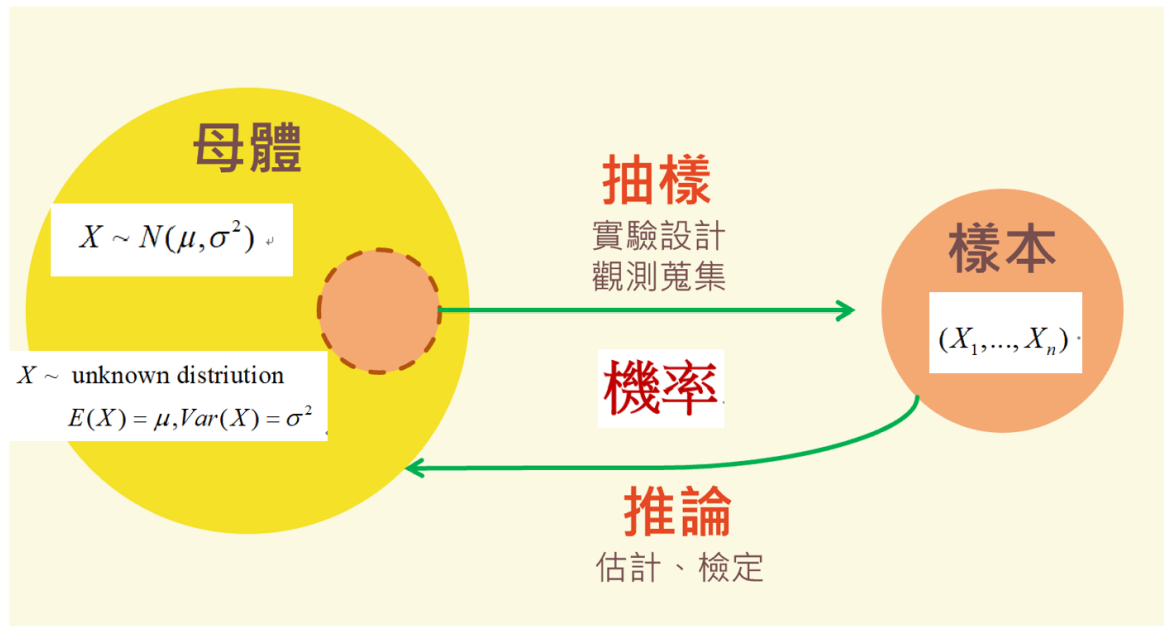


Lecture 8 11/3/2021

1. Review: from probability to statistics

Random sample:

X_1, \dots, X_n form a random sample (隨機樣本) from a population with mean μ and variance σ^2 .



We can write $X_i \sim^{iid} E(X_i) = \mu, \text{Var}(X_i) = \sigma^2$.

Properties of \bar{X} :

a. $E(\bar{X}) = \mu$

b. $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$,

$$\sqrt{\text{Var}(\bar{X})} = \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

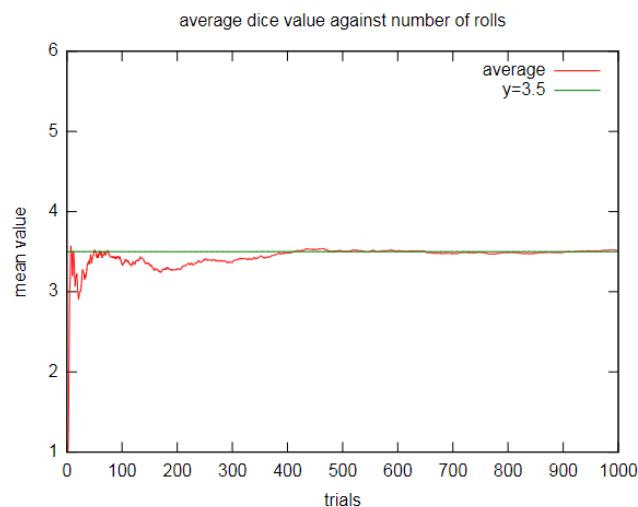
Note: Taking the average of repeated measurements can reduce the error.

Parameter of interest: μ

Point estimator of μ : $\hat{\mu} = \bar{X}$

To make inference of μ , we need to know the probabilistic properties of $\hat{\mu} = \bar{X}$.

Law of Large Number (L.L.N., 大數法則) $\bar{X} \rightarrow \mu$ as $n \rightarrow \infty$.



Remarks:

- You can see that when $n \rightarrow \infty$, $Var(\bar{X}) = \frac{\sigma^2}{n} \rightarrow 0$.
- There are two versions of LLN: strong law and weak law.
- $\sum_{i=1}^n g(X_i) / n \rightarrow E[g(X)] = \int g(x)f(x)dx$

Central Limit Theorem: $X_i \sim^{iid} E(X_i) = \mu, Var(X_i) = \sigma^2$.

$$\bar{X}_n \underset{n \rightarrow \infty}{\sim} N(\mu, Var(\bar{X}_n) = \sigma^2 / n)$$

$$\frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \underset{n \rightarrow \infty}{\sim} N(0,1)$$

Graphical example:

There are 5 different types of random variables, their sample averages ($n = 5$) become very alike.

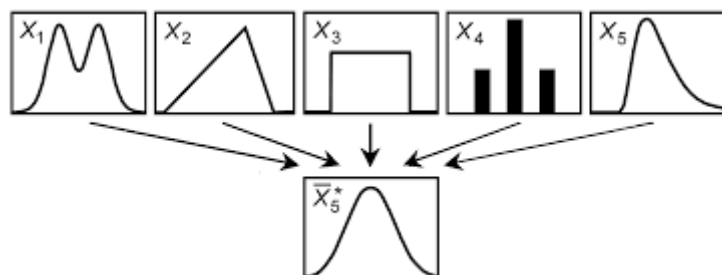


Exhibit 3.30: The central limit theorem is illustrated in the case of five arbitrarily selected independent random variables. Random variables X_1 , X_2 , X_3 , and X_5 are continuous, so their PDFs are shown; X_4 is discrete so its PF is shown. The normalized average \bar{X}_5^* is approximately $N(0,1)$. All graphs indicate the interval $[-3,3]$ on the x-axis.

Random Variable	Mean	Standard Deviation	Skewness	Kurtosis	Description
X_1	0.00	1.00	0.00	1.89	continuous
X_2	0.00	1.00	-0.41	2.41	continuous
X_3	0.00	1.00	0.00	1.80	continuous
X_4	0.00	1.00	0.00	2.00	discrete
X_5	0.00	1.00	1.62	7.89	continuous
\bar{X}_5^*	0.00	1.00	0.11	3.03	continuous

Comparison: LLN and CLT

Law of Large Number:

$$\bar{X}_n \xrightarrow[n \rightarrow \infty]{} \mu \Leftrightarrow \bar{X}_n - \mu \xrightarrow[n \rightarrow \infty]{} 0$$

Central Limit Theorem

$$\bar{X}_n \underset{n \rightarrow \infty}{\sim} N(\mu, \text{Var}(\bar{X}_n) = \sigma^2 / n) \Leftrightarrow \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \underset{n \rightarrow \infty}{\sim} N(0,1)$$

$$LLN : \bar{X}_n - \mu \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence to a constant})$$

$$CLT : \sqrt{n}(\bar{X}_n - \mu) \underset{n \rightarrow \infty}{\sim} N(0, \sigma^2) \quad (\text{Enlarged version: convergence to a distribution})$$

$$\text{Divergence: } n(\bar{X}_n - \mu) \xrightarrow[n \rightarrow \infty]{} \infty$$

▼ An actual photo to take by cell phone as follows:

Samsung S3



1X

2X

3X

4X

▼ An actual photo to take by cell phone & Micro-lens as follows:

Samsung S3+DMX i95 (ZOOM 4~50X)



12X

25X

35X

50X

Case 1: Normal population $X_i \sim^{iid} N(\mu, \sigma)$, $\frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ any n

1. A bottling company uses a filling machine to fill plastic bottles with a popular cola. The bottles are supposed to contain 300 milliliters. In fact, the contents vary according to a normal distribution with mean $\mu = 298$ ml and standard deviation

= 3 ml. (i.e. $X_i \sim^{iid} N(298, \sigma = 3)$)

- a. What is the probability that an individual bottle contains less than 295 ml?

$$\Pr(X_i < 295) = \Pr\left(\frac{X_i - \mu}{\sqrt{\text{Var}(X_i)}} < \frac{295 - 298}{3}\right) = \Pr(Z < -1) = 0.1587.$$

- b. What is the probability that the **mean contents** of the bottles in a **six-pack** is less than 295 ml?

$$n = 6$$

$$\bar{X} = (X_1 + \dots + X_6) / 6 \rightarrow \bar{X} \sim N(\mu = 298, \text{Var}(\bar{X}) = \frac{3^2}{6})$$

$$\begin{aligned} \Pr(\bar{X} < 295) &= \Pr\left(\frac{\bar{X} - \mu}{\sqrt{\text{Var}(\bar{X})}} < \frac{295 - 298}{3/\sqrt{6}}\right) \\ &= \Pr(Z < -\sqrt{6}) = \Pr(Z < -2.45) = 0.0071 \end{aligned}$$

Case 2: Central Limit Theorem

1. X = number of accidents per week = a discrete random variable

$$E(X) = 2.2 \quad \sigma = \sqrt{\text{Var}(X)} = 1.4,$$

$$\text{Random sample: } (X_1, \dots, X_{52}), \quad n = 52, \quad \bar{X} = \frac{X_1 + \dots + X_{52}}{52}$$

- a. What is the approximate distribution of $\bar{X} = \frac{X_1 + \dots + X_{52}}{52}$?

$$E(\bar{X}) = 2.2, \quad \sqrt{\text{Var}(\bar{X})} = \sigma / \sqrt{n} = 1.4 / \sqrt{52} = 0.194$$

$$\bar{X} \sim^{\text{approximately}} N(2.2, \sigma_{\bar{X}} = 0.194)$$

- b. $\Pr(\bar{X} < 2)$ = the average number of accidents is smaller than 2

$$\Pr(\bar{X} < 2) = \Pr\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} < \frac{2 - 2.2}{1.4 / \sqrt{52}}\right) \approx \Pr(Z < -1) = 0.1587$$

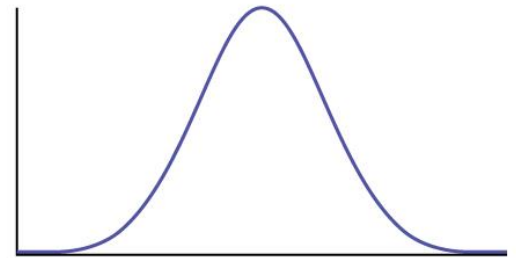
- c. $\Pr(X_1 + \dots + X_{52} < 100)$

= the total number of accidents within a year is smaller than 100

Note: You need to convert “total” to “average”

$$\Pr(X_1 + \dots + X_{52} < 100) = \Pr(\bar{X} < 100 / 52) = \Pr(\bar{X} < 1.92)$$

$$\begin{aligned}\Pr(\bar{X} < 1.92) &= \Pr\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} < \frac{1.92 - 2.2}{1.4 / \sqrt{52}}\right) \\ &\approx \Pr(Z < -1.44) \\ &= 0.0749\end{aligned}$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681

Topic: Normal approximation to Binomial distribution (CLT 重要應用)

$Y \sim N(n, p)$, sometimes it is difficult to compute

$$\Pr(a \leq Y \leq b) = \sum_{y=a}^{y=b} \Pr(Y = y) = \sum_{y=a}^{y=b} \binom{n}{y} p^y (1-p)^{n-y}$$

Poisson to approximation for Binomial distribution:

When n large but p small (rare events),

$$\Pr(a \leq Y \leq b) = \sum_{y=a}^{y=b} \Pr(Y = y) \approx \sum_{y=a}^{y=b} \frac{e^{-np} (np)^y}{y!}$$

● Normal approximation for Binomial distribution

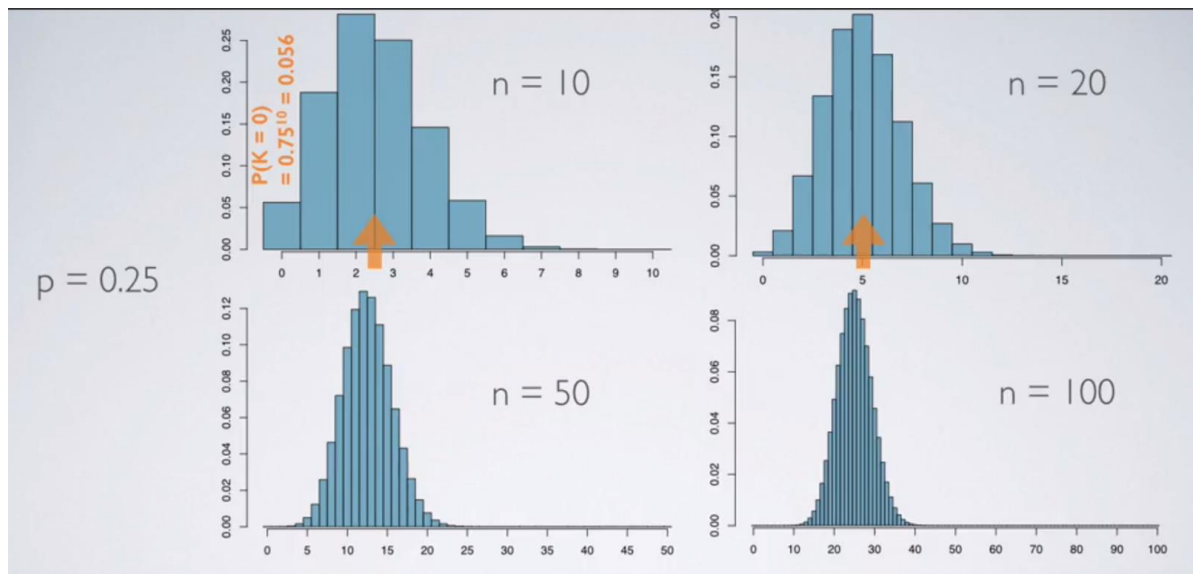
When $n \cdot p \geq 10$ & $n \cdot (1-p) \geq 10$, we can approximate

$$Y \sim N(n, p) \text{ by } X \sim N(\mu = np, \sigma^2 = np(1-p)).$$

Derivations:

$$\begin{aligned}\Pr(a \leq Y \leq b) &= \Pr(a - np \leq Y - np \leq b - np) \\ &= \Pr\left(\frac{a - np}{\sqrt{np(1-p)}} \leq \frac{Y - np}{\sqrt{np(1-p)}} \leq \frac{b - np}{\sqrt{np(1-p)}}\right) \\ &\approx \Pr\left(\frac{a - np}{\sqrt{np(1-p)}} \leq \frac{X - \mu}{\sigma} \leq \frac{b - np}{\sqrt{np(1-p)}}\right) \text{ (by CLT)} \\ &= \Pr\left(\frac{a - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{b - np}{\sqrt{np(1-p)}}\right)\end{aligned}$$

Plot: fix $p = 0.25$, change n



CLT \rightarrow distributional property of the sample mean when the sample size is large

$$Y = \sum_{i=1}^n B_i = \text{total number of successes} = \text{sum of Bernoulli random variables}$$

(i.e. $B_i \sim \text{Bernoulli}(p)$, with $E(B_i) = p$ and $\text{Var}(B_i) = p(1-p)$)

$$\Pr(a \leq Y \leq b) = \Pr(a \leq \sum_{i=1}^n B_i \leq b) = \Pr\left(\frac{a}{n} \leq \frac{\sum_{i=1}^n B_i}{n} \leq \frac{b}{n}\right) = \Pr\left(\frac{a}{n} \leq \bar{B} \leq \frac{b}{n}\right)$$

By the Central Limit Theorem,

$$\bar{B} = \hat{p} \sim_{\text{approximately}} N\left(p, \text{Var}(\bar{B}) = \frac{\text{Var}(B_i)}{n} = \frac{p(1-p)}{n}\right)$$

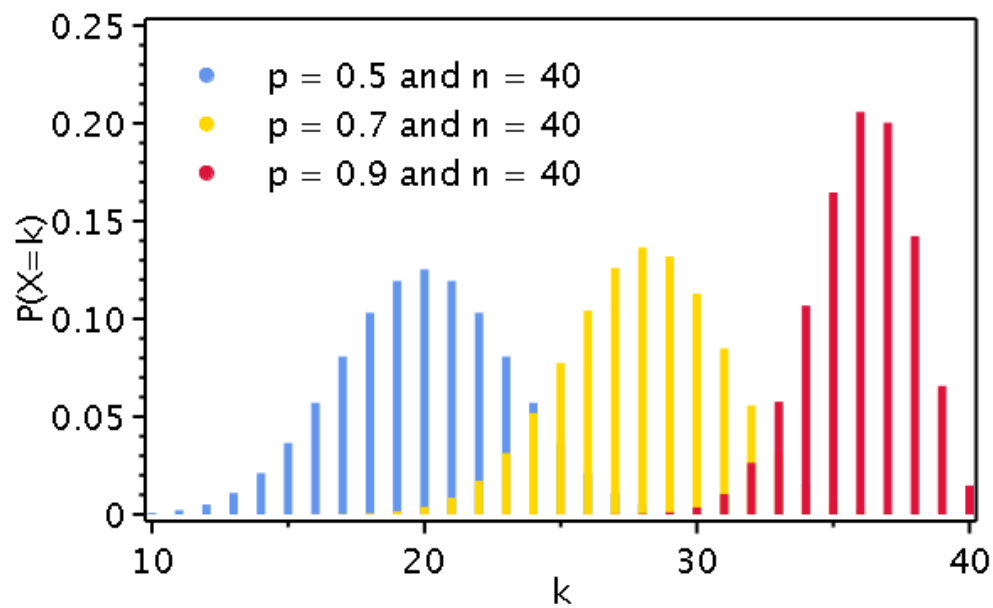
$$\begin{aligned} \Pr\left(\frac{a}{n} \leq \hat{p} \leq \frac{b}{n}\right) &= \Pr\left(\frac{\frac{a}{n} - p}{\sqrt{\frac{p(1-p)}{n}}} \leq \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \leq \frac{\frac{b}{n} - p}{\sqrt{\frac{p(1-p)}{n}}}\right) \\ &= \Pr\left(\frac{a - np}{n\sqrt{\frac{p(1-p)}{n}}} \leq \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \leq \frac{b - np}{n\sqrt{\frac{p(1-p)}{n}}}\right) \\ &\approx \Pr\left(\frac{a - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{b - np}{\sqrt{np(1-p)}}\right) \end{aligned}$$

Remark: requirement for good approximation $\rightarrow n \cdot p \geq 10$ & $n \cdot (1 - p) \geq 10$

When $p = 0.5$, $n \geq 20$; (original Binomial is already symmetric)

When $p = 0.01$, $n \geq 1000$ (original Binomial is very skew)

Plot: fix $n = 40$, change p

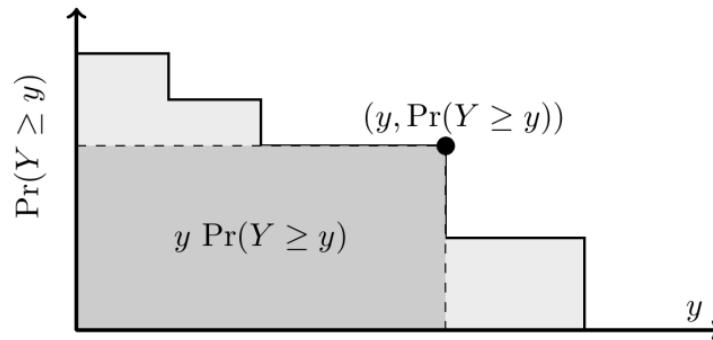


* *History of Normal distribution and Central Limit Theorem*

Markov Inequality – for positive random variables

Let $X > 0$ be a positive random variable.

Markov Inequality: For $a > 0$, $\Pr(X > a) \leq \frac{E(X)}{a}$.



Let $S(x) = \Pr(X > x) = 1 - \Pr(X \leq x) = 1 - F(x)$ = the survival function

For $X > 0$,

$$\begin{aligned}\mu = E(X) &= \int_0^{\infty} xf(x)dx = -\int_0^{\infty} xdS(x) \\ &= -\left(xS(x)\Big|_0^{\infty} - \int_0^{\infty} S(x)dx\right) \\ &= -\{\infty S(\infty) - 0 \cdot S(0)\} + \int_0^{\infty} S(x)dx \\ &= \int_0^{\infty} S(x)dx\end{aligned}$$

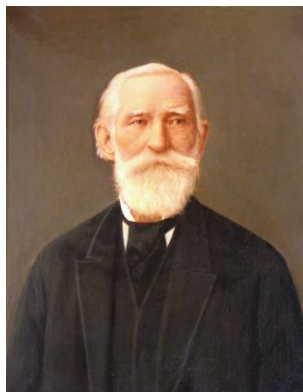
In the graph ($X \rightarrow Y, a \rightarrow y$), the area under $S(y) = \Pr(Y \geq y)$ is

$$E(Y) = \int_0^{\infty} S(y)dy = \int_0^{\infty} \Pr(Y \geq y)dy,$$

which is larger than the area of the rectangle.

Chebyshev (1821-1894)

Markov (1852-1922)

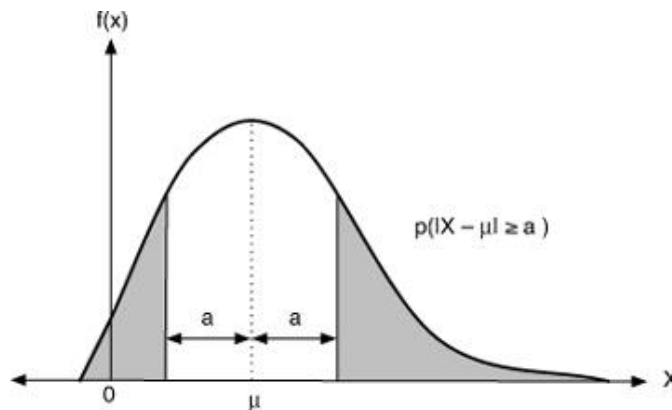


Chebyshev Inequality – for ANY random variables

Version 1: Let X be a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$.

For $k > 0$

$$\begin{aligned} \Pr(|X - \mu| \geq k) &= \Pr(|X - \mu|^2 \geq k^2) \\ &\leq \frac{E(|X - \mu|^2)}{k^2} = \frac{\sigma^2}{k^2} \end{aligned}$$



Version 2: useful for statistics

Let $X_i \sim^{iid} E(X_i) = \mu$ and $Var(X_i) = \sigma^2$

$$\begin{aligned} \Pr(|\bar{X} - \mu| \geq k) &\leq \frac{Var(\bar{X})}{k^2} \\ &= \frac{1}{k^2} \frac{\sigma^2}{n} \end{aligned}$$

Statistical meaning of $\Pr(|\bar{X} - \mu| \geq k)$

- \bar{X} is an estimator of μ
- $|\bar{X} - \mu|$ = estimation error which is a random variable
- $|\bar{X} - \mu| \geq k \rightarrow$ the error is at least k which is a pre-specified standard
 \rightarrow a bad thing

Remarks:

- We want the value of error term $|\bar{X} - \mu|$ as small as possible
- Taking the randomness into account, we want $\Pr(|\bar{X} - \mu| \geq k)$ as small as possible.

- Chebyshev's inequality tells you the upper bound of $\Pr(|\bar{X} - \mu| \geq k)$ is $\frac{1}{k^2} \frac{\sigma^2}{n}$.

Example: $X = \#$ of items produced in a factory during a week with $E(X) = 50$

Solution: Fact: $X > 0 \rightarrow$ suitable for the Markov inequality

$$1. \Pr(X \geq 75) \leq \frac{E(X)}{75} = \frac{50}{75}$$

The probability that the production exceeds 75 will be no greater than 2/3.

$$2. \Pr(40 < X < 60) = ?$$

Step 1: Re-express the above two-side inequality in terms of "one-side"

$$\begin{aligned}\Pr(40 < X < 60) &= \Pr(-10 < X - \mu < 10) \\ &= \Pr(|X - \mu| < 10)\end{aligned}$$

Step 2: Calculate the tail probability first

Based on Chebyshev's inequality

$$\begin{aligned}\Pr(|X - \mu| \geq 10) &= \Pr(|X - \mu|^2 \geq 100) \\ &\leq \frac{E[|X - \mu|^2]}{100} \\ &= \frac{\text{Var}(X)}{100} = \frac{25}{100}\end{aligned}$$

Step 3: Then calculate the bound for the "within" probability

$$\Pr(|X - \mu| < 10) > 1 - \frac{25}{100} = \frac{3}{4}$$

The probability that the production is between 40 and 60 will be at least 3/4.

$$\text{Note: } \Pr(|X - \mu| \geq 10) + \Pr(|X - \mu| < 10) = 1$$

Remarks on Chebyshev's inequality:

Strength:

It can be applied to any distribution and any sample size \rightarrow very weak assumption. Hence it has very broad applications.

Weakness: The bound may be too wide and not practical.

Other alternatives to calculate the tail probability

Case 1: $X_i \sim^{iid}$ a known distribution \rightarrow exact calculation

Case 2: $X_i \sim^{iid}$ unknown distribution

\rightarrow apply the central limit theorem for approximation

Example: Chebyshev's approximation vs. exact calculation

(1) *Chebyshev vs. Uniform distribution*

Given: $E(X) = 5$ and $Var(X) = \frac{25}{3} \rightarrow$ many possibilities

a. Chebyshev's inequality \rightarrow suitable for any distribution

$$\Pr(|X - 5| > 4) \leq \frac{Var(X)}{4^2} = \frac{1}{16} \cdot \frac{25}{3} = \frac{25}{48}$$

b. Given $X \sim Uniform(0,10)$, $E(X) = \frac{0+10}{2}$

$$\begin{aligned} Var(X) &= \int_0^{10} x^2 \frac{1}{10} dx - 5^2 \\ &= \frac{1}{10} \cdot \frac{1}{3} x^3 \Big|_0^{10} - 25 = \frac{100}{3} - \frac{75}{3} = \frac{25}{3} \end{aligned}$$

$$\begin{aligned} \Pr(|X - 5| > 4) &= \Pr(X > 9) + \Pr(X < 1) \\ &= \frac{1}{10} + \frac{1}{10} = \frac{2}{10} \end{aligned}$$

$$\text{Note: } \frac{2}{10} << \frac{25}{48}$$

Conclusion: The bound provided by Chebyshev's inequality is correct but too rough.

(2) *Chebyshev vs. normal distribution*

Given $E(X) = 5$ and $Var(X) = \sigma^2$

a. Chebyshev's inequality

suitable for any distribution with $E(X) = 5$, $Var(X) = \sigma^2$

$$\begin{aligned} \Pr(|X - \mu| > 2\sigma) &\leq \frac{Var(X)}{(2\sigma)^2} \\ &= \frac{\sigma^2}{4\sigma^2} = \frac{1}{4} \end{aligned}$$

b. Normal

$$\begin{aligned}\Pr(|X - \mu| > 2\sigma) &= \Pr\left(\frac{|X - \mu|}{\sigma} > \frac{2\sigma}{\sigma}\right) \\ &= \Pr(|Z| > 2) = \Pr(Z > 2) + \Pr(Z < -2) \\ &= 2 \Pr(Z < -2) = 0.0456\end{aligned}$$

Note: $0.0456 \ll \frac{1}{4} \rightarrow$ The bound is correct but too rough.

Remark: Chebyshev's inequality requires very weak assumption and has very broad applications. However it does not provide precise results.

There is no free lunch!!

Example: The tail probability under normality

Case 1: normal population

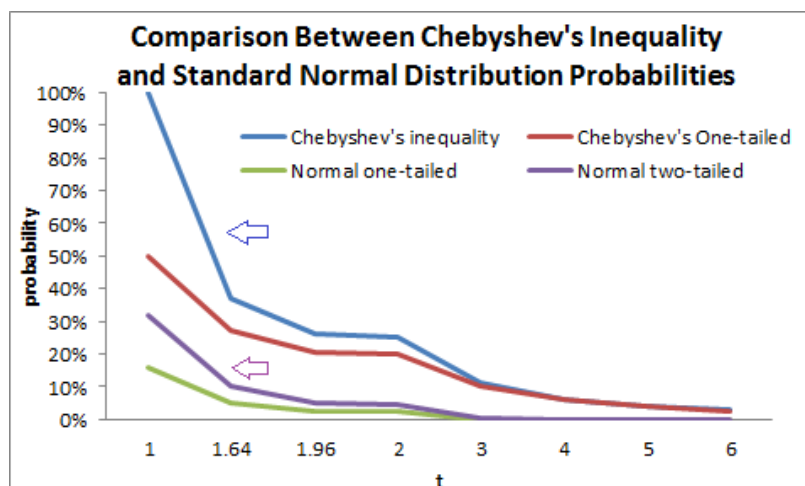
Under normal population: $X_i \sim^{iid} N(\mu, \sigma^2)$:

$$\begin{aligned}\Pr(|\bar{X} - \mu| > k) &= \Pr\left(\frac{|\bar{X} - \mu|}{\sigma/\sqrt{n}} > \frac{k}{\sigma/\sqrt{n}}\right) \quad \text{for any } n \\ &= \Pr(|Z| > \frac{k}{\sigma/\sqrt{n}})\end{aligned}$$

Case 2: non-normal population but large sample

For any population but with large sample, $X_i \sim^{iid} E(X_i) = \mu, \text{Var}(X_i) = \sigma^2$

$$\Pr\left(\frac{|\bar{X} - \mu|}{\sigma/\sqrt{n}} > \frac{k}{\sigma/\sqrt{n}}\right) \approx \Pr(|Z| > \frac{k}{\sigma/\sqrt{n}}) \quad \text{for large } n$$



Implication of Chebyshev's inequality for statistical inference

$$\begin{aligned}\Pr(|\bar{X} - \mu| \geq k) &\leq \frac{\text{Var}(\bar{X})}{k^2} \\ &= \frac{\sigma^2}{n} \frac{1}{k^2}\end{aligned}$$

- \bar{X} is an estimator of μ
- $|\bar{X} - \mu| \geq k \Leftrightarrow$ estimation error is at least $k \rightarrow$ a bad thing
- $\Pr(|\bar{X} - \mu| \geq k) \rightarrow$ the smaller, the better
- The probability of the bad thing is bounded by $\frac{\sigma^2}{n} \frac{1}{k^2}$
- Upper bound = $\frac{\sigma^2}{n} \frac{1}{k^2}$

1. k = the standard (人定標準)

small $k \rightarrow$ standard is strict

large $k \rightarrow$ standard is loose

$$k \uparrow \Rightarrow \frac{\sigma^2}{n} \frac{1}{k^2} \downarrow$$

When you loosen the standard, it becomes less likely to violate it.

2. σ^2 = the variation of the original population (天生)

$$\sigma^2 \downarrow \Rightarrow \frac{\sigma^2}{n} \frac{1}{k^2} \downarrow \text{ (修改箭頭)}$$

If the original population is less variable, it becomes easier to achieve the standard.

3. n = sample size (後天)

$$n \uparrow \Rightarrow \frac{\sigma^2}{n} \frac{1}{k^2} \downarrow$$

If you get more sample observations, it becomes easier to achieve the standard.

達成目標(降低誤差)的三要素

- 天生麗質 small σ
- 後天努力 large n
- 人定標準寬鬆 k

History of Normal distribution

http://onlinestatbook.com/2/normal_distribution/history_normal.html

Discovery from astronomical observation:

One of the first applications of the normal distribution was to the analysis of errors of measurement made in astronomical observations, errors that occurred because of imperfect instruments and imperfect observers. Galileo (1561-1642, Italian) noted that these errors were symmetric and that small errors occurred more frequently than large errors.

Discovery from coin flip: de Moivre (棣美弗, 1667–1754) noted that when the number of events (coin flips) increased, the shape of the binomial distribution approached a very smooth curve.

Formulation: Independently, the mathematicians Adrain in 1808 (USA) and Gauss in 1809 (Germany) developed the

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

formula for the normal distribution and showed that errors were fit well by this distribution. This same distribution had been discovered by Laplace in 1778 (France) when he derived the extremely important *central limit theorem*.

Laplace



Gauss



Adrain

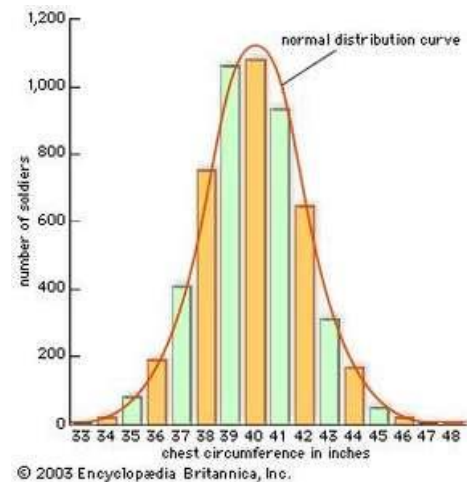


History of the Central Limit Theorem

The de Moivre–Laplace theorem:

The earliest version (1810) states that the normal distribution may be used as an approximation to the binomial distribution.

Towards the end of the 19th century, the mathematical discussion was turning increasingly from computational mathematics to a more fundamental analysis, to "pure" mathematics. This had a big impact on probability theory as it had been considered more as "common sense" than a rigorous mathematical theory.



<http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.530.4669&rep=rep1&type=pdf>

Several Versions of the Central Limit Theorem

- *Lindeberg–Lévy CLT*:

$\{X_1, \dots, X_n\}$ is a sequence of *iid* sample with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$

- *Lyapunov CLT*: Suppose $\{X_1, \dots, X_n\}$ is a sequence of *independent* sample with

$E(X_i) = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2 < \infty$. Lyapunov derived a condition to show that

$$\left(\sum_{i=1}^n \sigma_i^2 \right)^{-1/2} \sum_{i=1}^n (X_i - \mu_i)$$

- *Lindeberg CLT*: Lindeberg (1920) derived a weaker condition (stronger result) for the theorem. Lindeberg's work was unknown to Alan Turing, who proved CLT in his dissertation in 1935.

Subsequent work: The classical CLT assumes identically distributed random variables with finite variance. The Lindeberg–Lévy–Feller central limit theorem showed that we can weaken the condition of identically distributed random variables so long as they satisfy the Lindeberg condition. It is natural to ask what happens if instead of weakening the identically distributed hypothesis, we weaken dependence.

Exercises: Normal approximation to Binomial distribution

Ex 1: Suppose a fair coin is tossed 100 times. What is the probability there will be more than 60 heads?

$$X_i \sim \text{Bernoulli}(n = 100, p = 0.5)$$

Solution:

$$\mu = np = 50, \quad \sigma = \sqrt{np(1-p)} = 5$$

$$\text{We have } \sigma = \sqrt{np(1-p)} = 5$$

$$\begin{aligned} \Pr(X_1 + \dots + X_{100} > 60) &= \Pr\left(\frac{(X_1 + \dots + X_{100}) - np}{\sqrt{np(1-p)}} > \frac{60 - 50}{5}\right) \\ &\approx \Pr(Z > 2) = 0.0228 \end{aligned}$$

Ex 2: Suppose a die is rolled 180 times.

What is the probability a 3 will be showing more than 50 times?

Solution:

.

Ex 3: Suppose a drug is supposed to be 75% effective. It is tested on 100 people.

What is the probability more than 70 people will be helped?

Solution: