ \varphi : G \longrightarrow G''$.*

Proof. Let $u, v \in V$ be given. Assume that $(u, v) \in E$. We need to show that $(|\varphi'| \circ |\varphi|(u), |\varphi'| \circ |\varphi|(v)) \in E''$. Since φ is a graph homomorphism, $(|\varphi|(u), |\varphi|(v)) \in E'$. And since φ' is a graph homomorphism, the result follows. □

3. GRAPH ISOMORPHISMS

Definition 3.1. Let G and G' be graphs. A graph homomorphism $\varphi : G \longrightarrow G'$ is a *graph isomorphism* if there is a graph homomorphisms $\varphi^{-1} : G' \longrightarrow G$ such that $\varphi \circ \varphi^{-1} = \mathrm{id}_{G'}$ and $\varphi^{-1} \circ \varphi = \mathrm{id}_G$.

(3·1) **(2·4)** has demonstrated that not every bijection is a graph isomorphism because D and I are not isomorphic. However, the underlying function of every graph isomorphism is a bijection.

Notation 3.2. We write $G \cong G'$ when there is a graph isomorphism between the two graphs.

4. GRAPH COLORING

Definition 4.1. Let $G = (V, E)$ be a simple undirected graph, a *k-coloring* is a function $\chi : V \to \{ 1, 2, \ldots, k \}$ such that for any $u, v \in V$, if $\{ u, v \} \in E$ then $\chi(u) \neq \chi(v)$.

Lemma 4.2. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A graph homomorphism $\varphi : G \to G'$ together with a k-coloring χ of G' induce a k-coloring of G.*

$$\begin{tikzcd} V \arrow[rr, "|\varphi|"] \arrow[dr, "\chi\circ|\varphi|"'] & & V' \arrow[dl, "\chi"] \\ & \{ 1, 2, \ldots, k \} & \end{tikzcd}$$

Proof. **(4·1)** Suppose that $\{ u, v \} \in E$. We need to verify that $\chi \circ |\varphi|(u) \neq \chi \circ |\varphi|(v)$. Since φ is a graph homomorphism, $\{ \ |\varphi|(u), |\varphi|(v) \} \in E'$. Since χ is a k-coloring of G', the result follows.

□

Definition 4.3. Let G be a simple undirected graph. The *chromatic number* χ_G of G is the minimum number such that G has a χ_G-coloring.

Lemma 4.4. *Chromatic number is a graph theoretic property, i.e., if $G \cong G'$ and G has chromatic number k, then G' also has chromatic number k.*

Proof. **(4·2)** By Lemma 4.2, a k-coloring of G and an isomorphism $\varphi : G' \to G$ induces a k-coloring of G'. It suffices to show that k is the minimal number that makes G' colorable.

(4·3) Suppose that there is some $k' < k$ such that G' is k' colorable. Any k'-coloring of G' together with the inverse isomorphism $\varphi^{-1} : G \to G'$ induce a k'-coloring of G. This contradicts the minimality of k.

□