

# P versus NP

Frank Vega 

Joysonic, Uzun Mirkova 5, Belgrade, 11000, Serbia  
vega.frank@gmail.com

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## Abstract

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P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency. However, a precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. Another major complexity classes are L and NL. Whether  $L = NL$  is another fundamental question that it is as important as it is unresolved. We demonstrate if L is not equal to NL, then  $P = NP$ . In addition, we show if L is equal to NL, then  $P = NP$ . In this way, we prove the complexity class P is equal to NP. Furthermore, we demonstrate the complexity class NL is equal to NP.

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## 1 Introduction

In previous years there has been great interest in the verification or checking of computations [19]. Interactive proofs introduced by Goldwasser, Micali and Rackoff and Babai can be viewed as a model of the verification process [19]. Dwork and Stockmeyer and Condon have studied interactive proofs where the verifier is a space bounded computation instead of the original model where the verifier is a time bounded computation [19]. In addition, Blum and Kannan has studied another model where the goal is to check a computation based solely on the final answer [19]. More about probabilistic logarithmic space verifiers and the complexity class  $NP$  has been investigated on a technique of Lipton [19]. In this work, we show some results about the logarithmic space verifiers applied to the class  $NP$  and logarithmic space disqualifiers applied to the class  $coNP$  which solve one of the most important open problems in computer science, that is  $P$  versus  $NP$ .

The  $P$  versus  $NP$  problem is a major unsolved problem in computer science [7]. This is considered by many to be the most important open problem in the field [7]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US\$1,000,000 prize for the first correct solution [7]. The precise statement of the  $P = NP$  problem was introduced in 1971 by Stephen Cook in a seminal paper [7]. In 2012, a poll of 151 researchers showed that 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore impossible to prove or disprove, 8 (5%) said either do not know or do not care or don't want the answer to be yes nor the problem to be resolved [13].

The  $P = NP$  question is also singular in the number of approaches that researchers have brought to bear upon it over the years [10]. From the initial question in logic, the focus moved to complexity theory where early work used diagonalization and relativization techniques [10]. It was showed that these methods were perhaps inadequate to resolve  $P$  versus  $NP$  by demonstrating relativized worlds in which  $P = NP$  and others in which  $P \neq NP$  [4]. This shifted the focus to methods using circuit complexity and for a while this approach was deemed the one most likely to resolve the question [10]. Once again, a negative

result showed that a class of techniques known as “Natural Proofs” that subsumed the above could not separate the classes  $NP$  and  $P$ , provided one-way functions exist [23]. There has been speculation that resolving the  $P = NP$  question might be outside the domain of mathematical techniques [10]. More precisely, the question might be independent of standard axioms of set theory [10]. Some results have showed that some relativized versions of the  $P = NP$  question are independent of reasonable formalizations of set theory [14].

It is fully expected that  $P \neq NP$  [22]. Indeed, if  $P = NP$  then there are stunning practical consequences [22]. For that reason,  $P = NP$  is considered as a very unlikely event [22]. Certainly,  $P$  versus  $NP$  is one of the greatest open problems in science and a correct solution for this incognita will have a great impact not only in computer science, but for many other fields as well [1]. Whether  $P = NP$  or not is still a controversial and unsolved problem [1]. We show some results that prove this outstanding problem with the unexpected solution of  $P = NP$ .

## 2 Theory and Methods

### 2.1 Preliminaries

In 1936, Turing developed his theoretical computational model [25]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [25]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [25]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [25].

Let  $\Sigma$  be a finite alphabet with at least two elements, and let  $\Sigma^*$  be the set of finite strings over  $\Sigma$  [3]. A Turing machine  $M$  has an associated input alphabet  $\Sigma$  [3]. For each string  $w$  in  $\Sigma^*$  there is a computation associated with  $M$  on input  $w$  [3]. We say that  $M$  accepts  $w$  if this computation terminates in the accepting state, that is  $M(w) = \text{“yes”}$  [3]. Note that  $M$  fails to accept  $w$  either if this computation ends in the rejecting state, that is  $M(w) = \text{“no”}$ , or if the computation fails to terminate, or the computation ends in the halting state with some output, that is  $M(w) = y$  (when  $M$  outputs the string  $y$  on the input  $w$ ) [3].

Another relevant advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet [8]. A complexity class is a set of problems, which are represented as a language, grouped by measures such as the running time, memory, etc [8]. The language accepted by a Turing machine  $M$ , denoted  $L(M)$ , has an associated alphabet  $\Sigma$  and is defined by:

$$L(M) = \{w \in \Sigma^* : M(w) = \text{“yes”}\}.$$

We denote by  $t_M(w)$  the number of steps in the computation of  $M$  on input  $w$  [3]. For  $n \in \mathbb{N}$  we denote by  $T_M(n)$  the worst case run time of  $M$ ; that is:

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

where  $\Sigma^n$  is the set of all strings over  $\Sigma$  of length  $n$  [3]. We say that  $M$  runs in polynomial time if there is a constant  $k$  such that for all  $n$ ,  $T_M(n) \leq n^k + k$  [3]. In other words, this means the language  $L(M)$  can be decided by the Turing machine  $M$  in polynomial time. Therefore,  $P$  is the complexity class of languages that can be decided by deterministic Turing

machines in polynomial time [8]. A verifier for a language  $L_1$  is a deterministic Turing machine  $M$ , where:

$$L_1 = \{w : M(w, c) = \text{"yes"} \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of  $w$ , so a polynomial time verifier runs in polynomial time in the length of  $w$  [3]. A verifier uses additional information, represented by the symbol  $c$ , to verify that a string  $w$  is a member of  $L_1$ . This information is called certificate.  $NP$  is the complexity class of languages defined by polynomial time verifiers [22]. If  $NP$  is the class of problems that have succinct certificates, then the complexity class  $coNP$  must contain those problems that have succinct disqualifications [22]. That is, a “no” instance of a problem in  $coNP$  possesses a short proof of its being a “no” instance [22].

► **Definition 1.** We will extend the definition of succinct disqualification for an element  $w \in L_2$  when  $L_2 \in coNP$  as the polynomially bounded string  $c$  such that  $M(w, c) = \text{"no"}$  and  $M$  is the polynomial time verifier of the complement of  $L_2$  in  $NP$ .

## 2.2 First Hypothesis

A function  $f : \Sigma^* \rightarrow \Sigma^*$  is a polynomial time computable function if some deterministic Turing machine  $M$ , on every input  $w$ , halts in polynomial time with just  $f(w)$  on its tape [25]. Let  $\{0, 1\}^*$  be the infinite set of binary strings, we say that a language  $L_1 \subseteq \{0, 1\}^*$  is polynomial time reducible to a language  $L_2 \subseteq \{0, 1\}^*$ , written  $L_1 \leq_p L_2$ , if there is a polynomial time computable function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  such that for all  $x \in \{0, 1\}^*$ :

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

An important complexity class is  $NP$ -complete [12]. A language  $L_1 \subseteq \{0, 1\}^*$  is  $NP$ -complete if:

- $L_1 \in NP$ , and
- $L' \leq_p L_1$  for every  $L' \in NP$ .

If  $L_1$  is a language such that  $L' \leq_p L_1$  for some  $L' \in NP$ -complete, then  $L_1$  is  $NP$ -hard [8]. Moreover, if  $L_1 \in NP$ , then  $L_1 \in NP$ -complete [8]. A principal  $NP$ -complete problem is  $SAT$  [12]. An instance of  $SAT$  is a Boolean formula  $\phi$  which is composed of:

1. Boolean variables:  $x_1, x_2, \dots, x_n$ ;
2. Boolean connectives: Any Boolean function with one or two inputs and one output, such as  $\wedge$ (AND),  $\vee$ (OR),  $\neg$ (NOT),  $\Rightarrow$ (implication),  $\Leftrightarrow$ (if and only if);
3. and parentheses.

A truth assignment for a Boolean formula  $\phi$  is a set of values for the variables in  $\phi$ . A satisfying truth assignment is a truth assignment that causes  $\phi$  to be evaluated as true. A Boolean formula with a satisfying truth assignment is satisfiable. The problem  $SAT$  asks whether a given Boolean formula is satisfiable [12]. We define a  $CNF$  Boolean formula using the following terms:

A literal in a Boolean formula is an occurrence of a variable or its negation [8]. A Boolean formula is in conjunctive normal form, or  $CNF$ , if it is expressed as an AND of clauses, each of which is the OR of one or more literals [8]. A Boolean formula is in 3-conjunctive normal form or  $3CNF$ , if each clause has exactly three distinct literals [8].

For example, the Boolean formula:

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in  $3CNF$ . The first of its three clauses is  $(x_1 \vee \neg x_1 \vee \neg x_2)$ , which contains the three literals  $x_1$ ,  $\neg x_1$ , and  $\neg x_2$ . Another relevant  $NP$ -complete language is  $3CNF$  satisfiability, or  $3SAT$  [8]. In  $3SAT$ , it is asked whether a given Boolean formula  $\phi$  in  $3CNF$  is satisfiable.

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [25]. The work tapes may contain at most  $O(\log n)$  symbols [25]. In computational complexity theory,  $L$  is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [22].  $NL$  is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [22].

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and read/write work tapes [25]. The work tapes must contain at most  $O(\log n)$  symbols [25]. A logarithmic space transducer  $M$  computes a function  $f : \Sigma^* \rightarrow \Sigma^*$ , where  $f(w)$  is the string remaining on the output tape after  $M$  halts when it is started with  $w$  on its input tape [25]. We call  $f$  a logarithmic space computable function [25]. We say that a language  $L_1 \subseteq \{0, 1\}^*$  is logarithmic space reducible to a language  $L_2 \subseteq \{0, 1\}^*$ , written  $L_1 \leq_l L_2$ , if there exists a logarithmic space computable function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  such that for all  $x \in \{0, 1\}^*$ :

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

The logarithmic space reduction is used in the definition of the complete languages for the classes  $L$  and  $NL$  [22]. A Boolean formula is in 2-conjunctive normal form, or  $2CNF$ , if it is in  $CNF$  and each clause has exactly two distinct literals. There is a problem called  $2SAT$ , where we asked whether a given Boolean formula  $\phi$  in  $2CNF$  is satisfiable.  $2SAT$  is complete for  $NL$  [22]. Another special case is the class of problems where each clause contains  $XOR$  (i.e. exclusive or) rather than (plain)  $OR$  operators. This is in  $P$ , since an  $XOR SAT$  formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [21]. We denote the  $XOR$  function as  $\oplus$ . The  $XOR 2SAT$  problem will be equivalent to  $XOR SAT$ , but the clauses in the formula have exactly two distinct literals.  $XOR 2SAT$  is in  $L$  [2], [24].

We can give a certificate-based definition for  $NL$  [3]. The certificate-based definition of  $NL$  assumes that a logarithmic space Turing machine has another separated read-only tape [3]. On each step of the machine, the machine's head on that tape can either stay in place or move to the right [3]. In particular, it cannot reread any bit to the left of where the head currently is [3]. For that reason this kind of special tape is called "read-once" [3].

► **Definition 2.** A language  $L_1$  is in  $NL$  if there exists a deterministic logarithmic space Turing machine  $M$  with an additional special read-once input tape polynomial  $p : \mathbb{N} \rightarrow \mathbb{N}$  such that for every  $x \in \{0, 1\}^*$ :

$$x \in L_1 \Leftrightarrow \exists u \in \{0, 1\}^{p(|x|)} \text{ such that } M(x, u) = \text{"yes"}$$

where by  $M(x, u)$  we denote the computation of  $M$  where  $x$  is placed on its input tape and the certificate  $u$  is placed on its special read-once tape, and  $M$  uses at most  $O(\log |x|)$  space on its read/write tapes for every input  $x$  where  $|\dots|$  is the bit-length function [3].  $M$  is called a logarithmic space verifier [3].

We state the following Hypothesis:

▷ **Hypothesis 3.** Given a nonempty language  $L_1 \in L$ , there is a language  $L_2$  in *NP-complete* with a deterministic Turing machine  $M$ , where:

$$L_2 = \{w : M(w, u) = y, \exists \text{ string } u \text{ such that } y \in L_1\}$$

when  $M$  runs in logarithmic space in the length of  $w$ ,  $u$  is placed on the special read-once tape of  $M$ , and  $u$  is polynomially bounded by  $w$ . In this way, there is an *NP-complete* language defined by a logarithmic space verifier  $M$  such that when the input is an element of the language with its certificate, then  $M$  outputs a string which belongs to a single language in  $L$ .

From the early days of automata and complexity theory, two different models of Turing machines are considered, the offline and online machines [18]. Each model has a read-only input tape and some work tapes [18]. The offline machines may read their input two-way while the online machines are not allowed to move the input head to the left [18]. In the terminology of the (generalized) Turing machine models are called two-way and one-way Turing machines, respectively [18].

Hartmanis and Mahaney have investigated the classes  $1L$  and  $1NL$  of languages recognizable by deterministic one-way logarithmic space Turing machine and nondeterministic one-way logarithmic space Turing machine, respectively [15]. They have shown that  $1L \neq 1NL$  (by looking at a uniform variant of the string non-equality problem from communication complexity theory) and have defined a natural complete problem for  $1NL$  under deterministic one-way logarithmic space reductions [15]. Furthermore, they have proven that  $1NL \subseteq L$  if and only if  $L = NL$  [15].

▶ **Theorem 4.** *If the Hypothesis 3 is true, therefore when  $L \neq NL$ , then  $P = NP$ .*

**Proof.** We can simulate the computation  $M(w, u) = y$  in the Hypothesis 3 by a nondeterministic logarithmic space Turing machine  $N$  such that  $N(w) = y$ , since we can read the certificate string  $u$  within the read-once tape by a work tape in a nondeterministic logarithmic space generation of symbols contained in  $u$  [22]. Certainly, we can simulate the reading of one symbol from the string  $u$  into the read-once tape just nondeterministically generating the same symbol in the work tapes using a logarithmic space [22].

If we suppose that  $L \subset 1NL$ , then we can accept the elements of the language  $L_1 \in L$  by a nondeterministic one-way logarithmic space Turing machine  $M'$ . In this way, there is a nondeterministic logarithmic space Turing machine  $M''(w) = M'(N(w))$  which will accept when  $w \in L_2$ . Consequently,  $M''$  is a nondeterministic logarithmic space Turing machine which decides the language  $L_2$ . The reason is because we can simulate the output string of  $N(w)$  within a read-once tape and thus, we can compute in a nondeterministic logarithmic space the logarithmic space composition using the same techniques of the logarithmic space composition reduction, but without any reset of the computation [22]. Certainly, we do not need to reset the computation of  $N(w)$  for the reading at once of a symbol in the output string of  $N(w)$  by the nondeterministic one-way logarithmic space Turing machine  $M'$ . Therefore,  $L_2$  is in  $NL$  and thus,  $L_2 \in P$  due to  $NL \subseteq P$  [22]. If any single *NP-complete* problem can be solved in polynomial time, then  $P = NP$  [8]. Since  $L_2 \in P$  and  $L_2 \in NP\text{-complete}$ , then we obtain the complexity class  $P$  is equal to  $NP$  under the assumption that  $L \subset 1NL$ .

Hartmanis and Mahaney have also shown with their result that if  $1NL \subseteq L$  or even  $1NL \subset L$ , then  $L = NL$ , because they proved there is a complete problem for both  $1NL$  and  $NL$  at the same time [15]. If this way, if  $L \neq NL$ , then  $L \subset 1NL$  by contraposition [22]. Since we already obtained that  $P = NP$  under the assumption that  $L \subset 1NL$ , therefore if  $L \neq NL$ , then  $P = NP$ . ◀

### 2.3 Second Hypothesis

An important complexity class is *coNP-complete* [12]. A language  $L_1 \subseteq \{0, 1\}^*$  is *coNP-complete* if:

- $L_1 \in \text{coNP}$ , and
- $L' \leq_p L_1$  for every  $L' \in \text{coNP}$ .

If  $L_1$  is a language such that  $L' \leq_p L_1$  for some  $L' \in \text{coNP-complete}$ , then  $L_1$  is *coNP-hard* [8]. Moreover, if  $L_1 \in \text{coNP}$ , then  $L_1 \in \text{coNP-complete}$  [8]. A principal *coNP-complete* problem is *UNSAT* [12]. A Boolean formula without any satisfying truth assignment is unsatisfiable. The problem *UNSAT* asks whether a given Boolean formula is unsatisfiable [12].

*coNL* is the complexity class containing the languages such that their complements belong to *NL* [22]. We can give a disqualification-based definition for *coNL* [3]. The disqualification-based definition of *coNL* assumes that a logarithmic space Turing machine has another separated read-only tape, that is the same kind of special tape called “read-once” that we use in the certificate-based definition for *NL* [3]. Besides, in the disqualification-based definition of *coNL*, we assume the disqualification string is appropriated for the instance [22]. For example, a truth assignment for a Boolean formula  $\phi$  in *3CNF* is appropriated for the instance when every possible variable in  $\phi$  could be evaluated in that truth assignment string, but we cannot affirm the same for every possible binary string.

► **Definition 5.** A language  $L_1$  is in *coNL* if there exists a deterministic logarithmic space Turing machine  $M$  with an additional special read-once input tape polynomial  $p : \mathbb{N} \rightarrow \mathbb{N}$  such that for every  $x \in \{0, 1\}^*$ :

$$x \in L_1 \Leftrightarrow \forall \text{ appropriated } u \in \{0, 1\}^{p(|x|)} \text{ then } M(x, u) = \text{“yes”}$$

where by  $M(x, u)$  we denote the computation of  $M$  where  $x$  is placed on its input tape and the disqualification  $u$  is placed on its special read-once tape, and  $M$  uses at most  $O(\log |x|)$  space on its read/write tapes for every input  $x$  where  $|\dots|$  is the bit-length function [3].  $M$  is called a logarithmic space disqualifier.

For example, there is a well-known *coNL* problem that states: Given a directed graph  $G = (V, E)$  and two nodes  $s, t \in V$ , is there no possible path from  $s$  to  $t$ ? In that problem, an appropriated disqualification  $u$  is a sequence of nodes contained in  $V$  when  $s$  is the first node and  $t$  is the last one such that this sequence of nodes is not a path: There is at least two consecutive pair of nodes in the sequence where they are not connected by an edge.

We state the following Hypothesis:

▷ **Hypothesis 6.** Given a nonempty language  $L_1 \in 1NL$ , there is a language  $L_2$  in *coNP-complete* with a deterministic Turing machine  $M$ , where:

$$L_2 = \{w : M(w, u) = y, \forall \text{ appropriated string } u \text{ such that } y \in L_1\}$$

when  $M$  runs in logarithmic space in the length of  $w$ ,  $u$  is placed on the special read-once tape of  $M$ , and  $u$  is polynomially bounded by  $w$ . In this way, there is a *coNP-complete* language defined by a logarithmic space disqualifier  $M$  such that when the input is an element of the language with any of its appropriated disqualification, then  $M$  always outputs a string which belongs to a single language in *1NL*.

► **Theorem 7.** If the Hypothesis 6 is true, therefore when  $L = NL$ , then  $P = NP$ .

**Proof.** We can accept the elements of the language  $L_1 \in 1NL$  by a nondeterministic one-way logarithmic space Turing machine  $M'$ . In this way, there is a nondeterministic logarithmic space Turing machine  $M''(w, u) = M'(M(w, u))$  which will accept when  $w \in L_2$  for all the appropriated disqualification  $u$ , where  $u$  is placed on the special read-once tape of  $M''$ . The reason is because we can simulate the output string of  $M(w, u)$  within a read-once tape and thus, we can compute in a nondeterministic logarithmic space the logarithmic space composition using the same techniques of the logarithmic space composition reduction, but without any reset of the computation [22]. Certainly, we do not need to reset the computation of  $M(w, u)$  for the reading at once of a symbol in the output string of  $M(w, u)$  by the nondeterministic one-way logarithmic space Turing machine  $M'$ . Consequently,  $M''$  can be converted into a logarithmic space disqualifier for the language  $L_2$  just assuming that  $L = NL$ , because of the nondeterministic logarithmic space Turing machine  $M''$  could be simulated by a deterministic logarithmic space Turing machine. Therefore,  $L_2$  is in  $coNL$  and thus,  $L_2 \in P$  due to  $coNL \subseteq P$  [22]. If any single  $coNP$ -complete problem can be solved in polynomial time, then  $P = NP$  [22]. Since  $L_2 \in P$  and  $L_2 \in coNP$ -complete, then we obtain the complexity class  $P$  is equal to  $NP$  under the assumption that  $L = NL$ . ◀

### 3 Results

#### 3.1 First Hypothesis

We show a previous known  $NP$ -complete problem:

► **Definition 8. NAE 3SAT**

*INSTANCE:* A Boolean formula  $\phi$  in 3CNF.

*QUESTION:* Is there a truth assignment for  $\phi$  such that each clause has at least one true literal and at least one false literal?

*REMARKS:*  $NAE\ 3SAT \in NP$ -complete [12].

We define a new problem:

► **Definition 9. MAXIMUM EXCLUSIVE-OR 2-SATISFIABILITY**

*INSTANCE:* A positive integer  $K$  and a Boolean formula  $\phi$  that is an instance of XOR 2SAT.

*QUESTION:* Is there a truth assignment in  $\phi$  such that at most  $K$  clauses are unsatisfied?

*REMARKS:* We denote this problem as  $MAX \oplus 2SAT$ .

► **Theorem 10.**  $MAX \oplus 2SAT \in NP$ -complete.

**Proof.** It is trivial to see  $MAX \oplus 2SAT \in NP$  [22]. Given a Boolean formula  $\phi$  in 3CNF with  $n$  variables and  $m$  clauses, we create three new variables  $a_{c_i}$ ,  $b_{c_i}$  and  $d_{c_i}$  for each clause  $c_i = (x \vee y \vee z)$  in  $\phi$ , where  $x$ ,  $y$  and  $z$  are literals, in the following formula:

$$P_i = (a_{c_i} \oplus b_{c_i}) \wedge (b_{c_i} \oplus d_{c_i}) \wedge (a_{c_i} \oplus d_{c_i}) \wedge (x \oplus a_{c_i}) \wedge (y \oplus b_{c_i}) \wedge (z \oplus d_{c_i}).$$

We can see  $P_i$  has at most one unsatisfied clause for some truth assignment if and only if at least one member of  $\{x, y, z\}$  is true and at least one member of  $\{x, y, z\}$  is false for the same truth assignment. Hence, we can create the Boolean formula  $\psi$  as the conjunction of the  $P_i$  formulas for every clause  $c_i$  in  $\phi$ , such that  $\psi = P_1 \wedge \dots \wedge P_m$ . Finally, we obtain that:

$$\phi \in NAE\ 3SAT \text{ if and only if } (\psi, m) \in MAX \oplus 2SAT.$$



Consequently, we prove  $NAE\ 3SAT \leq_p MAX \oplus 2SAT$  where we already know the language  $NAE\ 3SAT \in NP\text{-complete}$  [12]. To sum up, we show  $MAX \oplus 2SAT \in NP\text{-hard}$  and  $MAX \oplus 2SAT \in NP$  and thus,  $MAX \oplus 2SAT \in NP\text{-complete}$ . ◀

► **Theorem 11.** *There is a deterministic Turing machine  $M$ , where:*

$$MAX \oplus 2SAT = \{w : M(w, u) = y, \exists \text{ string } u \text{ such that } y \in XOR\ 2SAT\}$$

when  $M$  runs in logarithmic space in the length of  $w$ ,  $u$  is placed on the special read-once tape of  $M$ , and  $u$  is polynomially bounded by  $w$ .

**Proof.** Given a valid instance  $(\psi, K)$  for  $MAX \oplus 2SAT$  when  $\psi$  has  $m$  clauses, we can create a certificate array  $A$  which contains  $K$  different natural numbers in ascending order which represents the indexes of the clauses in  $\psi$  that we are going to remove from the instance. We read at once the elements of the array  $A$  and we reject whether this is not an appropriated certificate: That is when the numbers are not sorted in ascending order, or the array  $A$  does not contain exactly  $K$  elements, or the array  $A$  contains a number that is not between 1 and  $m$ . While we read the elements of the array  $A$ , we remove the clauses from the instance  $(\psi, K)$  for  $MAX \oplus 2SAT$  just creating another instance  $\phi$  for  $XOR\ 2SAT$  where the Boolean formula  $\phi$  does not contain the  $K$  different indexed clauses  $\psi$  represented by the numbers in  $A$ . Therefore, we obtain the array  $A$  would be valid according to the Theorem 11 when:

$$(\psi, K) \in MAX \oplus 2SAT \text{ if and only if } \phi \in XOR\ 2SAT.$$

Furthermore, we can make this verification in logarithmic space such that the array  $A$  is placed on the special read-once tape, because we read at once the elements in the array  $A$  and we assume the clauses in the input  $\psi$  are indexed from left to right. Hence, we only need to iterate from the elements of the array  $A$  to verify whether the array is an appropriated certificate and also remove the  $K$  different clauses from the Boolean formula  $\psi$  when we write the final clauses to the output. This logarithmic space verification will be the Algorithm 1. We assume whether a value does not exist in the array  $A$  into the cell of some position  $i$  when  $A[i] = \text{undefined}$ . In addition, we reject immediately when the following comparisons:

$$A[i] \leq max \vee A[i] < 1 \vee A[i] > m$$

hold at least into one single binary digit. Note, in the loop  $j$  from  $min$  to  $max - 1$ , we do not output any clause when  $max - 1 < min$ . ◀

► **Theorem 12.** *The Hypothesis 3 is true.*

**Proof.** This is a consequence of Theorems 10 and 11. ◀

### 3.2 Second Hypothesis

We show a previous known *coNP-complete* problem:

► **Definition 13.** *3UNSAT*

*INSTANCE:* A Boolean formula  $\phi$  in 3CNF.

*QUESTION:* Is  $\phi$  unsatisfiable?

*REMARKS:*  $3UNSAT \in coNP\text{-complete}$  [12].

We define a new problem:



**Algorithm 1** Logarithmic space verifier

---

```

1: /*A valid instance for  $MAX \oplus 2SAT$  with its certificate*/
2: procedure VERIFIER(( $\psi, K$ ),  $A$ )
3:   /*Initialize minimum and maximum values*/
4:    $min \leftarrow 1$ 
5:    $max \leftarrow 0$ 
6:   /*Iterate for the elements of the certificate array  $A^*$ */
7:   for  $i \leftarrow 1$  to  $K + 1$  do
8:     if  $i = K + 1$  then
9:       /*There exists a  $K + 1$  element in the array*/
10:      if  $A[i] \neq \text{undefined}$  then
11:        /*Reject the certificate*/
12:        return "no"
13:      end if
14:      /* $m$  is the number of clauses in  $\psi^*$ */
15:       $max \leftarrow m + 1$ 
16:      else if  $A[i] = \text{undefined} \vee A[i] \leq max \vee A[i] < 1 \vee A[i] > m$  then
17:        /*Reject the certificate*/
18:        return "no"
19:      else
20:         $max \leftarrow A[i]$ 
21:      end if
22:      /*Iterate for the clauses of the Boolean formula  $\psi^*$ */
23:      for  $j \leftarrow min$  to  $max - 1$  do
24:        /*Output the indexed  $j^{th}$  clause in  $\psi^*$ */
25:        output " $\wedge c_j$ "
26:      end for
27:       $min \leftarrow max + 1$ 
28:    end for
29: end procedure

```

---

► **Definition 14. SUM ZERO**

*INSTANCE:* A collection of integers  $C$  such that  $0 \notin C$  and every integer in  $C$  has the same bit-length of the number that represents the cardinality of  $C$  multiplied by 3 (we do not take into account the symbol minus in counting the bit-length of the negative integers).

*QUESTION:* Are there two elements  $a, b \in C$ , such that  $a + b = 0$ ?

*REMARKS:* We denote this problem as *SUM-ZERO*.

► **Theorem 15. SUM-ZERO  $\in 1NL$ .**

**Proof.** Given a collection of integers  $C$ , we can read its elements from left to right, verify that every element is not equal to 0, check that every element in  $C$  has the same bit-length and count the amount of elements in  $C$  to finally multiply it by 3 and compare its bit-length with the single bit-length from the elements in  $C$ . In addition, we can nondeterministically pick two elements  $a$  and  $b$  from  $C$  and accept in case of  $a + b = 0$  otherwise we reject. We can make all this computation in a nondeterministic one-way using logarithmic space. Certainly, the calculation and store of the bit-length of the elements in  $C$  could be done in logarithmic space since this is a unique value. On the one hand, we can count and store the number of elements that we read from the input and multiply it by 3 to finally compare its bit-length with the stored unique bit-length from the elements of the collection, since the cardinality of  $C$  multiplied by 3 could be stored in a binary number of bit-length that is logarithmic in relation to the encoded length of  $C$ . On the other hand, the two elements  $a$  and  $b$  that we pick from  $C$  have a logarithmic space in relation to the encoded length of  $C$ , because of every integer in  $C$  has the same bit-length of the number that represents the cardinality of  $C$  multiplied by 3. Indeed, we never need to read to the left on the input for the acceptance of the elements in *SUM-ZERO* in a nondeterministic logarithmic space. ◀

► **Theorem 16. There is a deterministic Turing machine  $M$ , where:**

$$3UNSAT = \{w : M(w, u) = y, \forall \text{ appropriated string } u \text{ such that } y \in \text{SUM-ZERO}\}$$

when  $M$  runs in logarithmic space in the length of  $w$ ,  $u$  is placed on the special read-once tape of  $M$ , and  $u$  is polynomially bounded by  $w$ .

**Proof.** Given a Boolean formula  $\phi$  in *3CNF* with  $n$  variables and  $m$  clauses, we can create a disqualification array  $A$  which contains  $m$  positive integers between 1 and 3 which represents the literals of the clauses in  $\phi$  which appear from left to right. We read at once the elements of the array  $A$  and we reject whether this is not an appropriated disqualification: That is when the array  $A$  does not contain exactly  $m$  elements, or the array  $A$  contains a number that is not between 1 and 3. While we read the elements of the array  $A$ , we select from the clauses  $\phi$  the literals such that these ones occupy the position that represents the number between 1 and 3, that is the first, second or third place within the clause from left to right. In this way, we output the selected literals that are represented by a positive or negative (in case of a negated variable) integer just creating another instance  $C$  for *SUM-ZERO* where the collection  $C$  contains those integers which are the selected literals for each clause in  $\phi$ . Therefore, we obtain that all the appropriated array  $A$  would be valid according to the Theorem 16 when:

$$\phi \in 3UNSAT \text{ if and only if } C \in \text{SUM-ZERO}$$

since we assume the positive and negated literals of some variable in the input  $\phi$  correspond to a positive integer and its negative value. Furthermore, we can make this disqualification in

logarithmic space such that the array  $A$  is placed on the special read-once tape, because we read at once the elements in the array  $A$ . Hence, we only need to iterate from the elements of the array  $A$  to verify whether the array is an appropriated disqualification and pick the  $m$  literals from the Boolean formula  $\phi$  when we write the final integers that represent these literals to the output. This logarithmic space disqualification will be the Algorithm 2. We assume whether a value does not exist in the array  $A$  into the cell of some position  $i$  when  $A[i] = \text{undefined}$ . In addition, we reject immediately when the following comparisons:

$$A[i] < 1 \vee A[i] > 3$$

hold at least into one single binary digit. Note, that every possible literal in  $\phi$  could have a representation by an integer between  $-3 \times m$  and  $3 \times m$  with the exception of 0, where  $m$  is the cardinality of the collection  $C$ . In this way, we guarantee the output collection  $C$  is an appropriated instance of *SUM-ZERO* just filling with zeroes to the left the elements with bit-length lesser than  $|3 \times m|$  where  $|\dots|$  is the bit-length function.

---

**Algorithm 2** Logarithmic space disqualifier

---

```

1: /*A valid instance for 3UNSAT with its disqualification*/
2: procedure DISQUALIFIER( $\phi, A$ )
3:   /*Initialize an index*/
4:    $j \leftarrow 0$ 
5:   /* $m$  is the number of clauses in  $\phi$ */
6:   /*Iterate for the elements of the disqualification array  $A$ */
7:   for  $i \leftarrow 1$  to  $m + 1$  do
8:     if  $i = m + 1$  then
9:       /*There exists an  $m + 1$  element in the array*/
10:      if  $A[i] \neq \text{undefined}$  then
11:        /*Reject the disqualification*/
12:        return "no"
13:      end if
14:      /*Break the for loop*/
15:      break
16:    else if  $A[i] = \text{undefined} \vee A[i] < 1 \vee A[i] > 3$  then
17:      /*Reject the disqualification*/
18:      return "no"
19:    else
20:       $j \leftarrow A[i]$ 
21:    end if
22:    /*From the indexed  $i^{\text{th}}$  clause  $c_i = (x_j \vee y_k \vee z_r)$  in  $\phi$ */
23:    /*Where  $x, y$  and  $z$  are literals with local indexes  $\{j, k, r\} = \{1, 2, 3\}$  in  $c_i$ */
24:    /*Output the integer representation of the  $j^{\text{th}}$  literal, that is  $n(x_j)$ */
25:    /*Filled with zeroes to the left until a total of  $|3 \times m|$  bits including the literal*/
26:    /*But, the bit-length of the symbol minus is ignored in filling the negated literals*/
27:    output " ,  $n(x_j)$ "
28:  end for
29: end procedure

```

---



► **Theorem 17.** *The Hypothesis 6 is true.*



**Proof.** This is a consequence of Theorems 15 and 16. ◀

### 3.3 Consequences

► **Theorem 18.** *If  $L \neq NL$  then  $P = NP$ .*

**Proof.** This is a direct consequence of Theorems 4 and 12. ◀

► **Theorem 19.** *If  $L = NL$  then  $P = NP$ .*

**Proof.** This is a direct consequence of Theorems 7 and 17. ◀

► **Theorem 20.**  $P = NP$ .

**Proof.** Since we have either  $L \neq NL$  or  $L = NL$  is true, then the complexity class  $P$  is equal to  $NP$ . ◀

► **Theorem 21.**  $NL = NP$ .

**Proof.** Since the polynomial time reduction in Theorem 10 could be easily transformed in a logarithmic space reduction, then the  $NP$ -complete problem in Theorem 4, that would be  $MAX \oplus 2SAT$ , is necessarily in  $NL$  and thus all the problems in  $NP$  when  $L \neq NL$ , because of the Cook's Theorem can also be transformed in a logarithmic space reduction [12]. Certainly, every  $NP$  problem could be logarithmic space reduced to  $SAT$  by the Cook's Theorem algorithm and  $SAT$  can be indeed logarithmic space reduced to  $NAE\ 3SAT$  [12]. For the same reason, the  $coNP$ -complete problem in Theorem 7, that would be  $3UNSAT$ , is necessarily in  $coNL$  and thus all the problems in  $coNP$  when  $L = NL$ , because of the problem  $UNSAT$  can be actually logarithmic space reduced to  $3UNSAT$  [12]. Since we have either  $L \neq NL$  or  $L = NL$  is true, then the complexity class  $NL$  is equal to  $NP$ . ◀

## 4 Conclusions

No one has been able to find a polynomial time algorithm for any of more than 300 important known  $NP$ -complete problems [12]. A proof of  $P = NP$  will have stunning practical consequences, because it leads to efficient methods for solving some of the important problems in  $NP$  [7]. The consequences, both positive and negative, arise since various  $NP$ -complete problems are fundamental in many fields [7]. The following consequences are assuming that we have a practical solution for the  $NP$ -complete problems where such existence was proven with our nonconstructive result:

Cryptography, for example, relies on certain problems being difficult. A constructive and efficient solution to an  $NP$ -complete problem such as  $3SAT$  will break most existing cryptosystems including: Public-key cryptography [16], symmetric ciphers [20] and one-way functions used in cryptographic hashing [9]. These would need to be modified or replaced by information-theoretically secure solutions not inherently based on  $P$ - $NP$  equivalence.

There are enormous positive consequences that will follow from rendering tractable many currently mathematically intractable problems. For instance, many problems in operations research are  $NP$ -complete, such as some types of integer programming and the traveling salesman problem [12]. Efficient solutions to these problems have enormous implications for logistics [7]. Many other important problems, such as some problems in protein structure prediction, are also  $NP$ -complete, so this will spur considerable advances in biology [5].

Since all the  $NP$ -complete optimization problems become easy, everything will be much more efficient [11]. Transportation of all forms will be scheduled optimally to move people

and goods around quicker and cheaper [11]. Manufacturers can improve their production to increase speed and create less waste [11]. Learning becomes easy by using the principle of Occam's razor: We simply find the smallest program consistent with the data [11]. Near perfect vision recognition, language comprehension and translation and all other learning tasks become trivial [11]. We will also have much better predictions of weather and earthquakes and other natural phenomenon [11].

There would be disruption, including maybe displacing programmers [17]. The practice of programming itself would be more about gathering training data and less about writing code [17]. Google would have the resources to excel in such a world [17].

But such changes may pale in significance compared to the revolution an efficient method for solving  $NP$ -complete problems will cause in mathematics itself [7]. Research mathematicians spend their careers trying to prove theorems, and some proofs have taken decades or even centuries to find after problems have been stated [1]. For instance, Fermat's Last Theorem took over three centuries to prove [1]. A method that is guaranteed to find proofs to theorems, should one exist of a "reasonable" size, would essentially end this struggle [7].

We also prove that  $NL = NP$ . There are several important consequences from this result, but we will only mention one of the principal outcomes, that would be the application of Theorem Buhrman-Fortnow-van Melkebeek-Torenvliet when  $NL = NP$  [6].

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