Liouville Sectors

1. Motivation: Kontsevich's cosheaf conjecture

This talk (and the next one or two) will be about some of the results from the papers [GPS20] and [GPS24b]. Let me begin by explaining the motivation for these two papers (and also the third one [GPS24a]). The idea was to formulate and prove a version of a 2008 conjecture of Kontsevich [Kon09] called the cosheaf conjecture. To state the conjecture, consider a Weinstein manifold X. Recall that the *core* $\mathfrak{c}_X$ of X is by definition the set of all points in X which do not escape to infinity under the positive Liouville flow. Figure 1 illustrates two possible cores that may arise from equivalent Weinstein structures on an infinite pair of pants.

Conjecture 1.1 (Kontsevich's cosheaf conjecture)**.** There exists a natural homotopy cosheaf of A_∞-categories on $\mathfrak{c}_X$ whose global sections gives the wrapped Fukaya category of X.

In plainer language, the conjecture says that there should be a natural way of associating to each open subset $U \subseteq \mathfrak{c}_X$ an A_∞-category $\mathcal{C}(U)$ and to each inclusion of open subsets $U \subseteq V$ an A_∞-functor $\mathcal{C}(U) \to \mathcal{C}(V)$. The category $\mathcal{C}(\mathfrak{c}_X)$ should be the wrapped Fukaya category of X. Moreover, these categories should satisfy a "descent" property: given any collection of open subsets $\{U_i\}_{i \in I}$ of $\mathfrak{c}_X$ with union $U = \bigcup U_i$, the induced functor

$$\operatorname*{hocolim}_{J \subseteq I} \mathcal{C}\left(\bigcap_J U_i\right) \to \mathcal{C}(U)$$

should be a pre-triangulated equivalence of A_∞-categories. Thus, the conjecture essentially states that the wrapped Fukaya category of X can be computed from local information. It may help to compare this statement to the Seifert–van Kampen or Mayer–Vietoris theorems from algebraic topology.

Exercise 1.2 (Wrapped Fukaya categories of cotangent bundles)**.** The cotangent bundle T^*Q of a smooth manifold is a Weinstein manifold with core $\mathfrak{c}_{T^*Q} = Q$. Assuming the cosheaf conjecture, use the fact that $\mathsf{Perf}\,\mathcal{C}(B) \simeq \mathsf{Perf}\,\mathbb{Z}$ for any open ball $B \subseteq Q$ to prove

$$\mathsf{Perf}\,\mathcal{W}(T^*Q) \simeq \mathsf{Perf}\,C_{-*}(\Omega Q).$$

This result is originally due to [AS06] and [Abo12] (proven without the cosheaf conjecture) and can be thought of as an "open string" version of the Viterbo isomorphism.

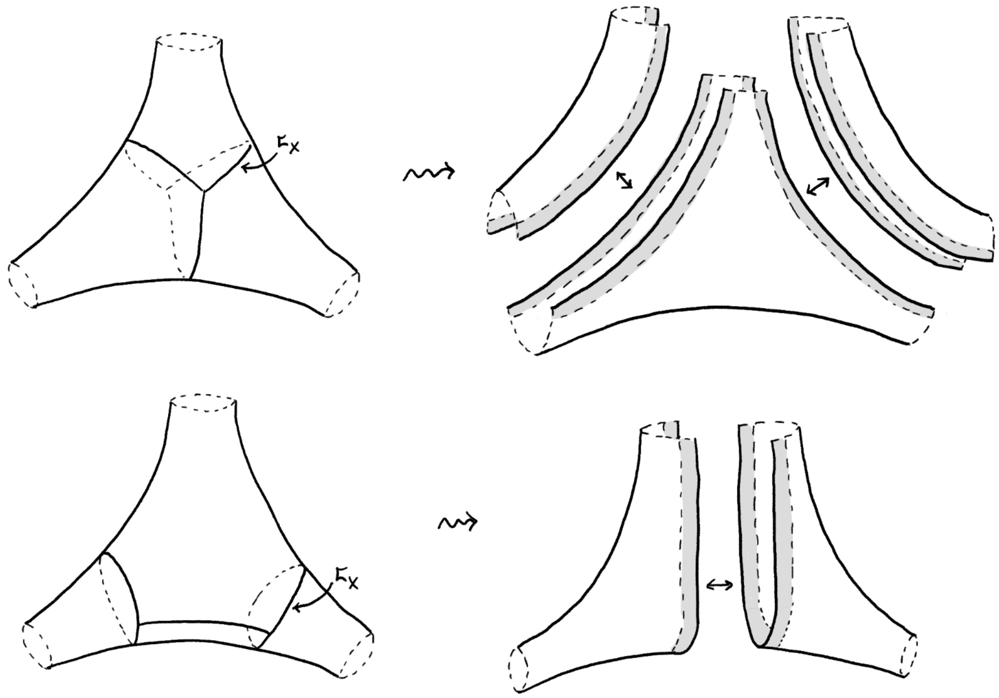

Figure 1: Two (equivalent) Weinstein structures on the infinite pair of pants and corresponding sectorial decompositions. The individual sectors should be thought of as preimages under the Liouville flow of open subsets in the corresponding core.

The formulation of the cosheaf conjecture proven in [GPS24b] is slightly different from the statement above. Indeed, from the perspective of Floer theory it is not entirely clear how one should define the category $\mathcal{C}(U)$ for an arbitrary open subset U (of course, this is possible with sheaf-theoretic techniques). Thus, instead of working with open subsets of the core $\mathfrak{c}_X$, we consider "nice" codimension zero submanifolds with boundary $X' \subseteq X$ called *Liouville sectors*. Figure 1 shows two different ways in which a pair of pants can be decomposed into Liouville sectors. The first paper [GPS20] defines the wrapped Fukaya category $\mathcal{W}(X')$ of a Liouville sector and proves it is covariantly functorial with respect to inclusions. The descent property is now stated in terms of Liouville sectors.

Theorem 1.3 (Sectorial descent)**.** Given a Weinstein sectorial covering $\{X_i\}_{i\in I}$ of X (defined in Definition 3.20), the induced functor

$$\operatorname*{hocolim}_{J\subseteq I} \mathcal{W}(X_i) \to \mathcal{W}(X)$$

is a pre-triangulated equivalence.

Exercise 1.4 (Mirror symmetry for a pair of pants)**.** We expect from SYZ mirror symmetry that the infinite pair of pants X is mirror to the union of two complex lines $\hat{X} = \{xy = 0\}$. Let us outline a local-to-global way of verifying this claim. On the A-side we can decompose X into sectors X_1 and X_2 as in the bottom of Figure 1, and on the B-side we can decompose $\hat{X}$ into the union of two complex lines. These decompositions should be thought of as mirror to each other; indeed, the sectors X_1, X_2 are mirror to complex lines and the sector $X_1 \cap X_2$ is mirror to a point. From these two decompositions we obtain commutative squares

$$\begin{tikzcd} \mathcal{W}(X_1 \cap X_2) \arrow[r] \arrow[d] & \mathcal{W}(X_1) \arrow[d] \\ \mathcal{W}(X_2) \arrow[r] & \mathcal{W}(X) \end{tikzcd} \qquad \begin{tikzcd} D^b\mathsf{Coh}(*) \arrow[r] \arrow[d] & D^b\mathsf{Coh}(\mathbb{A}^1) \arrow[d] \\ D^b\mathsf{Coh}(\mathbb{A}^1) & D^b\mathsf{Coh}(\hat{X}) \end{tikzcd}$$

Both of these diagrams are pushout squares; the first by sectorial descent and the second can be checked by hand. One can further define a morphism of spans

$$\begin{tikzcd} \mathcal{W}(X_2) \arrow[d] & \mathcal{W}(X_1 \cap X_2) \arrow[l] \arrow[r] \arrow[d] & \mathcal{W}(X_1) \arrow[d] \\ D^b\mathsf{Coh}(\mathbb{A}^1) & D^b\mathsf{Coh}(*) \arrow[l] \arrow[r] & D^b\mathsf{Coh}(\mathbb{A}^1) \end{tikzcd}$$

and explicitly show that each vertical map is a pre-triangulated equivalence. This induces a pre-triangulated equivalence $\mathcal{W}(X) \to D^b\mathsf{Coh}(\hat{X})$, thus verifying mirror symmetry for X and $\hat{X}$.

2. Stable Hamiltonian hypersurfaces

Although the goal of this talk will be to introduce the notion of a Liouville sector, let us first discuss something which may at first seem completely unrelated: stable Hamiltonian structures. This is a generalization of a contact structure which originally arose in [HZ94] as a setting in which the Weinstein conjecture could still be proven. The reason we are interested in them is that they provide a general setting in which SFT still works; this will help motivate the somewhat abstruse definition of a Liouville sector in the following section.

Definition 2.1 (Stable Hamiltonian structures)**.** A *stable Hamiltonian structure* (SHS) on Y^{2n-1} is a pair (ω, λ) consisting of:

- a closed 2-form ω of maximal rank,
- a 1-form λ such that $\lambda|_{\ker \omega} \neq 0$ and $\ker \omega \subseteq \ker d\lambda$.

The *Reeb vector field* of Y is the unique vector field tangent to $\ker \omega$ such that $\lambda(R) = 1$.

Example 2.2 (Contact forms)**.** If α is a contact form on Y, then $(d\alpha, \alpha)$ is a stable Hamiltonian structure whose Reeb vector field corresponds with the Reeb vector field of α.

We will primarily be interested in *stable Hamiltonian hypersurfaces*, i.e., hypersurfaces $Y^{2n-1} \subseteq (X^{2n}, \omega)$ which admit an SHS of the form $(\omega|_Y, \lambda)$. Note that in this case $\omega|_Y$ is always closed and of maximal rank, so we just need to find a 1-form λ satisfying the desired properties. Also recall that the 1-dimensional distribution $C := \ker(\omega|_Y)$ is called the *characteristic foliation* of Y.

Example 2.3 (Contact type hypersurfaces)**.** Recall that Y is a *contact type hypersurface* if in a neighborhood of Y the symplectic form ω admits a primitive λ whose Liouville vector field Z is transverse to Y. Then $(\omega|_Y, \lambda|_Y)$ is an SHS.

Example 2.4 (Sectorial hypersurfaces, sort of)**.** If H is a Hamiltonian on X, then $(\omega|_Y, dH|_Y)$ is an SHS on Y iff X_H is transverse to Y. This example will be important later when we define a Liouville sector. Note that H must either strictly increase or decrease in the direction of C, which implies the leaves C are all embedded copies of $\mathbb{R}$. In particular, the symplectic reduction $F = Y/C$ is well-defined.

Example 2.5 (Regular energy surfaces of Hamiltonian circle actionsyeah)**.** Suppose X has a Hamiltonian circle action with moment map H, and let $Y = H^{-1}(0)$ be a regular energy surface. Then C is an S^1-foliation and the symplectic reduction $F = Y/C$ is defined. Let λ be any connection 1-form on the principal S^1-bundle $Y \to F$. Then $(\omega|_Y, \lambda)$ is a stable Hamiltonian structure on Y.

Exercise 2.6. Suppose Y is a stable Hamiltonian hypersurface in X, and let Z be the dual vector field of λ. For a small time t, let Y_t be the image of Y under the time t flow of Z. Show that the flow of Z sends the characteristic foliation of Y to the characteristic foliation of Y_t.

As mentioned earlier, stable Hamiltonian structures are a good setting for SFT. Let us very briefly outline how one of the main ideas in SFT, the "stretching of the neck" procedure, works. Suppose Y is a stable Hamiltonian hypersurface which separates X into two pieces X_- and X_+. The SHS on Y implies the existence of a tubular neighborhood of Y which is symplectomorphic to

$$((-\varepsilon, \varepsilon)_r \times Y, \omega|_Y + d(r\lambda)),$$

see [Wen16] for a proof. The idea will be to make this "neck" region longer and longer until the manifold X splits into two pieces. More precisely, for a large number $T > 0$, consider coordinates on the neck given by a level-preserving diffeomorphism

$$(-T, T) \times Y \to (-\varepsilon, \varepsilon) \times Y,$$

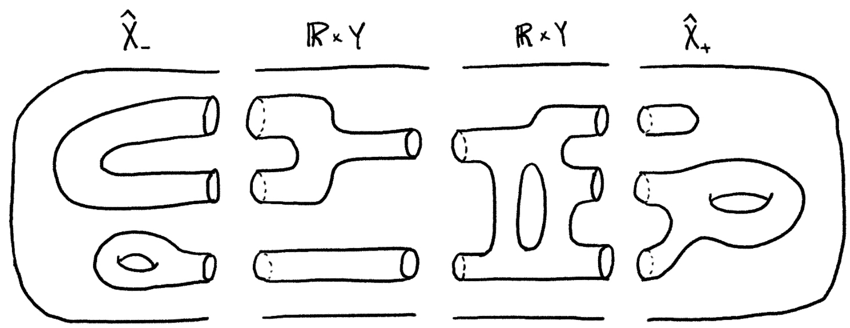

Figure 2: A holomorphic building resulting from stretching the neck

and equip X with an almost complex structure J_T which is cylindrical with respect to these coordinates. As we take $T \to \infty$, the almost complex structures J_T will degenerate (it is worthwhile to think about this statement in the dimension $n = 1$ case), and we imagine that X splits into two completed halves $\hat{X}_-$ and $\hat{X}_+$. By placing some additional assumptions on J_T (which can be satisfied precisely because Y has a SHS), a version of the SFT compactness theorem [BEH+03] now implies that any sequence of holomorphic curves u_T in (X, J_T) will admit a subsequence u_{T_n} with $T_n \to \infty$ converging to a *holomorphic building* u_∞, i.e., a holomorphic curve with multiple levels, consisting of a bottom level in $\hat{X}_-$, several middle levels in different copies of the symplectization of Y, and a top level in $\hat{X}_+$. One very important fact to know is that the different components of u_∞ tend asymptotically to Reeb orbits of Y. See Figure 2 for a picture.

3. Liouville sectors

3.1. The definition

We first recall the definition of a Liouville manifold. We emphasize that our definition of a Liouville manifold does not come equipped with a fixed cylindrical end. Some common definitions are different with this coordinate-free approach (see for instance the definition of a linear Hamiltonian in the first bullet point of Definition 3.3), but the arguments tend to be cleaner this way.

Definition 3.1 (Liouville manifolds)**.** A *Liouville manifold* is an exact symplectic manifold (X, λ) for which a neighborhood at infinity is diffeomorphic to the positive half of a symplectization

via a diffeomorphism respecting Liouville forms. A *Liouville manifold with boundary* is defined in the same way, except the manifolds X and Y are now allowed to have boundary. The *Liouville vector field* of X is the dual vector field Z of the 1-form λ.

Exercise 3.2. Show that the contact manifold Y above is independent of the choice of cylindrical end. We will refer to Y as the *boundary of X at infinity* and denote it by $\partial_\infty X$.

Definition 3.3 (Liouville sectors)**.** A *Liouville sector* is a Liouville manifold with boundary X for which there exists a function $I : \partial X \to \mathbb{R}$ such that

- I is linear at infinity, i.e., $ZI = I$ near infinity (note this is different from another common definition of a linear Hamiltonian, found for example in [Abo15]),
- the Hamiltonian vector field X_I is outward pointing along ∂X.

Strictly speaking, the second bullet point makes sense only after extending I to a neighborhood of ∂X, but it turns out that the extension does not matter. This follows, for instance, from the following exercise.

Exercise 3.4. Orient the characteristic foliation of ∂X so that the positive direction C satisfies $\omega(N, C) > 0$ for an inward pointing vector field N. Show that the second condition in Definition 3.3 is equivalent to the condition that I is strictly increasing in the positive direction of the characteristic foliation. (This is how Liouville sectors are defined in [GPS20].)

You may also notice that the second condition in Definition 3.3 implies by Example 2.4 that $(\omega|_{\partial X}, dI)$ is an SHS on ∂X. In fact, the example shows that the leaves of the characteristic foliation of ∂X are embedded copies of $\mathbb{R}$, so the leaf space is a symplectic manifold by symplectic reduction. We summarize the structure of ∂X in the following proposition.

Proposition 3.5 (Structure of ∂X)**.** Let $F = I^{-1}(0)$. Then:

(i) $(F, \lambda|_F)$ is a Liouville manifold.

(ii) F is symplectomorphic to the symplectic reduction of X by its characteristic foliation.

(iii) The quotient map $\partial X \to F$ and the function I specify a diffeomorphism $\partial X \cong \mathbb{R} \times F$ sending the characteristic foliation of ∂X to the horizontal foliation on $\mathbb{R} \times F$.

See Figure 3.

Proof. We will prove (i), leaving (ii) and (iii) as exercises. Fix a choice of cylindrical end $[1, \infty)_r \times \partial_\infty X$ of X. On this cylindrical end, we can write $I(r, y) = b(y)r$ for some function $b : \partial_\infty X \to \mathbb{R}$ (this is precisely what it means for I to be linear at infinity). Observe that the intersection of F with our fixed cylindrical end is given

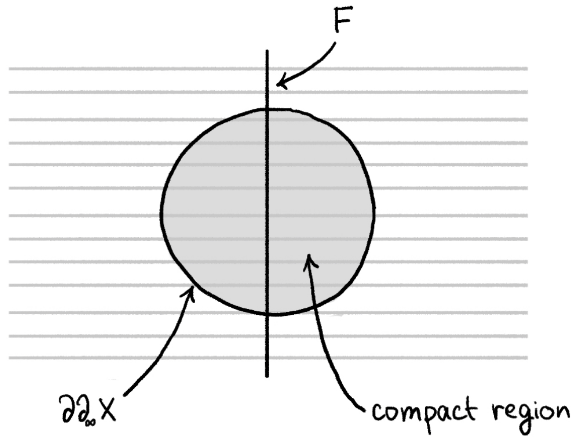

Figure 3: The boundary of a Liouville sector. The Liouville vector field (not shown) points in the outward radial direction. Note this figure is only a cartoon since the boundary of a sector is never 2-dimensional.

by $[1, \infty)_r \times b^{-1}(0)$. In particular, we see that the Liouville vector field $r\partial_r$ is tangent to F, so it must also be the Liouville vector field of F itself. This exhibits a cylindrical end of F on which the Liouville flow is complete, so we are done. □

The observation that ∂X is a stable Hamiltonian hypersurface allows us to use ideas from SFT to give the following motivation for Definition 3.3. Imagine performing a neck stretch on X along the hypersurface ∂X. Since ∂X has no closed characteristics, any holomorphic building obtained in the limit cannot have components in $\mathbb{R} \times \partial X$, and thus has only a single level contained in the interior of X. This means that holomorphic curves should stay away from ∂X once the neck has been sufficiently stretched; more precisely, there should exist an almost complex structure on X for which holomorphic curves avoid a neighborhood of ∂X. While this is purely motivation, a precise statement along these lines can be proven, which we will see in Proposition 3.13. Having this strong control on holomorphic curves near ∂X is crucial for several reasons when defining the wrapped Fukaya category of X, for instance in ensuring that Gromov compactness still holds.

3.2. Examples

Example 3.6 (Cotangent bundles)**.** The cotangent bundle T^*Q of a manifold with boundary is a Liouville sector. To see this, consider the decomposition near the boundary

$$T^*Q \cong T^*\partial Q \times T^*[0, \varepsilon) = T^*\partial Q \times [0, \varepsilon)_s \times \mathbb{R}_t.$$

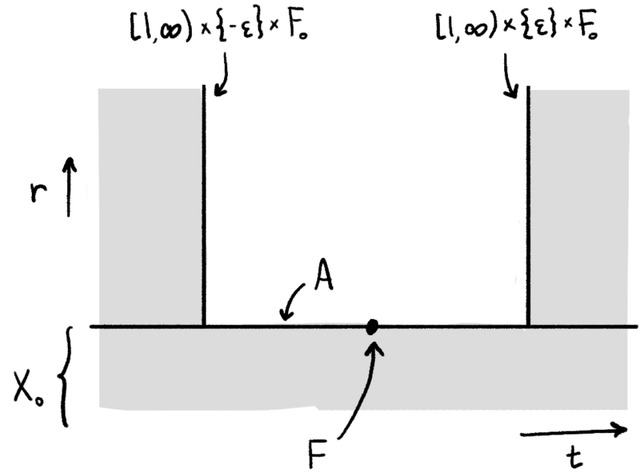

Figure 4: The Liouville sector obtained from a sutured Liouville domain.

The function $I = t$ is linear at infinity and has the outward pointing Hamiltonian vector field $-\partial_s$, as required.

Example 3.7 (Punctured bordered Riemann surfaces)**.** Let X be a compact bordered Riemann surface punctured at a finite number of points. Then X is a Liouville sector iff each boundary component of X is homeomorphic to $\mathbb{R}$. Some examples are pictured in Figure 1.

Example 3.8 (Sutured Liouville domains)**.** Let (X_0, λ) be a Liouville domain, and let $F_0 \subseteq \partial X_0$ be a codimension one submanifold with boundary such that $(F_0, \lambda|_{F_0})$ is a Liouville domain. The condition that $d\lambda|_{F_0}$ is a symplectic form on F_0 is equivalent to the Reeb vector field of ∂X_0 being transverse to F_0, so the Reeb flow exhibits an embedding

$$A := (-\varepsilon, \varepsilon)_t \times F_0 \hookrightarrow \partial X_0$$

Consider the completion of X_0 in the complement of A given by

$$X := X_0 \cup_{\partial X_0 - A} ([1, \infty)_r \times (\partial X_0 - A))$$

with the usual Liouville 1-form (see Figure 4). Note that X is a manifold with corners, but let us ignore this issue for now. In fact, the corner structure provides a convenient decomposition of ∂X into the four faces

$$\partial X = A \cup ([0, \infty) \times (-\varepsilon, \varepsilon) \times \partial F_0) \cup ([1, \infty) \times \{-\varepsilon\} \times F_0) \cup ([1, \infty) \times \{\varepsilon\} \times F_0).$$

To show that X is a Liouville sector, consider the linear Hamiltonian $I = -tr$ on ∂X, where t and r are the coordinates in the above definitions of A and X. We now show that the Hamiltonian vector field X_I is outward pointing on each of the four faces:

- On A, the function I is given by the negative Reeb coordinate $-t$. Thus X_I is the Liouville vector field of X_0, which is outward pointing because X_0 is a Liouville domain.
- On $[1, \infty) \times \{-\varepsilon\} \times F_0$, the function I is given by the symplectization coordinate r. Thus, X_I is precisely the Reeb vector field ∂_t, which again is outward pointing. Similar reasoning works for the face $[1, \infty) \times \{\varepsilon\} \times F_0$.
- The remaining face $[0, \infty)_r \times (-\varepsilon, \varepsilon)_t \times \partial F_0$ is a bit tricky. Let us denote this face by V. First, observe that the submanifold $S = \{t = 0\}$ equipped with the restriction of λ is precisely the symplectization of ∂F_0. One can show that there is an isomorphism respecting 1-forms

  $$(V, \lambda|_V) \cong ((-\varepsilon, \varepsilon)_t \times S, \lambda|_S + r dt).$$

  In view of this isomorphism, it is straightforward to check that the positive direction of the characteristic foliation of V (in the sense of Exercise 3.4) is given by the vector field $R_{\partial F_0} - \partial_t$, where $R_{\partial F_0}$ is the Reeb vector field of the contact manifold $(\partial F_0, \lambda)$. Since I is increasing along this vector field V, Exercise 3.4 implies X_I is outward pointing along V.

We have shown that X satisfies the definition of a Liouville sector as long as we ignore the corners. It turns out the corners do not pose an actual issue. Roughly, the idea is that we can smooth out the corners in a "convex" way, so that the positive direction of the characteristic foliation in the smoothed boundary lies in the convex hull of the positive directions of the characteristic foliation in the original boundary. In particular, this means the condition of Exercise 3.4 is preserved by this smoothing process. This completes the proof that X is a Liouville sector. (See Lemma 2.13 and Definition 2.14 in [GPS20] for a different and arguably simpler proof of this result.)

Exercise 3.9. Show that Example 3.7 is a special case of Example 3.8. In fact, there is a precise sense in which every Liouville sector is equivalent to one obtained from Example 3.8, cf. Section 2.7 in [GPS20].

Exercise 3.10. Let X and F_0 be as in Example 3.8. Show that the symplectic reduction of ∂X by its characteristic foliation is given by the completion of F_0.

Example 3.11 (Legendrian stops)**.** Let X_0 be a Liouville domain and $\Lambda \subseteq \partial X$ a Legendrian in its contact boundary. The Weinstein tubular neighborhood theorem tells us that a neighborhood of Λ in ∂X_0 is given by an open subset of the jet bundle $J^1\Lambda = T^*\Lambda \times \mathbb{R}$. Under this identification, the disk bundle $D^*\Lambda \times \{0\}$ is a Liouville domain sitting inside ∂X_0, so we may apply the above construction to obtain a Liouville sector X.

This construction is essentially the same as adding a Legendrian *stop* to X_0, which we will discuss more in the next talk. Namely, we will consider the *partially wrapped Fukaya category* of (the completion of) X_0 with a stop at Λ, which is defined by the usual wrapping procedure, except we no longer allow Lagrangians to wrap through Λ. This is essentially equivalent to the wrapped Fukaya category of the Liouville *sector* X, since Lagrangians wrapping past Λ would now have to leave X.

3.3. Holomorphic curves near the boundary

We now return to the idea that holomorphic curves must stay away from the boundary of a Liouville sector. The idea will be to strengthen the result of Proposition 3.5 by showing a neighborhood of the boundary admits a particularly nice product structure.

Proposition 3.12 (Product decomposition near ∂X)**.** Let X be a Liouville sector with a choice of function I, and let $F = I^{-1}(0)$ be as in Proposition 3.5. Then there exists a cylindrical neighborhood of ∂X which is symplectomorphic to $F \times \mathbb{C}_{0\leq \mathrm{Re}\leq \varepsilon}$ with the product symplectic form.

Proof. We denote a small cylindrical neighborhood of ∂X by $\mathrm{Nbd}\,\partial X$. Let R (where the letter 'R' stands for "real," not "Reeb") be the unique function defined on $\mathrm{Nbd}\,\partial X$ satisfying

$$R|_{\partial X} = 0, \qquad X_I R = -1$$

Since dR vanishes on ∂X, the vector field X_R must lie in the symplectic complement of $T\partial X$, hence X_R is tangent to the characteristic foliation of ∂X. Moreover, the computation

$$[X_I, X_R] \lrcorner\, \omega = [X_I, X_R] \lrcorner\, \omega + X_R \lrcorner\, \mathcal{L}_{X_I}\omega = \mathcal{L}_{X_I}(X_R \lrcorner\, \omega) = \mathcal{L}_{X_I}(dR) = d(\mathcal{L}_{X_I}R) = 0$$

shows that $[X_I, X_R] = 0$. Thus, the flow of $(1/2)X_I$ and X_R define a diffeomorphism

$$F \times \mathbb{C}_{0\leq \mathrm{Re}\leq \varepsilon} \cong \mathrm{Nbd}\,\partial X.$$

We leave it as an exercise to show this is a symplectomorphism. (The factor of $1/2$ is necessary to match the symplectic form on $\mathbb{C}$.) □

To prevent holomorphic curves from approaching ∂X, we consider almost complex structures J such that the projection

$$\pi : \mathrm{Nbd}\,\partial X \to \mathbb{C}_{0\leq \mathrm{Re}\leq \varepsilon}$$

is J-holomorphic. We call such almost complex structures *adapted* to ∂X. In view of Proposition 3.12, it is clear that adapted almost complex structures exist.

Proposition 3.13 (Holomorphic curves stay away from ∂X)**.** Suppose J is adapted, and let $u : \Sigma \to X$ be a connected holomorphic curve (possibly with boundary) such that $u(\Sigma) \cap \operatorname{Nbd} \partial X$ is compact and disjoint from $u(\partial\Sigma)$. Then either u is constant or disjoint from $\operatorname{Nbd} \partial X$.

Proof. Our hypotheses together with the open mapping theorem from complex analysis imply that the image of

$$u^{-1}(\operatorname{Nbd} \partial X) \xrightarrow{u} \operatorname{Nbd} \partial X \xrightarrow{\pi} \mathbb{C}_{0 \leq \operatorname{Re} \leq \varepsilon}$$

is either empty or a point. □

Example 3.14 (Holomorphic disks with Lagrangian boundary)**.** When defining the wrapped Fukaya category of X, we will consider cylindrical at infinity Lagrangians $L_1, \ldots, L_k \subseteq X$ and holomorphic disks with boundary conditions on these Lagrangians. By choosing $\operatorname{Nbd} \partial X$ to be disjoint from the L_i, any holomorphic disk with boundary on the L_i must be disjoint from $\operatorname{Nbd} \partial X$ by Proposition 3.13. This, together with a geometric boundedness argument, will imply the moduli space of such disks is compact.

Remark 3.15. One issue with working with adapted almost complex structures is that it is not possible to guarantee that J is both adapted and contact type at infinity. This is a real issue (since the maximum principle no longer holds) and is discussed in [GPS20], but we will mostly ignore it.

3.4. Sectorial coverings

We now return to our original motivation for Liouville sectors and discuss a way of decomposing a Liouville manifold X into sectors so that Theorem 1.3 can be applied. The idea will be to cut X along hypersurfaces satisfying a condition which generalizes Definition 3.3.

Example 3.16 (Cutting along a single hypersurface)**.** As a warm-up, let us consider the case of a single hypersurface H which divides X into two cylindrical pieces X_- and X_+. The picture to keep in mind is the second decomposition in Figure 1. For X_- and X_+ to be sectors, we want the hypersurface H to admit a linear at infinity function $I : H \to X$ such that X_I is transverse to H and points from X_+ to X_-. Then I realizes X_+ as a Liouville sector and $-I$ realizes X_- as a Liouville sector. For $\{X_-, X_+\}$ to form a cover of X for which the statement of Theorem 1.3 makes any sense, we need to enlarge X_- and X_+ so that the intersection $X_- \cap X_+$ is also a

Liouville sector. This is certainly possible in view of Proposition 3.12. In fact, the proposition implies that one can arrange for the resulting intersection to be of the form

$$X_- \cap X_+ \cong F \times \mathbb{C}_{|\mathrm{Re}| \le \varepsilon} \cong F \times T^*[0,1]$$

where $F = I^{-1}(0)$ is as usual.

The key to making Example 3.16 work was the local model given by Proposition 3.12. In general, if we want to cut X along a (not necessarily disjoint) collection of hypersurfaces $H_1, \dots, H_n$, we will need a condition on the multiple intersections $H_{i_1} \cap \cdots \cap H_{i_k}$ to guarantee the existence of an analogous product neighborhood.

Definition 3.17 (Sectorial collection of hypersurfaces)**.** A collection $H_1, \dots, H_n$ of cylindrical hypersurfaces in X is *sectorial* if all multiple intersections $H_{i_1} \cap \cdots \cap H_{i_k}$ are coisotropic and there exist functions $I_i : \mathrm{Nbd}\, H_i \to \mathbb{R}$ such that

$$dI_i|_{C_i} \neq 0, \qquad dI|_{C_j} = 0 \text{ for } i \neq j, \qquad \{I_i, I_j\} = 0,$$

where C_i is the characteristic foliation of H_i.

We refer to Section 12 of [GPS24b] for a much more detailed treatment of sectorial collections. The following result (Lemma 12.8 and Remark 12.9 from [GPS24b]) is proven in essentially the same way as Proposition 3.12.

Proposition 3.18 (Local model near multiple intersections)**.** For a sectorial collection of hypersurfaces $H_1, \dots, H_n$, any multiple intersection $H_{i_1} \cap \cdots \cap H_{i_k}$ admits a cylindrical neighborhood which is symplectomorphic to a cylindrical neighborhood of $F \times \mathbb{R}^k$ in $F \times T^*\mathbb{R}^k$. In this local model, the hypersurfaces H_{i_j} are simply preimages of the coordinate hyperplanes in $\mathbb{R}^k$ under the projection $F \times T^*\mathbb{R}^k \to \mathbb{R}^k$. In fact, any mutually transverse collection of hypersurfaces in $\mathbb{R}^k$ lifts to a sectorial collection in this local model.

We can now generalize Example 3.16 to a sectorial collection of hypersurfaces $H_1, \dots, H_n$ in X. We require that these hypersurfaces split the manifold into pieces $X_1, \dots, X_m$ whose closures are embedded (this is analogous to the condition in Example 3.16 that H splits X into two pieces). As before, we would like to enlarge the X_i in a way so that they form a cover of X whose multiple intersections are Liouville sectors. However, we immediately run into an issue; unlike in Example 3.16, the X_i may now have corners corresponding to multiple intersections of the hypersurfaces H_i. In particular, it is not even clear that the X_i are Liouville sectors at all (what would their I-functions be?). To remedy this issue, we observe that the boundary faces of X_i form a sectorial collection, allowing us to apply the following result.

Exercise 3.19. Let X be a Liouville manifold with corners (defined by replacing every instance of the word "boundary" in the definition of a Liouville manifold with boundary with the word "corner") whose boundary faces $\partial^1 X, \ldots, \partial^n X$ form a sectorial collection realized by functions $I_1, \ldots, I_n$. We splice these functions together to define $I : \partial X \to \mathbb{R}$ as follows. On the face $\partial^i X$ and away from the corners, set $I = I_i$. Near the corners of ∂X, where we have a local model given by Proposition 3.18, we define

$$I = \sum_i \varphi(t_i) I_i,$$

where φ is a bump function supported near zero and $t_1, \ldots, t_n$ are the cotangent coordinates of $T^*\mathbb{R}^k$. Show that I realizes X as a Liouville sector. (The fact X has corners is an issue, see the argument at the end of Example 3.8.)

Thus, the X_i are indeed Liouville sectors. In fact, we claim that, for appropriate enlargements of the X_i, the multiple intersections $X_{i_1} \cap \cdots \cap X_{i_k}$ form Liouville manifolds which also satisfy the hypotheses of Exercise 3.19, and are thus Liouville sectors. The idea needed to choose these enlargements is that perturbing the sectorial collection H_i is easy to do by the last sentence in Proposition 3.18. Thus, we obtain the desired cover of X. On such a cover, Theorem 1.3 can be applied (after also assuming a Weinstein condition), which is stated for the following more general class of covers:

Definition 3.20 (Sectorial coverings)**.** A $X_1, \ldots, X_n$ cover of a Liouville manifold X by manifolds with boundary is *sectorial* if the hypersurfaces $\partial X_1, \ldots, \partial X_n$ form a sectorial collection.

Exercise 3.21. Apply Exercise 3.19 to show the multiple intersections $X_{i_1} \cap \cdots \cap X_{i_k}$ of a sectorial cover is a Liouville sector.

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