Unit 15: Seasonal ARMA Models

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Readings for Unit 15

Textbook chapter 3.9.

Last Unit

1. Integrated models for nonstationary data.
2. Building ARIMA models (Exploratory data analysis, Model estimation, Model diagnostics, Model selection).

This Unit

Seasonal ARMA models for seasonal time series.

Motivation

So far, we've avoided seasonal data. The ARIMA models that we've discussed do not allow for skipping lags. However, we may wish to have a model for monthly observations which depends on both the previous month and the same month one year ago. SARMA models will allow us to do that.

SARMA Model

We can write the pure seasonal ARMA model, ARMA$(P, Q)_s$, using backshift operators in the following way.

$$\Phi_P(B^s)x_t = \Theta_Q(B^s)w_t \tag{1}$$

where

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \dots - \Phi_P B^{Ps}$$

and

$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \dots + \Theta_Q B^{Qs}.$$

The first polynomial is the seasonal autoregressive operator and the second is the seasonal moving average operator.

Seasonal ARMA Model

Suppose you have quarterly data and want to think about an $\text{ARMA}(1,1)_4$. This would be

$$(1 - \Phi_1 B^4)x_t = (1 + \Theta_1 B^4)w_t$$

or

$$x_t = \Phi_1 x_{t-4} + w_t + \Theta w_{t-4}.$$

This is essentially an ARMA model, except lags between zero and four are omitted.

Seasonal ARMA Model

Just like the nonseasonal ARMA models, the pure seasonal ARMA$(P, Q)_s$ is causal only when the roots of $\Phi_P(z^s)$ lie outside the unit circle, and is invertible only when the roots of $\Theta_Q(z^s)$ lie outside the unit circle.

ACF for Seasonal MA

Let's consider monthly data and look at a seasonal MA(1). The model would be written as

$$x_t = w_t + \Theta_1 w_{t-12}.$$

The variance will be $\sigma_w^2(1 + \Theta_1^2)$.

ACF for Seasonal MA

The autocovariance would then be (for $h > 0, h \neq 12$)

$$\gamma(h) = E(x_t x_{t-h}) = E(w_t + \Theta_1 w_{t-12})(w_{t-h} + \Theta_1 w_{t-h-12}) = 0$$

since none of the white noise terms are in common.

ACF for Seasonal MA

However, when $h = 12$,

$$\gamma(12) = E(x_t x_{t-12}) = E(w_t + \Theta_1 w_{t-12})(w_{t-12} + \Theta_1 w_{t-24}) = \Theta_1 \sigma_w^2.$$

ACF for Seasonal MA

In general, for the MA$(Q)_s$ model

$$x_t = w_t + \Theta_1 w_{t-s} + \Theta_2 w_{t-2s} + \cdots + \Theta_Q w_{t-Qs},$$

- $\gamma(h) = 0$ for $h \neq ks, k = 1, 2, \ldots$.
- $\gamma(0), \gamma(s), \gamma(2s), \ldots, \gamma(Qs)$ are non-zero.
- $\gamma(ks) = 0$ for $k \geq Q + 1$.

ACF for Seasonal AR

Now, let's think about a pure seasonal $AR(1)_{12}$ process. This would be

$$x_t = \Phi_1 x_{t-12} + w_t.$$

Iterate recursively to obtain

$$x_t = \sum_{k=0}^{\infty} \Phi_1^k w_{t-12k}$$

for $|\Phi_1| < 1$.

ACF for Seasonal AR

The autocovariance would then be (for $h > 0, h \neq 12k$)

$$\gamma(h) = E(x_t x_{t-h}) = E\Big(\sum_{k=0}^{\infty} \Phi_1^k w_{t-12k}\Big)\Big(\sum_{k=0}^{\infty} \Phi_1^k w_{t-h-12k}\Big) = 0.$$

ACF for Seasonal AR

Using the same reasoning, we obtain

$$\gamma(12k) = \sigma_w^2 \frac{\Phi_1^k}{1 - \Phi_1^2}$$

for $k = 1, \dots$. We will also get an autocorrelation of

$$\rho(12k) = \Phi_1^k.$$

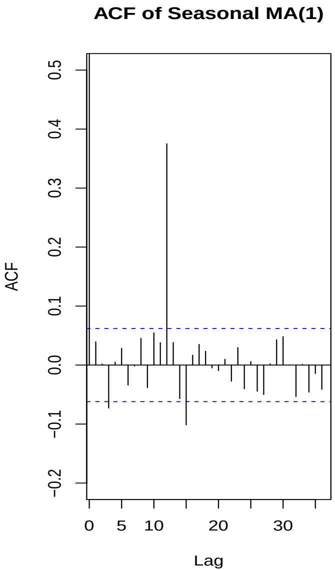
ACF for Seasonal MA, AR

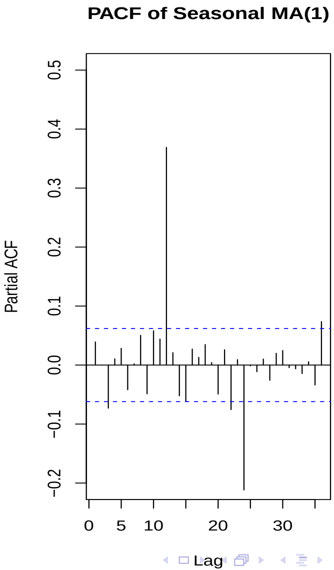
When looking at ACF and PACF plots, we are going to use the same criteria as before but looking only at the lags that are a multiple of the period. A pure seasonal MA(1) should have a significant value for the ACF at the lag of the period and roughly zero otherwise. A pure seasonal AR(1) should tail off exponentially at the lag of the period and be roughly zero otherwise.

PACF for Seasonal MA, AR

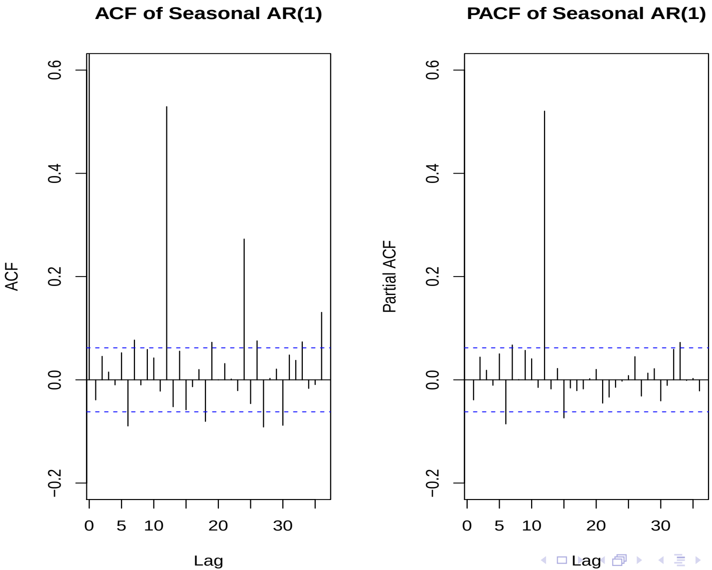
The PACF of a pure seasonal MA(1) should tail off exponentially at multiples of the period and be zero otherwise. The PACF of a pure seasonal AR(1) should cut off after the lag of one period and should be zero for all other values.

ACF & PACF for Seasonal MA(1)





ACF & PACF for Seasonal AR(1)



ACF and PACF for Seasonal ARMA

For ARMA$(P, Q)_s$, both ACF and PACF tail off exponentially at multiples of the period.

ACF and PACF for SARMA

(From page 155, Table 3.3 of text)

	AR$(P)_s$	**MA**$(Q)_s$	**ARMA**$(P, Q)_s$
ACF	Tails off at lags ks	0 after lag Qs	Tails off at lags ks
PACF	0 after lag Ps	Tails off at lags ks	Tails off at lags ks

Note: The values at nonseasonal lags $h \neq ks$ for $k = 1, 2, \cdots$ are 0.

1. SARMA Model
2. Exploratory Data Analysis
3. Multiplicative Seasonal ARMA Models

Multiplicative Seasonal ARMA Models

We can also combine the seasonal aspects and the regular ARMA models to get multiplicative seasonal autoregressive moving average models denoted ARMA$(p, q) \times (P, Q)_s$. We may write the model as

$$\Phi_P(B^s)\phi(B)x_t = \Theta_Q(B^s)\theta(B)w_t. \tag{2}$$

Multiplicative Seasonal ARMA Models

The properties that we observe for pure seasonal ARMA models are not strictly true of the multiplicative seasonal ARMA models ARMA$(p, q) \times (P, Q)_s$. For the multiplicative models, we should expect to see a mix of patterns that we observe in non-seasonal and pure seasonal ARMA models.

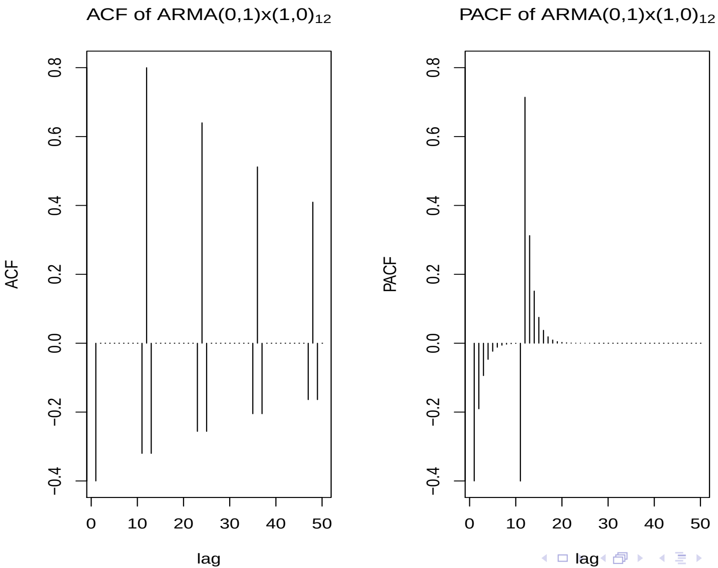
Multiplicative Seasonal ARMA Models

Question: How do we write ARMA$(0, 1) \times (1, 0)_{12}$?

Multiplicative Seasonal ARMA Models

Question: How does the ACF of ARMA$(0,1) \times (1,0)_{12}$ look like?

ACF & PACF for Multiplicative SARMA



Next...

Next we will look at removing seasonal non-stationarity and the full SARIMA model.