



# Worksheet 1

HoTTEST Summer School 2022

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## 1 (★)

State the introduction and elimination rules for

1.  $\times$ -types 
$$\frac{\Gamma \vdash a : \tau_1 \quad \Gamma \vdash b : \tau_2}{\Gamma \vdash (a, b) : \tau_1 \times \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2}$$
2.  $\rightarrow$ -types 
$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1 e_2 : \tau_2}$$
3.  $\prod$ -types

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2(x)}{\Gamma \vdash \lambda x : \tau_1. e : \prod_{x : \tau_1} \tau_2(x)}$$

$$\frac{\Gamma \vdash f : \prod_{x : A} B(x) \quad \Gamma \vdash a : A}{\Gamma \vdash f a : B[x \mapsto a] = B(a)}$$

## 2 (★)

Fill in this proof tree:

$$\frac{\frac{\frac{a : A, b : B \vdash a : A}{a : A, b : B} \quad \frac{a : A, b : B \vdash b : B}{(a, b) : A \times B}}{a : A \vdash \lambda(b : B).(a, b) : B \rightarrow A \times B} \quad \frac{}{\cdot \vdash \lambda(a : A).\lambda(b : B).(a, b) : A \rightarrow B \rightarrow A \times B}$$

## 3 (\*\*)

Write a proof tree ending with a term of type  $A \times B \rightarrow B \times A$  in the empty context.

$$\begin{array}{c}
 \frac{}{x:A \times B \vdash x:A \times B} \quad \frac{}{x:A \times B \vdash x:A \times B} \\
 \frac{}{x:A \times B \vdash \pi_2 x : B} \quad \frac{}{x:A \times B \vdash \pi_1 x : A} \\
 \frac{}{x:A \times B \vdash (\pi_2 x, \pi_1 x) : B \times A} \\
 \hline
 \vdash \lambda x:A \times B. (\pi_2 x, \pi_1 x) : A \times B \rightarrow B \times A
 \end{array}$$

## 4 (\*\*)

For problems 2 and 3, what is the *logical* content of the proof tree? That is, under the “types are theorem” interpretation of Curry-Howard, what theorems have we proven?

2. We've proved that if A then if B then A and B.

3. We've proved that if A and B then B and A.

Next, what is the *computational* content of the proof tree? That is, under the “programs are proofs” interpretation of Curry-Howard, what programs have we written?

2. We've written a function that takes an argument of type A, produces a function that takes an argument of type B and produce a pair.

3. We've written swap.

## 5 (\*\*\*)

Define the **swap** function  $\sigma_{A,B}$  of type

$$\sigma_{A,B} : \left( \prod_{x:A} \prod_{y:B} C(x,y) \right) \rightarrow \left( \prod_{y:B} \prod_{x:A} C(x,y) \right)$$

and show that  $\sigma_{B,A} \circ \sigma_{A,B}$  is (definitionally) equal to the identity.

$$\sigma_{A,B} = \lambda p : \prod_{x:A} \prod_{y:B} C(x,y). \lambda y:B. \lambda x:A. p \ x \ y$$

$$\begin{aligned}
 \sigma_{B,A} \circ \sigma_{A,B} &= \lambda p : \prod_{x:A} \prod_{y:B} C(x,y). \sigma_{B,A} (\sigma_{A,B} p) \equiv \lambda p : \prod_{x:A} \prod_{y:B} C(x,y). \sigma_{B,A} p \ x \ y \\
 &\equiv \lambda p : \prod_{x:A} \prod_{y:B} C(x,y). p
 \end{aligned}$$