Unit 17: Introduction to Spectral Analysis

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Readings for Unit 17

Textbook chapter 4.1, 4.2.

Last Unit

1. Seasonal Differencing.
2. Building the Seasonal ARIMA model.

Motivation

There are two primary approaches to time series. One is the **time domain** approach, which we covered in Units 8 to 16. This approach focuses on the rules for a time series to move forward.

The other approach is the **frequency domain** approach. This approach tries to understand how differing oscillations can contribute to current observations.

1. Introduction to Spectral Analysis

2. Aliasing

3. Periodic Time Series

Time Domain Approach

Time domain approach: models which give an explicit formula for the current observation in terms of past observations and past white noise terms. "Regression of the present on the past."

Frequency Domain Approach

Frequency domain approach: current observation as a combination of waves. "Regression of the current time on sines and cosines of various frequencies."

Idea: decompose a stationary time series $\{x_t\}$ into a combination of sinusoids, with random and uncorrelated coefficients. This is also referred to spectral analysis.

Spectral Analysis

- Identify dominant frequencies within the data.
- Periodogram: ______________ at a given sequence of frequencies.
- Power spectrum: ___________ version of the periodogram.

Spectral Analysis

Time (period) and frequency are inversely related. With quarterly data, there are four data points per year (cycle). This corresponds to 0.25 cycles per data point. The notation is

$$T = \frac{1}{\omega}$$

where ω is the frequency.

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2. Aliasing
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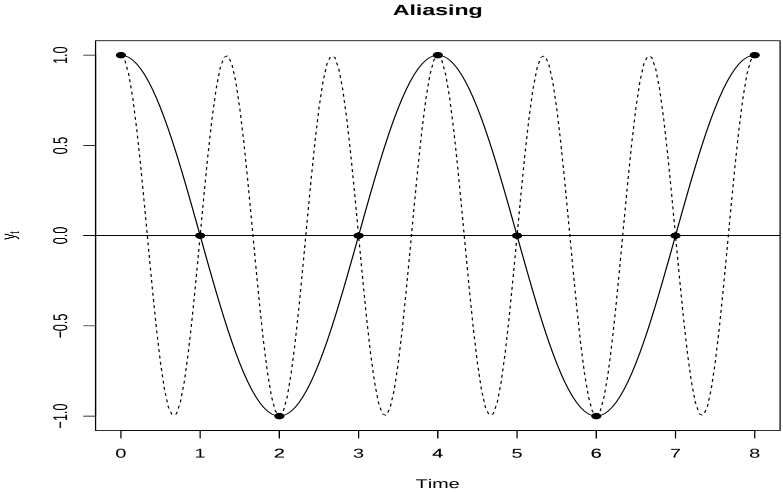
Aliasing

- When $\omega = 1$, the time series makes one cycle per time unit.
- When $\omega = 0.5$, the time series makes one cycle every two time units.
- When $\omega = 0.25$, the time series makes one cycle every four time units.

Aliasing

Consider cosine curves with $\omega = \frac{1}{4}$ (bold) and $\omega = \frac{3}{4}$ (dashed).



Aliasing

Notice that at the discrete time points $0, 1, 2, 3, \cdots$ the two cosine curves have identical values. With ____________ observations, we would not be able to ___________ between the two curves. So, the frequencies $\frac{1}{4}$ and $\frac{3}{4}$ are ______ with one another.

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Periodic Time Series

In Unit 6, we discussed periodic functions on the integers that have the following form

$$x(t) = A\cos(2\pi\omega t + \phi)$$

for $t = 0, \pm 1, \pm 2, \cdots$ where ω is the frequency, A is the amplitude, and ϕ is the phase.

Some Trigonometric Identities

$$
\begin{aligned}
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}, \\
\sin^2(\theta) + \cos^2(\theta) &= 1, \\
\sin(a \pm b) &= \sin(a)\cos(b) \pm \cos(a)\sin(b), \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b).
\end{aligned}
$$

Periodic Time Series

Having the ϕ inside the cosine function can be problematic since if we want to do a regression, the ϕ makes this a non-linear regression. This issue is worked around using a trig identity

$$\cos(\alpha \mp \beta) = \cos(\alpha)\cos(\beta) \pm \sin(\alpha)\sin(\beta)$$

to rewrite the periodic function as

$$x_t = A\cos(\phi)\cos(2\pi\omega t) - A\sin(\phi)\sin(2\pi\omega t). \tag{1}$$

Periodic Time Series

Now let's consider (1) differently from before. Re-write the periodic function (1) as

$$x_t = U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t). \tag{2}$$

where $U_1 = A\cos(\phi)$ and $U_2 = -A\sin(\phi)$. We now assume U_1, U_2 are iid Gaussian with zero mean and fixed variance.

Periodic Time Series

Generalize (2) to include ________ frequencies and amplitudes with

$$x_t = \sum_{k=1}^{q} U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t) \tag{3}$$

where the U_{k1} and the U_{k2} are independent and $N(0, \sigma_k^2)$ and the ω_k are distinct frequencies.

Periodic Time Series

A consequence of the representation given by (3) is that any stationary time series may be thought of, approximately, as the random superposition of sines and cosines oscillating at various frequencies.

Periodic Time Series

Let's derive the moments of (3).

Periodic Time Series

Periodic Time Series

Periodic Time Series

Question: What is the implication of these derivations?

Next...

Spectral Density: Fourier transformation of the autocovariance.