History of the golden ratio, the Fibonacci sequence, continued fractions, and their relations.

MATH 446
Final Paper
Frank Yan

Introduction

In the long meandering course of human history, the pursuit for beauty and the inquiring of the natural world have been a guiding force. In the early days of civilizations, humans are drawn to the beauty of the natural world and the mysticals of its phenomenons. This desire and curiosity to understand the natural world and explain its mysteries, led to incredible discoveries and innovations. Science as we know it today, stemmed from this fascination and curiosity of the natural world. The pursuit for beauty inspired painters, musicians, and artisans to create captivating works that draw this very human fascination and appreciation for beauty. From the remarkable religious art pieces of the Renaissance period to Monet's impressionist use of shifting colors and natural light, there is an inherent human trait to capture and express beauty of the natural world. Concurrently, the pursuit of science allowed humanity to unlock the wonders of the natural world and acquire knowledge to push human progress. More than 3000 years ago, ancient Assyro-Babylonians observed shifting movements of celestial bodies and realized its relations in the natural seasons on Earth. This led to the formal development of recorded astronomy and use in agricultural and navigational purposes (*A History of Astrometry - Part I Mapping the Sky From Ancient to Pre-Modern Times*, 2019). Together, the pursuit for beauty and science has shaped the course of human history and this curiosity continues to push human understanding and exploration of the natural world.

The Golden Ratio or the 'Divine Proportion'

The golden ratio, first coined as the 'extreme and mean ratio' by Euclid of Megara in his work *The Elements* around the year 300 B.C.E, has been studied and valued by mathematicians, artists, and scholars for thousands of years (Euclid, 300 B.C.E.). Euclid was not the first polymath to have discovered or implemented this ratio in this course of human history, as ancient Egyptians have used this proportion in the construction of great wonders such the Great Pyramid of Giza or the Great Sphinx. In ancient Greece, Hippasus, an early follower of Pythagoras, took a geometric approach to this unique ratio and discovered its freaky irrational nature. This of course shocked the school of Pythagoras, who followed the code that 'everything is a [rational] number' (*The Prime Glossary: Pythagoras*, n.d.). However they later realized the bizarre nature of their own theory of rationality, but they buried this discovery within themselves. In the Middle Ages, the golden ratio was studied by the mathematician Leonardo of Pisa, who is known as Fibonacci. He used the golden ratio to describe the growth of a population of rabbits and published his findings in his book "Liber Abaci." In the present, the golden ratio is continued to be used to describe both natural and man-made phenomena.

Euclid's Book of elements

The *Elements* is a collection of 13 texts associated with the works of Euclid of Megara, in the ancient city of Alexandria around the period of 300 B.C.E.. It covers a myriad of mathematical topics, elementary number theory, algebra, and provides the basis for Euclidean

Geometry. Within the text, mathematical concepts are elaborated using definitions, theorems, propositions, and proofs of propositions; it's often considered to be the basis for mathematical studies. While Euclid does not explicitly go into the discussion of the term ‘golden ratio’, he did go extensively over the geometric principles that relate to this topic. To begin, the golden ratio is the mathematical concept in which, given two uniquely different numbers, the ratio of the sum of these two numbers to the larger number is equivalent to the ratio of the larger number to the smaller number. Notably, this ratio is represented mathematically with the Greek letter ϕ or φ (Livio, 2003, Chapter 1).

In Book 6 of Euclid’s *Elements*, its third definition states that “A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less (Euclid, 300 B.C.E.).” Euclid first coined the concept of a golden ratio as the ‘extreme and mean’ ratio in which two line segments can exist sequentially one end to another. Here the line segment AB is cut at point C into two line segments, the larger AC and the smaller CB.

The ratio of the whole line segment AB to that of the larger segment AC is equivalent to the ratio of the larger AC segment to CB segment. Mathematically we can consider AC line segment as a and CB line segment as b, such that AB is $a + b$. Thus, ratio can be represented

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

$\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi}$ if we substitute $\frac{b}{a}$ with $\frac{1}{\varphi}$.

$$1 + \frac{1}{\varphi} = \varphi$$

$$\varphi + 1 = \varphi^2$$

$$\varphi^2 - \varphi - 1 = 0$$

Apply the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ yields $\frac{1 \pm\sqrt{1+4}}{2}$. Which we get $\frac{1+\sqrt{5}}{2} = 1.618033...$ $\frac{1-\sqrt{5}}{2} = -0.618033...$ Since the ratio cannot be a negative value, naturally only the positive value of 1.618033… is considered.

Additionally, we see an interpretation of this extreme and mean ratio in proposition II.11 and VI.30. Book two states, “To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment” (Euclid, 300 B.C.E.). Similar to the previous proposition, it begins with the two line segments AH and HB. It suggests that with the bisection of AC at E, it is able to create two equivalent line segments EF and EB. Thus, the two rectangles AEGH and HBDK are equivalent in size. Here the ratio of $\frac{AH+BH}{AH} = \frac{AH}{BH}$ can be derived using the ratio of between the size of rectangles. By assigning the line segment AH as a and BH as b, we can get the exact ratio of extreme and mean proposed in Book VI. 3 $\frac{a+b}{a} = \frac{a}{b} = \varphi$.

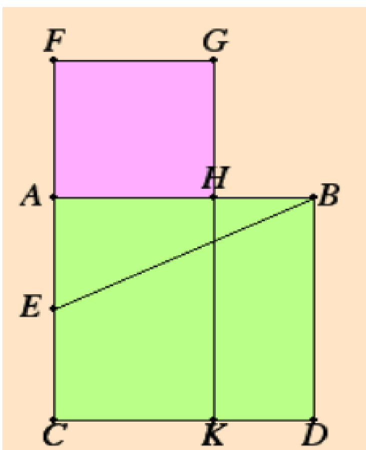

Figure 1. Geometric Representation of Euclid’s Element II.11

<u>Renaissance</u>

In the Middle Ages between Ancient Greece and the Renaissance, much of the mathematical development on continental Europe stalled. Euclid’s Element was reintroduced and circulated in Latin around 1120 (Meisner, 2018, 58). Many polymaths, including the such of Leonado Di Vinci and Luca Pacioli, were enthralled and continued to develop further techniques relating to past work. Luca Pacioli (c.1447 - c.1517) was a peer of da Vinci and a polymath in the Republic of Florence. He developed some of the earliest modern day accounting and bookkeeping techniques and pushed mathematical works such as *Summa de Arithmetica*

(Summary of Arithmetic) (Meisner, 2018, 59). He carried on the works of Ancient Greek mathematicians and coined Euclid's 'extreme and mean ratio' to the term 'De Divina Proportione', the Divine Proportion. He suggested that the divine proportion to be a "work necessary for all the clear-sighted and inquiring human minds" which would be transformative in finding beauty and harmony in all aspects of arts and sciences (Meisner, 2018, 60).

As the divine proportion provided a sense of serenity and harmony, arguably it suited for expressive works in the religious context. One of the most famous examples of the use of the golden ratio in Renaissance art is Leonardo da Vinci's "The Last Supper." The painting depicts the moment when Jesus announces that one of his Twelve Apostles will betray him, and is considered one of the greatest works of art in history. The golden ratio can be seen in the composition of the scene, with the main figures arranged according to the golden ratio. Jesus, being the center axis of the painting, his hands are at the golden ratio of half the height of the composition. The golden ratio can also be seen in the shapes and proportions of the objects in the painting, such as the dining table and the room architecture in the background (Meisner, 2014).

Another example of the use of the golden ratio in Renaissance art is Michelangelo's "The Creation of Adam" fresco on the ceiling of the Sistine Chapel. Michelangelo, also a Forentian native in the period of da Vinci, is a monumental sculpture, painter and architect who also incorporated mathematics extensively in his works (da Volterra et al., n.d.). In this painting, through the link of finger fits, God ignites life to the first man, Adam. The golden ratio can be seen in the composition of the figures and the overall design of the scene. The golden ratio can also be seen in the proportions of the God and Adam, and the positioning of their limbs. Additionally, this unique anatomic proportion is found in several other Michelangelo's works in Sistine Chapel, where it depicted other scenes of the Creation from the Book of Genesis (Meisner, 2016). Thus, there is clear evidence that Pacioli's of the 'Divine Proportion' heavily influenced artists' illustration styles where the proportion was used extensively in religious arts. Renaissance artists were able to devote particular harmonious and aesthetically pleasing compositions extensively in the paintings and sculptures of the time, as well as in the construction of buildings. The continued use of the golden ratio in art and architecture today is a testament to its enduring importance and relevance.

In the interlude between Ancient Greece and the European Renaissance, the Islamic Golden Age (8th to 14th century) flourished in mathematics and natural sciences. Egyptian mathematician, Abu Kamil (c. 850- c. 930), was one of the first to use irrational numbers as solutions and coefficients to mathematical equations (Herz-Fischler, 1998, 124). In one of his works, *On the Pentagon and Decagon* (Kitāb al-mukhammas wa'al-mu'ashshar), (Meisner, 2018, 39), Kamil directly referred Euclid as the foundational work for many of his geometric and algebra theories.

Fibonacci Sequence

Leonardo Bonacci (c. 1170 - c.1250), commonly known as Fibonacci, was an Italian mathematician from the Republic of Pisa (Gies, n.d.). Born in a merchant family, he acquired knowledge of mathematics, including the Hindu-Arabic numeral system on journeys around the Mediterranean. His most important contribution is perhaps the *Liber Abbaci* (*Book of Calculations*), where he introduced the Hindu-Arabic system and algebraic problems to Medieval Europe (Katz, 2009, 336). Many of the algebraic notations in mathematics today, including the fraction notation, can trace their origins to *Liber Abbaci*. The book introduced the Fibonacci sequence and explained how to use it to solve mathematical problems. Fibonacci's contributions to mathematics were not limited to the Fibonacci sequence. He also made significant contributions to the field of arithmetics, trigonometry, and geometry, including the Fibonacci spiral, which is a curve formed by drawing quarter-circle arcs connecting the opposite corners of squares in the Fibonacci sequence. The Fibonacci sequence and the Fibonacci spiral can be found in many areas of science and nature, including the arrangement of leaves on a stem, the branching of trees, and the structure of pinecones. While the *Liber Abbaci* "contained no particular advance over mathematical works then current in the Islamic world" (Katz, 2009, 346). It was the necessary bridge that allowed European mathematics to continue the works from Islamic Golden Ages and fill the void of knowledge during the Middle Ages. Fibonacci's works in mathematics arguably laid the foundation for modern mathematics.

Fibonacci Sequence Relation to the Golden Ratio

The Fibonacci sequence is a series of numbers in which each number is the sum of the two preceding ones. The Fibonacci sequence commonly begins with 0 and 1, and each subsequent number is the sum of the previous two. For example, the first few numbers in the Fibonacci sequence are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, and so on. While Fibonacci was not the first mathematician to present the concept of the Fibonacci sequence, as Indian mathematics proposed this concept centuries prior, he introduced this concept in the *Liber Abaci* (Singh, 1985, 230). In the *Liber Abaci*, he proposed the a rabbit reproduction problem,

> "How pairs of rabbits will be produced each month, beginning with a single pair, if every month each 'productive' pair bears a new pair which becomes productive from the second month on?" (Debnath, 2011, 354)

In this context, we start out at F1 = 1, as a pair of rabbits, and since it takes a month for the rabbit to mature and reproduce, F2 is still when this pair begins to reproduce. By F3 the new pair of rabbits is born and there are now two pairs of rabbits. Immediately, the original pair continues to reproduce and another pair is both at F4. The original pair continues to reproduce every month thereforth. At F5, the pair born in F3 begin to reproduce and give birth to another pair. Thus, this increases sequentially with each pair maintaining its reproduction month by month.

Another example is the Fibonacci rectangle, which can incorporate Golden Spirals, which is a logarithmic spiral with growth factor ϕ . A Fibonacci rectangle is a rectangle whose sides are in the ratio of consecutive Fibonacci numbers. The Fibonacci sequence is a series of numbers in which each number is the sum of the previous two numbers, starting with 0 and 1.
The Fibonacci rectangle is often implemented in art and design to create visually appealing compositions and to illustrate the principles of the Fibonacci sequence and the Golden Ratio.

To create a Fibonacci rectangle, you can start by drawing a rectangle with sides in the ratio of 1:1. This rectangle represents the first two numbers in the Fibonacci sequence (0 and 1). Another way it can be represented is drawing a square with side-length of 1.
Then, you can add a rectangle with sides in the ratio of 1:2 (representing the next two numbers in the Fibonacci sequence: 1 and 1) next to the first rectangle. You can continue this process, adding rectangles with sides in the ratio of 2:3, 3:5, or with the ratio $\frac{1}{\phi}$ or $\frac{1}{\phi^2}$ and so on, to create a series of rectangles that form a spiral pattern.

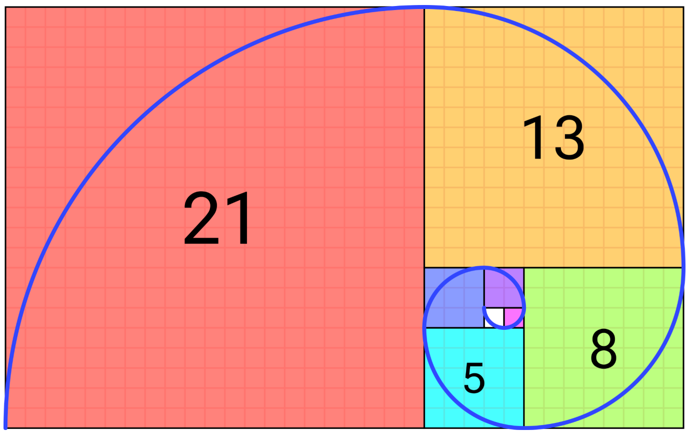

Figure 2. Fibonacci spiral derived from the terms of Fibonacci Sequence

Fibonacci Sequence can be defined using the recurrence relation of
$F_0 = 0, F_1 = 1 = F_2$ <- Initial Conditions
$F_n = F_{n-1} + F_{n-2}$ $for\ n > 1$. <- Recurrence Relation

F0	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
0	1	1	2	3	5	8	13	21	34	55	89

Table 1. First 12 terms of the Fibonacci Sequence

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \phi$$

Furthermore, we can deduce that the

$$F_{n+1} = F_n + F_{n-1} \; s.t.$$

$$Suppose\ that\ a_n = \frac{F_{n+1}}{F_n}$$

$$\phi = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{a_{n-1}}$$

$$1 + \frac{1}{\phi} = \phi \; s.t.$$

$$\phi^2 - \phi - 1 = 0$$

Which also yields $\frac{1+\sqrt{5}}{2} = 1.618033...\ \frac{1-\sqrt{5}}{2} = -0.618033...$

Indian Mathematicians and Sequences

While the Fibonacci sequence is closely associated with Leonardo Bonacci and *Liber Abbaci*, majority of his work stem from translation and consolidating the works of al-Khawarizmi and Abu Kamil who were influenced by Indian mathematicians (Livio, 2003, 92). Indian mathematicians, Virahanka (c.600 ~ c.800), Pingala (c. 450 BC ~ 200 BC) proposed similar proposed numeral sequences centuries before Fibonacci. The most notable example can be found in Sanskrit prosody, a poetic metering device, where the “basic units in Sanskrit prosody are a letter having a single mātrā (syllable) called laghu (light) and two mātrās called guru (heavy)” (Singh, 1985, 230). This can be interpreted as having two syllables of one and two units in length, respectively. As the prosody continues, these combinatorics can be expanded in the forms of the nth mātrā can be obtained by combining the (n -1) and (n-2) mātrās to the left. The number of syllables of the first few mātrās are obtained as 1, 2, 3, 5, 8, 13, which is the same as the first few Fibonacci Numbers (Singh, 1985, 232). In Sanskrit and Hindu texts, sequence was referred to as mātrā-vrttas.

A translation of Virahanka's earlier works solidified the existence of such a metering device where: "The variations of two earlier meters being mixed, the number is obtained. That is a direction for knowing the number of (variations) of the next mātrā-vrttas" (Singh, 1985, 233). While mātrā-veritas sequence is most commonly used in the context of poetry and religious texts, it's possible to suggest it was also used in architecture and art. It is interesting to note that the Fibonacci sequence was discovered independently in different parts of the world. As India and the Arabic world engaged extensively during the Islamic Golden period, mathematical and scientific knowledge such the Hindu-Arabic numeral system allowed Arabic mathematicians such as Abu Kamil to make significant contributions to algebra and geometry. Hence, while the sequence existed in India and the Arab world, it was Fibonacci who introduced the sequence to the Western world and popularized its use in mathematics.

Applications of Fibonacci Sequence in Nature

One of the most famous examples of the Fibonacci sequence in nature is the spiral pattern that can be found in many plants, such as pineapples, sunflowers, and pine cones. This pattern is formed by the arrangement of the seeds in the plant's fruit or cone, and it is thought to be an example of the plant's attempt to maximize the amount of seeds it can produce while minimizing the amount of space they occupy (Katz, 2009, 43). The Fibonacci sequence can also be found in the arrangement of leaves on a stem, where the leaves are often arranged in a logarithmic spiral pattern as a way for the plant to maximize the amount of sunlight it can capture.

Similar to Libre Abbacci's original proposition on rabbit reproduction, the Fibonacci Sequence is present in the human genome replication process (Liu, 2018, 2). The human genome is composed of 23 chromosome pairs and repetitive sequences are patterns of DNA or RNA that occupy the bulk of the human genome (Liehr, 2021). Researchers proposed that the growth "of repetitive DNAs is analogous to the pattern described by the Fibonacci process" (Liu, 2018, 2). While there are certainly limitations to using the Fibonacci process as a scientifically accurate model to interpret and predict human genome activities, it does show that some of the physical chemistry behind repetitive sequences follow a form of natural optimization growth (Liu, 2018, 16). Arguably, Fibonacci Sequence and the Golden Ratio are present in most optimization growth phenomena in the natural world. Thus, it is possible to suggest that many of the human biological and physiological functions operate similarly to aspects in the natural world, where it seeks to optimize many of its repetitive processes for efficiency.

The Fibonacci sequence is a beautiful and fascinating example of the ways in which mathematics and nature intersect and influence each other. It serves as a reminder of the underlying unity and order that exists in the natural world, and it reminds us of the power of mathematics to help us understand and explain the world around us. Whether we are studying the arrangement of leaves on a stem or the patterns of a spiral galaxy, the Fibonacci sequence is a testament to the ways in which mathematics and nature are inextricably linked.

Lucas Numbers and Sequence

The Lucas numbers are a sequence of integers that are closely related to the Fibonacci numbers(Koshy & Koshy, 2001, 30). François Lucas (c.1842 - c.1891), a French Mathematician studied works of Fibonacci, who proposed a sequence with successive terms that also approaches the Golden Ratio similarly to Fibonacci sequence. Like the Fibonacci numbers, the Lucas numbers are defined by a recurrence relation, which means that each number in the sequence is calculated based on the previous numbers in the sequence. The Lucas numbers begin with 2 and 1 rather than 0 and 1.

The Lucas numbers are defined as follows:

$$L_0 = 2,\ L_1 = 1,\ s.t.\ L_n = L_{n-1} + L_{n-2}\ for\ n > 1$$

The first few Lucas numbers list as follow:

$$L_0 = 2,\ L_1 = 1,\ L_2 = 3,\ L_3 = 4,\ L_4 = 7,\ L_5 = 11,\ L_6 = 18,\ L_7 = 29...$$

The relationship between these two sets of numbers follow the same recurrence sequence with different L_0 and L_1. Additionally, this relationship can be proven using induction techniques.

$L_n = L_{n-1} + L_{n-2}\, for\, n > 1$ and $F_n = F_{n-1} + F_{n-2}\ for\ n > 1$

For all $n \in \mathbb{N}_{>0}$, $let\, P(n)$ be the proposition:

$L_n = F_{n-1} + F_{n+1}$

For Fibonacci Number,

$F_{n+1} = F_n + F_{n-1} = (F_{n-1} + F_{n-2}) + F_{n-1}$

$F_{n+1} = F_{n-1} + F_{n-1} + F_{n-2}$

$F_{n+1} = 2F_{n-1} + F_{n-2}$

For Lucas Number

$L_{n+1} = L_n + L_{n-1} = (L_{n-1} + L_{n-2}) + L_{n-1}$

$L_{n+1} = L_{n-1} + L_{n-1} + L_{n-2}$

$L_{n+1} = 2L_{n-1} + L_{n-2}$

Therefore, by induction, the recurrence relation holds for all positive integers n, and the Fibonacci numbers and Lucas numbers are well-defined.

<u>Continued Fractions</u>

A continued fraction is the iterative method of representing a number (usually irrational) as the sum of an integer and fraction. a_0, a_1, a_2... are coefficient as positive integers. While b_1, b_2, b_3... exist as numerators of fractions. In most circumstances, the numerators are 1, and this is referred to as generalized continued fractions. Many irrational numbers are approximated using this technique.

$$[x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}] \qquad [x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}]$$

Figure 4. Continued Fractions

While it can also be used to represent a real number. It is most frequently used to represent irrational numbers, $\sqrt{2}$ and π can be expressed as an approximation using the method of continued fractions. Every rational number can be represented with a simple continued fraction, where as shown above, the denominator b terms can be represented as 1. Additionally, a simple continued fraction can also be considered for irrational numbers, however due to the nature that it cannot be represented as a ratio of two integers, they are essential unique fractions.

$$[\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}] \qquad [\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}]$$

Figure 6. Representing Irrational Numbers with Continued Fractions

Continued fraction can trace its history to as early as Euclid and his book of *Elements*. In Book 7 proposition 2, Euclid proposed that, "to find the greatest common measure of two given numbers not relatively prime" (Euclid, 300 B.C.E.). This Euclidean algorithm is a method for finding the greatest common divisor (GCD) of two integers and works by dividing the larger of the two numbers by the smaller, and then taking the remainder of the division.
(*Euclid's Algorithm*, n.d.). This process is repeated with the smaller number and the remainder until the remainder is zero. The final remaining number is the GCD. In the context of continued

fractions, the remainder in each step of euclidean algorithm operations, is another iteration of fraction over a consistent numerator.

However, it wasn't until Swiss mathematician Leonhard Euler (c. 1707- c.1783) who presented the continued fraction as an equivalent form to infinite series, to fully incorporate both rational and irrational numbers in continued fractions. (Cretney, 2014, 140). In his book, *Introductio in analysin infinitorum*, the foundational text for mathematical analysis, he proved that rational numbers can be represented with a finite continued fraction while irrational numbers can be represented with an infinite continued fraction.

But how does that all tie together with the 'Golden Ratio' and Fibonacci Sequence?

Recall earlier that $\frac{F_{n+1}}{F_n} = \frac{F_n}{F_{n-1}} = \phi$, we know that the successive numbers in the Fibonacci sequence approaches the golden ratio ϕ with an increased n. There is a connection between the Fibonacci sequence and continued fractions. It turns out that the ratio of consecutive Fibonacci numbers approaches the golden ratio, which is a number that can be represented as a continued fraction. The golden ratio is approximately equal to 1.6180339887..., and it can be represented as a continued fraction as follows:

$\varphi = 1 + 1/(1 + 1/(1 + 1/(1 + ...)))$

This continued fraction representation of the golden ratio is an example of the simple continued fractions. Thus the connection between the Fibonacci sequence and continued fraction is often a harmonious explanation of both their connections to the golden ratio.

Conclusion

The golden ratio, Fibonacci sequence and continued fractions have a distinctive mathematical relation in which all can be used to study various concepts of mathematics. The golden ratio can be characterized using the extreme and mean ratio between a larger and a smaller quantity, where the ratio of the smaller quantity to the larger quantity is the same as the ratio of the larger quantity to the sum of the two quantities. The Fibonacci sequence is a series of numbers in which each number is the sum of the two preceding numbers, starting with 0 and 1. Both the golden ratio and Fibonacci sequence have appearances in natural phenomena such as plant structure and leaf branching of trees. The ratio of a Fibonacci term and its preceding term approach to the golden ratio of 1.618033 as size of n increases. Continued fractions can be used to represent the golden ratio, which itself is an irrational number, and this process has been used to study other mathematical phenomenons relating to the golden ratio.

It is clear that the development of mathematics is not linear nor is it exceptional to one civilization or geographical region. While for the centuries following Euclid continental Europe did not carry on his work, Arabic mathematicians such as al-Khwarizmi and Abu Kamil were influenced by Euclid's work and continued on works in geometry and algebra. Additionally, due to the cultural and trade linkages between the Arab world and India, they introduced the rest of the world to Indian mathematics and numeral systems. Indian mathematicians such as Virahanka arguably delved into golden ratio and sequences centuries prior to Fibonacci, however they did not receive as much recognition as the polymaths in Renaissance Europe. In all civilizations mathematics was used to observe and study natural phenomenons. The pursuit for beauty and science are notable in all cultures, however given inconsistencies with written history and representation of mathematical knowledge, many are now lost for anthropological studies. The oldest recorded mathematical artifact, the Ishango Bone has been dated to more than 20000-25000 years ago (Swetz, n.d.). Its possible use as a base 12 counting system continues to fascinate our understanding of prehistoric mathematics. As we examine the history of mathematics, we see that different cultures have approached the study and application of mathematics in unique ways, which has resulted in significant shifts in our understanding of the subject. This diversity of perspectives and approaches has broadened our understanding of mathematics and has contributed to the development of new ideas and theories.

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