FUNCTIONS

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Contents

1. Countable Sets and Uncountable Sets

Theorem 1.1. $\mathbb{N}^{\mathbb{N}}$ *is uncountable.*

Proof. Suppose that $\mathbb{N}^{\mathbb{N}}$ is countable, i.e., $\mathbb{N} \cong \mathbb{N}^{\mathbb{N}}$. One possible interpretations of this hypothesis is that every function $f : \mathbb{N} \to \mathbb{N}$ can be given a unique natural-number code. That is, there are functions

$$\begin{aligned} \mathsf{decode} &: \mathbb{N} \to \mathbb{N}^{\mathbb{N}} \\ \mathsf{encode} &: \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \end{aligned}$$

that are mutual inverses. Consider the function

$$\begin{aligned} k &: \mathbb{N} \to \mathbb{N} \\ k &: n \mapsto \mathsf{decode}(n)(n) + 1 \end{aligned}$$

Given a code n, the function k decodes n, yielding a function $\mathbb{N} \to \mathbb{N}$, then evaluates that function at n, and finally adds 1 to the result.

The function k has a unique code given by $\mathsf{encode}(k)$. Now, let's evaluate k at its own code:

$$\begin{aligned} k(\mathsf{encode}(k)) &= \mathsf{decode}(\mathsf{encode}(k))(\mathsf{encode}(k)) + 1 \\ &= k(\mathsf{encode}(k)) + 1 \end{aligned}$$

This is a contradiction.[1] $\square$

Theorem 1.1 tells us that some functions $f : \mathbb{N} \to \mathbb{N}$ are uncomputable: there are only countably many programs that one can write, but there are uncountably many endofunctions on $\mathbb{N}$. Thus, some of those functions do not have a corresponding program that computes it.

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[1] This proof is not a proof by contradiction. In fact, it is a constructively valid proof.