Representing Images In Frequency Domain

Spatial Frequency Concepts

- General, we use frequency to represent how fast a periodic signal varies over time, expressed in cycles-per-time-step (e.g., Hz for cycles/second).
- Spatial frequency represents how fast the signal (here the pixel values) vary spatially. Consider some 1-D examples:

$f = 1/4$

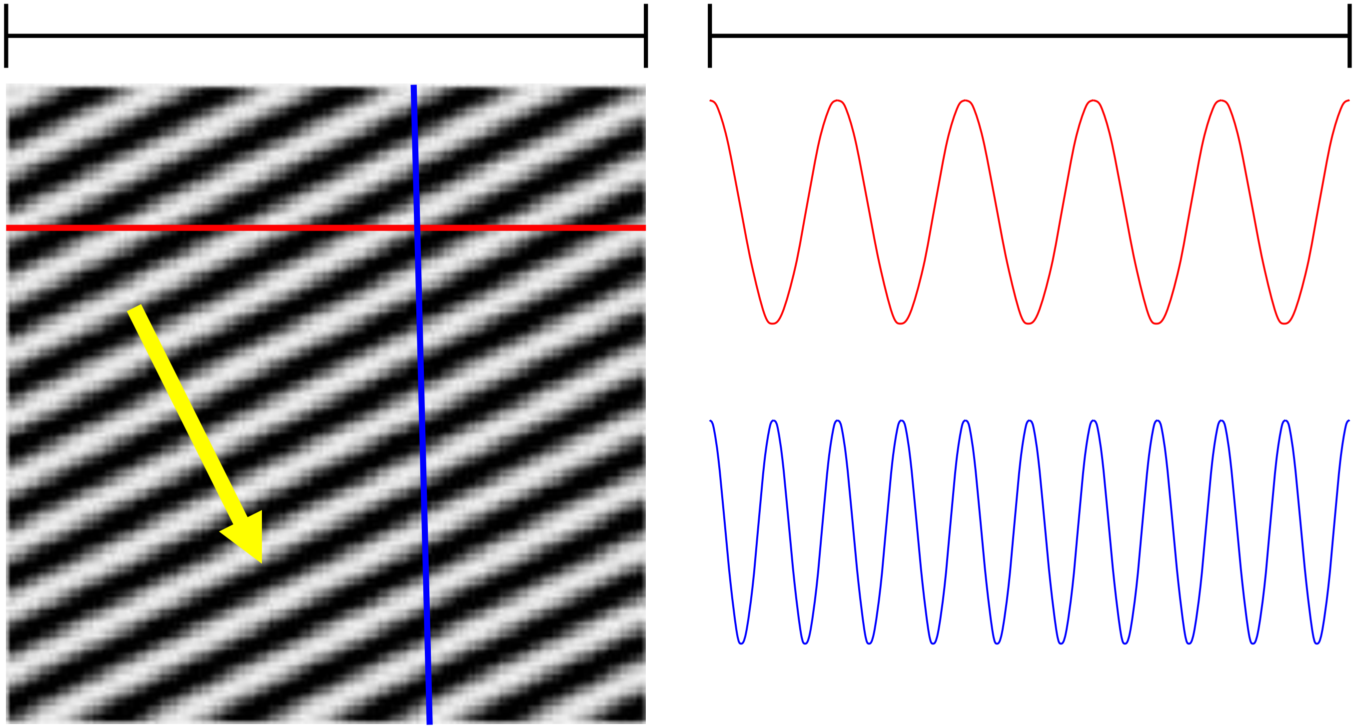
$f = 1/6$

Unit:
cycles per pixel

Spatial Frequency Concepts

Spatial frequency for 2-D signals:

- One frequency for each dimension.
- The overall spatial frequency is a vector (wave vector).



$f_x \approx 1/20$

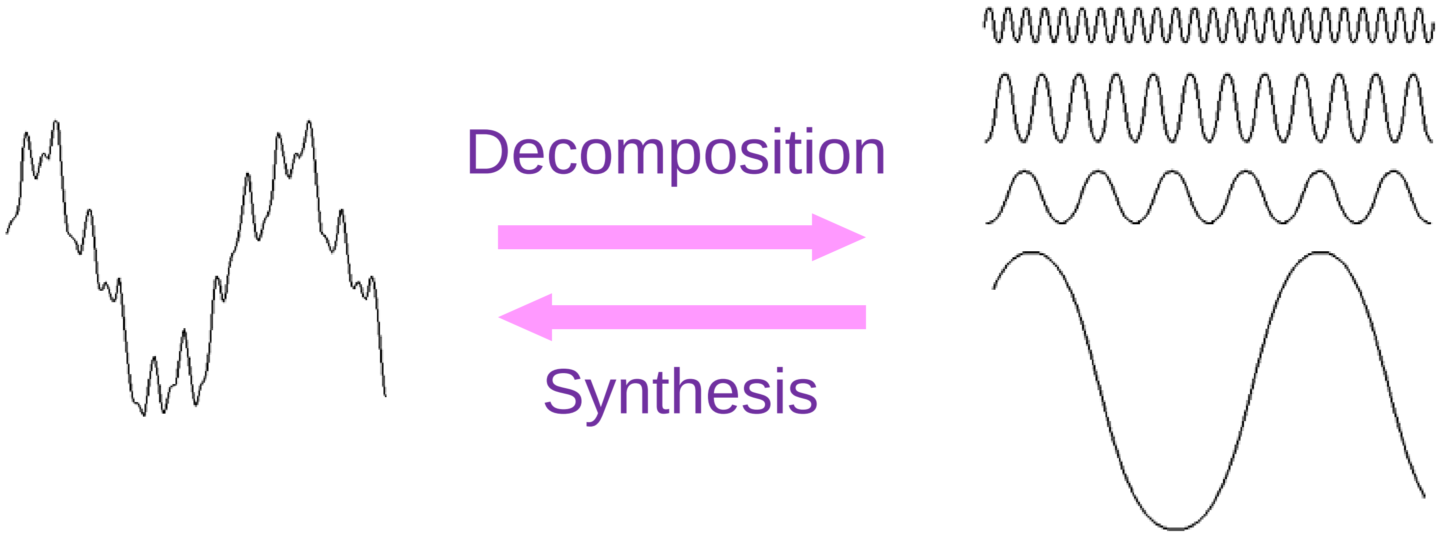
$f_y \approx 1/10$

Assuming image size of 100x100

Overall: $\sqrt{f_x^2 + f_y^2}$

Spectrum

- In general, the "waveform" of a signal consists of a **mixture** of many sinusoidal waves:



- The information regarding the **frequencies**, **amplitudes**, and **phases** of these sinusoidal waves is collectively called the **spectrum** of the signal.

Fourier Series

Fourier Series: Any periodic function of period T can be expressed as the combination of a series of sine and/or cosine functions multiplied by appropriate coefficients:

Synthesis

$$f(t) = \sum_{n=-\infty}^{\infty} \left[c_n \exp(j \frac{2\pi n}{T} t) \right] \quad \text{while} \quad f(t) = f(t+T)$$

Decomposition

To get the (complex) coefficients: $c_n = \dfrac{1}{T} \displaystyle\int_{-T/2}^{T/2} f(t) \exp(-j \frac{2\pi n}{T} t) dt$

This is the projection of the function $f(t)$ on a set of orthogonal basis functions.

Continuous Fourier Transform

Fourier transform: Any function (periodic or not) with finite total area under the curve can be expressed as the integral of sine and/or cosine functions multiplied by different coefficients:

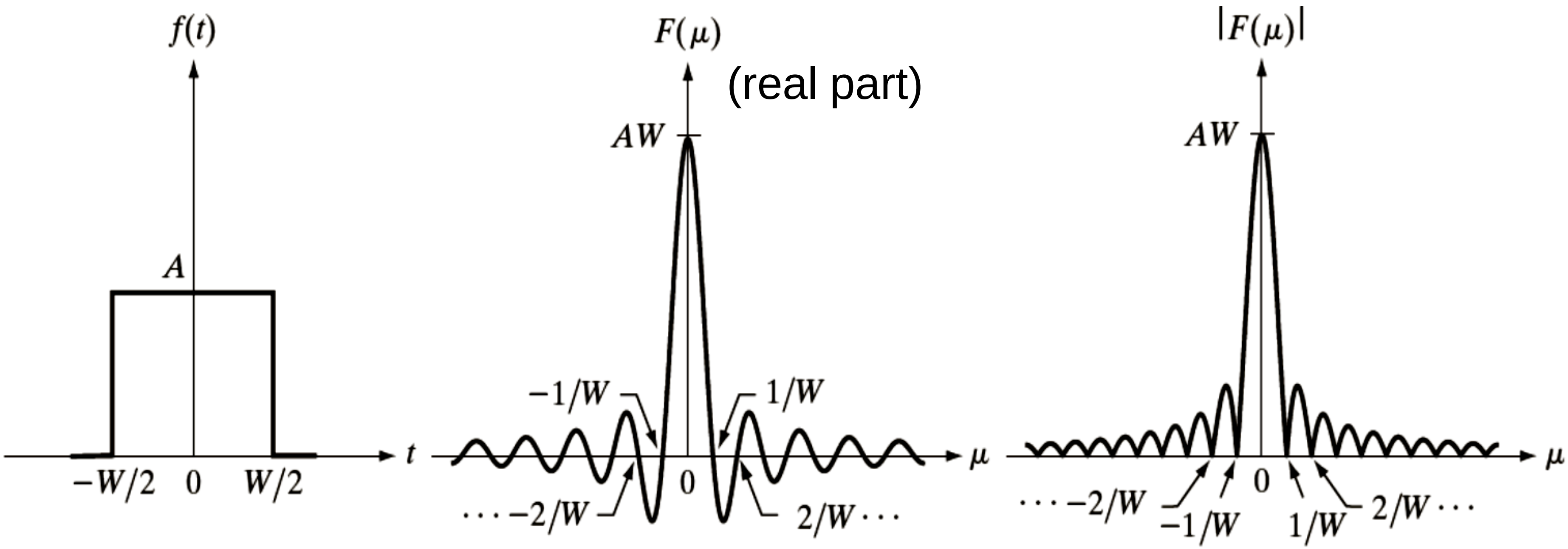
The Fourier Transform pair:

FT: $$F(\mu) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi\mu t) dt$$ $\boxed{\text{Decomposition}}$

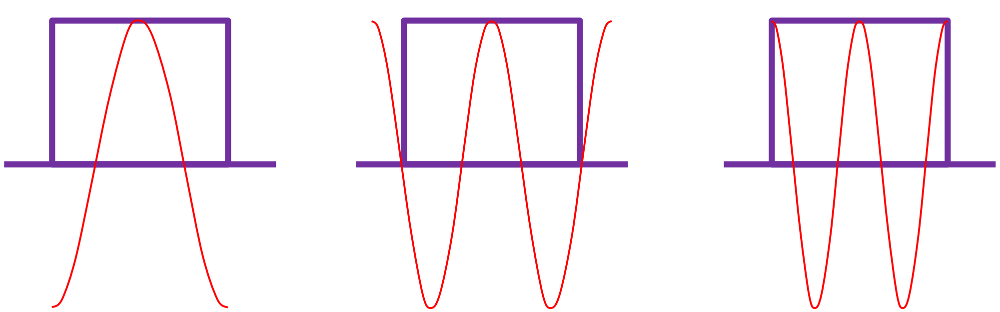
IFT: $$f(t) = \int_{-\infty}^{\infty} F(\mu) \exp(j2\pi\mu t) d\mu$$ $\boxed{\text{Synthesis}}$

Here μ represents frequency (in cycles per unit of space or time).

Fourier Spectrum



To understand this so-called **sinc** function:



Fourier Spectrum

- We call the domain of μ the frequency domain.
- $F(\mu)$ is complex: $F(\mu) = R(\mu) + jI(\mu) = |F(\mu)| \exp(j\phi(\mu))$
- **Phase angle**: $\phi(\mu) = \tan^{-1}[I(\mu)/R(\mu)]$
- The **power spectrum** of $f(t)$ is $P(\mu) = |F(\mu)|^2$
- The **spectral density** at a frequency μ is just $|F(\mu)|^2$
- $F(0)$ is the DC term. The other terms are AC terms.
- For real-valued $f(t)$: $F(\mu) = F^*(-\mu)$ and $\left|F(\mu)\right| = \left|F(-\mu)\right|$

Frequency-Domain Image Processing

- The general process:
  - Convert the original image to its frequency domain representation (the spectrum) – Fourier transform.
  - Modify the spectrum.
  - Convert the spectrum back to an image (spatial domain) – Inverse Fourier transform.
- Two main issues for applying frequency-domain processing to images:
  - Spatially discrete (sampled) signals.
  - Finite spatial ranges.

Delta (Impulse) Function

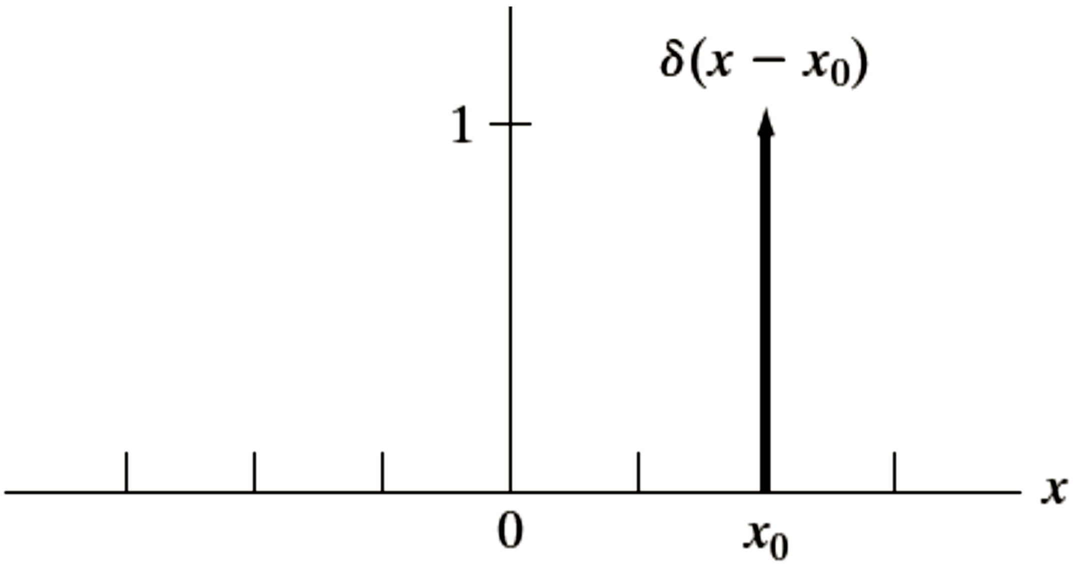
We introduce the concept of **delta functions** here in order to model the sampling operation.

$$\delta(t) = \infty \quad \text{for } t = 0, \quad 0 \quad \text{otherwise}$$

$$\int \delta(t)dt = 1$$

$$\int f(t)\delta(t - t_0)dt = f(t_0)$$

$$\int \exp(j2\pi\mu t)dt = \delta(\mu)$$



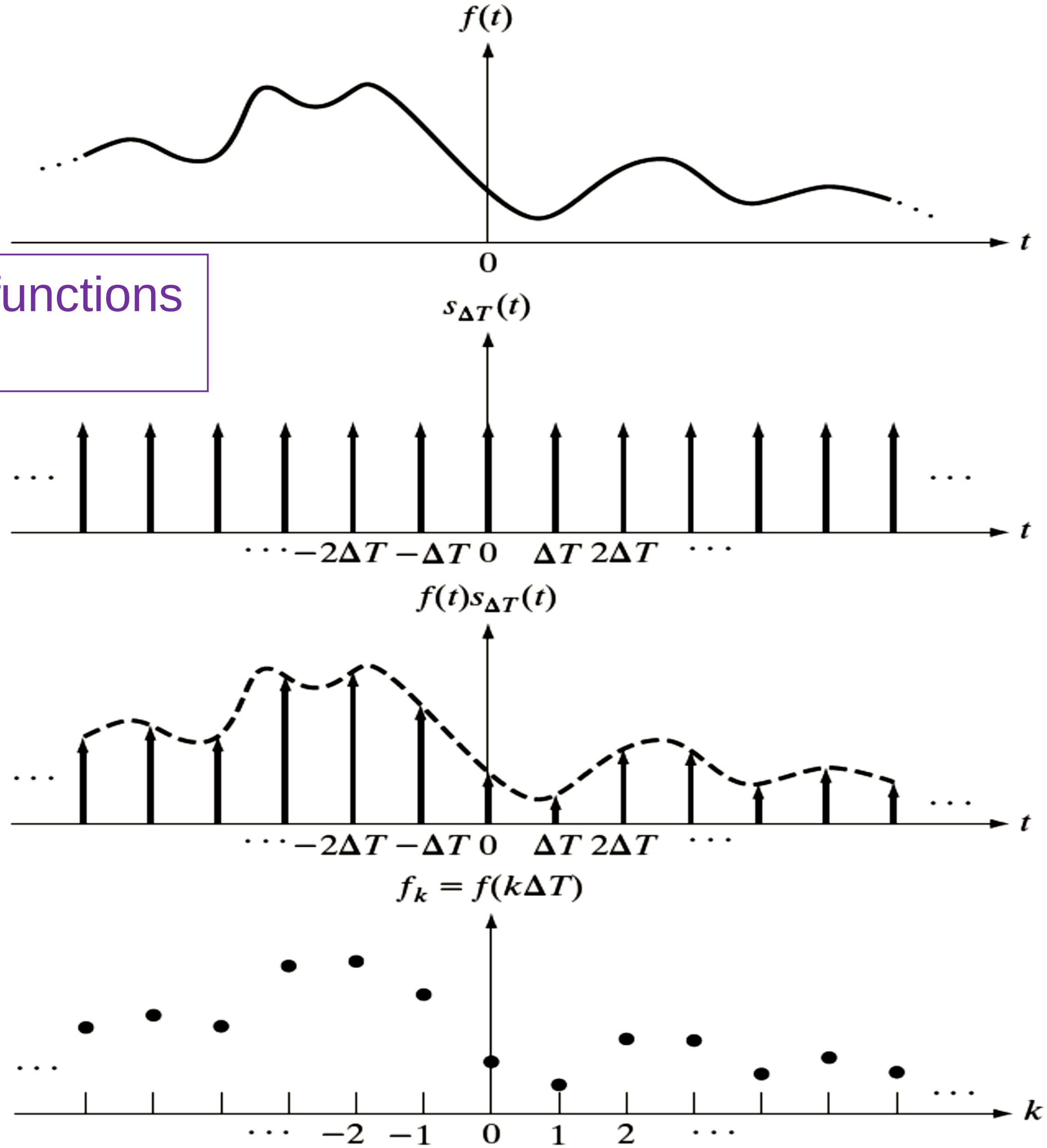
Sampled Signal

Continuous Signal

A periodic series of delta functions (an impulse train)

Multiplication

Sampled Signal (Infinite-height impulses with areas given by the sampled values.)



FT of Impulse Train

The impulse train for sampling is a periodic function with period of ΔT.

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

Therefore, we can express it using Fourier series:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \left[c_n \exp(j \frac{2\pi n}{\Delta T} t) \right] \quad \Rightarrow \quad c_n = \frac{1}{\Delta T}$$

$$\Rightarrow s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \exp(j \frac{2\pi n}{\Delta T} t)$$

By applying FT to this expression of impulse train, we get

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

This is an impulse train, with period of $(\Delta T)^{-1}$, in the frequency domain.

Convolution

Continuous Space, 1-D

Convolution: $$f(t)*h(t)=\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Convolution kernel

Convolution Theorem:

$$\phi(t)=f(t)*h(t) \quad \Leftrightarrow \quad \Phi(\mu)=F(\mu)H(\mu)$$

$$\phi(t)=f(t)h(t) \quad \Leftrightarrow \quad \Phi(\mu)=F(\mu)*H(\mu)$$

Convolution theorem allows us to understand the effect of temporal/spatial domain processing in the frequency domain, and vice versa.

We introduce convolution theorem here in order to derive the FT of sampled signals.

FT of Sampled Signal

Sampled Signal

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$

Convolution theorem

FT of Sampled Signal

$$\tilde{F}(\mu) = F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)\,d\tau$$

$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta(\mu-\tau-\frac{n}{\Delta T})\,d\tau$$

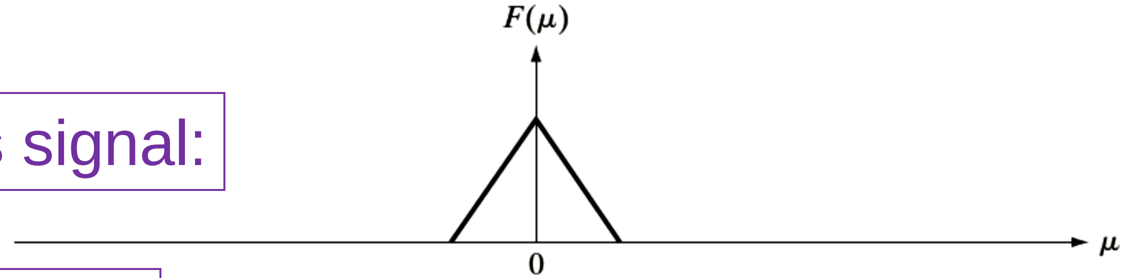
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta(\mu-\tau-\frac{n}{\Delta T})\,d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(\mu-\frac{n}{\Delta T})$$

We get multiple copies of $F(\mu)$, FT of the continuous signal, evenly space by $(\Delta T)^{-1}$.

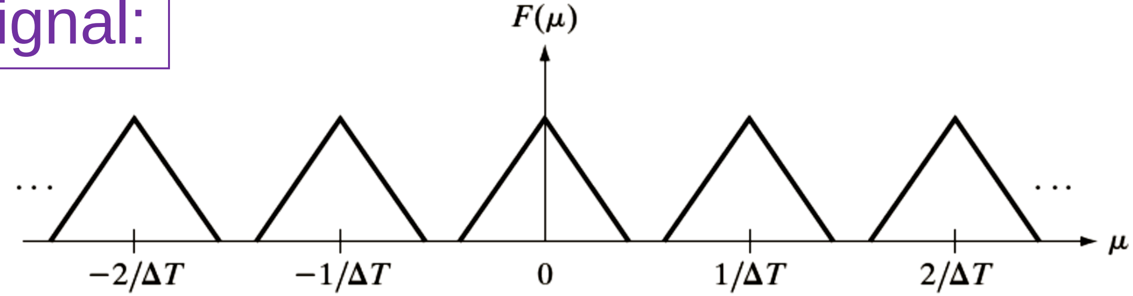
FT of Sampled Signal: Examples

Spectrum of continuous signal:

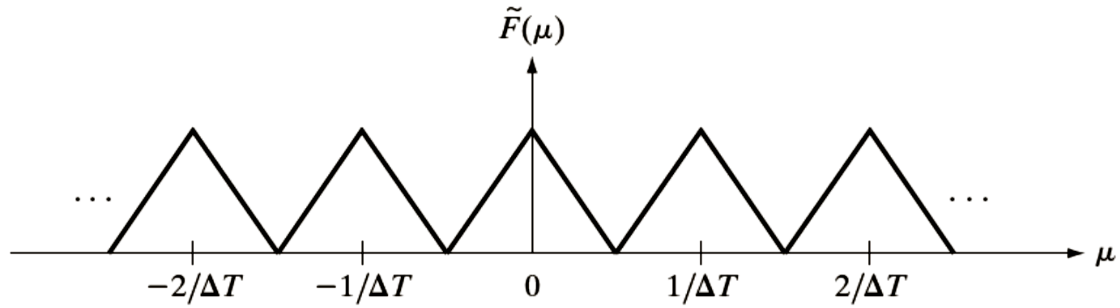


Spectrum of sampled signal:

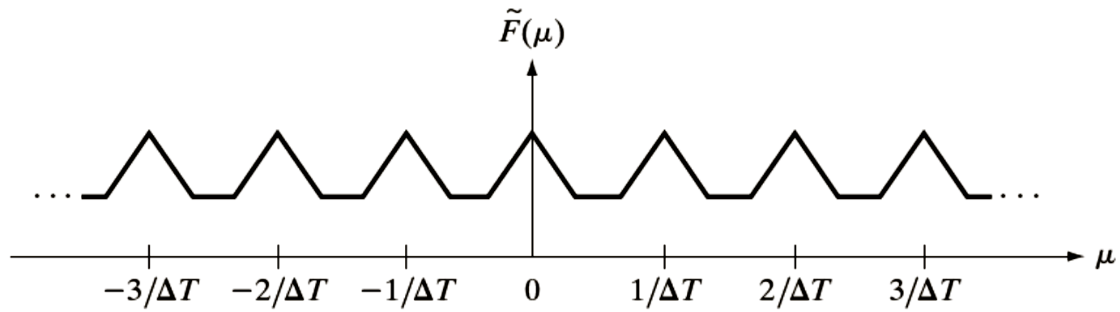
Over-sampling:



Critically-sampling:

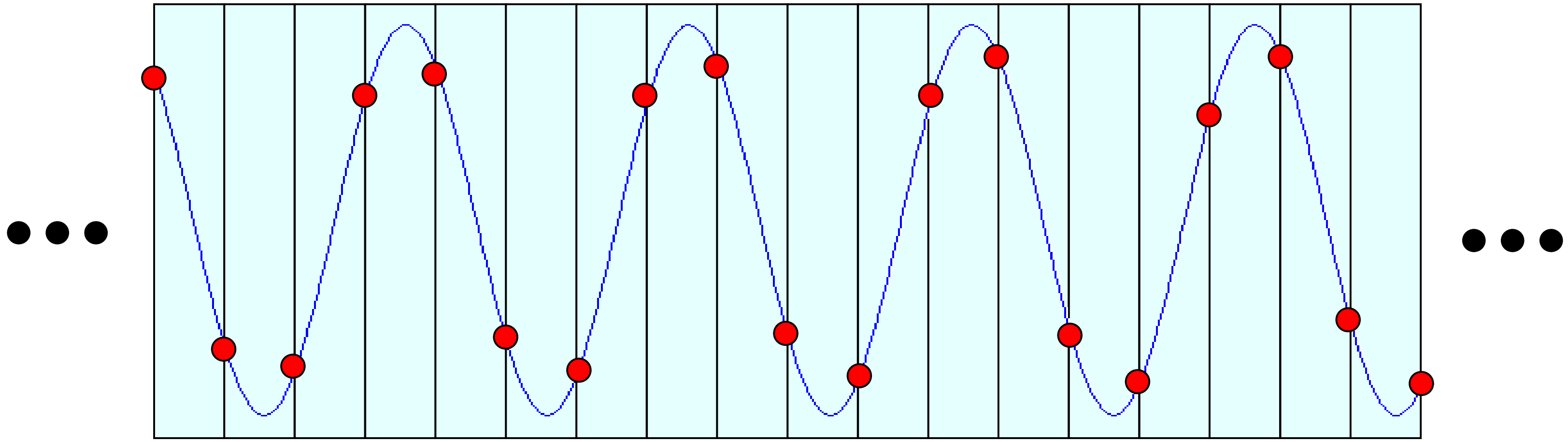


Under-sampling:

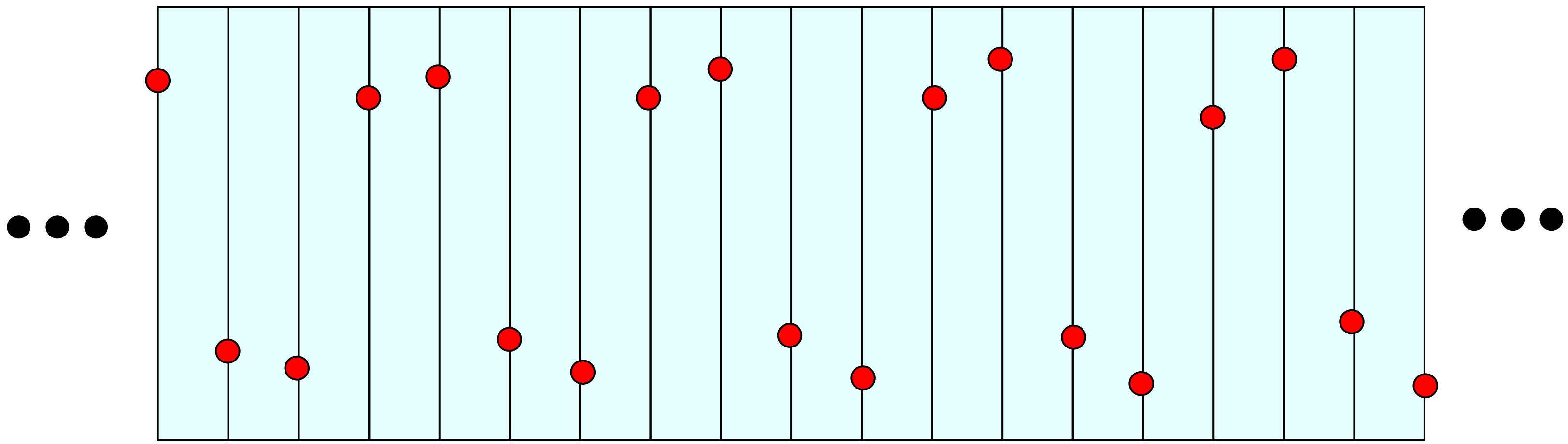


Illustrations of Sampling

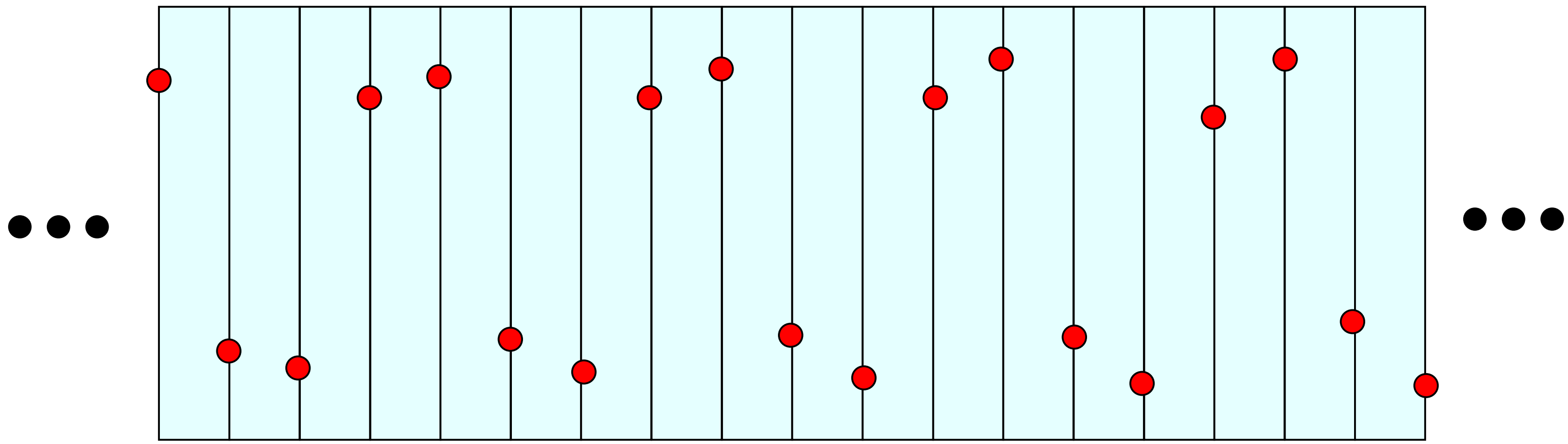
Sampling a sinusoidal function:



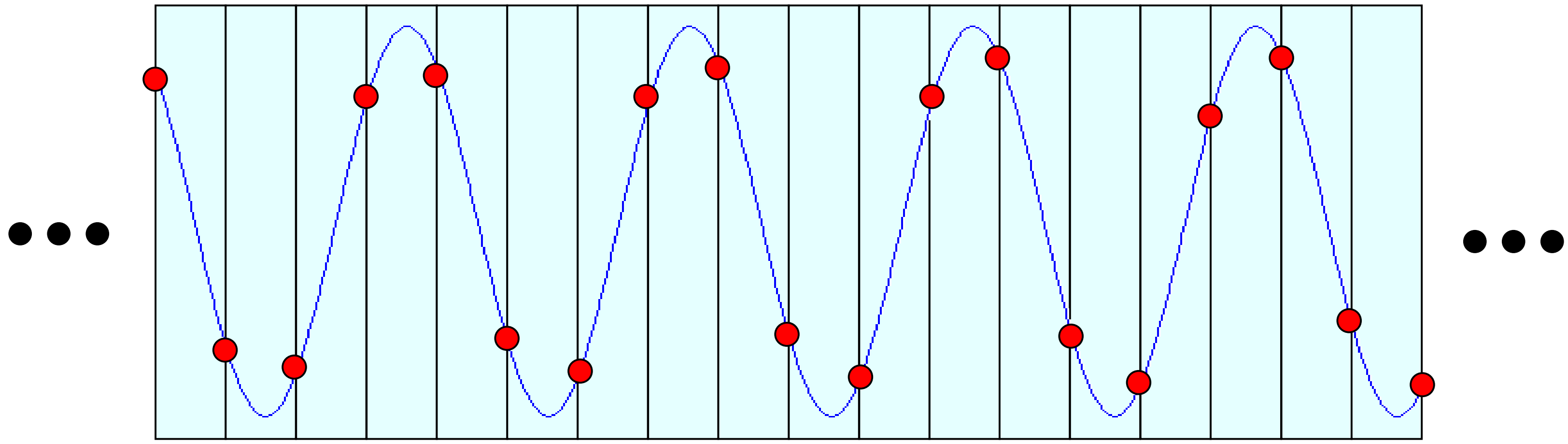
Assume that sampling is done on a 1-D regular grid. We get the sampled values:



Illustrations of Sampling

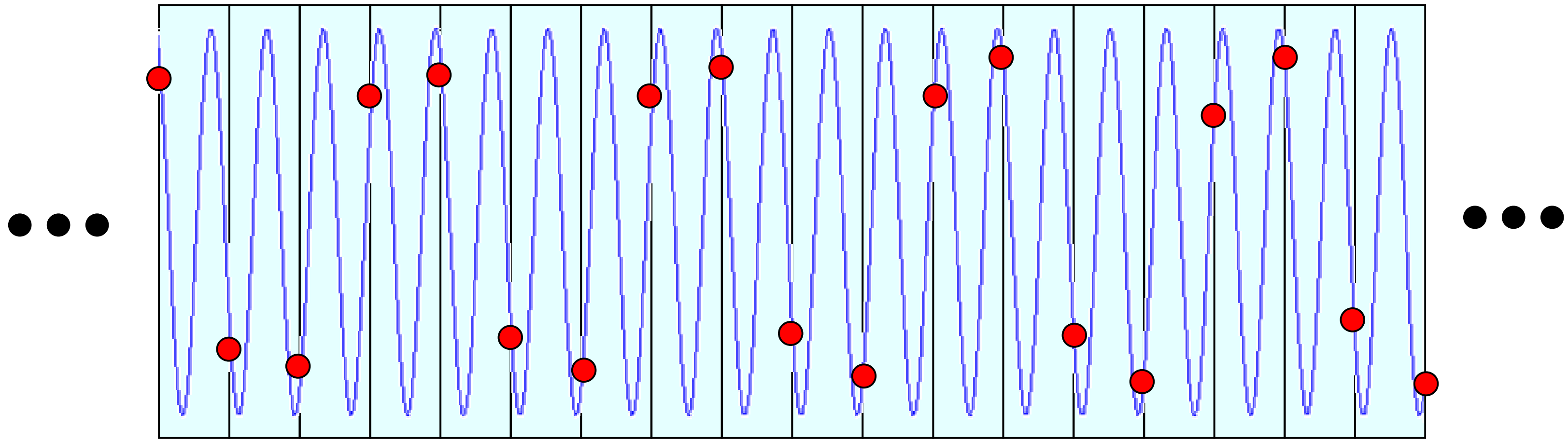


Now if we try to "fit" a sinusoidal function to these sampled points, we can expect to get this, which is the original signal:

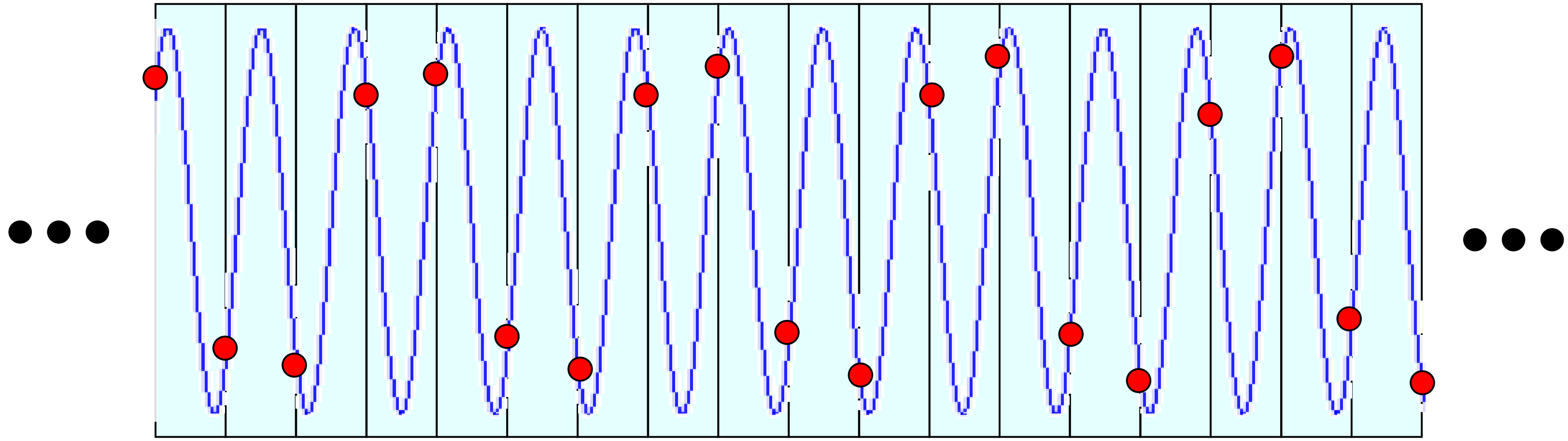


Illustrations of Sampling

However, we can also fit the sampled data to another sinusoidal function of much higher frequency:

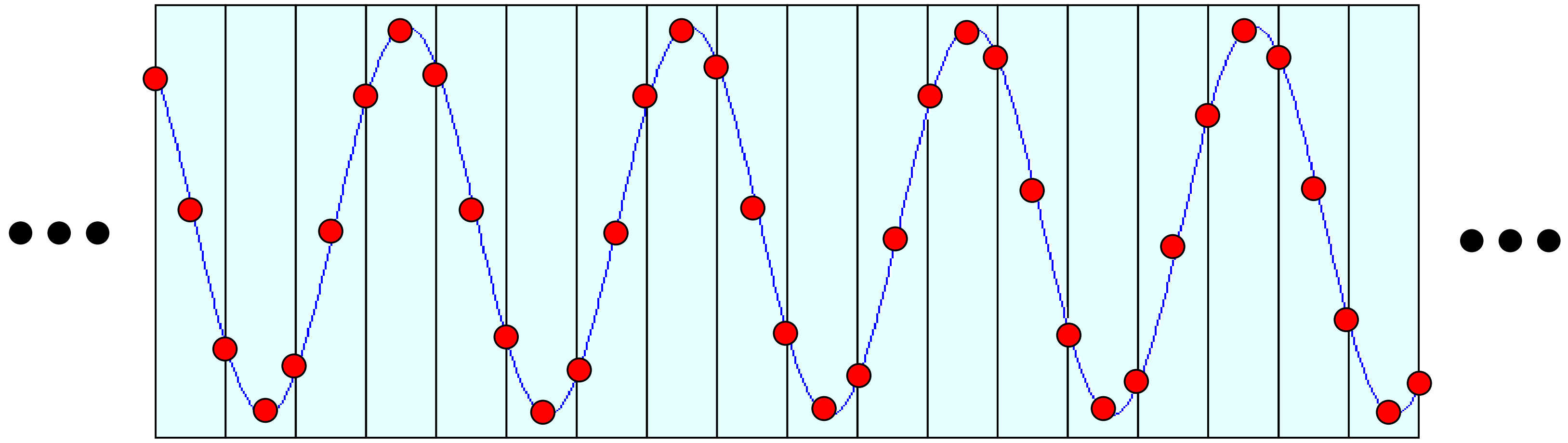


This is yet another solution:

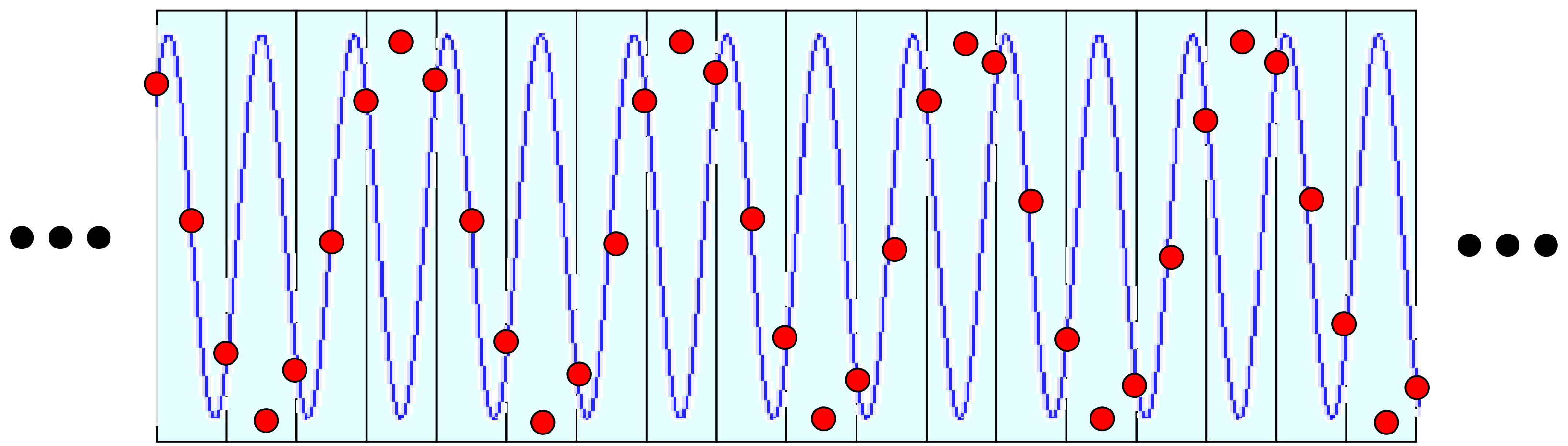


Illustrations of Sampling

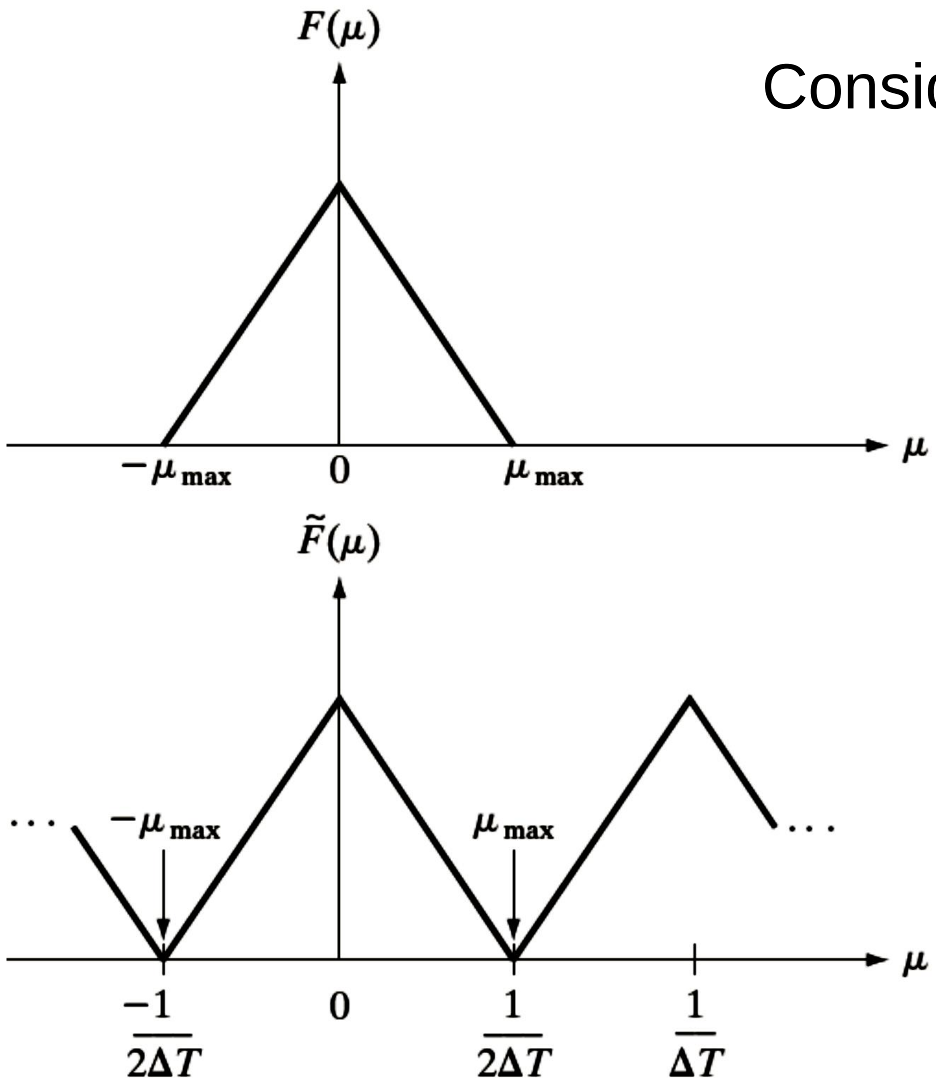
Now, if we double the sampling rate:



We won't get this solution …



Sampling Theorem



Consider a **band-limited** function $f(x)$:

To avoid overlapping between "copies" of the original $F(\mu)$, it is necessary that

$$\frac{1}{\Delta T} \geq 2\mu_{\max}$$

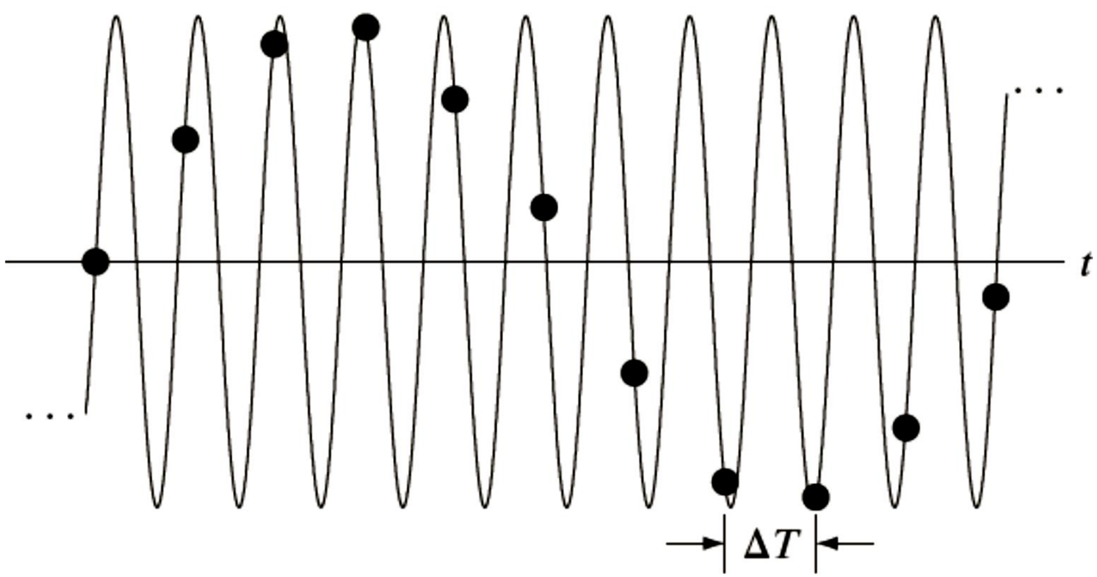
In other words, we need at least two samples per period in the original signal.

Aliasing

Simply said, **aliasing** is what happens when the condition

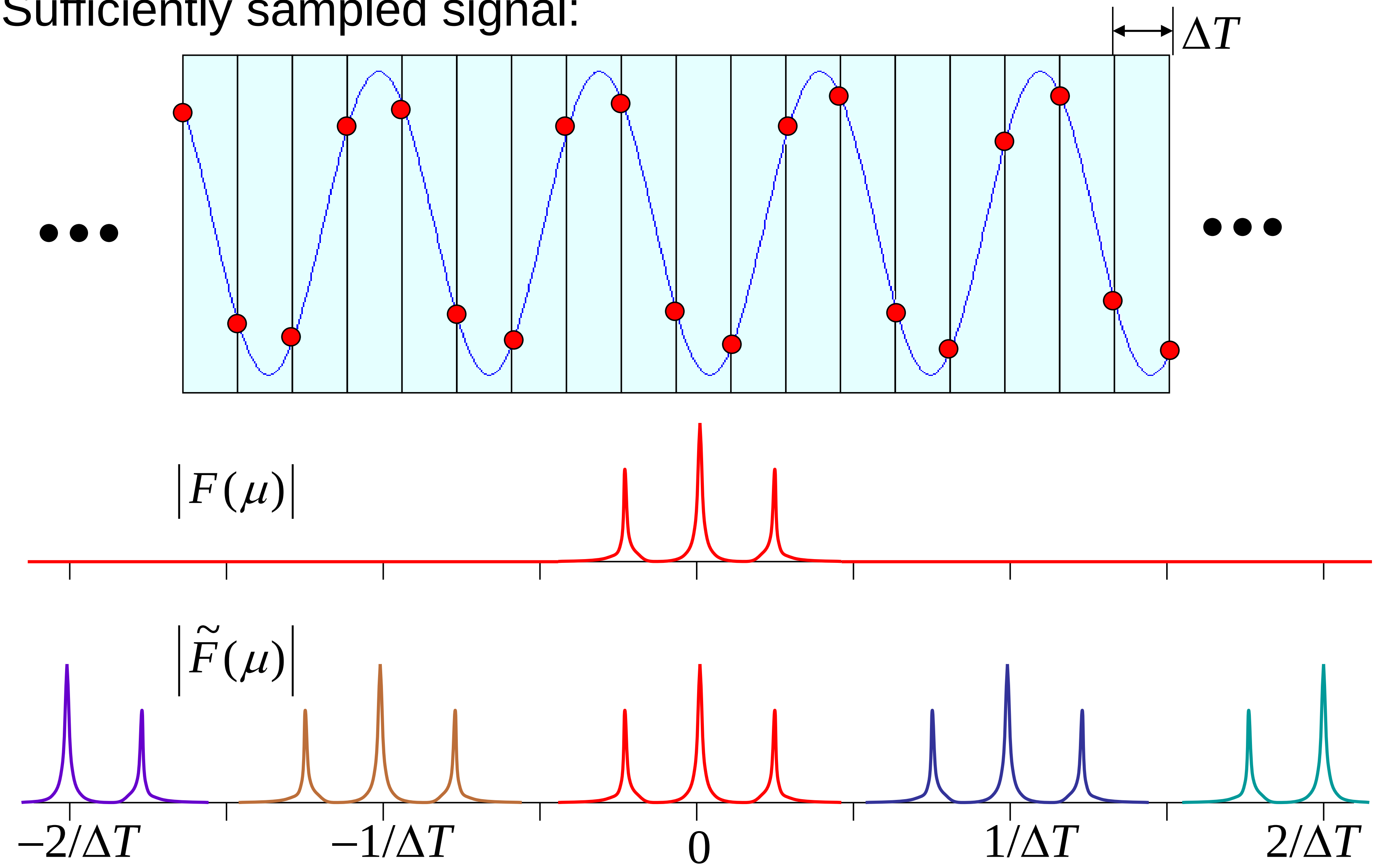
$$\frac{1}{\Delta T} \geq 2\mu_{\max}$$

is not satisfied. We can not faithfully reconstruct the original signal from the sampled data in this case.



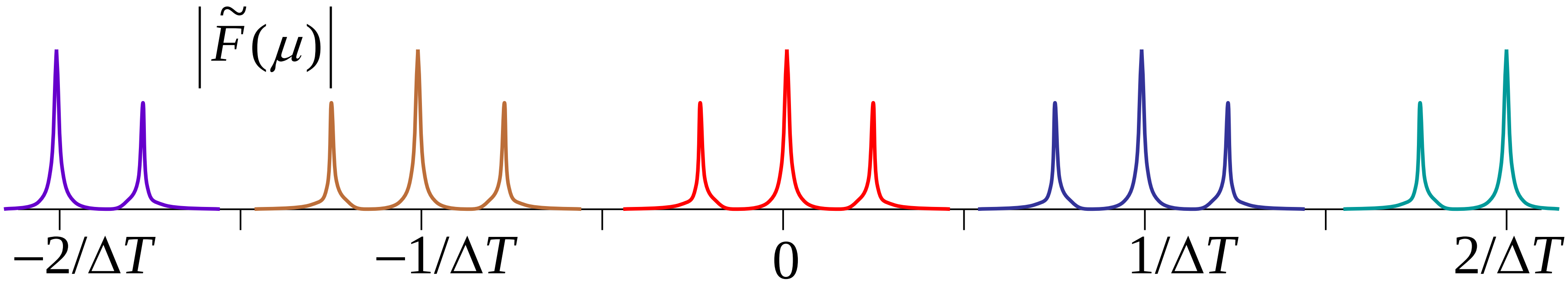
Aliasing – More Illustrations

Sufficiently sampled signal:



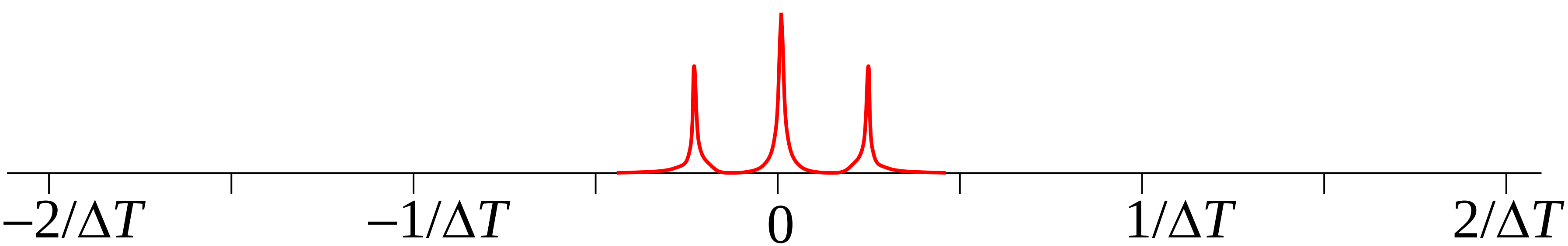
Aliasing – More Illustrations

Reconstruction of the sufficiently sampled signal:



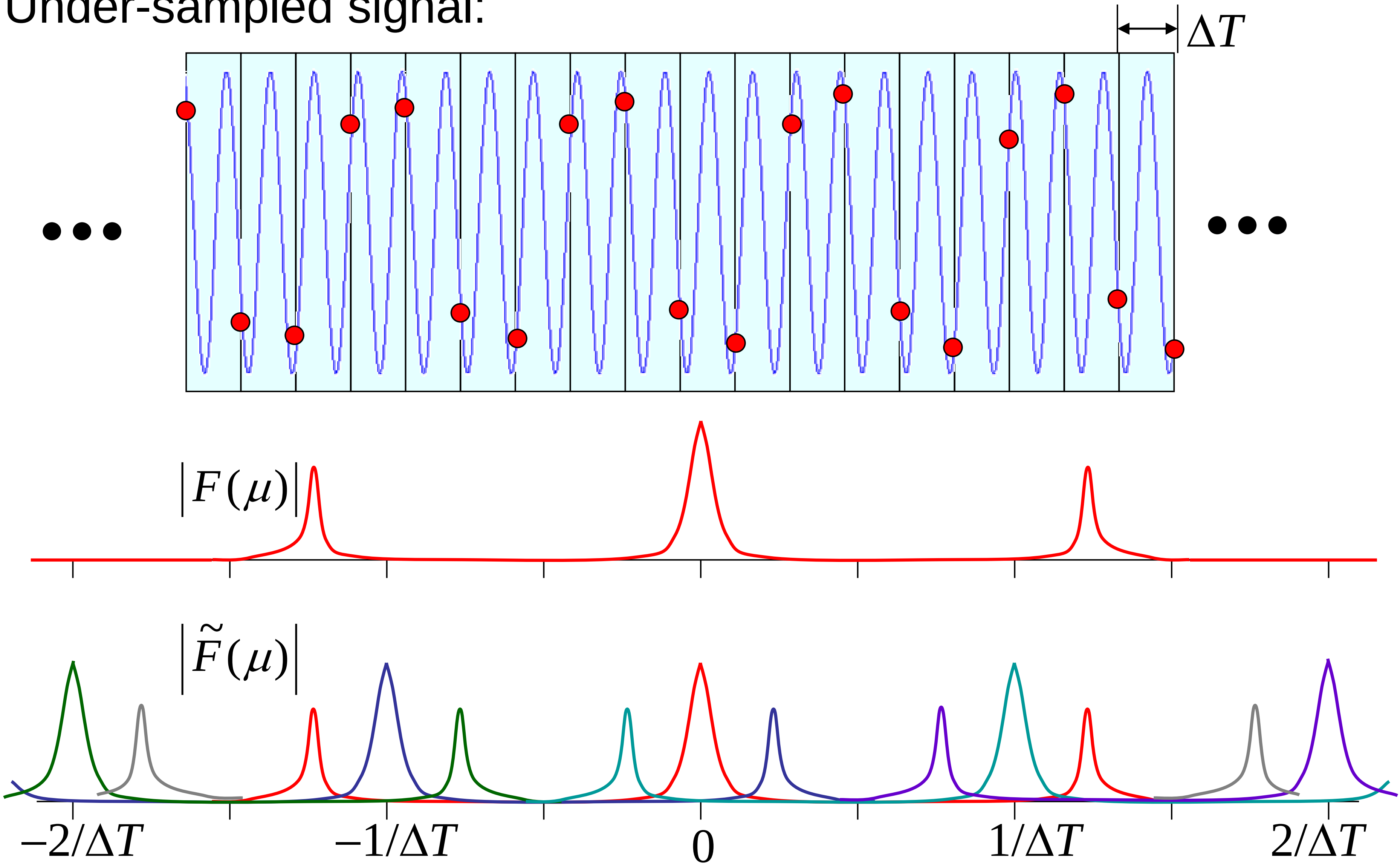
Limit the spectrum to include only one copy of $F(\mu)$ by multiplying with this:

This is what is used for reconstruction (correct signal frequency):



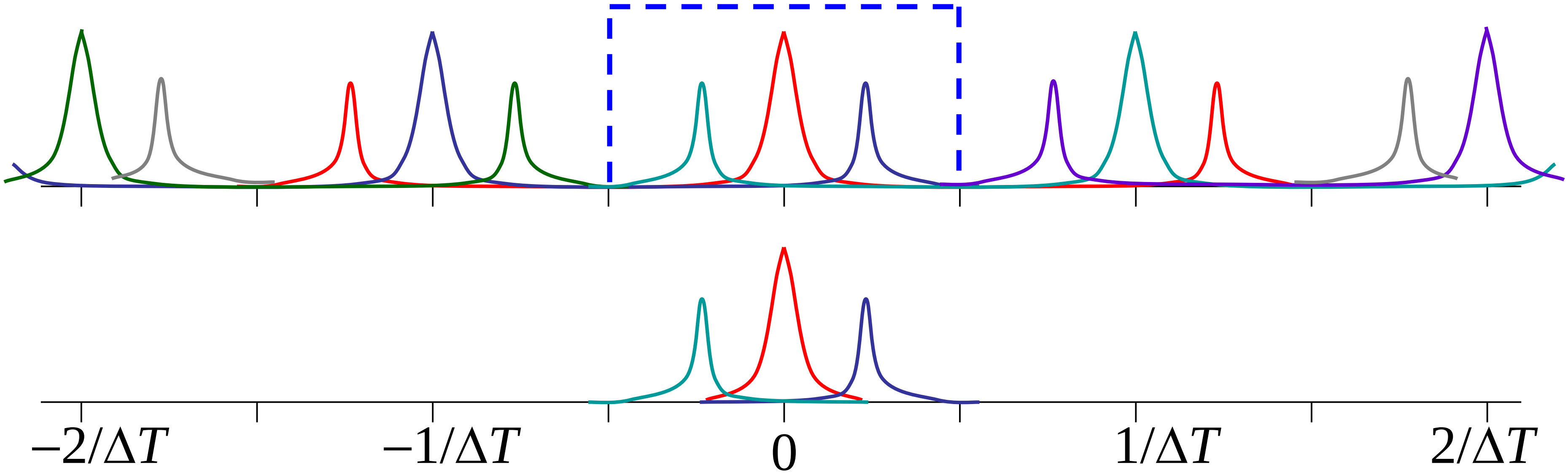
Aliasing – More Illustrations

Under-sampled signal:

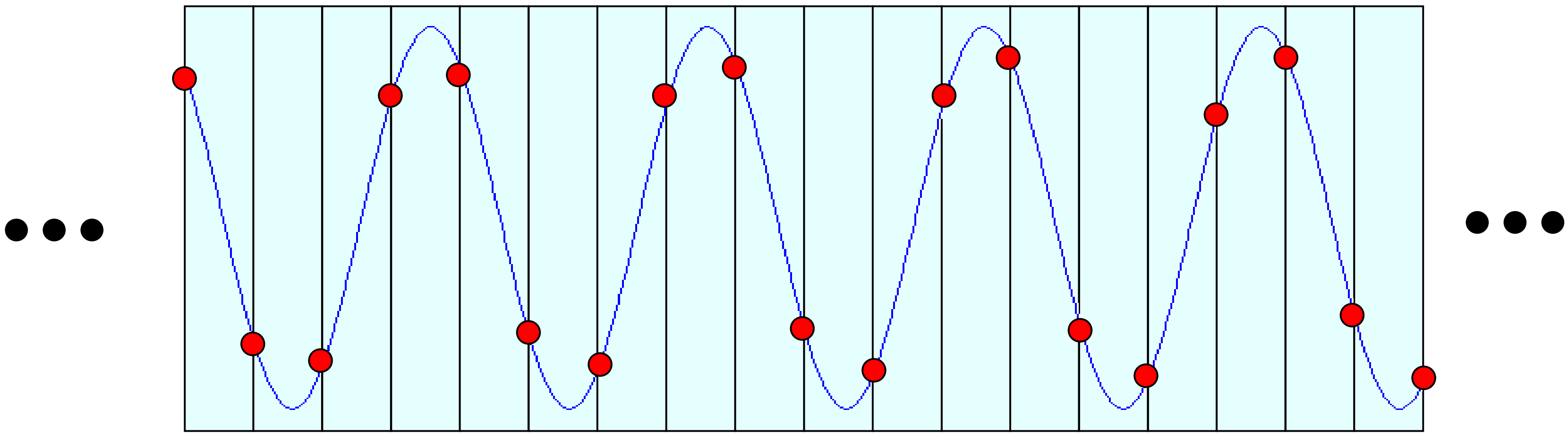


Aliasing – More Illustrations

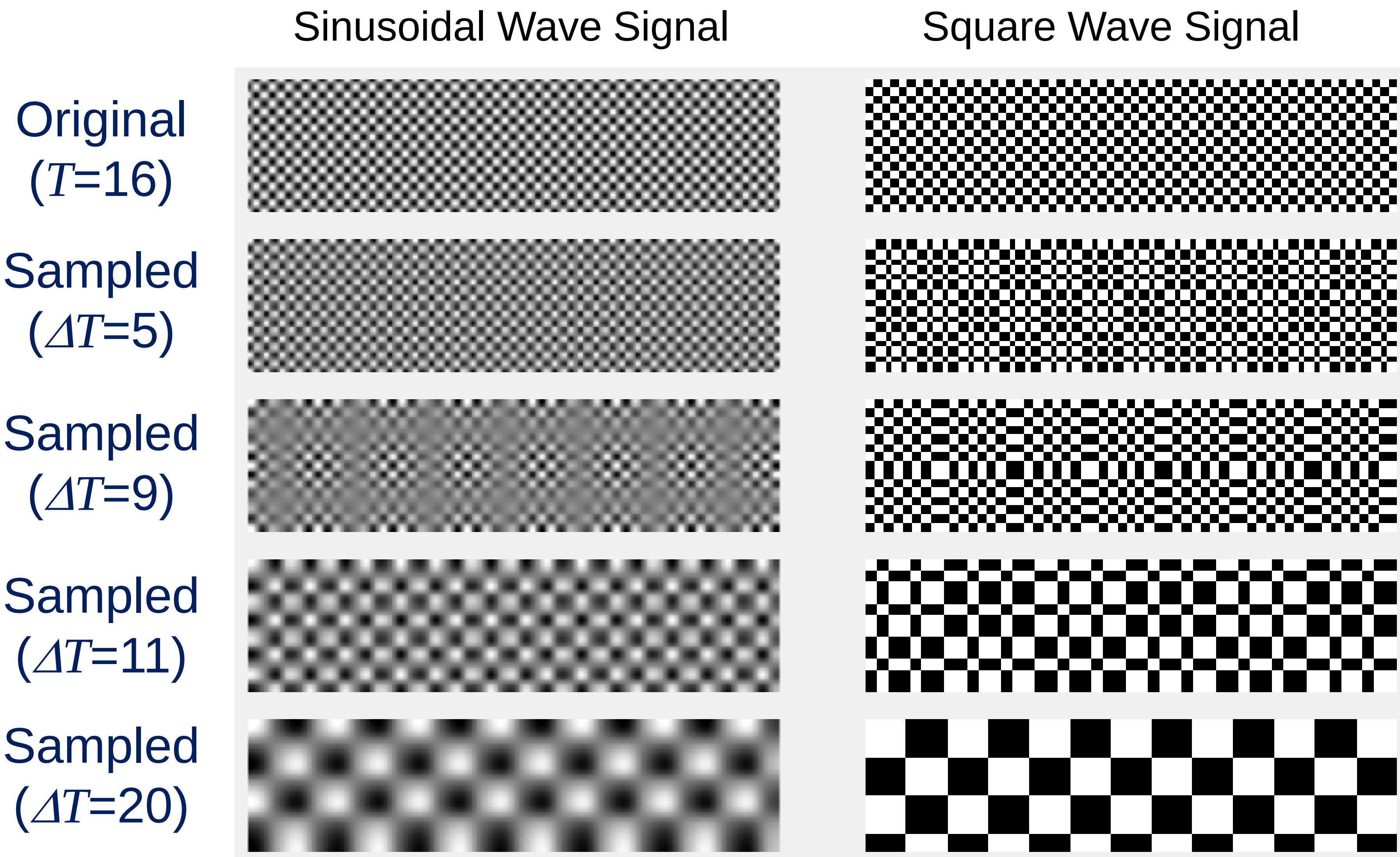
Reconstruction of the under-sampled signal:



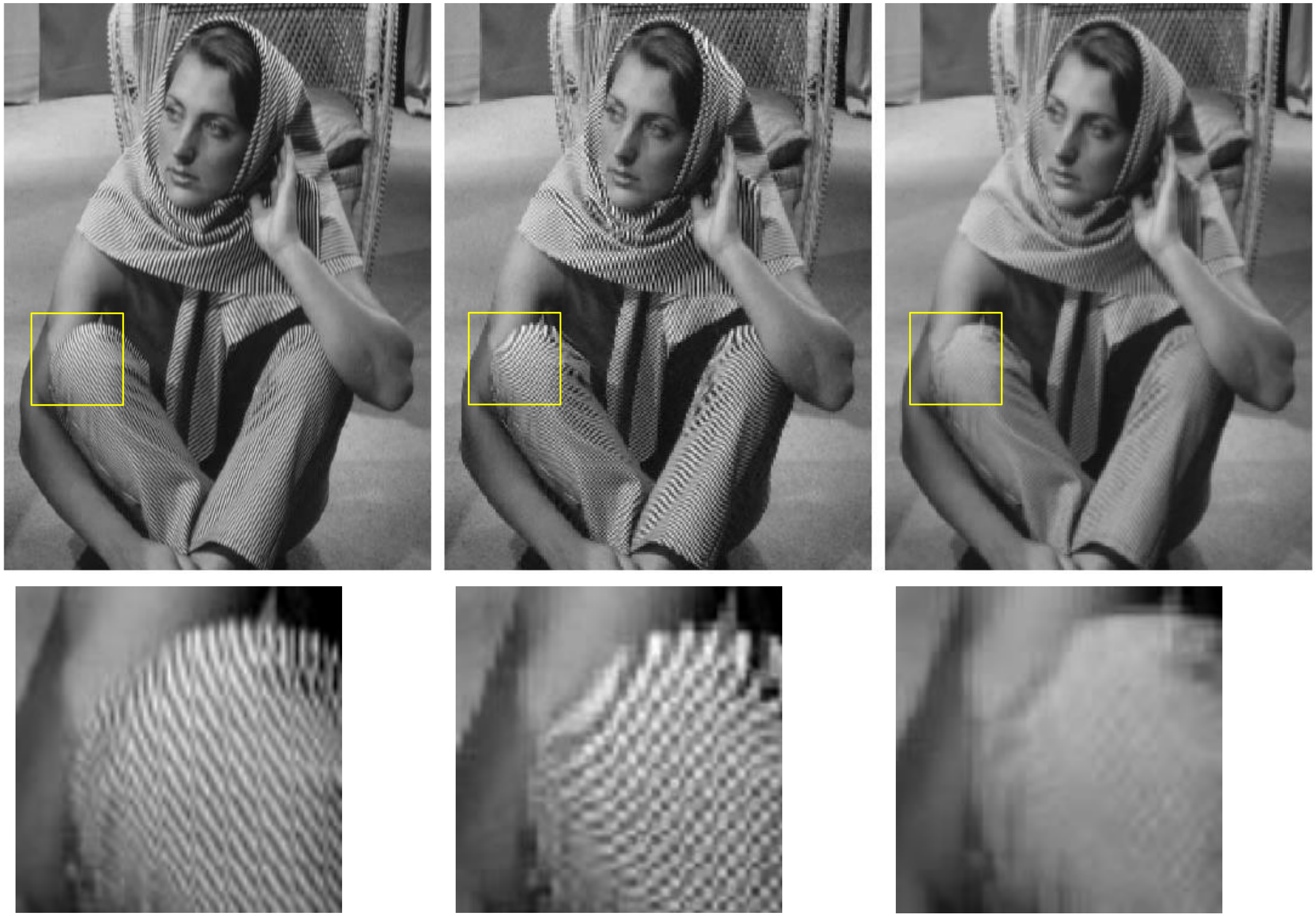
The reconstructed signal:



Aliasing: Examples



Anti-aliasing by Smoothing



Original

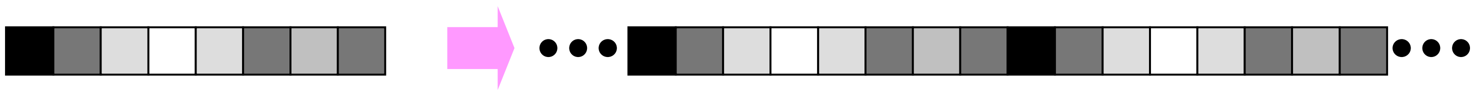
50% size reduction

50% size reduction after smoothing

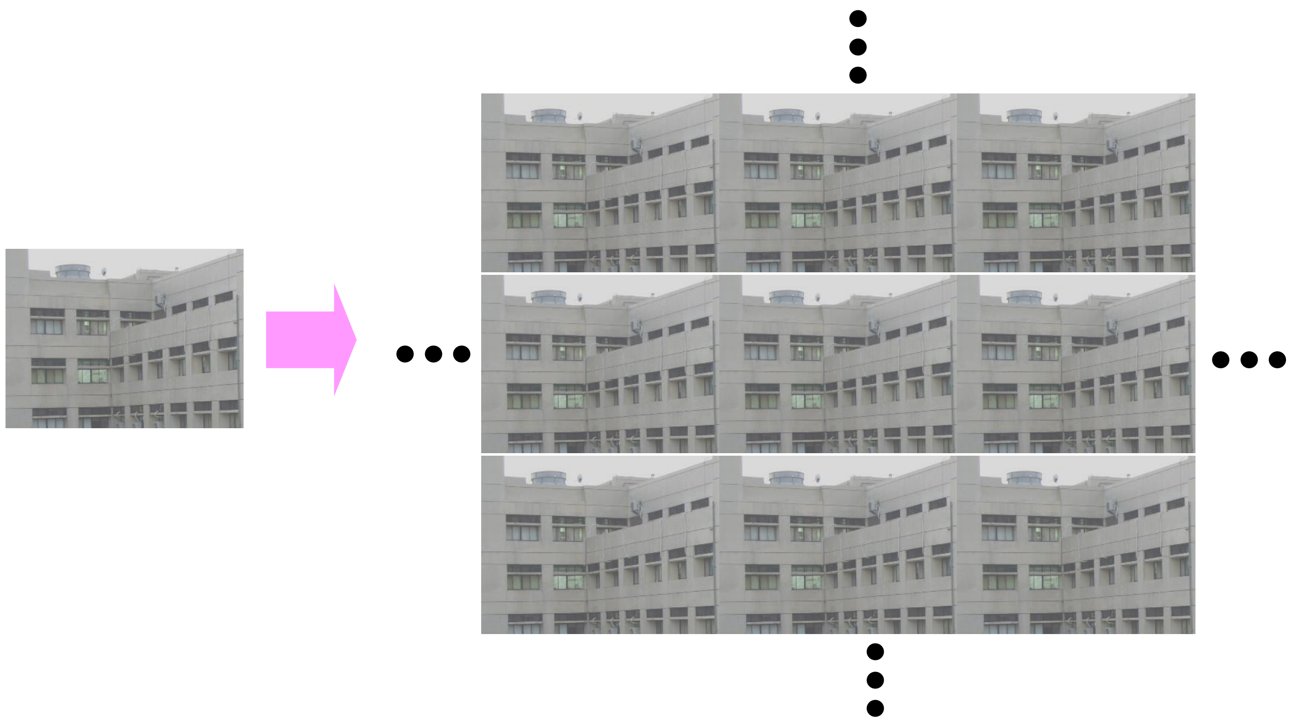
Discrete Fourier Transform (DFT)

Since now we have finite spatial ranges, we can think of discrete Fourier transform as the Fourier series of a periodic function consisting of repeating copies of the original finite-range sampled signal (i.e., the image).

1-D:



2-D:



Summary of 1-D DFT and IDFT

We consider the FT of $f(x)$ defined at a finite (total M samples), discrete, and evenly-spaced set of x in its domain

$$\text{DFT:} \quad F(u) = \sum_{x=0}^{M-1} f(x) \exp(-j2\pi ux/M), \quad u = 0, 1, ..., M-1$$

$$\text{IDFT:} \quad f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) \exp(j2\pi ux/M), \quad x = 0, 1, ..., M-1$$

Note that $f(x)$ and $F(u)$ have the same number of elements.

Since both $f(x)$ and $F(u)$ are periodic (with period M), we only need M points in each domain for computing the transforms.

Deriving DFT from Fourier Series

Slightly modified equations of Fourier series (to make the results consistent with the textbook):

$$\tilde{f}(t) = \frac{1}{M}\sum_{u=-\infty}^{\infty}\left[F(u)\exp(j\frac{2\pi ut}{M})\right]$$

Here t is a continuous variable. Both x and u are discrete (integral) variables.

$$F(u) = \int_{0}^{M-}\tilde{f}(t)\exp(-j\frac{2\pi ut}{M})dt$$

$$= \int_{0}^{M-} f(t)\sum_{x=-\infty}^{\infty}\delta(t-x)\exp(-j\frac{2\pi ut}{M})dt$$

$$= \sum_{x=0}^{M-1} f(x)\exp(-j\frac{2\pi ux}{M})$$

This is periodic: $F(u+M) = F(u)$

Proof of Inverse DFT

$$\frac{1}{M}\sum_{u=0}^{M-1} F(u)\exp(j\frac{2\pi ux}{M})$$

$$=\frac{1}{M}\sum_{u=0}^{M-1}\sum_{x'=0}^{M-1} f(x')\exp(-j\frac{2\pi ux'}{M})\exp(j\frac{2\pi ux}{M})$$

$$=\frac{1}{M}\sum_{x'=0}^{M-1} f(x')\sum_{u=0}^{M-1}\exp(-j\frac{2\pi ux'}{M})\exp(j\frac{2\pi ux}{M}) = f(x)$$

Note:

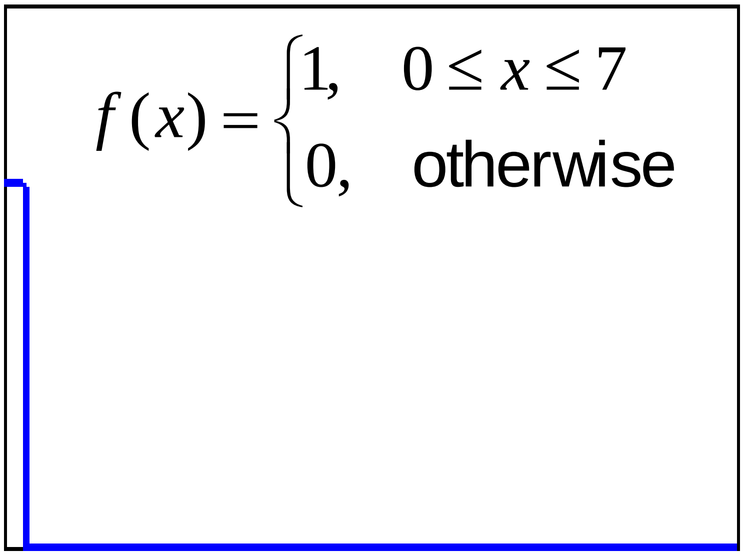
$$\sum_{u=0}^{M-1}\exp(-j\frac{2\pi ux'}{M})\exp(j\frac{2\pi ux}{M}) = \begin{cases} M & \text{for } x = x' \\ 0 & \text{otherwise} \end{cases}$$

$$x, x' = 0, 1, \cdots, M-1$$

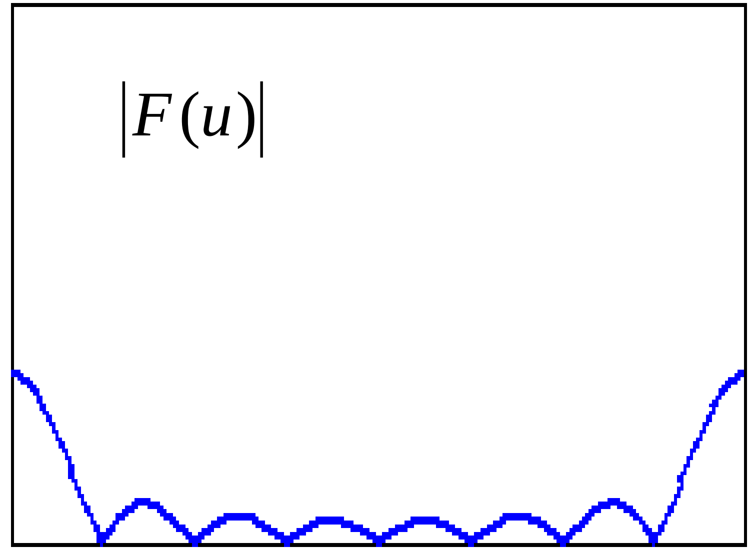
Spectrum - Example

M=256

$$f(x) = \begin{cases} 1, & 0 \le x \le 7 \\ 0, & \text{otherwise} \end{cases}$$



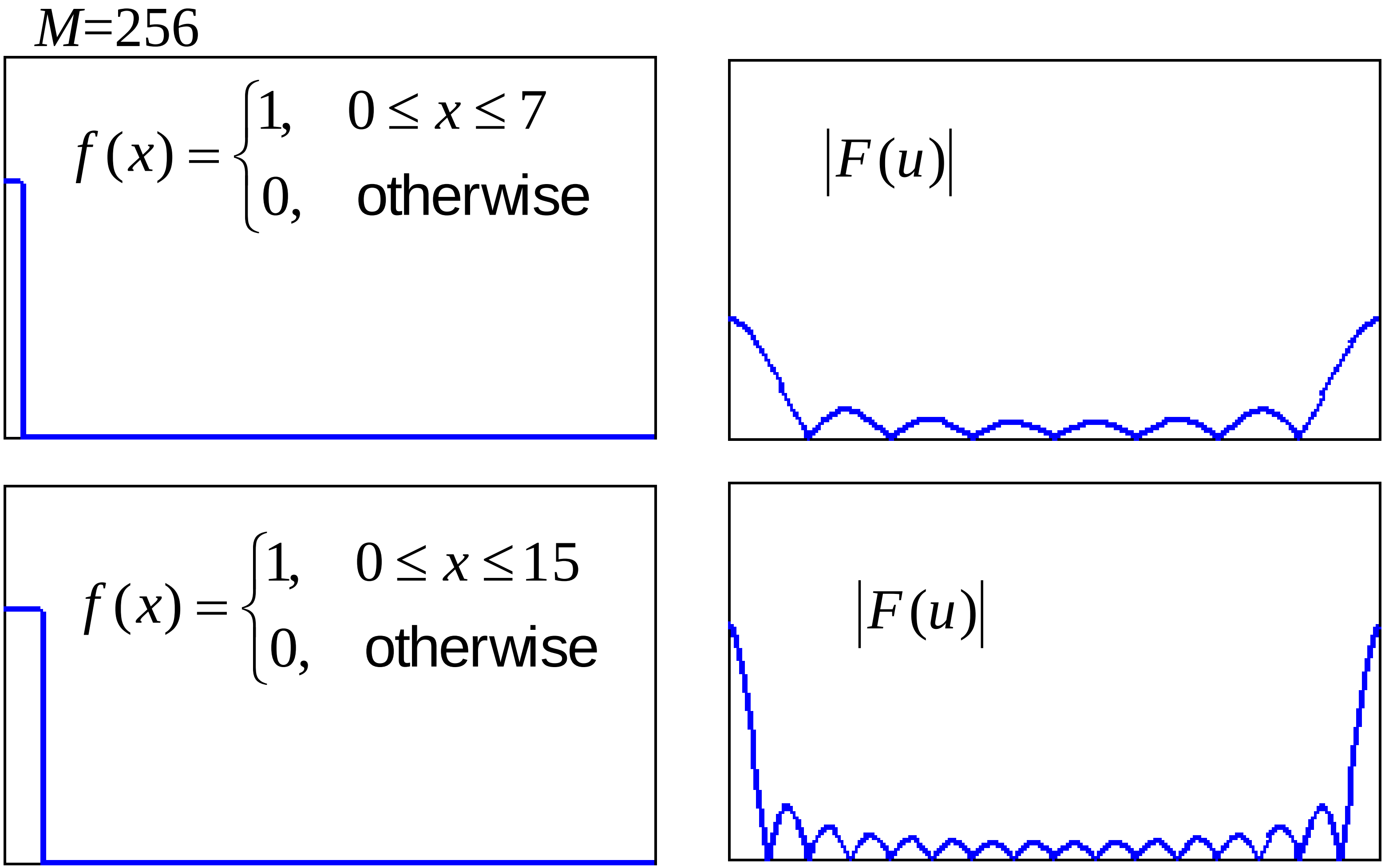
$|F(u)|$



Interesting points:

- What happens at $|F(0)|$?
- What happens at $|F(M-1)|$?
- What happens if we change M but leaves $f(x)$ unchanged?
- What happens if we just shift the non-zero part of $f(x)$ all by a constant Δx?

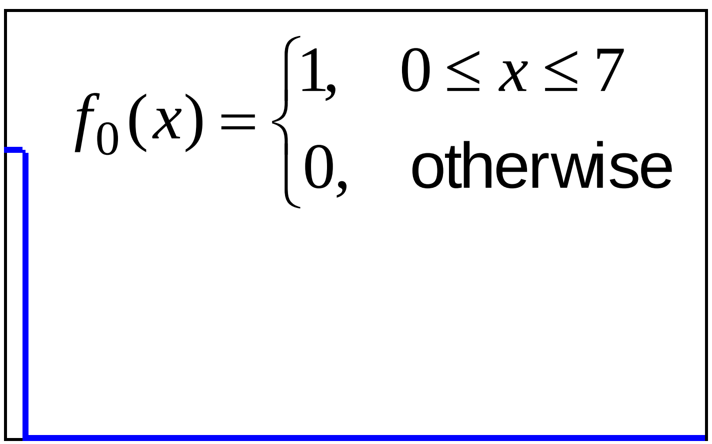
Space-Frequency Reciprocity



Note in $|F(u)|$: (1) peak height, (2) peak width, (3) # zeros

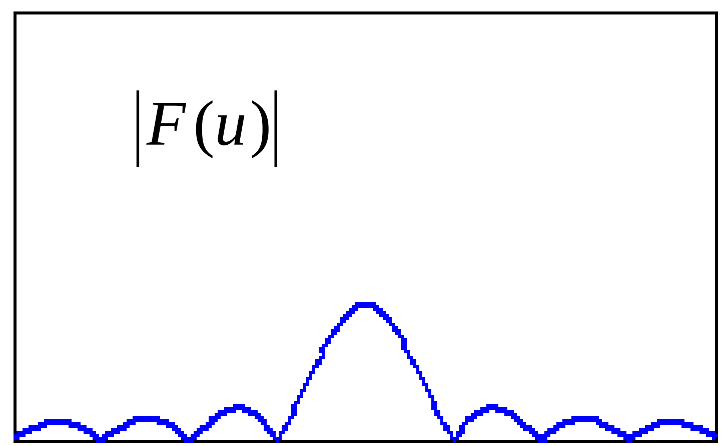
Centering the Spectrum

When visualizing the spectrum, it is a common practice to multiply $f(x)$ by $(-1)^x$ first before applying DFT, so that the DC term of $|F(u)|$ appears at the center ($M/2$) of the resulting spectrum.



$$f_0(x) = \begin{cases} 1, & 0 \le x \le 7 \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = (-1)^x f_0(x)$$



$$|F(u)|$$

2-D DFT and IDFT

The discrete 2-D FT/IFT (sample size MxN):

DFT: $$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M + vy/N)]$$

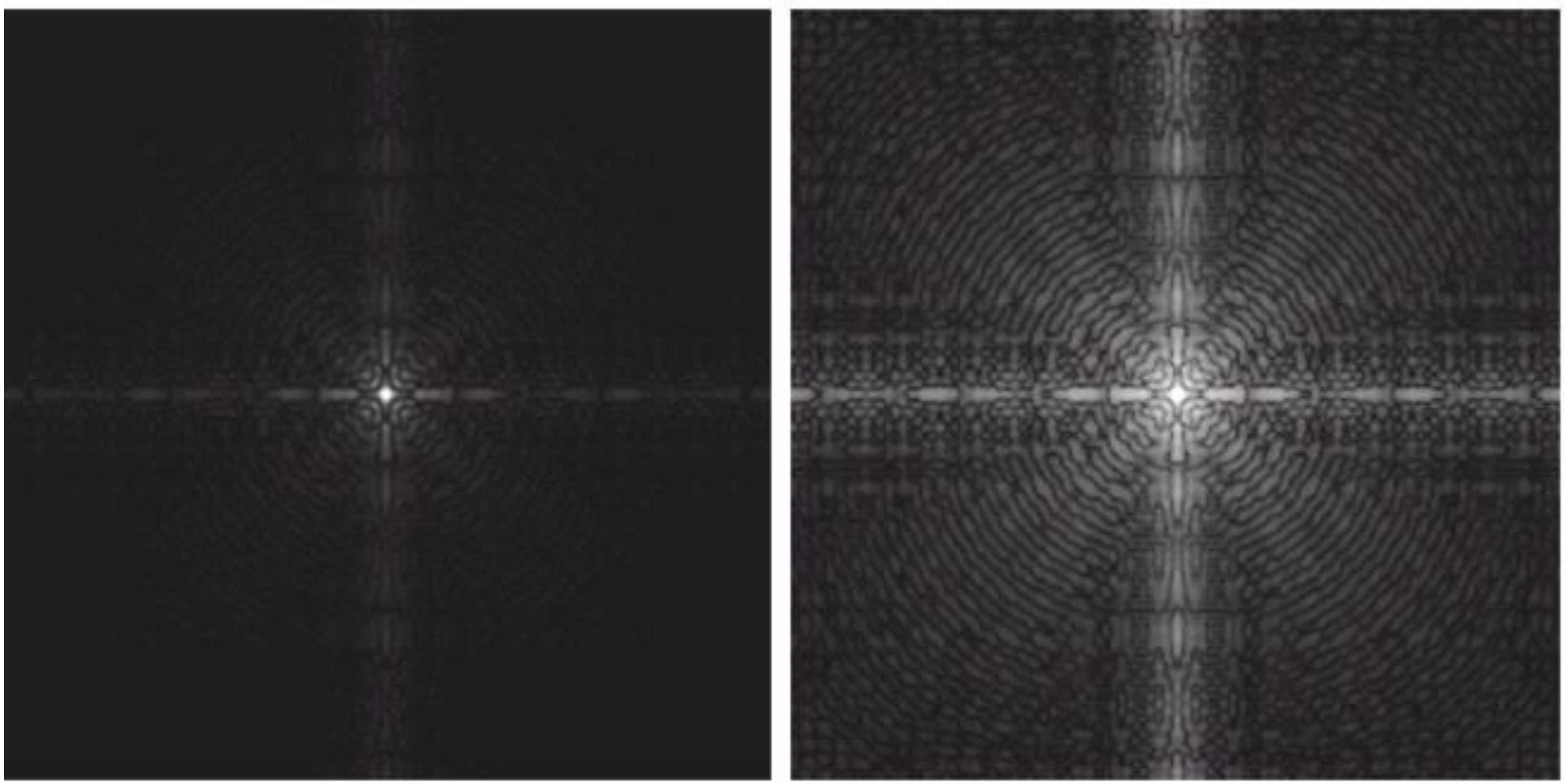
IDFT: $$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M + vy/N)]$$

To center the spectrum: multiply $f(x,y)$ by $(-1)^{x+y}$.

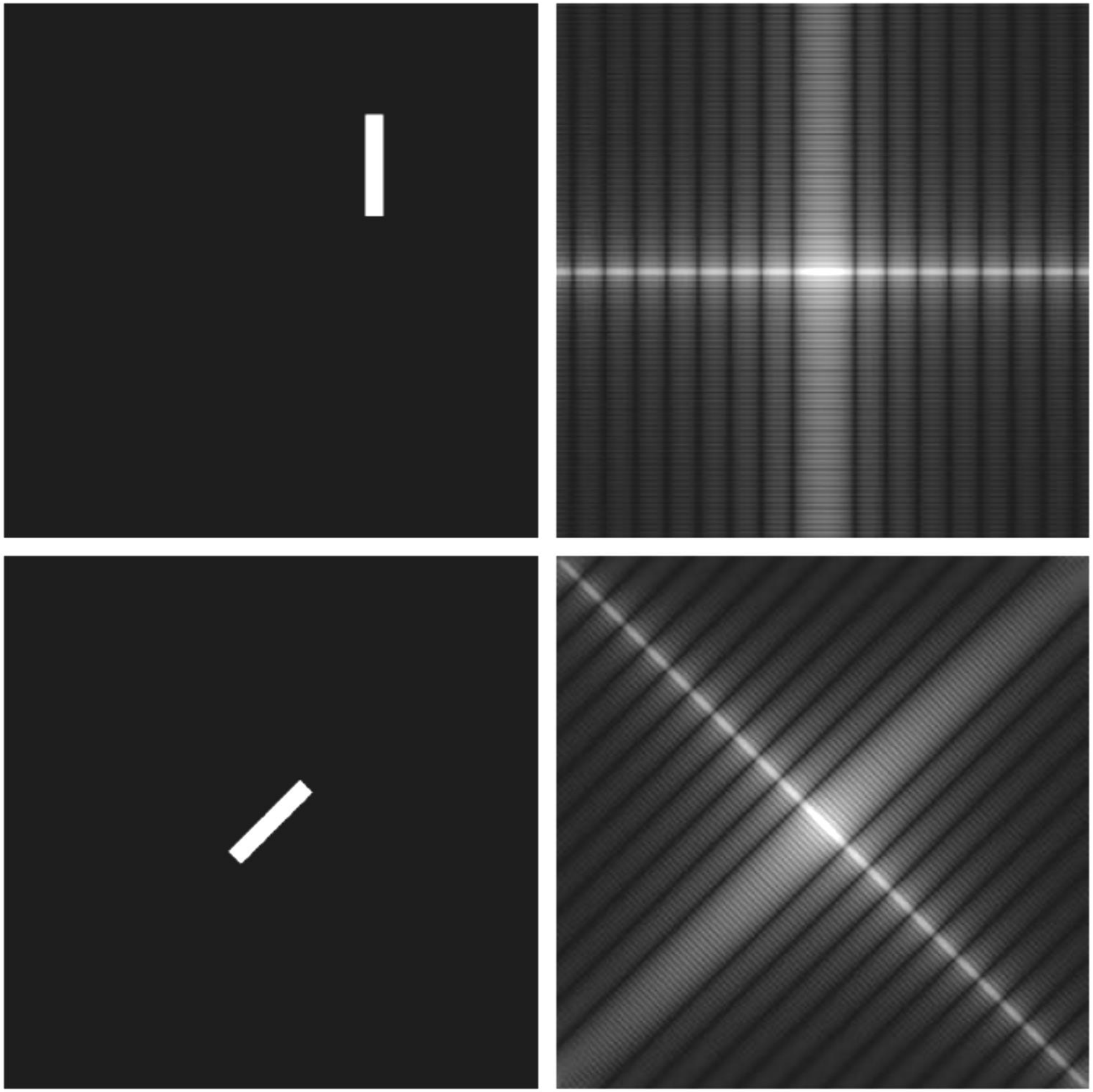
Log Transformations

The energy of a spectrum is usually highly concentrated at the low-frequency region. A very strong contrast enhancement method, such as log transformation, is usually used to show details of the spectrum.

$$s = c\log(1+r)$$



2-D Spectrum - Example



2-D Spectrum - Example

Note the effect of strong intensity variations along particular directions: