Solution:

For simplicity, assume that log has base e, i.e. $\log = \ln$ (the solution is the same no matter what base we assume). The optimization problem we are trying to solve is:

$$\begin{aligned} \underset{\vec{p}}{\operatorname{argmin}} \quad & \sum_{i=1}^{d} p_i \log p_i \\ \text{s.t.} & \sum_{i=1}^{d} p_i = 1 \end{aligned}$$

Formulating the Lagrangian, we get

$$\mathcal{L}(\vec{p}, \lambda) = \sum_{i=1}^{d} p_i \log p_i + \lambda \left(1 - \sum_{i=1}^{d} p_i\right)$$

Taking the derivative w.r.t. p_i and λ:

$$\begin{aligned} \frac{\partial}{\partial p_i}\mathcal{L}(\vec{p}, \lambda) &= \log p_i + \frac{p_i}{p_i} - \lambda \implies \lambda - 1 = \log p_i \\ \frac{\partial}{\partial \lambda}\mathcal{L}(\vec{p}, \lambda) &= \sum_{i=1}^{d} p_i = 1 \end{aligned}$$

This says that $\log p_i = \log p_j, \forall i, j$, which implies that $p_i = p_j, \forall i, j$. Combining this with the constraint, we get that $p_i = \frac{1}{d}$, which is the uniform distribution.