Solution: (a) We have

$$J(\mathbf{v_2}, \mathbf{z_2}) \;=\; \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x_i^T x_i} - z_{i1}\mathbf{x_i^T v_1} - z_{i2}\mathbf{x_i^T v_2} - z_{i1}\mathbf{v_1^T x_i} + z_{i1}^2\mathbf{v_1^T v_1} + \tag{1}$$

$$z_{i1}z_{i2}\mathbf{v_1^T v_2} - z_{i2}\mathbf{v_2^T x_i} + z_{i1}z_{i2}\mathbf{v_2^T v_1} + z_{i2}^2\mathbf{v_2^T v_2}) \tag{2}$$

Take derivative respect to $\mathbf{z_2}$, we have

$$\frac{\partial J}{\partial z_{i2}} = \frac{1}{n}(-\mathbf{x_i^T v_2} + z_{i1}\mathbf{v_1^T v_2} - \mathbf{v_2^T x_i} + z_{i1}\mathbf{v_2^T v_1} + 2z_{i2}\mathbf{v_2^T v_2}) = \frac{1}{n}(-2\mathbf{x_i^T v_2} + 2z_{i2}\mathbf{v_2^T v_2})$$

Set the derivative to 0 and we have

$$z_{i2}\mathbf{v_2^T v_2} = \mathbf{x_i^T v_2}$$

Since $\mathbf{v_2^T v_2} = 1$, we have $z_{i2} = \mathbf{x_i^T v_2}$

(b) Plug in z_{i2} into $J(\mathbf{v_2}, \mathbf{z_2})$, we have

$$J(\mathbf{v_2}) = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x_i^T x_i} - z_{i1}\mathbf{x_i^T v_1} - z_{i2}\mathbf{x_i^T v_2} - z_{i1}\mathbf{v_1^T x_i} + z_{i1}^2\mathbf{v_1^T v_1} - z_{i2}\mathbf{v_2^T x_i} + z_{i2}^2\mathbf{v_2^T v_2})$$

$$= \frac{1}{n}\sum_{i=1}^{n}(const - 2z_{i2}\mathbf{x_i^T v_2} + z_{i2}^2) = \frac{1}{n}\sum_{i=1}^{n}(-2\mathbf{v_2^T x_i x_i^T v_2} + \mathbf{v_2^T x_i x_i^T v_2} + const)$$

$$= -\mathbf{v_2^T C v_2} + const$$

In order to minimize J with constraints $\mathbf{v_2^T v_2} = 1$, we have Langrage $L = -\mathbf{v_2^T C v_2} + \lambda(\mathbf{v_2^T v_2} - 1)$ and take derivative of $\mathbf{v_2}$, we have

$$\frac{\partial L}{\partial \mathbf{v_2}} = -2\mathbf{C v_2} + 2\lambda v_2 = 0$$

Then, we have

$$\mathbf{C v_2} = \lambda \mathbf{v_2}$$