Unit 9: ARMA Models: MA(1)

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Readings for Unit 9

Textbook chapter 3.2.

Last Unit

1. AR(p) process
2. AR(1) process
3. AR(1) in terms of backshift operator
4. AR(1) and causality

This Unit

1. Identifiability
2. MA(1) in terms of backshift operator
3. MA(1) and invertibility
4. ARMA model

Motivation

In the previous unit, we noted a condition for an AR(1) model to be causal. In this unit, we will explore another issue, this time regarding MA models. This issue is called identifiability.

1. MA(1) and Invertibility
2. ARMA Model

MA(q) Process

Recall the MA(q) model

$$x_t = w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}.$$

We can rewrite this using backshift operator as

$$x_t = \theta(B) w_t \tag{1}$$

where $\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q$ which can be called the ____________________________. We already know that an MA(q) process is stationary.

MA(1) Process

Let's look at the $MA(1)$ model, i.e.

$$x_t = w_t + \theta w_{t-1}$$

We can then calculate the autocovariance function.

$$\gamma(0) = (1 + \theta^2)\sigma_w^2$$

and

$$\gamma(1) = \theta\sigma_w^2$$

and zero for larger lags. Assume that we simulate from a model using $\theta = a$ and $\sigma_w^2 = b$, then look at the autocovariance

$$\gamma(0) = b + a^2 b, \quad \gamma(1) = ab$$

.

Identifiability

Question: Suppose we have another MA(1) model with $\theta = 1/a$ and $\sigma_w^2 = a^2 b$, what is the autocovariance function for this model?

Identifiability

Which one is "correct"? Both are correct: this is an **identifiability** problem as we have MA models that are not unique. We'll use the one that gives us an infinite order AR process.

Identifiability

Rewrite the MA(1) model as

$$w_t = -\theta w_{t-1} + x_t.$$

This is just like the AR(1) that we saw last time but now the roles for x_t, w_t are reversed. So, we could write this as

$$\begin{aligned} w_t &= \sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \\ &= \sum_{j=0}^{\infty} (-\theta)^j B^j x_t, \end{aligned} \tag{2}$$

provided that $|\theta| < 1$.

MA(1) in Terms of Backshift Operator

(2) can be written as

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = \pi(B) x_t, \tag{3}$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$ and $\pi_j = (-\theta)^j$.

Identifiability

Question: What's an intuitive explanation as to why we want $|\theta| < 1$ in (2)?

Invertible MA(1)

We could write an MA(1) as

$$\begin{aligned} x_t &= w_t + \theta w_{t-1} \\ &= w_t + \theta \sum_{j=0}^{\infty} (-\theta)^j x_{t-1-j} \\ &= w_t - \sum_{j=1}^{\infty} (-\theta)^j x_{t-j}. \end{aligned}$$

So, we have written the MA(1) as an infinite order AR process. An MA model which can be written like this is called an ________ MA. We require $|\theta| < 1$ for invertibility. We will later extend and formalize a condition on the parameters $\theta_1, \theta_2, \cdots, \theta_q$ for an MA(q) process to be invertible.

1. MA(1) and Invertibility
2. ARMA Model

ARMA Model

Recall that an AR(p) model is given by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots \phi_p x_{t-p} + w_t$$

or in terms of backshift operator

$$\phi(B)x_t = w_t, \quad \text{where} \quad \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p.$$

ARMA Model

An MA(q) model is given by

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

or in terms of backshift operator

$$x_t = \theta(B) w_t, \quad \text{where} \quad \theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

ARMA Model

x_t is called an ARMA (p,q) model if x_t is stationary and can be written as

$$\phi(B)x_t = \theta(B)w_t.$$

Thus far, we have seen a number of issues with the general definition of ARMA(p,q) models:

- Stationary AR models that depend on the _____.
- MA models that are not _____ (identifiability issue).

We will next look at one more issue with ARMA(p,q) models before formalizing conditions for causality and invertibility.