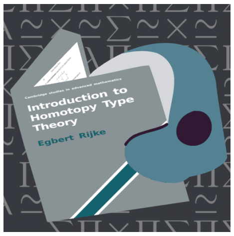


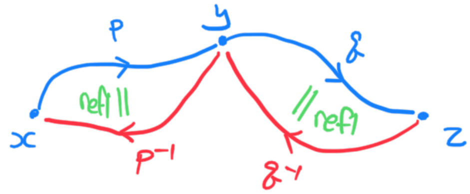
Worksheet 3

HoTTEST Summer School 2022

The HoTTEST TAs, and

11 July 2022

1 (⋆)



Let A be a type, and $x, y, z : A$. Show that path inversion distributes over concatention, i.e., construct a term of the following type:

$$\Pi_{p:x=y}\Pi_{q:y=z}\underbrace{(\overbrace{p \cdot q}^{x=z})^{-1}}_{z=x} = \underbrace{\overbrace{q}^{z=y}{}^{-1} \cdot \overbrace{p}^{y=x}{}^{-1}}_{z=x}$$

2 (⋆⋆)

Let A and B be types. We can define the product type as $A \times B := \Sigma_A B$, where B is considered a constant type family over A. The resulting elimination principle is:

$$\frac{\Gamma, z : A \times B \vdash D(z) \text{ type} \quad \Gamma, a : A, b : B \vdash d : D(a, b)}{\Gamma, z : A \times B \vdash \mathsf{ind}_\times(d, z) : D(z)} \times\text{-Elim}$$

Define the two projections $\mathsf{pr}_1 : A \times B \to A$ and $\mathsf{pr}_2 : A \times B \to B$ using the elimination principle.

$, b : B \vdash b : B$

B

$)$

$\mathsf{ind}_\times(a, z)$

Now use the elimination principle to give a term of the following type:

$$\Pi_{z:A\times B}\, \underset{(a,b)}{z} = (\underset{a}{\mathsf{pr}_1 z}, \underset{b}{\mathsf{pr}_2 z})$$

3 (⋆)

Let A and B be types. Give an informal construction of a term of the following type:

$$\Pi_{a,a':A}\Pi_{b,b':B}(a \overset{\mathsf{refl}_a}{=_A} a') \times (b \overset{\mathsf{refl}_b}{=_B} b') \to \big((a,b) =_{A\times B} (a',b')\big)$$

(We will later see that this map is an *equivalence*, and this map *characterises* paths in product types as pairs of paths between the components.)

4 (⋆⋆)

Let A be a type and $a : A$. Show that refl_a is unique among paths starting at a but with the other endpoint free. That is, for any $z : \Sigma_{x:A}(a = x)$ construct a term

$$(a, \mathsf{refl}_a) =_{\Sigma_{x:A}(a=x)} z \quad (\underbrace{a'}_{\overset{\shortparallel}{a}}, \underbrace{a = a'}_{\mathsf{refl}_a})$$

(We emphasize that this does not mean that refl_a is unique as a loop at a!)

(This is Prop. 5.5.1 in Egbert's book.)

5 (⋆⋆⋆)

Let A be a type. In the third lecture, a function concat of the following type was constructed:

$$\mathsf{concat} : \Pi_{x,y,z:A}(x = y) \to ((y = z) \to (x = z))$$

Call the above for concat_1. Define two different terms concat_2 and concat_3 of the same type.

Choose your favorite numbers $i, j \in \{1, 2, 3\}$ $(i \neq j)$. Give a term of the following type:

$$\Pi_{x,y,z:A}\Pi_{p:x=y}\Pi_{q:y=z}\mathsf{concat}_i(p, q) = \mathsf{concat}_j(p, q)$$

6 $(\star\star\star)$

Let A be a type with an element $a : A$. Can you construct a term of the following type? If not, what goes wrong?

$$\Pi_{p:(a=_A a)} p =_{(a=_A a)} \mathsf{refl}_a$$