BASIC SET THEORY

FRANK TSAI

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1. Sets

Set theory made its debut in Cantor's 1874 paper: "On a Property of the Collection of All Real Algebraic Numbers." Later, mathematicians such as Russell realized that the Cantorian set theory contained several contradictions, but mathematicians did not abandon set theory. The work of Zermelo, Fraenkel, and Skolem resulted in a well-known axiomatization of set theory.

In this class, we will not talk about axiomatic set theory. Instead, we will look at set theory naïvely.

Definition 1.1. A *set* is a collection of elements. These elements have no internal structures, so you can think of a set as a collection of dots.

The language of (ZF) set theory contains a binary predicate symbol $\in$. This is known as the *membership relation.* $x \in y$ means x is an element of y.

Example 1.2.

(i) The empty set: $\varnothing$.
(ii) The set containing the empty set: $\{\varnothing\}$.
(iii) A set containing 3 elements: $\{a, b, c\}$.
(iv) The set of all natural numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$.
(v) The set of all integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
(vi) The set containing $\mathbb{N}$ and $\mathbb{Z}$: $\{\mathbb{N}, \mathbb{Z}\}$.

Example 1.3.

(i) Nothing is in the empty set: $x \notin \varnothing$.
(ii) The set containing the empty set has an element: $\varnothing \in \{\varnothing\}$.
(iii) $a \in \{a, b, c\}$, $b \in \{a, b, c\}$, $c \in \{a, b, c\}$.
(iv) $\mathbb{N} \in \{\mathbb{N}, \mathbb{Z}\}$, $\mathbb{Z} \in \{\mathbb{N}, \mathbb{Z}\}$, but $0 \notin \{\mathbb{N}, \mathbb{Z}\}$.

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2. Subsets

Definition 2.1. A set x is a subset of y, denoted $x \subseteq y$, if every element in x is also in y.

$$x \subseteq y \equiv \forall z.(z \in x \Rightarrow z \in y)$$

The relation $\subseteq$ is called *set inclusion*.

Lemma 2.2. *The empty set is a subset of any set.*

$$\forall y.\, \varnothing \subseteq y$$

Proof. Let y be any set. By definition, $\varnothing \subseteq y \equiv \forall z.(z \in \varnothing \Rightarrow z \in y)$. Let z be given. Assume that $z \in \varnothing$, but this is impossible since $z \notin \varnothing$.[1] □

Lemma 2.3. *Every set is a subset of itself.*

$$\forall x.\, x \subseteq x$$

Proof. Exercise. □

Example 2.4.
(i) $\varnothing$ has a subset $\varnothing$.
(ii) $\{\varnothing\}$ has subsets $\varnothing$ and $\{\varnothing\}$.
(iii) $\{a, b\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, and $\{a, b\}$.
(iv) $\{a, b, c\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

3. Equality

Two sets are equal when they contain the same elements. We can express this in terms of the set inclusion relation.

$$\forall x.\forall y.((x \subseteq y \land y \subseteq x) \Rightarrow x = y)$$

Given two sets x and y, to prove that $x = y$, it suffices to prove $x \subseteq y$ and $y \subseteq x$.

Example 3.1.
(i) $\{a, b, c, d, d\} = \{a, b, c, d\}$.
(ii) $\{a, b, c\} = \{c, b, a\}$.

4. Comprehension

Given a set w, there is a set of w whose elements satisfy certain property φ.

$$\{x \in w \mid \varphi(x)\}$$

Example 4.1.
(i) The set of all even natural numbers: $\{x \in \mathbb{N} \mid \mathrm{even}(x)\}$.
(ii) The set of all odd natural numbers: $\{x \in \mathbb{N} \mid \mathrm{odd}(x)\}$.
(iii) The set of all integers divisible by 2: $\{x \in \mathbb{Z} \mid x \equiv 0 \mod 2\}$.
(iv) The set of all real numbers between 0 and 1 (inclusive): $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

[1] This is not a proof by contradiction, but let's ignore the details for now.

5. Power Set

In Example 5.2, $\{a, b, c\}$ has subsets $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. These subsets form a set

$$\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Definition 5.1. Let x be a set. The *power set* of x, denoted $\mathcal{P}(x)$, is the set of all subsets of x.

Example 5.2.
(i) $\mathcal{P}(\varnothing) = \{\varnothing\}$.
(ii) $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$.
(iii) $\mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$.
(iv) $\mathcal{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 5.3 (Cantor's Theorem)**.** *For any set x, there is no surjective function $f : x \to \mathcal{P}(x)$.*

6. Union

Definition 6.1. Given a nonempty set of sets $\mathcal{F}$, the *union*, denoted $\cup\mathcal{F}$, is a set whose elements are those elements of *some* set in $\mathcal{F}$. [2]

Example 6.2.
(i) If $\mathcal{F} = \{\{1, 2, 3\}, \{a, b, c\}\}$ then $\cup\mathcal{F} = \{1, 2, 3, a, b, c\}$.
(ii) If $\mathcal{F} = \{\{a, b, c\}, \{b, c, d\}, \{c, d, e\}\}$ then $\cup\mathcal{F} = \{a, b, c, d, e\}$.

Notation 6.3. When $\mathcal{F}$ has just a few elements, for example $\mathcal{F} = \{\{1, 2, 3\}, \{a, b, c\}\}$, we write

$$\{1, 2, 3\} \cup \{a, b, c\}$$

for $\cup\mathcal{F}$.

Indeed, the union of any two sets can be expressed in terms of comprehension (see footnote):

$$x \cup y = \{z \in E \mid z \in x \vee z \in y\}$$

7. Intersection

Definition 7.1. Given a nonempty set of sets $\mathcal{F}$, the *intersection*, denoted $\cap\mathcal{F}$, is a set whose elements are those elements of *every* set in $\mathcal{F}$.

Example 7.2.
(i) If $\mathcal{F} = \{\{1, 2, 3\}, \{a, b, c\}\}$ then $\cap\mathcal{F} = \varnothing$.
(ii) If $\mathcal{F} = \{\{a, b, c\}, \{b, c, d\}, \{c, d, e\}\}$ then $\cap\mathcal{F} = \{c\}$.

Notation 7.3. When $\mathcal{F}$ has just a few elements, for example $\mathcal{F} = \{\{1, 2, 3\}, \{a, b, c\}\}$, we write

$$\{1, 2, 3\} \cap \{a, b, c\}$$

for $\cap\mathcal{F}$.

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[2] The Axiom of Union does not assert the existence of $\cup\mathcal{F}$ directly. It only asserts that there is always a set E that contains *at least* those elements of some set in $\mathcal{F}$. One can then define $\cup\mathcal{F}$ using comprehension: $\cup\mathcal{F} = \{x \in E \mid \exists y. (y \in \mathcal{F} \wedge x \in y)\}$.

The intersection of any two sets can be expressed in terms of union and comprehension:

$$x \cap y = \{z \in x \cup y \mid z \in x \wedge z \in y\}$$

Definition 7.4. Two sets x and y are said to be *disjoint* if $x \cap y = \varnothing$.

8. Set Difference

Definition 8.1. Given two sets x and y, the *set difference*, denoted $x \setminus y$ (sometimes $x - y$), is a set containing exactly those elements in x but not in y. That is,

$$x \setminus y = \{z \in x \mid z \in x \wedge z \notin y\}$$

Example 8.2.
(i) $\{1, 2, 3\} \setminus \{a, b, c\} = \{1, 2, 3\}$.
(ii) $\{a, b, c\} \setminus \{1, 2, 3\} = \{a, b, c\}$.
(iii) $\{a, b, c\} \setminus \{c, d, e\} = \{a, b\}$.
(iv) $\{c, d, e\} \setminus \{a, b, c\} = \{d, e\}$.

Definition 8.3. Let $x \subseteq y$. The *complement* of x, denoted x^{c}, is the set $y \setminus x$.