PROOFS

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CONTENTS

1. Proofs

Logical symbols are explained by (1) how to prove it (introduction) and (2) how to use it (elimination). We use the capital Greek letter Γ to denote a list of hypotheses. A hypothesis is just a formula. The order and the number of occurrence of a given formula do not matter. For example,

$$\varphi, \psi, \chi \qquad \psi, \varphi, \chi \qquad \varphi, \varphi, \psi, \chi$$

are considered to be the same set of hypotheses.

We write

$$\Gamma \vdash \varphi$$

to mean "the hypotheses in Γ *entail* φ".

1.1. **Rules.**

The simplest rule is the *identity rule*. It says that we can conclude φ if it is already part of the set of hypotheses.

$$\frac{}{\Gamma, \varphi \vdash \varphi}\,\textsc{Id}$$

We are going to allow ourselves to do the following rewrites:

$$\neg\varphi \equiv \varphi \Rightarrow \bot \qquad \neg(\neg\varphi) \equiv \varphi$$

1.1.1. *Top.* We can always derive $\top$, but because we can't use $\top$ in any meaningful way, it has no elimination rules.

$$\frac{}{\Gamma \vdash \top}\,\top\text{-}\textsc{Intro}$$

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1.1.2. *Conjunction.* Conjunctions behave like pairs. To construct a pair, we need two elements (the first element and the second element). It has two elimination rules, one extracts the first element and the other extracts the second element.

$$\textsc{$\wedge$-Intro}\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \qquad \textsc{$\wedge$-Elim-L}\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \textsc{$\wedge$-Elim-R}\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

Proof Script.

(i) To prove $\varphi \wedge \psi$: we need to prove both φ and ψ.
(ii) To use $\varphi \wedge \psi$: since $\varphi \wedge \psi$, we may assume both φ and ψ.

1.1.3. *Implication.* Implications behave like functions. If I can write a program ψ with input φ, then I can abstract the input to get a function. The elimination rule is also known as *modus ponens*. It is function application.

$$\textsc{$\Rightarrow$-Intro}\ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi \Rightarrow \psi} \qquad \textsc{$\Rightarrow$-Elim}\ \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

Proof Script.

(i) To prove $\varphi \Rightarrow \psi$: suppose that φ ... therefore ψ.
(ii) To use $\varphi \Rightarrow \psi$: since $\varphi \Rightarrow \psi$, to prove ψ it suffices to prove φ.

1.1.4. *Universal Quantification.*

$$\textsc{$\forall$-Intro}\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi}(x \notin \mathrm{FV}(\Gamma)) \qquad \textsc{$\forall$-Elim}\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[t/x]}$$

Note that the side condition is crucial. Say, one of the hypotheses is "x is even" from which we can prove x is divisible by 2. It's incorrect to infer that for all x, x is divisible by 2 because the hypotheses constrain x to even numbers.
Proof Script.

(i) To prove $\forall x.\varphi$: let x be given, ... therefore φ.
(ii) To use $\forall x.\varphi$: since $\forall x.\varphi$, $\varphi[t/x]$.

1.1.5. *Bottom.* Bottom does not have an introduction rule but it has a powerful elimination rule. The elimination rule is also known as *ex falso quodlibet*.

$$\textsc{$\bot$-Elim}\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

1.1.6. *Disjunction.* Disjunctions behave like tagged unions. Its elimination rule is the `if ... then ... else ...` construct in programming languages.

$$\textsc{$\vee$-Intro-L}\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \textsc{$\vee$-Intro-R}\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \textsc{$\vee$-Elim}\ \frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Proof Script.

(i) To prove $\varphi \vee \psi$: prove either φ or ψ.
(ii) To use $\varphi \vee \psi$: case analysis.

1.1.7. *Existential Quantification.*

$$\exists\text{-Intro}\;\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x.\varphi} \qquad\qquad \exists\text{-Elim}\;\frac{\Gamma \vdash \exists x.\varphi}{\Gamma \vdash \varphi[y/x]}(y \notin \mathrm{FV}(\Gamma))$$

The side condition in the elimination rule is crucial because $\exists x.\varphi$ contains a *witness* that we know nothing about so the set of hypotheses Γ cannot assume anything about it.
Proof Script.

(i) To prove $\exists x.\varphi$: find a witness t for which $\varphi[t/x]$ holds.
(ii) To use $\exists x.\varphi$: we get a hypothetical witness y that we don't know anything about and $\varphi[y/x]$ holds.

2. Examples

Example 2.1.

$$(\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

Proof 1. Suppose $\varphi \Rightarrow \psi$. We need to prove $\neg\psi \Rightarrow \neg\varphi$. Suppose $\neg\psi$. We need to prove $\neg\varphi$. Suppose φ. We need to prove $\bot$. Since we know $\neg\psi$, it suffices to prove ψ. Since we know $\varphi \Rightarrow \psi$, it suffices to prove φ, which is already a hypothesis. ☐

Proof 2.

$$\Rightarrow\text{-Intro}\;\cfrac{\Rightarrow\text{-Intro}\;\cfrac{\Rightarrow\text{-Intro}\;\cfrac{\Rightarrow\text{-Elim}\;\cfrac{\text{Id}\;\cfrac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \neg\psi} \qquad \Delta}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \bot}}{\varphi \Rightarrow \psi, \neg\psi \vdash \neg\varphi}}{\varphi \Rightarrow \psi \vdash \neg\psi \Rightarrow \neg\varphi}}{\vdash (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)}$$

where $\Delta =$

$$\Rightarrow\text{-Elim}\;\cfrac{\text{Id}\;\cfrac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \varphi \Rightarrow \psi} \qquad \cfrac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \varphi}\;\text{Id}}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \psi}$$

☐

Example 2.2.

$$(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$$

Proof 1. Assume $\neg\psi \Rightarrow \neg\varphi$. We need to prove $\varphi \Rightarrow \psi$. Assume φ, or equivalently $\neg(\neg\varphi)$. We need to prove ψ, or equivalently $\neg(\neg\psi)$. Assume $\neg\psi$. We need to prove $\bot$. Since we know $\neg(\neg\varphi)$, it suffices to show $\neg\varphi$. Since we know $\neg\psi \Rightarrow \neg\varphi$, it suffices to show $\neg\psi$, which is already a hypothesis. ☐

Proof 2.

$$
\textsc{$\Rightarrow$-Intro}\;\dfrac{\textsc{$\Rightarrow$-Intro}\;\dfrac{\textsc{$\Rightarrow$-Intro}\;\dfrac{\textsc{$\Rightarrow$-Elim}\;\dfrac{\Delta \qquad \dfrac{\textsc{Id}\;\dfrac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\psi \Rightarrow \neg\varphi} \qquad \Xi}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\varphi}\;\textsc{$\Rightarrow$-Elim}}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \bot}}{\neg\psi \Rightarrow \neg\varphi, \varphi \vdash \neg(\neg\psi)}}{\neg\psi \Rightarrow \neg\varphi \vdash \varphi \Rightarrow \psi}}{\vdash (\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)}
$$

where $\Delta =$

$$
\textsc{Id}\;\dfrac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg(\neg\varphi)}
$$

and $\Xi =$

$$
\textsc{Id}\;\dfrac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\psi}
$$

□

Example 2.3.

$$\neg(\varphi \vee \psi) \Rightarrow \neg\varphi \wedge \neg\psi$$

Proof 1. Suppose that $\neg(\varphi \vee \psi)$. We need to prove $\neg\varphi \wedge \neg\psi$. It suffices to prove $\neg\varphi$ and $\neg\psi$ separately.

(i) Proof of $\neg\varphi$: Assume φ, we need to prove $\bot$. Since we know that $\neg(\varphi \vee \psi)$, it suffices to prove $\varphi \vee \psi$, which follows immediately from φ.
(ii) Proof of $\neg\psi$: Assume ψ, we need to prove $\bot$. Since we know that $\neg(\varphi \vee \psi)$, it suffices to prove $\varphi \vee \psi$, which follows immediately from ψ.

□

Proof 2.

$$
\textsc{$\Rightarrow$-Intro}\;\dfrac{\textsc{$\wedge$-Intro}\;\dfrac{\textsc{$\Rightarrow$-Intro}\;\dfrac{\textsc{$\Rightarrow$-Elim}\;\dfrac{\textsc{Id}\;\dfrac{}{\neg(\varphi \vee \psi), \varphi \vdash \neg(\varphi \vee \psi)} \qquad \Delta}{\neg(\varphi \vee \psi), \varphi \vdash \bot}}{\neg(\varphi \vee \psi) \vdash \neg\varphi} \qquad \dfrac{\Xi}{\neg(\varphi \vee \psi) \vdash \neg\psi}\;\textsc{$\Rightarrow$-Intro}}{\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi}}{\vdash \neg(\varphi \vee \psi) \Rightarrow \neg\varphi \wedge \neg\psi}
$$

where $\Delta =$

$$
\textsc{$\vee$-Intro-L}\;\dfrac{\textsc{Id}\;\dfrac{}{\neg(\varphi \vee \psi), \varphi \vdash \varphi}}{\neg(\varphi \vee \psi), \varphi \vdash \varphi \vee \psi}
$$

and $\Xi =$

$$
\textsc{$\Rightarrow$-Elim}\;\dfrac{\textsc{Id}\;\dfrac{}{\neg(\varphi \vee \psi), \psi \vdash \neg(\varphi \vee \psi)} \qquad \dfrac{\dfrac{}{\neg(\varphi \vee \psi), \psi \vdash \psi}\;\textsc{Id}}{\neg(\varphi \vee \psi), \psi \vdash \varphi \vee \psi}\;\textsc{$\vee$-Intro-R}}{\neg(\varphi \vee \psi), \psi \vdash \bot}
$$

□

Example 2.4.

$$\forall x.\varphi \Rightarrow \exists x.\varphi$$

Proof 1. Suppose that $\forall x.\varphi$. We need to prove $\exists x.\varphi$. It suffices to find a witness t so that $\varphi[t/x]$. Since we know $\forall x.\varphi$, any witness works. □

Proof 2.

$$\Rightarrow\text{-Intro}\;\dfrac{\exists\text{-Intro}\;\dfrac{\forall\text{-Elim}\;\dfrac{\text{Id}\;\dfrac{}{\forall x.\varphi \vdash \forall x.\varphi}}{\forall x.\varphi \vdash \varphi[t/x]}}{\forall x.\varphi \vdash \exists x.\varphi}}{\vdash \forall x.\varphi \Rightarrow \exists x.\varphi}$$

□

Example 2.5.

$$\neg(\exists x.\varphi) \Rightarrow \forall x.\neg\varphi$$

Proof 1. Suppose $\neg\exists x.\varphi$. We need to prove $\forall x.\neg\varphi$. Let any x be given. We need to prove $\neg\varphi$. Suppose φ. We need to prove $\bot$. Since we know $\neg\exists x.\varphi$, it suffices to prove $\exists x.\varphi$. Since we already know φ, we can choose x as the witness. □

Proof 2.

$$\Rightarrow\text{-Intro}\;\dfrac{\Rightarrow\text{-Intro}\;\dfrac{\Rightarrow\text{-Intro}\;\dfrac{\Rightarrow\text{-Elim}\;\dfrac{\text{Id}\;\dfrac{}{\neg(\exists x.\varphi), \varphi \vdash \neg(\exists x.\varphi)} \qquad \dfrac{\dfrac{}{\neg(\exists x.\varphi), \varphi \vdash \varphi[x/x]}\;\text{Id}}{\neg(\exists x.\varphi), \varphi \vdash \exists x.\varphi}\;\exists\text{-Intro}}{\neg(\exists x.\varphi), \varphi \vdash \bot}}{\neg(\exists x.\varphi) \vdash \neg\varphi}}{\neg(\exists x.\varphi) \vdash \forall x.\neg\varphi}}{\vdash \neg(\exists x.\varphi) \Rightarrow \forall x.\neg\varphi}$$

□

Example 2.6.

$$\neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi$$

Proof 1. By Example 2.2, it suffices to prove $\neg\exists x.\neg\varphi \Rightarrow \neg(\neg\forall x.\varphi)$, or equivalently, $\neg\exists x.\neg\varphi \Rightarrow \forall x.\varphi$. Assume $\neg\exists x.\neg\varphi$. We need to prove $\forall x.\varphi$. Let any x be given. We need to prove φ, or equivalently $\neg(\neg\varphi)$. Assume $\neg\varphi$, we need to prove $\bot$. Since we know $\neg\exists x.\neg\varphi$, it suffices to prove $\exists x.\neg\varphi$. Since we know $\neg\varphi$, choosing x as the witness completes the proof. □

Proof 2.

$$\Rightarrow\text{-Elim}\;\dfrac{\Delta \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Id}\;\dfrac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \neg\exists x.\neg\varphi} \qquad \dfrac{\dfrac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \neg\varphi[x/x]}\;\text{Id}}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \exists x.\neg\varphi}\;\exists\text{-Intro}}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \bot}\;\Rightarrow\text{-Elim}}{\neg\exists x.\neg\varphi \vdash \neg(\neg\varphi)}\;\Rightarrow\text{-Intro}}{\neg\exists x.\neg\varphi \vdash \forall x.\varphi}\;\Rightarrow\text{-Intro}}{\vdash \neg\exists x.\neg\varphi \Rightarrow \forall x.\varphi}\;\Rightarrow\text{-Intro}}{\vdash \neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi}$$

where $\Delta =$

$$\dfrac{\vdots}{\vdash (\neg(\exists x.\neg\varphi) \Rightarrow \forall x.\varphi) \Rightarrow \neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi}$$

follows from Example 2.2. $\square$