COMPUTATIONS USING THEOREM 6.4

LINDSAY MARJANSKI, VINCENT SOLON, FRANK ZHENG, KATHLEEN ZOPFF

Abstract. We prove the results of [1] and [2] using Theorem 6.4. We also provide a Sage program that automatically carries out these computations.

1. Statement of Theorem 6.4

Theorem 6.4. Let Γ be a link-regular simplicial graph with maximum clique size d. Let $\ell_0 = |V(\Gamma)|$, and for $1 \le k \le d$, let $\ell_k = |\mathrm{Lk}(\sigma)|$, where σ is any k-clique (this is well-defined by link-regularity of Γ). For integers $0 \le m \le d$ and $0 \le k \le m$, set

$$N_{m,k} = \begin{cases} \left(\prod_{k<j<m} \ell_j\right) \sum_{j=k}^{m} \binom{m-k}{j-k} (-1)^{j-k} \ell_j & \text{if } k < m-1 \\ \ell_{m-1} - \ell_m - 1 & \text{if } k = m-1 \\ 1 & \text{if } k = m, \end{cases}$$

and for integers $0 \le i \le d$ and $0 \le j \le i$, set

$$\begin{cases} M_{i,j} = \displaystyle\sum_{\substack{j \le s_1 < t_1 \le \cdots \le s_n < t_n \le d \\ (t_1 - s_1) + \cdots + (t_n - s_n) = i - j}} (-1)^n \binom{t_1}{s_1} \cdots \binom{t_n}{s_n} N_{t_1,s_1} \cdots N_{t_n,s_n} & \text{if } i \ne j \\ M_{i,j} = 1 & \text{if } i = j. \end{cases}$$

Then the geodesic growth of RACG(Γ) is given by the rational function

$$\mathscr{G}(z) = \frac{\sum_{i=0}^{d} \left(\sum_{j=0}^{i} \ell_0 \cdots \ell_{j-1} M_{i,j}\right) z^i}{\sum_{i=0}^{d} M_{i,0} z^i}.$$

2. Calculation when $d = 2$

First, we prove the Theorem 5.1 in [1].

Theorem 2.1. Let Γ be an ℓ-regular triangle-free graph with n vertices. The geodesic growth of RACG(Γ) is given by the rational function

$$\mathscr{G}(z) = \frac{1 - (\ell - 3)z + 2z^2}{1 + (-n - \ell + 3)z + (-2n + 2 + n\ell)z^2}.$$

Proof. We apply Theorem 6.4 in the case $d = 2$. Note that $\ell_0 = n$, $\ell_1 = \ell$, and $\ell_2 = 0$. For the $N_{m,k}$, we have

$$\begin{aligned}
N_{m,k} &= 1 && \text{if } m = k, \\
N_{1,0} &= \ell_0 - \ell_1 - 1 = n - \ell - 1 \\
N_{2,1} &= \ell_1 - \ell_2 - 1 = \ell - 1 \\
N_{2,0} &= \ell_1 \sum_{j=0}^{2} \binom{2}{j} (-1)^j \ell_j = \ell(n - 2\ell).
\end{aligned}$$

Next, we compute the $M_{i,j}$. We have $M_{i,j} = 1$ when $i = j$ by definition, so we only need to consider when $i \neq j$. Let us first work through the computation of $M_{2,0}$ in detail. By definition, we have

$$M_{2,0} = \sum_{\substack{0 \le s_1 < t_1 \le \cdots \le s_n < t_n \le 2 \\ (t_1 - s_1) + \cdots + (t_n - s_n) = 2}} (-1)^n \binom{t_1}{s_1} \cdots \binom{t_n}{s_n} N_{t_1,s_1} \cdots N_{t_n,s_n}$$

The "partitions" $(s_1 < t_1) \le \cdots \le (s_n < t_n)$ that this sum ranges over are $(0 < 1) \le (1 < 2)$ and $(0 < 2)$. Thus,

$$\begin{aligned}
M_{2,0} &= (-1)^2 \binom{1}{0} \binom{2}{1} N_{1,0} N_{2,1} + (-1)^1 \binom{2}{0} N_{2,0} \\
&= 2(n - \ell - 1)(\ell - 1) - \ell(n - 2\ell) \\
&= n\ell - 2n + 2.
\end{aligned}$$

The other $M_{i,j}$ are computed similarly:

$$\begin{aligned}
M_{1,0} &= (-1)^1 \binom{1}{0} N_{1,0} + (-1)^1 \binom{2}{1} N_{2,1} = -n - \ell + 3 \\
M_{2,1} &= (-1)^1 \binom{2}{1} N_{2,1} = -2\ell + 2.
\end{aligned}$$

Finally, the numerator of the geodesic growth is computed as

$$M_{0,0} + (M_{1,0} + nM_{1,1})z + (M_{2,0} + nM_{2,1} + n\ell M_{2,2})z^2 = 1 + (-\ell + 3)z + 2z^2.$$

and the denominator is

$$M_{0,0} + M_{1,0} z + M_{2,0} z^2 = 1 + (-n - \ell + 3)z + (n\ell - 2n + 2)z^2,$$

which completes the proof. □

3. Calculation when $d = 3$

Next, we prove the main result of [2]. (Note that in [2], they give a formula of $\mathscr{G}(z) - 1$, so the numerator of the below rational function is slightly different.)

Theorem 3.1. Let Γ be a link-regular graph with n vertices and no 4-cliques, and let $\ell = \ell_1$ and $q = \ell_2$ (where ℓ_1, ℓ_2 are as in Theorem 6.4). Then the geodesic growth of RACG(Γ) is given by the rational function

$$\frac{1 + (-\ell - q + 6)z + (\ell q - 3\ell - q + 11)z^2 + 6z^3}{1 + (-n - \ell - q + 1)z + (n\ell + nq + \ell q - 5n - 3\ell - q + 11)z^2 + (-n\ell q + 3n\ell - 6n + 6)z^3}$$

Proof. We apply Theorem 6.4 in the case $d = 3$. For the $N_{m,k}$, we have

$$\begin{aligned}
N_{m,k} &= 1 && \text{if } m = k, \\
N_{1,0} &= \ell_0 - \ell_1 - 1 = n - \ell - 1 \\
N_{2,1} &= \ell_1 - \ell_2 - 1 = \ell - q - 1 \\
N_{2,0} &= \ell_1 \sum_{j=0}^{2} \binom{2}{j} (-1)^j \ell_j = \ell(n - 2\ell + q) \\
N_{3,2} &= \ell_2 - \ell_3 - 1 = q - 1 \\
N_{3,1} &= \ell_2 \sum_{j=1}^{3} \binom{2}{j-1} (-1)^{j-1} \ell_j = q(\ell - 2q) \\
N_{3,0} &= \ell_1 \ell_2 \sum_{j=0}^{3} \binom{3}{j} (-1)^j \ell_j = \ell q(n - 3\ell + 3q).
\end{aligned}$$

For the $M_{i,j}$, we have

$$\begin{aligned}
M_{i,j} &= 1 && \text{if } i = j, \\
M_{1,0} &= (-1)^1 \binom{1}{0} N_{1,0} + (-1)^1 \binom{2}{1} N_{2,1} + (-1)^1 \binom{3}{2} N_{3,2} \\
&= -(n - \ell - 1) - 2(\ell - q - 1) - 3(q - 1) \\
&= -n - \ell - q + 6 \\
M_{2,0} &= (-1)^2 \binom{1}{0} \binom{2}{1} N_{1,0} N_{2,1} + (-1)^2 \binom{1}{0} \binom{3}{2} N_{1,0} N_{3,2} \\
&\quad + (-1)^2 \binom{2}{1} \binom{3}{2} N_{2,1} N_{3,2} + (-1)^1 \binom{2}{0} N_{2,0} + (-1)^1 \binom{3}{1} N_{3,1} \\
&= 2(n - \ell - 1)(\ell - q - 1) + 3(n - \ell - 1)(q - 1) + 6(\ell - q - 1)(q - 1) \\
&\quad - \ell(n - 2\ell + q) - 3q(\ell - 2q) \\
&= n\ell + nq + \ell q - 5n - 3\ell - q + 11 \\
M_{2,1} &= (-1)^1 \binom{2}{1} N_{2,1} + (-1)^1 \binom{3}{2} N_{3,2} \\
&= -2(\ell - q - 1) - 3(q - 1) \\
&= -2\ell - q + 5
\end{aligned}$$

$$\begin{aligned}
M_{3,0} &= (-1)^3\binom{1}{0}\binom{2}{1}\binom{3}{2}N_{1,0}N_{2,1}N_{3,2} + (-1)^2\binom{1}{0}\binom{3}{1}N_{1,0}N_{3,1} \\
&\quad + (-1)^2\binom{2}{0}\binom{3}{2}N_{2,0}N_{3,1} + (-1)^1\binom{3}{0}N_{3,0} \\
&= -6(n-\ell-1)(\ell-q-1)(q-1) + 3(n-\ell-1)(q(\ell-2q)) \\
&\quad + 3\ell(n-2\ell+q)(q-1) - \ell q(n-3\ell+3q) \\
&= -n\ell q + 3n\ell - 6n + 6 \\
M_{3,1} &= (-1)^2\binom{2}{1}\binom{3}{2}N_{2,1}N_{3,2} + (-1)^1\binom{3}{1}N_{3,1} \\
&= 6(\ell-q-1)(q-1) + 3q(\ell-2q) \\
&= 3\ell q - 6\ell + 6 \\
M_{3,2} &= (-1)^1\binom{3}{2}N_{3,2} \\
&= -3q+3.
\end{aligned}$$

Finally, the numerator of the geodesic growth is computed as

$$\begin{aligned}
&M_{0,0} + (M_{1,0}+nM_{1,1})z + (M_{2,0}+nM_{2,1}+n\ell M_{2,2})z^2 \\
&\quad + (M_{3,0}+nM_{3,1}+n\ell M_{3,2}+n\ell q M_{3,3})z^3 \\
&= 1 + (-\ell-q+6)z + (\ell q - 3\ell - q + 11)z^2 + 6z^3.
\end{aligned}$$

and the denominator is

$$\begin{aligned}
&M_{0,0} + M_{1,0}z + M_{2,0}z^2 + M_{3,0}z^3 \\
&= 1 + (-n-\ell-q+6)z + (n\ell + nq + \ell q - 5n - 3\ell - q + 11)z^2 + (-n\ell q + 3n\ell - 6n + 6)z^3
\end{aligned}$$

which completes the proof. ∎

4. Sage Program

We have written a Sage program that computes the formula for the geodesic growth of a RACG with maximum clique size d. The setting for d can be changed by modifying in the second line of the program. For example, the program outputs

```
Numerator: -(l_1 - 3)*z + 2*z^2 + 1
Denominator: (l_0*l_1 - 2*l_0 + 2)*z^2 - (l_0 + l_1 - 3)*z + 1
```

when $d = 2$, and

```
Numerator: (l_1*l_2 - 3*l_1 - l_2 + 11)*z^2 + 6*z^3 - (l_1 + l_2 - 6)*z + 1
Denominator: -(l_0*l_1*l_2 - 3*l_0*l_1 + 6*l_0 - 6)*z^3 + (l_0*l_1 + l_0*l_2 +
    l_1*l_2 - 5*l_0 - 3*l_1 - l_2 + 11)*z^2 - (l_0 + l_1 + l_2 - 6)*z + 1
```

when $d = 3$. These formulas agree with the above results.

Here is the program text. We have also sent this program as a separate file.

```
# Clique size (CHANGE THIS)
d = 3


def generate_partitions(lower, upper, total):
    """
    Generator function that generates all partitions of the form
        lower <= s_1 < t_1 <= ... <= t_n < s_n <= upper
    such that (t_1 - s_1) + ... + (t_n - s_n) = total.
    """

    # Impossible case: yield no partitions
    if upper - lower < total:
        return

    # Base case: yield empty partition
    if total == 0:
        yield []

    # Chooses lower <= s_1 < t_1 <= upper such that t_1 - s_1 <= total.
    # Then uses recursion to choose remaining s_i and t_i.
    for s_1 in range(lower, upper - total + 1):
        for t_1 in range(s_1 + 1, min(s_1 + total + 1, upper + 1)):
            partitions = generate_partitions(
                t_1, upper, total - (t_1 - s_1))
            for partition in partitions:
                yield [(s_1, t_1)] + partition


if __name__ == '__main__':
    # Variables
    z = var('z')
    l = [var('l_' + str(i)) for i in range(0, d)]
    l.append(0)  # l[d] = 0

    # N_{m, k}
    N = {}
    for m in range(1, d + 1):
```

```
        N[(m, m)] = 1
        N[(m, m - 1)] = l[m - 1] - l[m] - 1
        for k in range(0, m - 1):
            N[(m, k)] = prod([l[j] for j in range(k + 1, m)]) * sum([
                binomial(m - k, j - k) * ((-1) ^ (j - k)) * l[j]
                for j in range(k, m + 1)
            ])

    # M_{i, j}
    M = {}
    for i in range(0, d + 1):
        for j in range(0, i + 1):
            M[(i, j)] = 0

            partitions = generate_partitions(j, d, i - j)
            for partition in partitions:
                n = len(partition)
                M[(i, j)] += (-1) ** n * prod([
                    binomial(t, s) for (s, t) in partition
                ]) * prod([
                    N[(t, s)] for (s, t) in partition
                ])

    # Geodesic growth
    numerator = sum([
        sum([
            prod(l[:j]) * M[(i, j)]
            for j in range(0, i + 1)
        ]) * (z ** i)
        for i in range(0, d + 1)
    ])
    numerator = numerator.expand().collect(z)  # Simplify
    denominator = sum([
        M[(i, 0)] * (z ** i)
        for i in range(0, d + 1)
    ])
    denominator = denominator.expand().collect(z)  # Simplify

```

```
    print('Numerator: ' + str(numerator))
    print('Denominator: ' + str(denominator))
```

References

[1] Yago Antolín and Laura Ciobanu, *Geodesic growth in right-angled and even Coxeter groups*, European Journal of Combinatorics **34** (2012), DOI 10.1016/j.ejc.2012.12.007.

[2] Yago Antolín and Islam Foniqi, *Geodesic Growth of some 3-dimensional RACGs*, arXiv preprint arXiv:2105.09751 (2021).