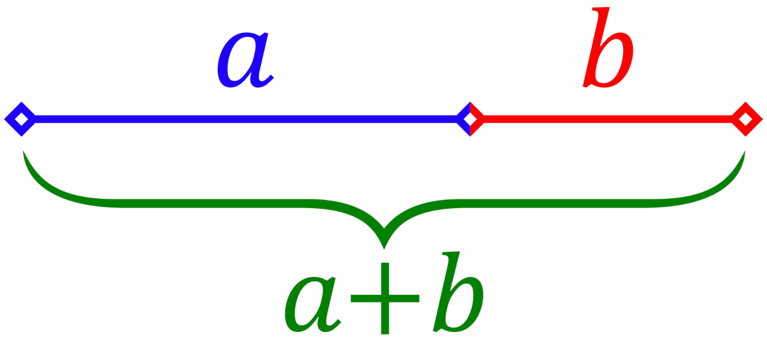
History of the golden ratio, the Fibonacci sequence, continued fractions, and their relations.

Frank Yan Math 446

Golden Ratio



$a+b$ is to a as a is to b

$$\frac{a+b}{a} = \frac{a}{b} = \varphi \quad \text{such that} a > b > 0$$

Here the two quantities a, b are in the golden ratio 'phi'

$$\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a} = \varphi \qquad 1 + \frac{1}{\varphi} = \varphi$$

$$\varphi + 1 = \varphi^2 \qquad \varphi^2 - \varphi - 1 = 0$$

$$\varphi = \frac{1 + sqrt(5)}{2} = 1.618033988..$$

Or

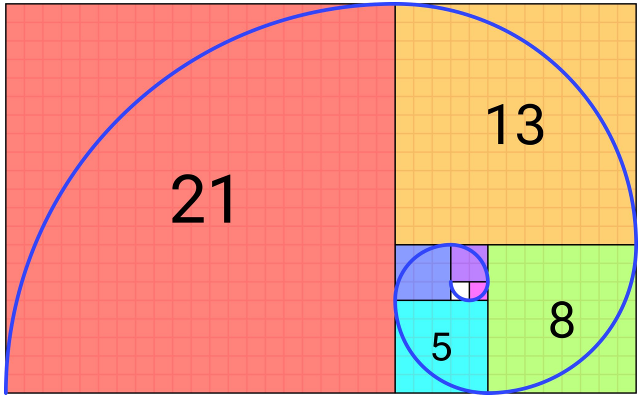
$$\varphi = \frac{1 - sqrt(5)}{2} = -0.618033988$$

Golden Ratio and the Fibonacci Sequence

Fibonacci Sequence is a *Recurrence sequence* where you have the base cases for n > 1

$$F_0 = 0, F_1 = 1, s.t F_n = F_{n-1} + F_{n-2}$$

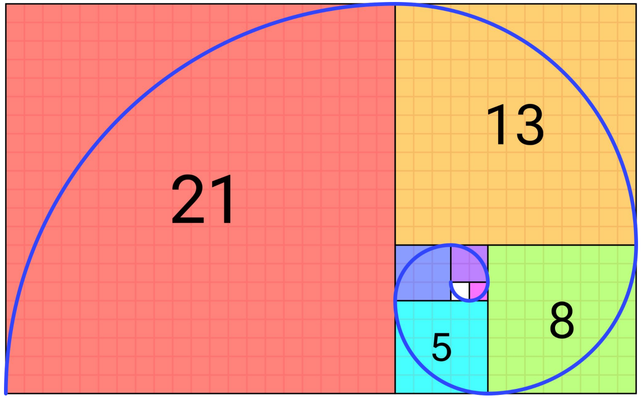
F0	F1	F2	F3	F4	F5	F6	F7	F8	F9	F1 0	F1 1
0	1	1	2	3	5	8	13	21	34	55	89



Golden Ratio and the Fibonacci Sequence

F0	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
0	1	1	2	3	5	8	13	21	34	55	89

F4/F3 = 3/2 = 1.4
F5/F4 = 5/3 = 1.666
F6/F5 = 8/5 = 1.6
F7/F6 = 13/8 = 1.625
F8/F7 = 21/13 = 1.61538
F9/F8 = 34/21 = 1.61905
F10/F9 = 55/34 = 1.617647
F11/F10 = 89/55 = 1.6181…
GOLDEN RATIO = 1.618033…



Continued Fractions

A continued fraction is a mathematical expression in which a number is represented as the sum of an integer and a fraction. Continued fractions are typically written in the form a0 + 1/(a1 + 1/(a2 + ...)),

Take a look at the the Iteration of

$$\varphi_{n+1} = 1 + \frac{1}{\varphi_n}, given \varphi_0 = 1$$

$$\varphi_{n+1} = \frac{\varphi_n + 1}{\varphi_n}$$

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Continued Fractions

Since we know that phi_0 = 1 in this instance, this is deduced to

$$\varphi_n = 1 + \frac{1}{\varphi_n}, given \varphi_0 = 1$$

$$\varphi_n^2 - \varphi_n - 1 = 0$$

Which provides us with the Golden Ratio as seen earlier

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}}$$

Significance & History

Golden Ratio or the ‘Divine Ratio’

- Euclid, Ancient Greece, Italian Renaissance implementation in the arts, science, architecture, etc.

Fibonacci Sequence

- Leonardo Binanci, Lucas Numbers, potential inspiration from Ancient India

History of Continued Fractions

- Euclid’s Elements, Euler’s Proof and Number Theory, better method of approximation for irrational numbers (golden ratio, pi, Euler’s number ‘e’)

Modern Applications

- Implementation of Golden Ratio and Fibonacci Sequence in finance, biology, CPSC, etc.