

ORBIFOLDS AS STACKS

Ex (Functor of points) Let \mathcal{C} be a category of geom. obj's. By the Yoneda lemma, \exists a fully faithful embedding

$$\mathcal{C} \hookrightarrow \text{Set}^{\mathcal{C}^{\text{op}}} \quad X \mapsto \text{Hom}(-, X).$$

↖ "generalized objects"

Ex ① $\text{Mfld} \rightarrow \text{Set}, U \mapsto \{\text{families of pseudo-hol curves in } (X, J) \text{ over } U\}$
should be repr. by $\mathcal{M}(X, J)$

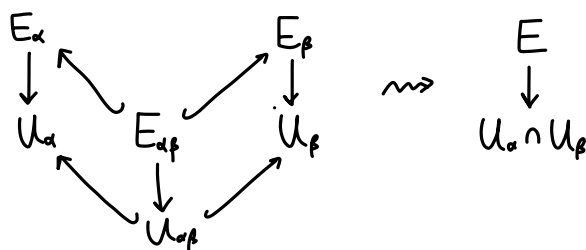
② $\text{Mfld} \rightarrow \text{Set}, U \mapsto \{\text{families of lines thru origin in } \mathbb{R}^n \text{ over } U\}$
is represented by $\mathbb{R}P^{n-1}$

③ $\text{Top} \rightarrow \text{Set}, U \mapsto \text{Vect}_k(U).$

Prob Isom. classes of vector bundles don't glue: "Vect_k(-) is not a sheaf"

$$E_\alpha \in \text{Vect}_k(U_\alpha) + E_\beta \in \text{Vect}_k(U_\beta) \not\sim E \in \text{Vect}_k(X)$$

Idea Consider functors $\mathcal{C}^{\text{op}} \rightarrow \text{Grpd}$. Now vector bundles glue:



Formally, $F: \mathcal{C}^{\text{op}} \rightarrow \text{Grpd}$ should satisfy **descent**:

$$F(X) \xrightarrow{\cong} \lim_{\leftarrow} (\prod_{\alpha} F(U_{\alpha}) \rightrightarrows \prod_{\alpha, \beta} F(U_{\alpha} \cap U_{\beta}))$$

↖ limit in 2-cat sense

Descent Let $F: \text{Top}^{\text{op}} \rightarrow \text{Grpd}$ be a functor. Then a **descent datum** consists of:

$$\begin{array}{ll} \text{① } P_{\alpha} \in F(U_{\alpha}) \quad \forall \alpha & \text{② } P_{\alpha}|_{U_{\alpha\beta}} \xrightarrow{\phi_{\alpha\beta}} P_{\beta}|_{U_{\alpha\beta}} \quad \text{s.t.} \quad \phi_{\alpha\gamma} = \phi_{\beta\gamma} \circ \phi_{\alpha\beta} \\ \downarrow & \text{"cocycle condition"} \\ P_{\alpha}|_{U_{\alpha\beta}} \in F(U_{\alpha\beta}) & \end{array}$$

Let $\mathcal{D}(\{U_{\alpha}\})$ be the category of descent data. Then F satisfies **descent** if

$$F(X) \xrightarrow{\cong} \mathcal{D}(\{U_{\alpha}\}).$$

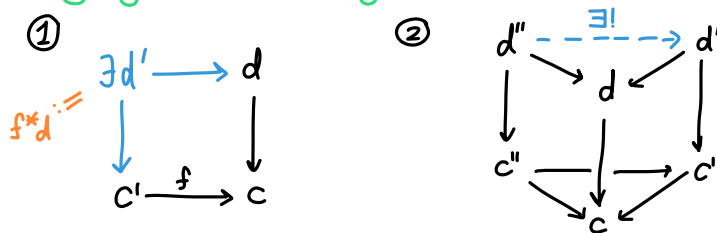
In other words,

- ① **Ess. surj.** \Rightarrow any descent datum glues,
- ② **Faithful** \Rightarrow morphisms are uniquely determined by a cover,
- ③ **Full** \Rightarrow morphisms glue

Def F is a **stack** if it satisfies descent.

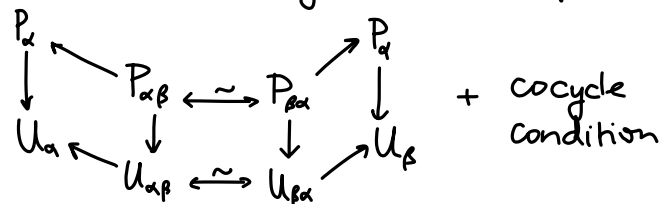
A more convenient language for stacks:

Def $\mathcal{D} \rightarrow \mathcal{C}$ is a category fibered in groupoids if



Rmk Given a CFG $\mathcal{D} \rightarrow \mathcal{C}$, we get a functor $\mathcal{C} \rightarrow \text{Grpd}$.

Def Descent data is defined similarly: (when $C = \text{Top}$)



$\mathcal{D} \xrightarrow{\pi} \text{Top}$ is a **stack** if

$$\pi^{-1}(X) \cong \text{descent cat for } \{U_\alpha\}$$

for any X and $\{U_\alpha\}$.

Ex Any $X \in \text{Top}$ determines a stack \underline{X} given by $Z \mapsto \text{Top}(Z, X) \in \text{Set} \subseteq \text{Grpd}$.

Ex Given a Lie grpd G , the category of principal G -bundles is a stack/Mfld.
This stack is called BG .

Ex For $G = \{\Gamma \times M \rightrightarrows M\}$, this is the category where the objects are maps $N \rightarrow M/\Gamma$ given by gluing maps $U_\alpha \rightarrow M \rightarrow M/\Gamma$.

Ex By the Yoneda lemma, a map $\underline{X} \rightarrow \mathcal{X}$ is an obj. of \mathcal{X} lying over X . This map sends a map $Z \rightarrow X$ to the pullback of the fixed object.

Ex Given a bibundle $P: G \rightarrow H$, define $BP: BG \rightarrow BH$ by composition of bibundles. If $F: BG \rightarrow BH$ is a morphism of stacks, take $P = F(G \rightarrow G_0)$.

Def (Fiber products) Given $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \Rightarrow \mathcal{C}$ CFG's, and maps $\mathcal{X}, \mathcal{Y} \Rightarrow \mathcal{Z}$, let $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ be the CFG

$$\text{Ob} = \left\{ \begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad} & \mathcal{Z} \\ & \searrow & \downarrow w \\ & & \mathcal{Y} \end{array} \right\}, \quad \text{Mor} = \left\{ \begin{array}{ccccc} \mathcal{X} & \xrightarrow{\quad} & \mathcal{X}' & \xrightarrow{\quad} & \mathcal{Z}' \\ & \searrow & \downarrow w & \searrow & \downarrow w' \\ & & \mathcal{Y} & \xrightarrow{\quad} & \mathcal{Y}' \end{array} \right\}$$

Def (Representable maps) A morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ of stacks is representable if $\forall Z \in \text{Top}, \mathcal{X} \times_{\mathcal{Y}} Z \in \text{Top}$.

For a property P satisfied by top \mathcal{X} spaces, f satisfies P if its pullback by every $Z \in \text{Top}$ satisfies P .

Ex $\mathcal{U} \hookrightarrow \mathcal{X}$ is an open embedding if $\forall Z, Z \times_{\mathcal{X}} \mathcal{U} \rightarrow Z$ is an open embedding.

Ex (Surprising fact) The map $G_0 \rightarrow BG$ is a surjective submersion. In fact, if $\underline{M} \rightarrow BG$ is given by the principal G -bundle P ,

$$\begin{array}{ccc} P & \longrightarrow & G_0 \\ \downarrow & \lrcorner & \downarrow \\ \underline{M} & \longrightarrow & BG \end{array}$$

Def An atlas for a stack \mathcal{X} over Mfld is a surjective submersion $\underline{M} \rightarrow \mathcal{X}$.

A stack admitting an atlas is called a geometric/differentiable stack. Topological stacks are defined similarly.

Thm If $\underline{M} \rightarrow \mathcal{X}$ is an atlas, then $\mathcal{X} \cong BG$ for some $G = \{M \times_{\mathcal{X}} M \rightrightarrows M\}$.