

PROOFS

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1. PROOFS

Logical symbols are explained by (1) how to prove it (introduction) and (2) how to use it (elimination). We use the capital Greek letter Γ to denote a list of hypotheses. A hypothesis is just a formula. The order and the number of occurrence of a given formula does not matter. For example,

$$\varphi, \psi, \chi \qquad \psi, \varphi, \chi \qquad \varphi, \varphi, \psi, \chi$$

are considered to be the same set of hypotheses.

We write

$$\Gamma \vdash \varphi$$

to mean “the hypotheses in Γ entail φ ”.

1.1. Rules. The simplest rule is the *identity rule*. It says that we can conclude φ if it is already part of the set of hypotheses.

$$\frac{\text{ID}}{\Gamma, \varphi \vdash \varphi}$$

We are going to allow ourselves to do the following rewrites:

$$\neg\varphi \equiv \varphi \Rightarrow \perp \qquad \neg(\neg\varphi) \equiv \varphi$$

1.1.1. Top. We can always derive \top , but because we can’t use \top in any meaningful way, it has no elimination rules.

$$\frac{\text{\top-INTRO}}{\Gamma \vdash \top}$$

1.1.2. *Conjunction*. Conjunctions behave like pairs. To construct a pair, we need two elements (the first element and the second element). It has two elimination rules, one extracts the first element and the other extracts the second element.

$$\begin{array}{c} \wedge\text{-INTRO} \\ \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \end{array} \qquad \begin{array}{c} \wedge\text{-ELIM-L} \\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \end{array} \qquad \begin{array}{c} \wedge\text{-ELIM-R} \\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \end{array}$$

Proof Script.

- (i) To prove $\varphi \wedge \psi$: we need to prove both φ and ψ .
- (ii) To use $\varphi \wedge \psi$: since $\varphi \wedge \psi$, we may assume both φ and ψ .

1.1.3. *Implication*. Implications behave like functions. If I can write a program ψ with input φ , then I can abstract the input to get a function. The elimination rule is also known as *modus ponens*. It is function application.

$$\begin{array}{c} \Rightarrow\text{-INTRO} \\ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi} \end{array} \qquad \begin{array}{c} \Rightarrow\text{-ELIM} \\ \frac{\Gamma \vdash \varphi \Rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \end{array}$$

Proof Script.

- (i) To prove $\varphi \Rightarrow \psi$: suppose that $\varphi \dots$ therefore ψ .
- (ii) To use $\varphi \Rightarrow \psi$: since $\varphi \Rightarrow \psi$, to prove ψ it suffices to prove φ .

1.1.4. *Universal Quantification*.

$$\begin{array}{c} \forall\text{-INTRO} \\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi} (x \notin \text{FV}(\Gamma)) \end{array} \qquad \begin{array}{c} \forall\text{-ELIM} \\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[t/x]} \end{array}$$

Note that the side condition is crucial. Say, one of the hypotheses is “ x is even” from which we can prove x is divisible by 2. It’s incorrect to infer that for all x , x is divisible by 2 because the hypotheses constrain x to even numbers.

Proof Script.

- (i) To prove $\forall x.\varphi$: let x be given, ... therefore φ .
- (ii) To use $\forall x.\varphi$: since $\forall x.\varphi$, $\varphi[t/x]$.

1.1.5. *Bottom*. Bottom does not have an introduction rule but it has a powerful elimination rule. The elimination rule is also known as *ex falso quodlibet*.

$$\begin{array}{c} \perp\text{-ELIM} \\ \frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi} \end{array}$$

1.1.6. *Disjunction*. Disjunctions behave like tagged unions. Its elimination rule is the **if ... then ... else ...** construct in programming languages.

$$\begin{array}{c} \vee\text{-INTRO-L} \\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \end{array} \qquad \begin{array}{c} \vee\text{-INTRO-R} \\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \end{array} \qquad \begin{array}{c} \vee\text{-ELIM} \\ \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma, \varphi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \end{array}$$

Proof Script.

- (i) To prove $\varphi \vee \psi$: prove either φ or ψ .
- (ii) To use $\varphi \vee \psi$: case analysis.

1.1.7. *Existential Quantification.*

$$\frac{\exists\text{-INTRO}}{\Gamma \vdash \varphi[t/x]} \quad \frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x.\varphi}$$

$$\frac{\exists\text{-ELIM}}{\Gamma \vdash \varphi[y/x]} \quad \frac{\Gamma \vdash \exists x.\varphi}{\Gamma \vdash \varphi[y/x]} (y \notin \text{FV}\Gamma)$$

The side condition in the elimination rule is crucial because $\exists x.\varphi$ contains a *witness* that we know nothing about so the set of hypotheses Γ cannot assume anything about it.

Proof Script.

- (i) To prove $\exists x.\varphi$: find a witness t for which $\varphi[t/x]$ holds.
- (ii) To use $\exists x.\varphi$: we get a hypothetical witness y that we don't know anything about and $\varphi[y/x]$ holds.

2. EXAMPLES

Example 2.1.

$$(\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

Proof 1. Suppose $\varphi \Rightarrow \psi$. We need to prove $\neg\psi \Rightarrow \neg\varphi$. Suppose $\neg\psi$. We need to prove $\neg\varphi$. Suppose φ . We need to prove \perp . Since we know $\neg\psi$, it suffices to prove ψ . Since we know $\varphi \Rightarrow \psi$, it suffices to prove φ , which is already a hypothesis. \square

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \neg\psi} \quad \Delta \\ \Rightarrow\text{-ELIM} \frac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \perp} \\ \Rightarrow\text{-INTRO} \frac{}{\varphi \Rightarrow \psi, \neg\psi \vdash \neg\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\varphi \Rightarrow \psi \vdash \neg\psi \Rightarrow \neg\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)} \end{array}$$

where $\Delta =$

$$\Rightarrow\text{-ELIM} \frac{\text{ID} \frac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \varphi \Rightarrow \psi} \quad \frac{}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \varphi} \text{ID}}{\varphi \Rightarrow \psi, \neg\psi, \varphi \vdash \psi}$$

\square

Example 2.2.

$$(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$$

Proof 1. Exercise. \square

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\psi \Rightarrow \neg\varphi} \quad \Xi \\ \Rightarrow\text{-ELIM} \frac{\Delta}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\varphi} \Rightarrow\text{-ELIM} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \perp} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\psi \Rightarrow \neg\varphi, \varphi \vdash \neg(\neg\psi)} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\psi \Rightarrow \neg\varphi \vdash \varphi \Rightarrow \psi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash (\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)} \end{array}$$

where $\Delta =$

$$\text{ID} \frac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg(\neg\varphi)}$$

and $\Xi =$

$$\text{ID} \frac{}{\neg\psi \Rightarrow \neg\varphi, \neg(\neg\varphi), \neg\psi \vdash \neg\psi}$$

□

Example 2.3.

$$\neg(\varphi \vee \psi) \Rightarrow \neg\varphi \wedge \neg\psi$$

Proof 1. Suppose that $\neg(\varphi \vee \psi)$. We need to prove $\neg\varphi \wedge \neg\psi$. It suffices to prove $\neg\varphi$ and $\neg\psi$ separately.

- (i) Proof of $\neg\varphi$: Assume φ , we need to prove \perp . Since we know that $\neg(\varphi \vee \psi)$, it suffices to prove $\varphi \vee \psi$, which follows immediately from φ .
- (ii) Proof of $\neg\psi$: Assume ψ , we need to prove \perp . Since we know that $\neg(\varphi \vee \psi)$, it suffices to prove $\varphi \vee \psi$, which follows immediately from ψ .

□

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\neg(\varphi \vee \psi), \varphi \vdash \neg(\varphi \vee \psi)} \quad \Delta \\ \Rightarrow\text{-ELIM} \frac{}{\neg(\varphi \vee \psi), \varphi \vdash \perp} \\ \Rightarrow\text{-INTRO} \frac{}{\neg(\varphi \vee \psi) \vdash \neg\varphi} \quad \frac{\Xi}{\neg(\varphi \vee \psi) \vdash \neg\psi} \Rightarrow\text{-INTRO} \\ \wedge\text{-INTRO} \frac{}{\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash \neg(\varphi \vee \psi) \Rightarrow \neg\varphi \wedge \neg\psi} \end{array}$$

where $\Delta =$

$$\text{V-INTRO-L} \frac{\text{ID} \frac{}{\neg(\varphi \vee \psi), \varphi \vdash \varphi}}{\neg(\varphi \vee \psi), \varphi \vdash \varphi \vee \psi}$$

and $\Xi =$

$$\Rightarrow\text{-ELIM} \frac{\text{ID} \frac{}{\neg(\varphi \vee \psi), \psi \vdash \neg(\varphi \vee \psi)} \quad \frac{\text{ID} \frac{}{\neg(\varphi \vee \psi), \psi \vdash \psi}}{\neg(\varphi \vee \psi), \psi \vdash \varphi \vee \psi} \text{V-INTRO-R}}{\neg(\varphi \vee \psi), \psi \vdash \perp}$$

□

Example 2.4.

$$\forall x.\varphi \Rightarrow \exists x.\varphi$$

Proof 1. Suppose that $\forall x.\varphi$. We need to prove $\exists x.\varphi$. It suffices to find a witness t so that $\varphi[t/x]$. Since we know $\forall x.\varphi$, any witness works. □

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\forall x.\varphi \vdash \forall x.\varphi} \\ \forall\text{-ELIM} \frac{}{\forall x.\varphi \vdash \varphi[t/x]} \\ \exists\text{-INTRO} \frac{}{\forall x.\varphi \vdash \exists x.\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash \forall x.\varphi \Rightarrow \exists x.\varphi} \end{array}$$

□

Example 2.5.

$$\neg(\exists x.\varphi) \Rightarrow \forall x.\neg\varphi$$

Proof 1. Suppose $\neg\exists x.\varphi$. We need to prove $\forall x.\neg\varphi$. Let any x be given. We need to prove $\neg\varphi$. Suppose φ . We need to prove \perp . Since we know $\neg\exists x.\varphi$, it suffices to prove $\exists x.\varphi$. Since we already know φ , we can choose x as the witness. □

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\neg(\exists x.\varphi), \varphi \vdash \neg(\exists x.\varphi)} \quad \frac{\text{ID} \frac{}{\neg(\exists x.\varphi), \varphi \vdash \varphi[x/x]}}{\neg(\exists x.\varphi), \varphi \vdash \exists x.\varphi} \text{ } \exists\text{-INTRO} \\ \Rightarrow\text{-ELIM} \frac{}{\neg(\exists x.\varphi), \varphi \vdash \neg(\exists x.\varphi)} \quad \frac{}{\neg(\exists x.\varphi), \varphi \vdash \exists x.\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\neg(\exists x.\varphi), \varphi \vdash \perp} \\ \Rightarrow\text{-INTRO} \frac{}{\neg(\exists x.\varphi) \vdash \neg\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\neg(\exists x.\varphi) \vdash \forall x.\neg\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash \neg(\exists x.\varphi) \Rightarrow \forall x.\neg\varphi} \end{array}$$

□

Example 2.6.

$$\neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi$$

Proof 1. By Example 2.2, it suffices to prove $\neg\exists x.\neg\varphi \Rightarrow \neg(\neg\forall x.\varphi)$, or equivalently, $\neg\exists x.\neg\varphi \Rightarrow \forall x.\varphi$. Assume $\neg\exists x.\neg\varphi$. We need to prove $\forall x.\varphi$. Let any x be given. We need to prove φ , or equivalently $\neg(\neg\varphi)$. Assume $\neg\varphi$, we need to prove \perp . Since we know $\neg\exists x.\neg\varphi$, it suffices to prove $\exists x.\neg\varphi$. Since we know $\neg\varphi$, choosing x as the witness completes the proof. □

Proof 2.

$$\begin{array}{c} \text{ID} \frac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \neg\exists x.\neg\varphi} \quad \frac{\text{ID} \frac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \neg\varphi[x/x]}}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \exists x.\neg\varphi} \text{ } \exists\text{-INTRO} \\ \Rightarrow\text{-ELIM} \frac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \neg\exists x.\neg\varphi} \quad \frac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \exists x.\neg\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\exists x.\neg\varphi, \neg\varphi \vdash \perp} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\exists x.\neg\varphi \vdash \neg(\neg\varphi)} \\ \Rightarrow\text{-INTRO} \frac{}{\neg\exists x.\neg\varphi \vdash \forall x.\varphi} \\ \Rightarrow\text{-INTRO} \frac{}{\vdash \neg\exists x.\neg\varphi \Rightarrow \forall x.\varphi} \\ \Rightarrow\text{-ELIM} \frac{\Delta}{\vdash \neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi} \end{array}$$

where $\Delta =$

$$\frac{\vdots}{\vdash (\neg(\exists x.\neg\varphi) \Rightarrow \forall x.\varphi) \Rightarrow \neg(\forall x.\varphi) \Rightarrow \exists x.\neg\varphi}$$

follows from Example 2.2. □