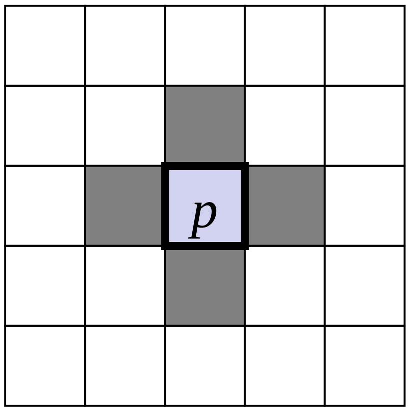
Binary Images Processing
and
Mathematical Morphology

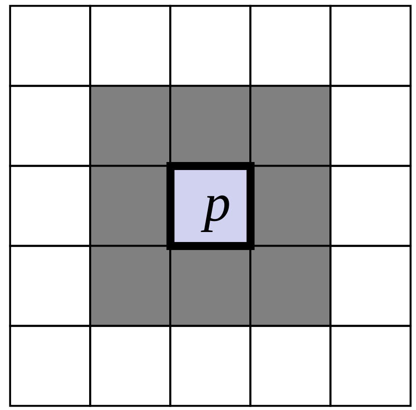
Relations Between Pixels

- Neighbors and Neighborhood:

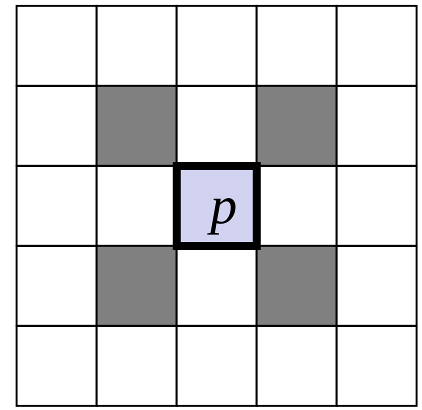
$N_4(p)$:



$N_8(p)$:



$N_D(p)$:



- Adjacency (for same-value pixels):

Examples for "1" pixels:

4-adjacency

0	1	1
0	1	0
0	0	1

8-adjacency

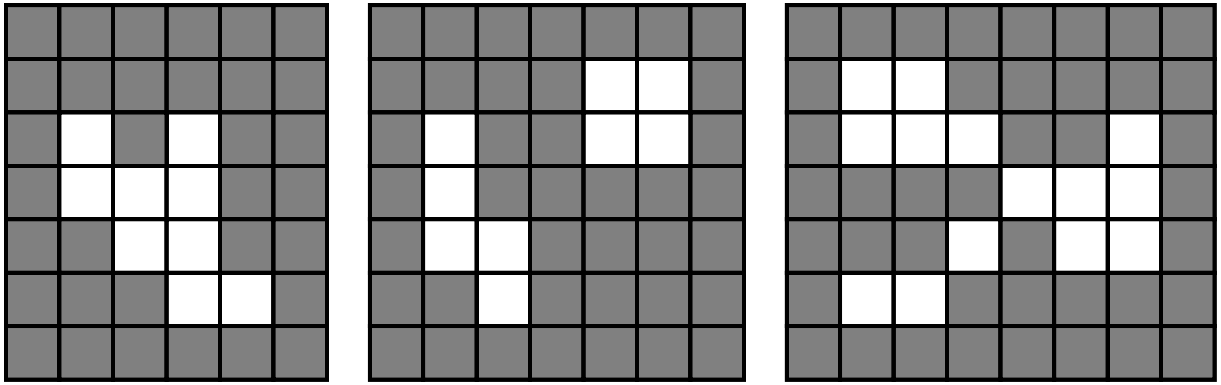
0	1	1
0	1	0
0	0	1

Connected Components (Regions)

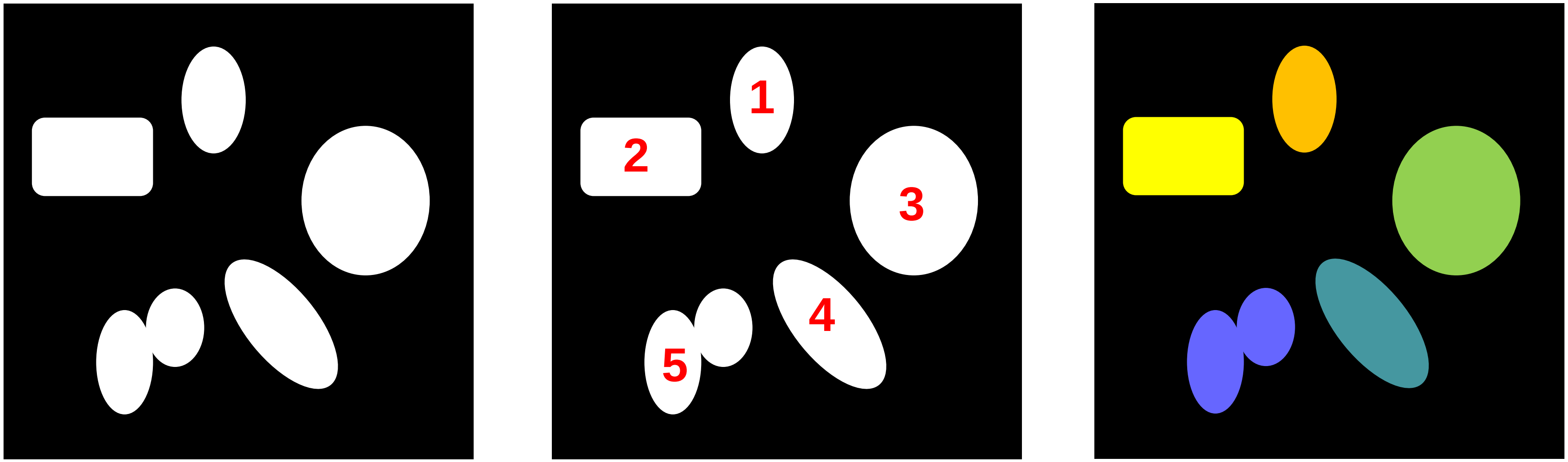
Definitions (the simple version):

- Path: A list of pixels, with each pixel adjacent to the next one.
- Two pixels p and q are connected if there exists a path connecting p and q.
- A **connected component** is a subset of pixels such that all the pixels in it are connected to one another.

How many connected components of white pixels are in these images? Note: This depends on the connectivity.



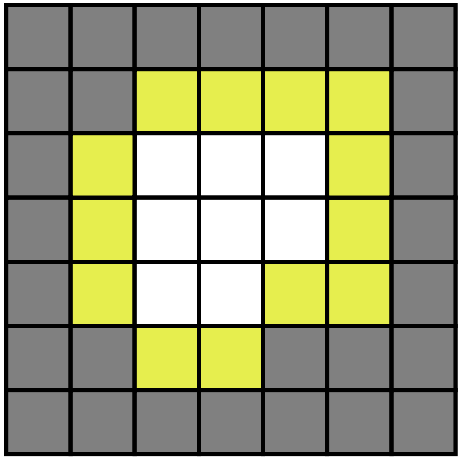
Connected Component Labeling

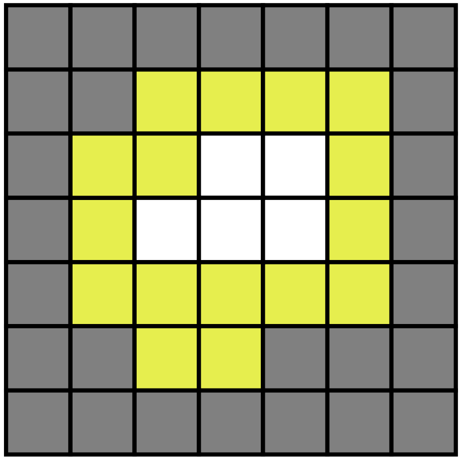


This is the process of identify separate foreground objects in a binary image. Each connected component is given an index or label.

Region Boundary

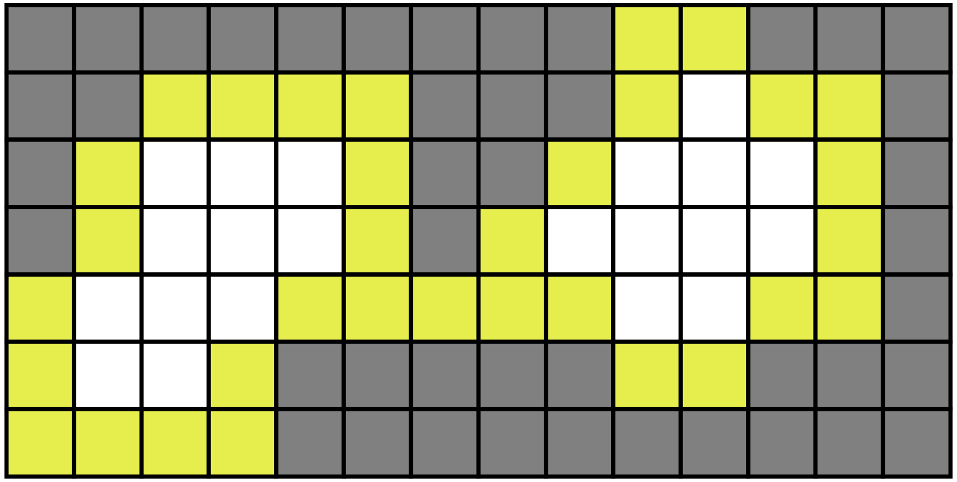
The **boundary** of a region R is the subset of pixels that (1) belong to R, and (2) have neighbors that are not in R.





These two are based on different adjacencies. Can you tell?

Pixels of a region R that are located on the image border are automatically considered part of the boundary of R.



Morphological Image Processing

- Morphology is the study of forms and structures, such as in biology.
- **Mathematical morphology**: How we represent and operate on “shapes”. We do this by treating regions of pixels as <u>sets</u>.
- The main practical applications of morphological operations in image processing are mainly in preprocessing and early-stage segmentation.
- The main discussions are focused on binary images, with a little extensions to gray-scale images.

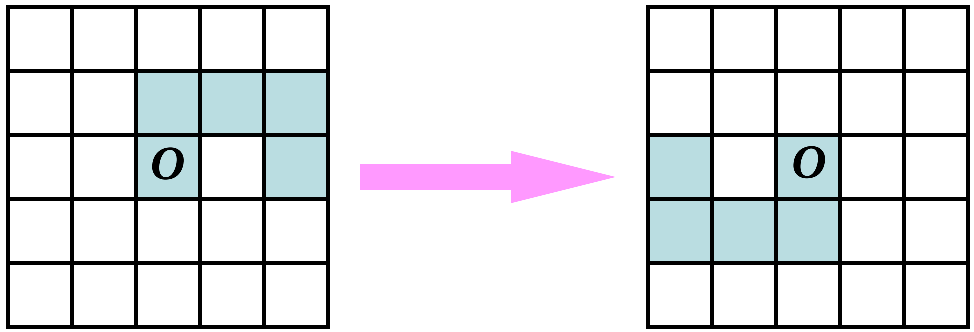
Set Operations

- Basic set relations and operations:
  - Empty (null) set
  - Subset
  - Intersection and Union
  - Complement
  - Set difference
- Specific set operations (for elements on a grid) that we need:
  - Set reflection: $\hat{B} = \{ w \mid w = -b, \quad b \in B \}$
  - Set translation: $(A)_z = \{ c \mid c = a + z, \quad a \in A \}$

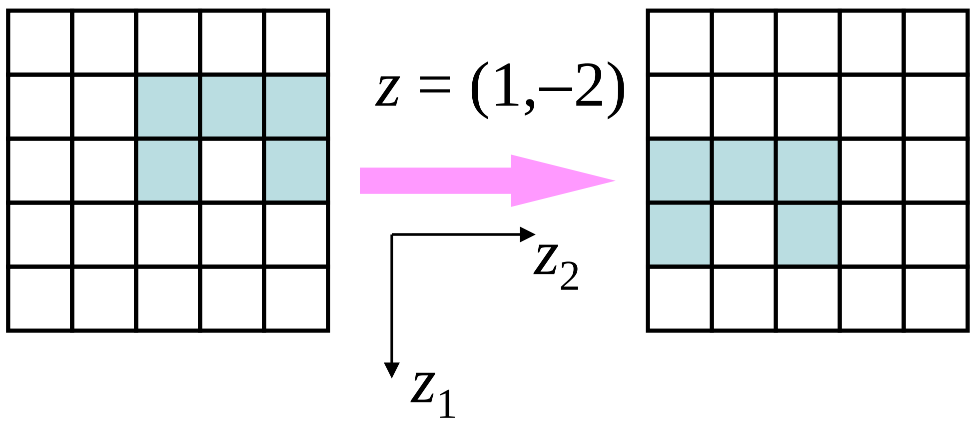
Set Operations

When we consider the set to contain the shaded pixels in an image:

Set reflection:

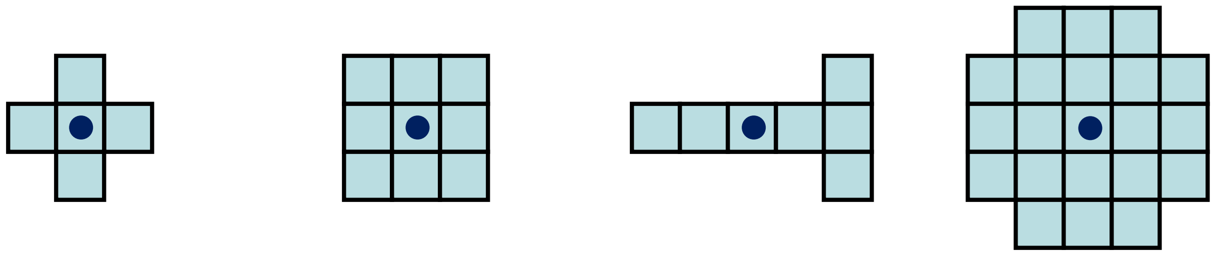


Set translation:

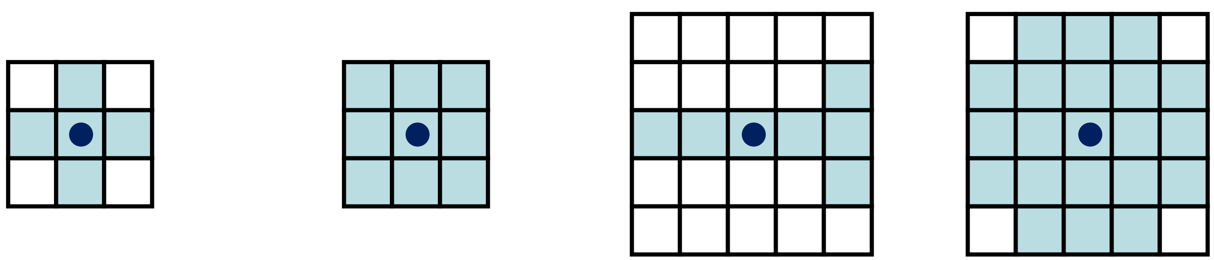


Structuring Element (SE)

Many morphological operations involve the interaction between an image and a **structuring element**, which is usually a small set of points that define a shape. A structuring element can be symmetric or non-symmetric. Examples:



They are often represented as part of a rectangular sub-image which looks like a spatial filter. But the operations are different.



(The ● dots represent the "centers" of the structuring elements.)

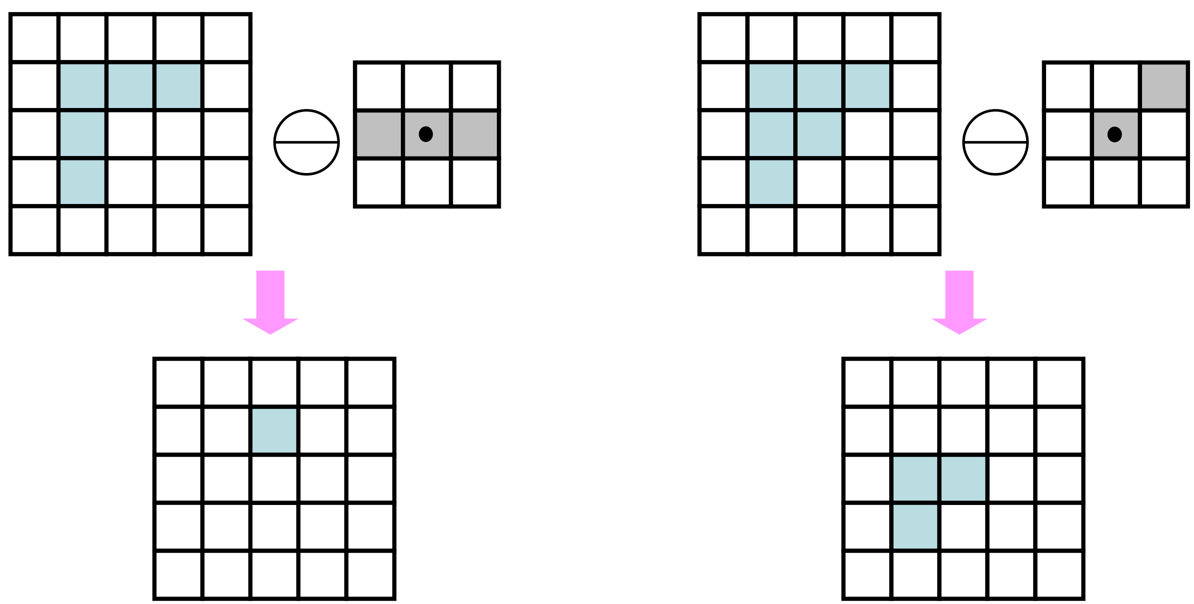
Erosion

Mathematical definition of **erosion**:

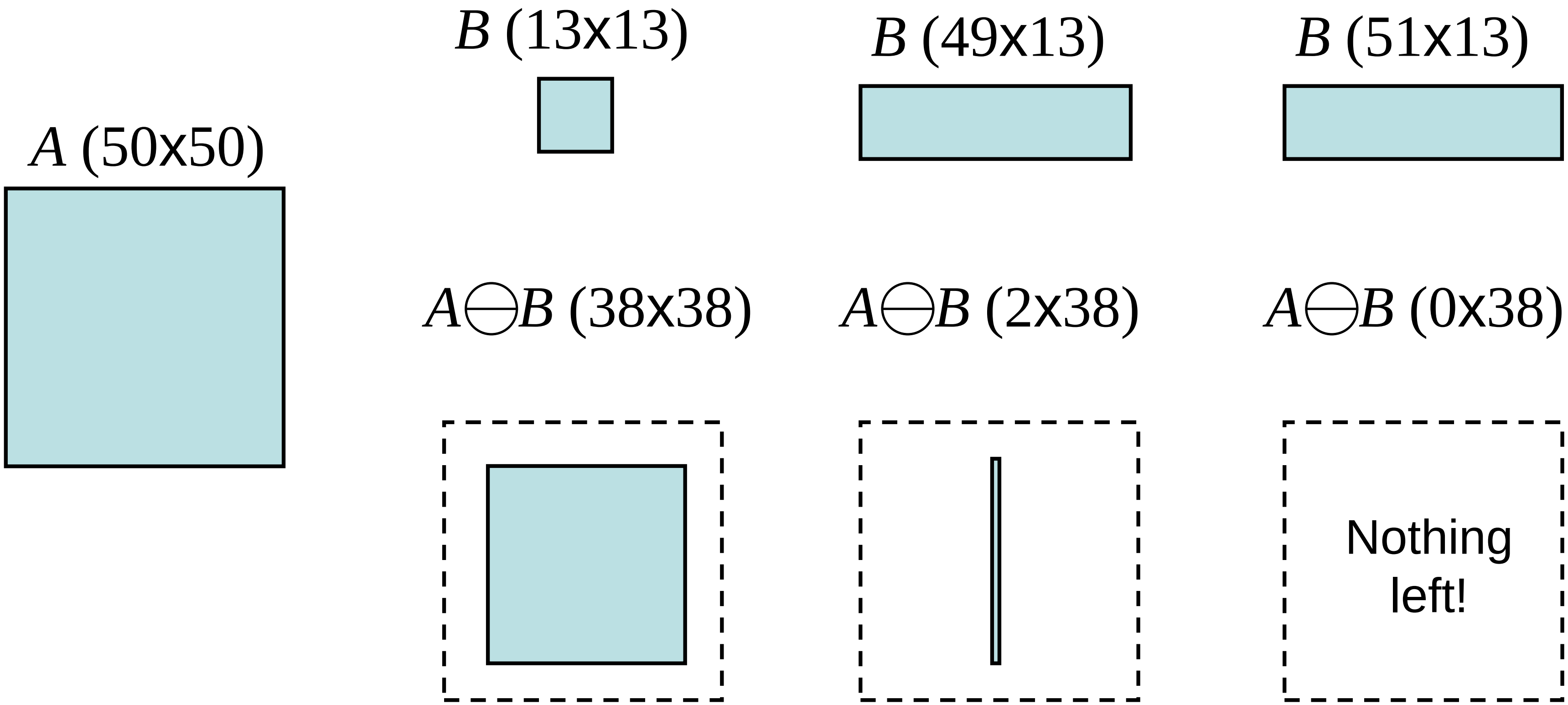
$$A \ominus B = \{ z \mid (B)_z \subseteq A \} = \left\{ z \mid (B)_z \cap A^c = \varnothing \right\}$$

Definition in plain English: The centers of all the translated B (the SE) that are contained in A.

Examples:



Erosion



A (50x50)
B (13x13)
B (49x13)
B (51x13)
$A \ominus B$ (38x38)
$A \ominus B$ (2x38)
$A \ominus B$ (0x38)
Nothing left!

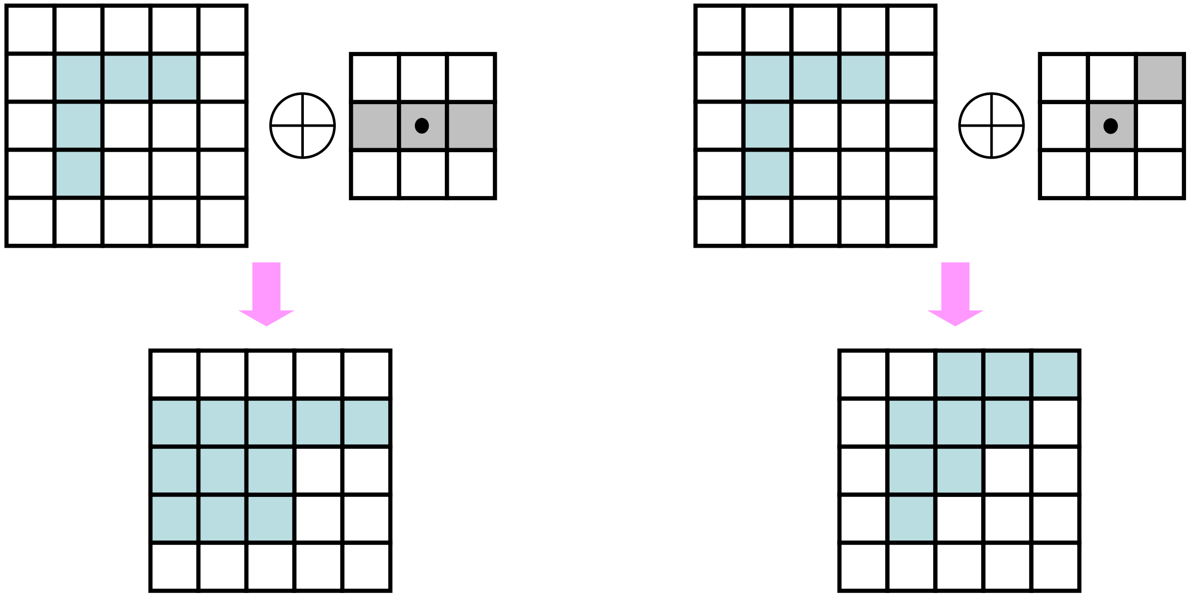
Dilation

Mathematical definition of **dilation**:

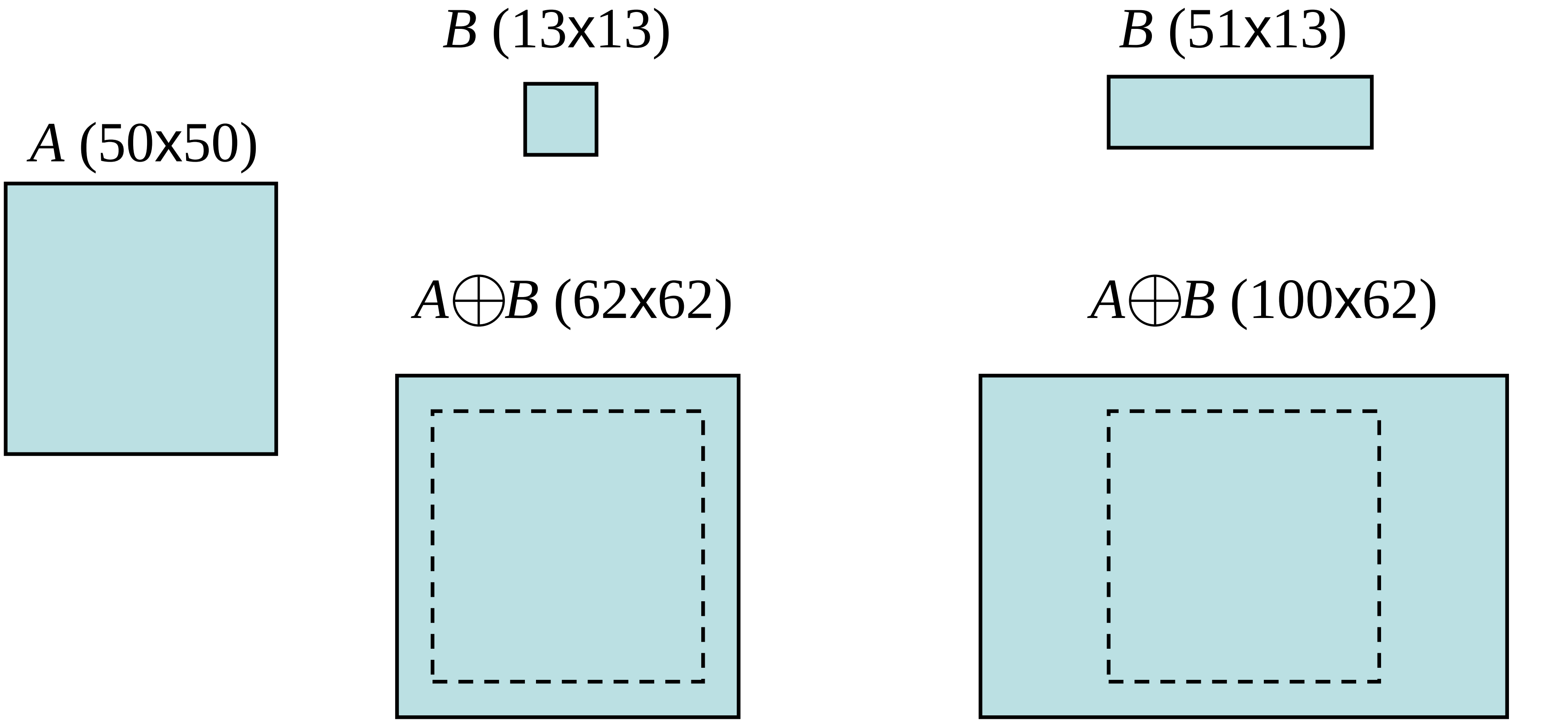
$$A \oplus B = \left\{ a + b \mid a \in A, b \in B \right\} = \bigcup_{z \in A} (B)_z$$

Definition in plain English: The union of all the translated B (the SE) whose centers are contained in A.

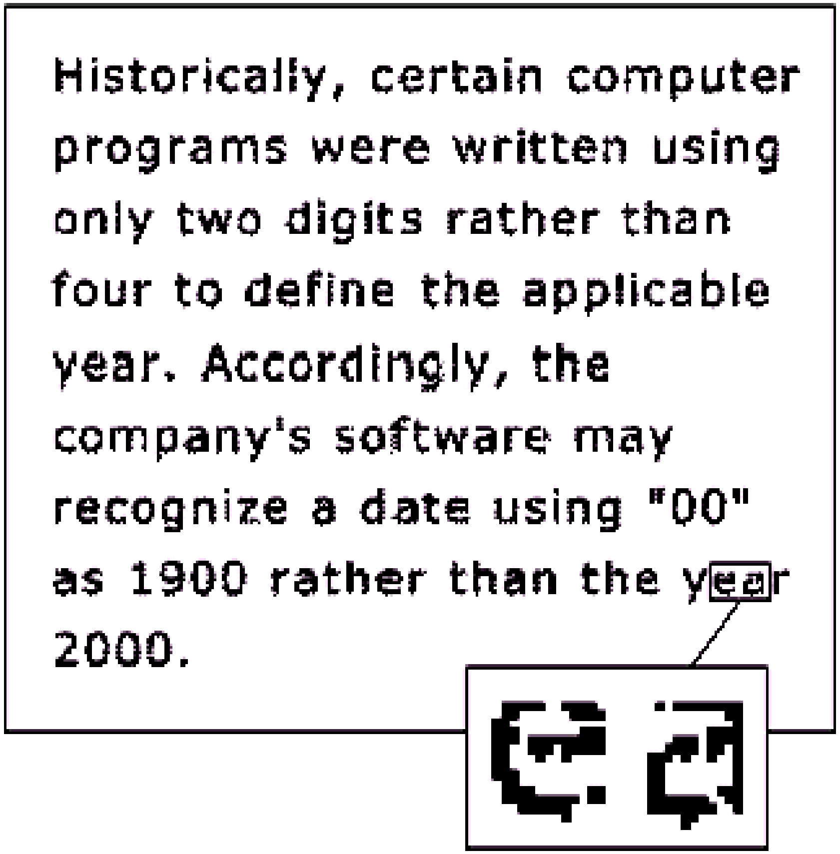
Examples:

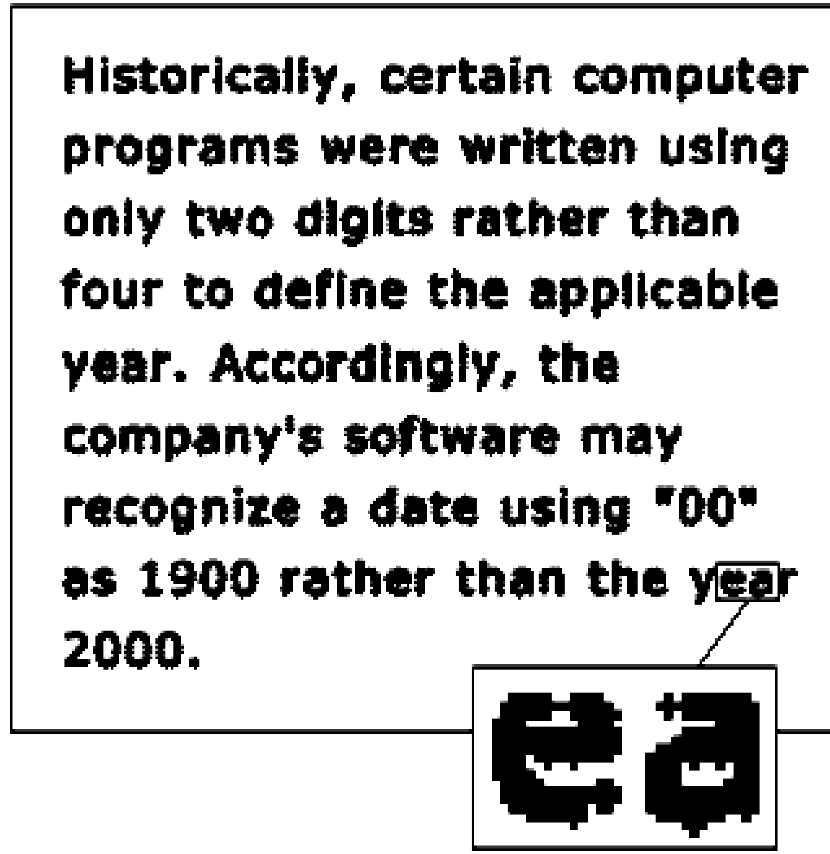


Dilation



Dilation

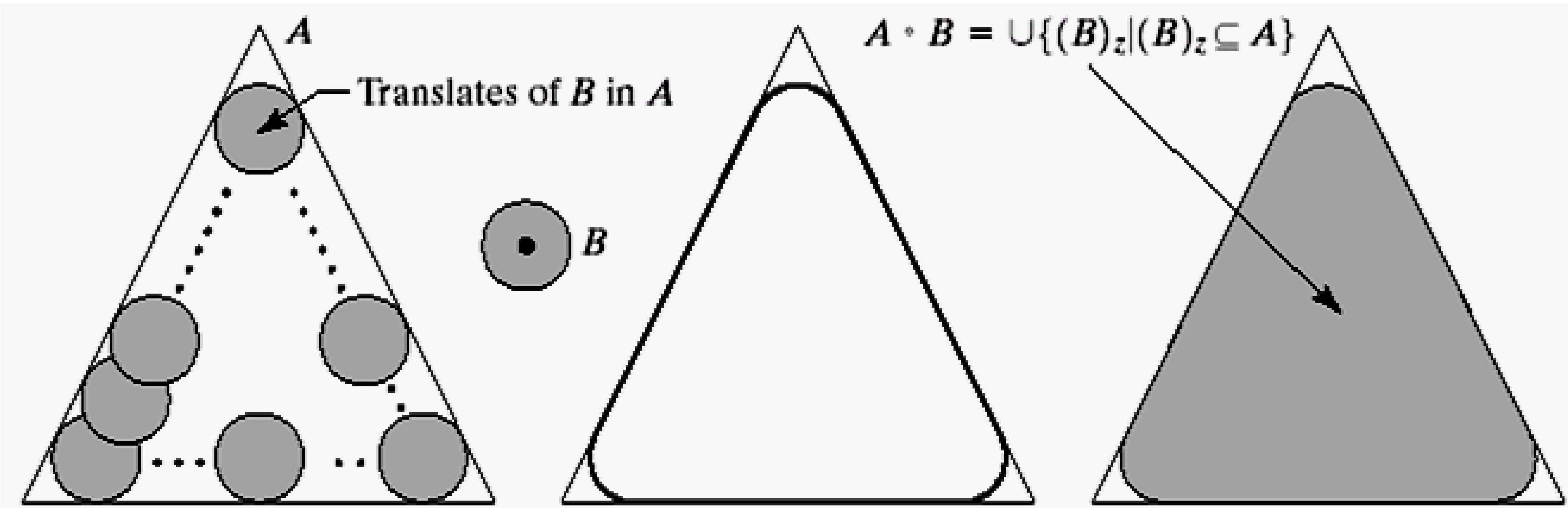




0	1	0
1	1	1
0	1	0

Opening

Definition: $A \circ B = (A \ominus B) \oplus B$

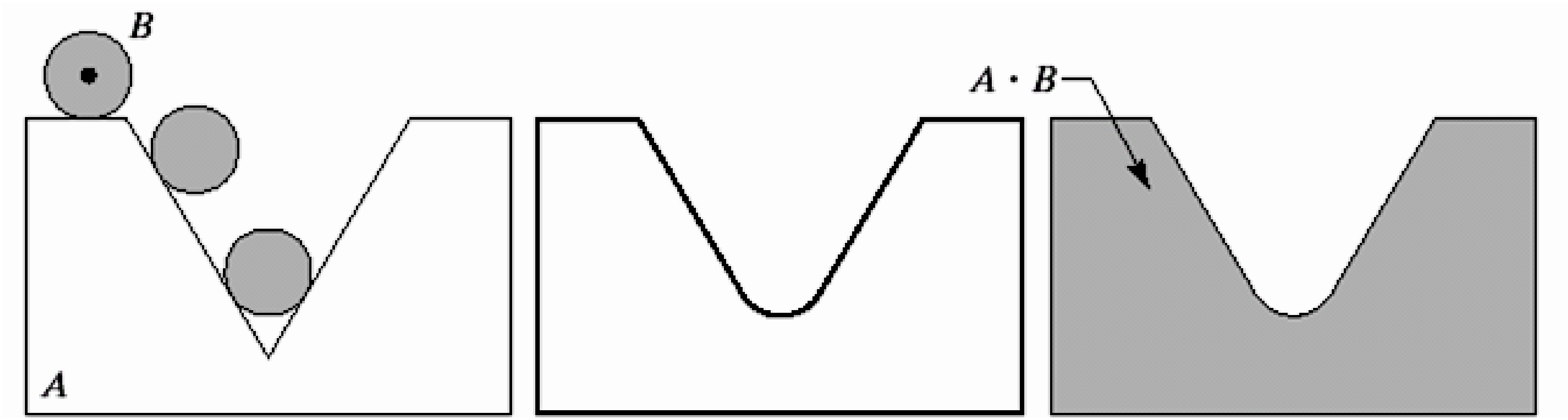


Effect: Preserving the part of A where B can be fitted in:

$$A \circ B = \bigcup_{(B)_z \subseteq A} (B)_z$$

Closing

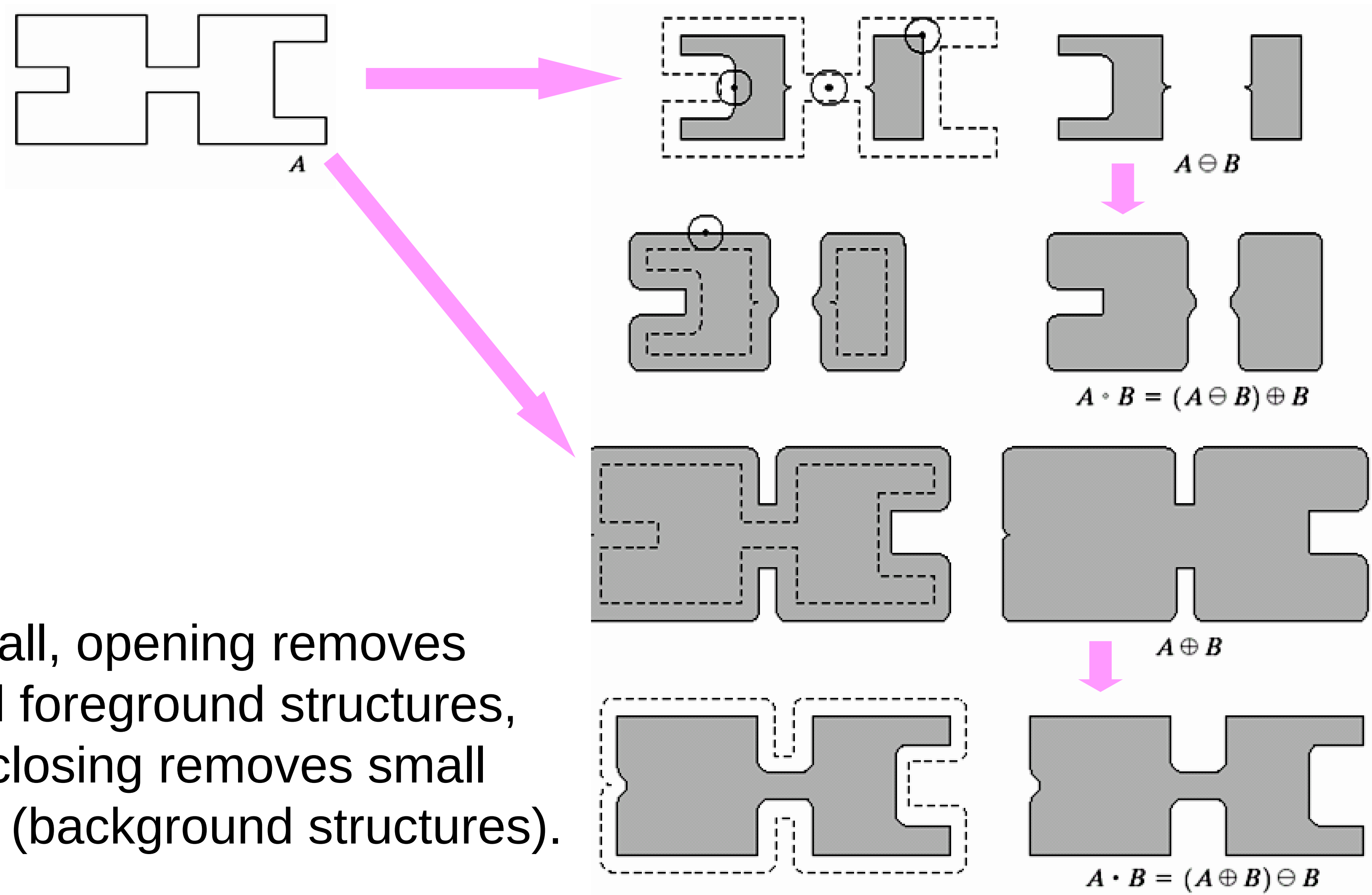
Definition: $A \bullet B = (A \oplus B) \ominus B$



Effect: Preserving the part of A^c where the reflection of B can be fitted in:

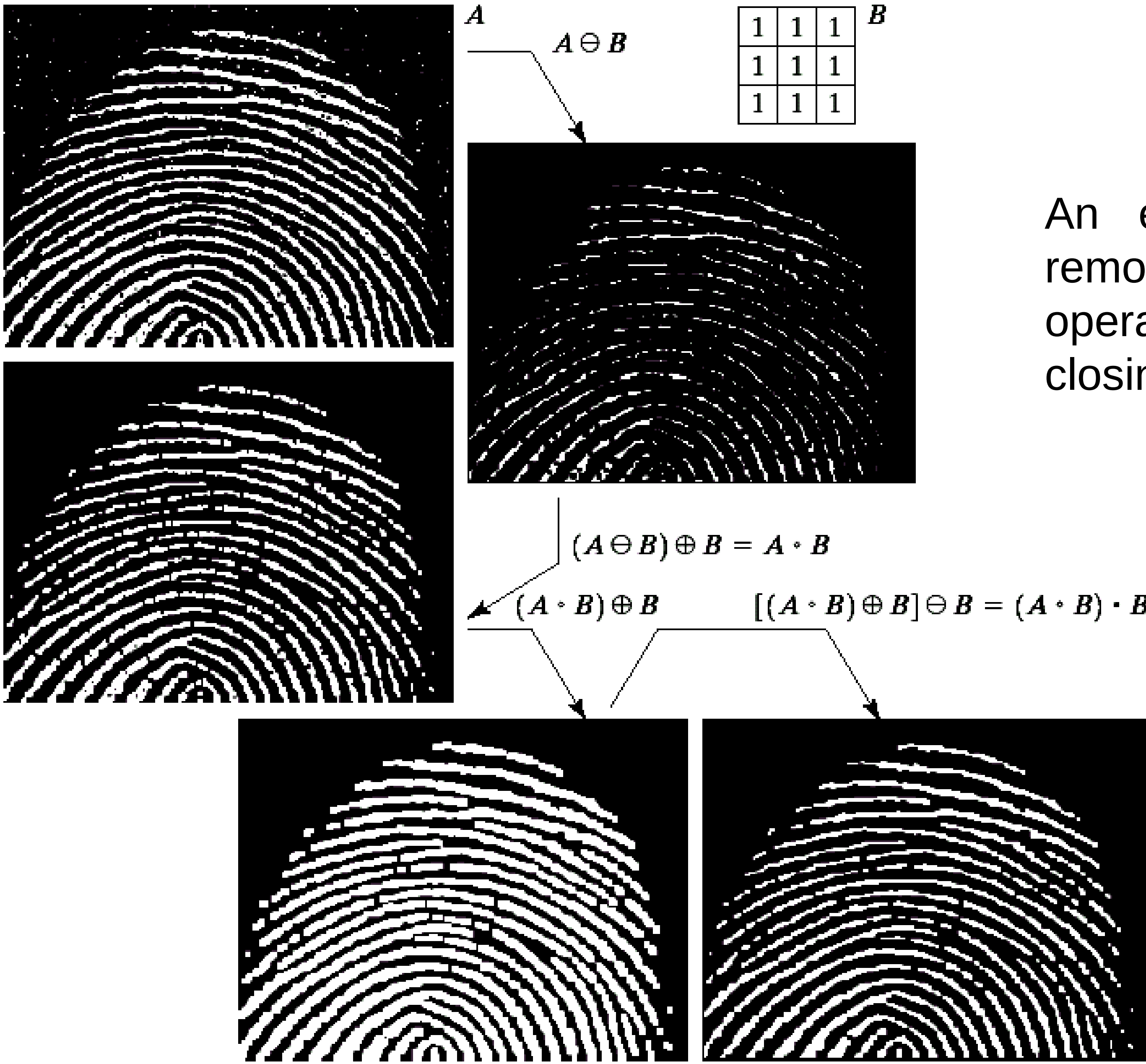
$$A \bullet B = \left[\bigcup_{(\hat{B})_z \not\subset A} (\hat{B})_z \right]^C$$

Opening/Closing Example



Overall, opening removes small foreground structures, and closing removes small gaps (background structures).

Opening/Closing Example



An example of noise removal with an opening operation followed by a closing operation.

More Properties of Operators

- A few additional properties:

$$A \circ B \subseteq A \subseteq A \bullet B$$

$$C \subseteq D \Rightarrow C \circ B \subseteq D \circ B \text{ and } C \bullet B \subseteq D \bullet B$$

$$(A \circ B) \circ B = A \circ B$$

$$(A \bullet B) \bullet B = A \bullet B$$

- De Morgan's Laws (Duality):

The erosion of A corresponds to the dilation of A^C, and vice versa.

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

$$(A \oplus B)^c = A^c \ominus \hat{B}$$

The opening of A corresponds to the closing of A^C, and vice versa.

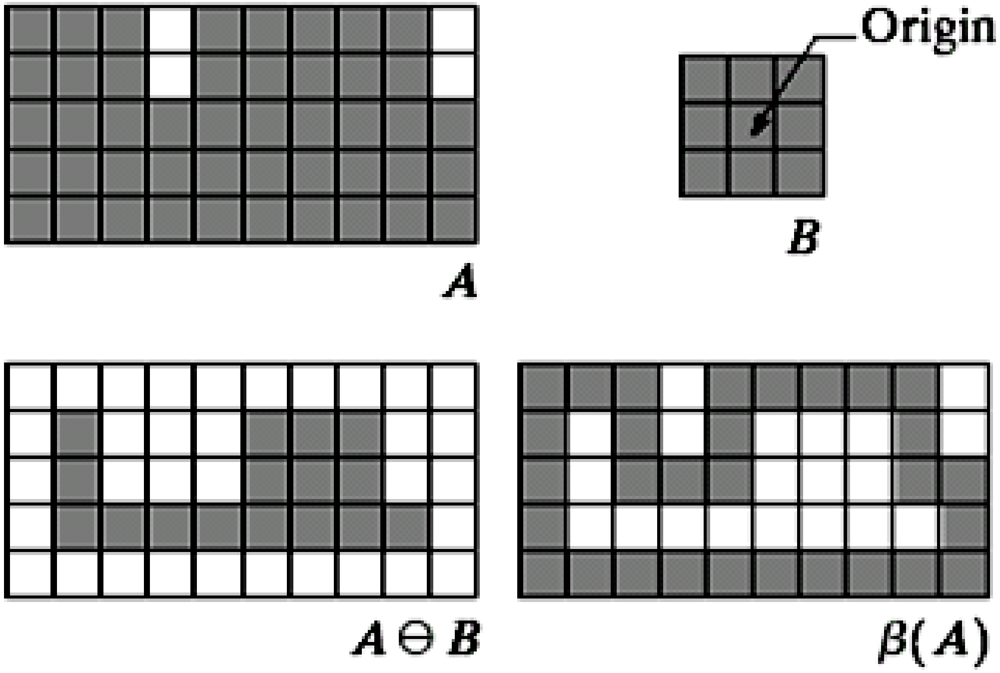
$$(A \bullet B)^c = A^c \circ \hat{B}$$

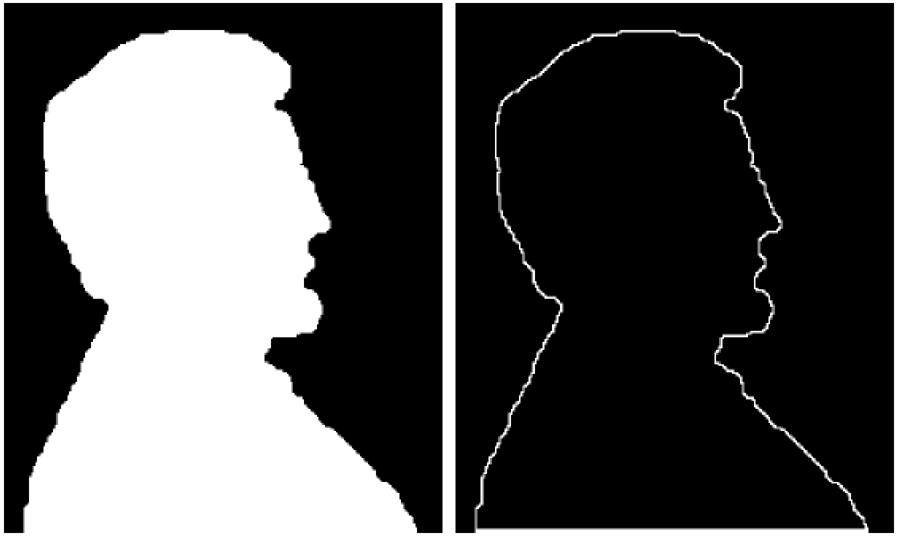
$$(A \circ B)^c = A^c \bullet \hat{B}$$

Boundary Extraction

$$\beta(A) = A - (A \ominus B)$$

The exact boundary extracted depends on B.



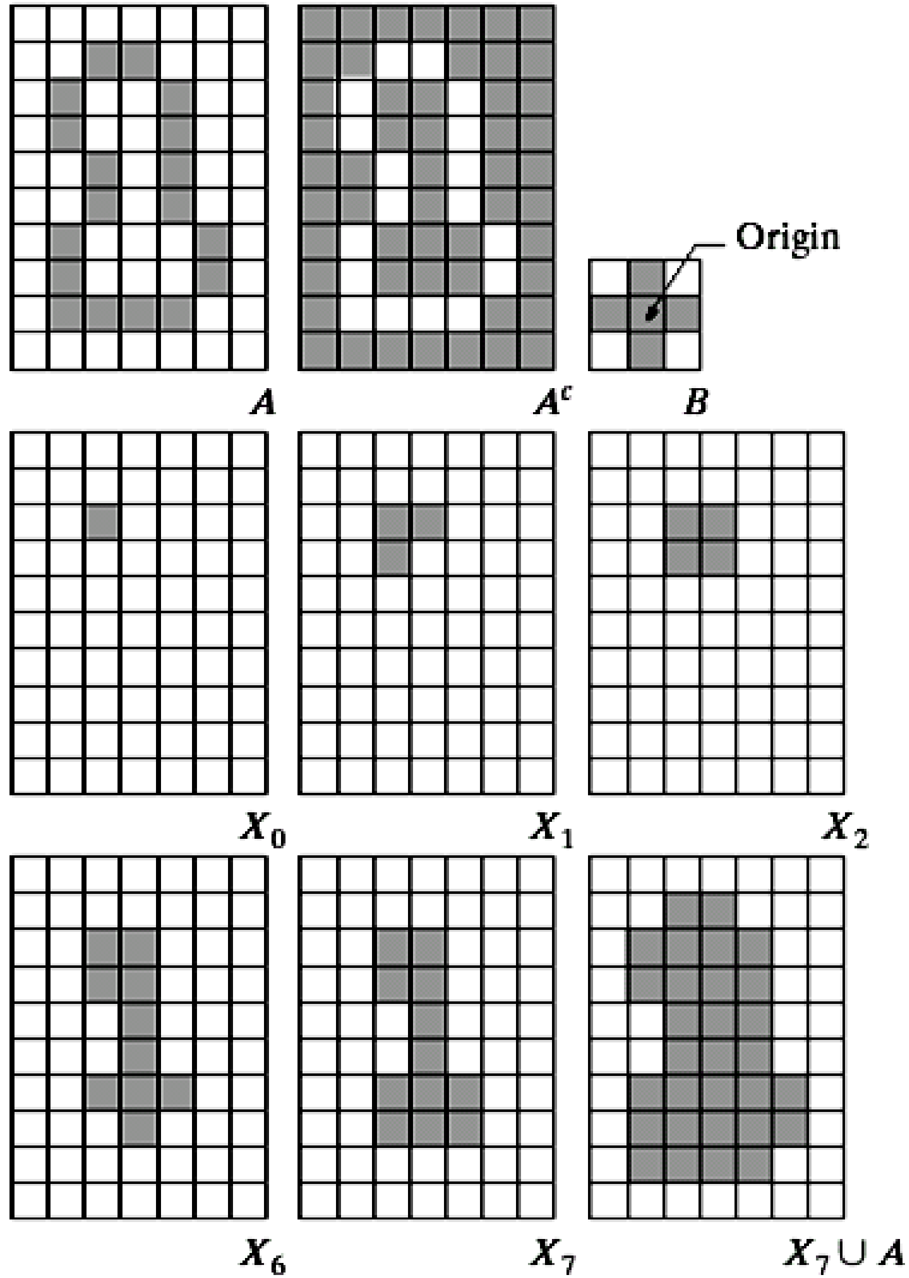


Region Filling

$$X_k = (X_{k-1} \oplus B) \cap A^c$$

$$(X_0 = \{p\})$$

This is an iterative method to fill a region (set to foreground) from a seed point p in A^c. The exact region filled depends on B.

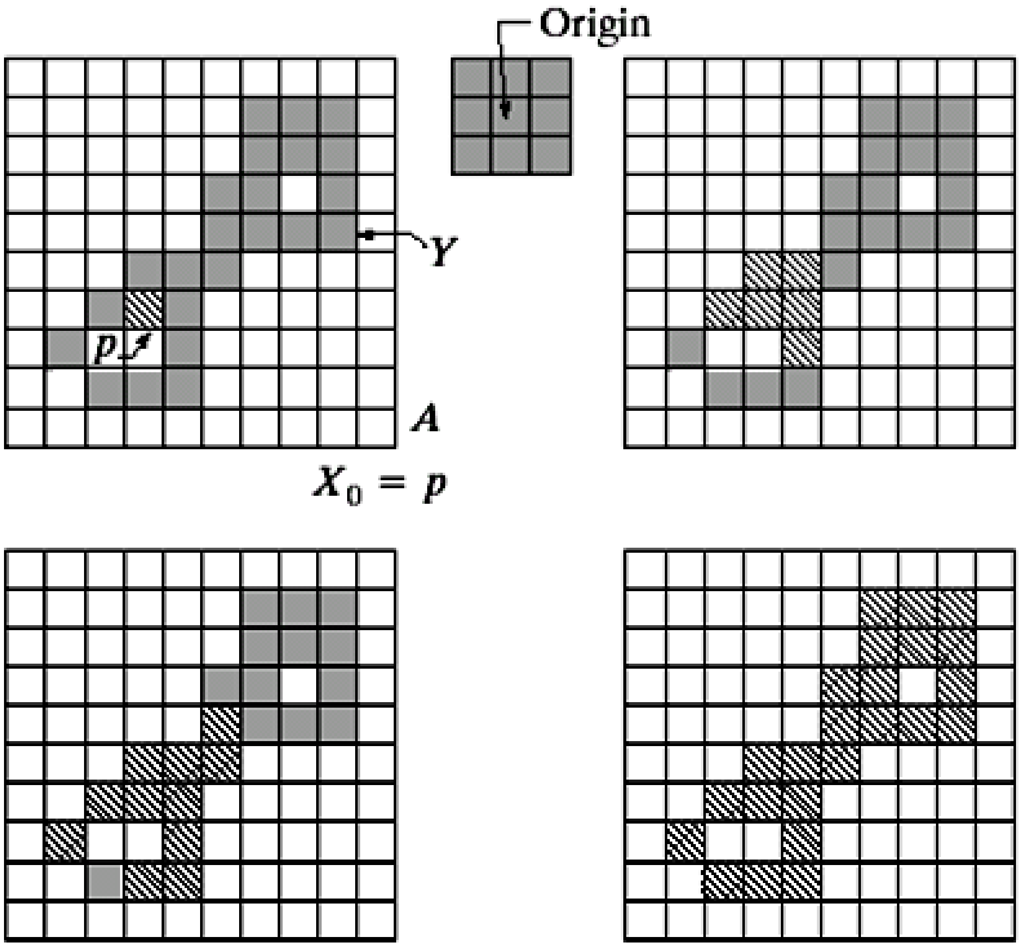


Connected Component Extraction

$$X_k = (X_{k-1} \oplus B) \cap A$$

$$(X_0 = \{p\})$$

This is an iterative method to find a connected component of A that contains a known seed point p in A. The exact connected component that is extracted depends on B.



Morphological Reconstruction

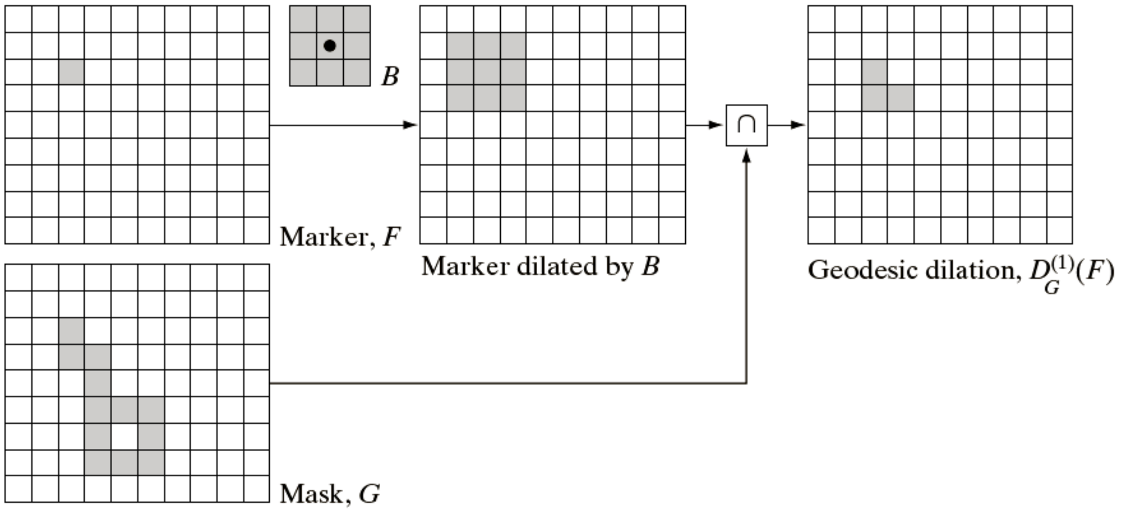
- Two images are involved:
  - Marker image: Where morphological operations are applied.
  - Mask image: This constraints the result of morphological operations.
- The structuring element is only used to define connectivity and is typically isotropic.
- The "shape" information is in the mask image.
- "Region filling" and "connected component extraction" are special cases of morphological reconstruction.

Geodesic Dilation and Erosion

- Geodesic Dilation:

$$D_G^{(1)}(F) = (F \oplus B) \cap G$$

$$D_G^{(n)}(F) = D_G^{(1)}[D_G^{(n-1)}(F)]$$

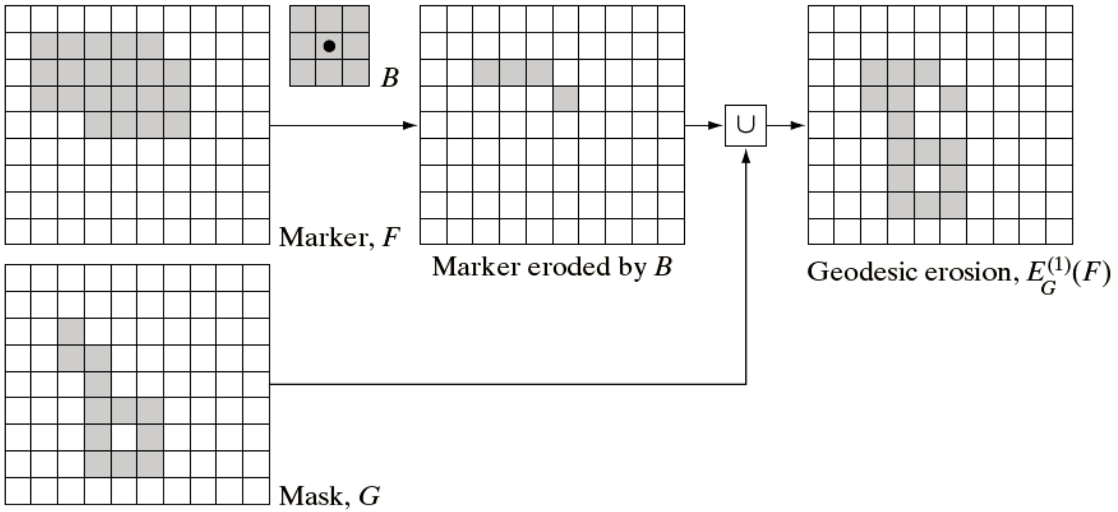


When the result converges, this is called "morphological reconstruction by dilation".

- Geodesic Erosion:

$$E_G^{(1)}(F) = (F \ominus B) \cup G$$

$$E_G^{(n)}(F) = E_G^{(1)}[E_G^{(n-1)}(F)]$$

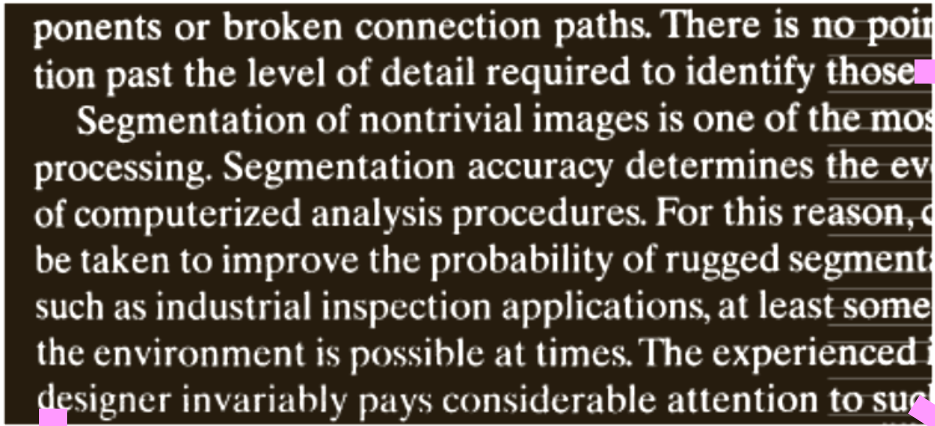


When the result converges, this is called "morphological reconstruction by erosion".

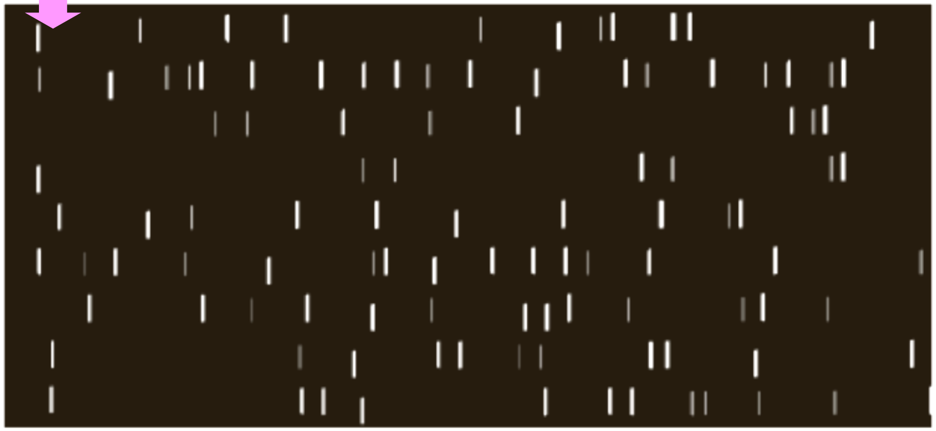
Example – Opening by Reconstruction

Goal: To find characters with long vertical strokes.

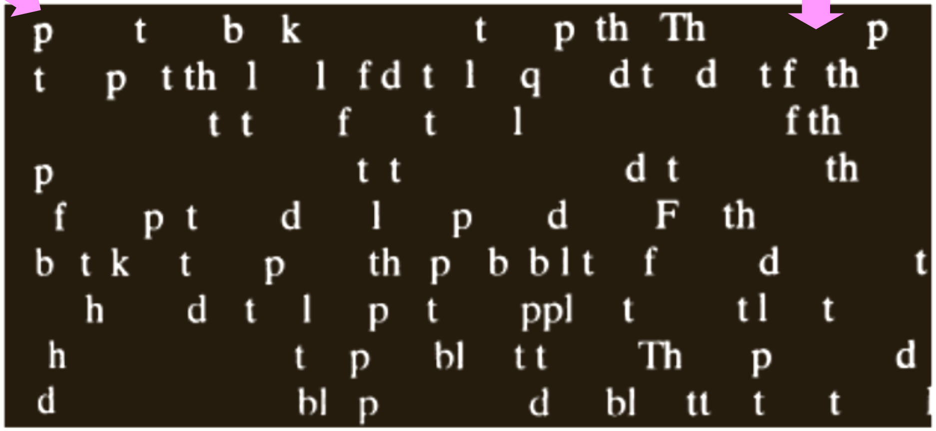
Original (also used as the mask image)



Erosion (marker image)



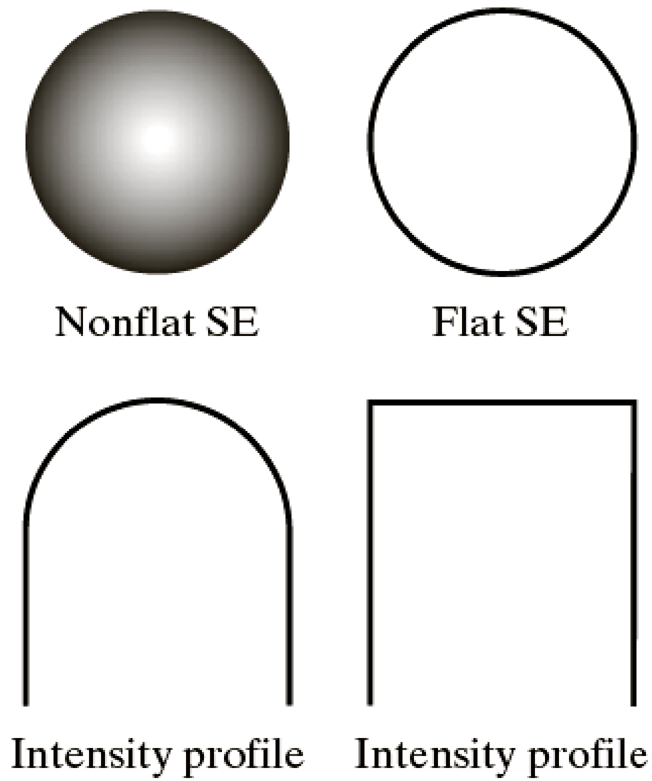
Result of regular opening



Morphological reconstruction

Gray-Scale Morphology

- The structuring element becomes 3-D:
  - Two spatial dimensions.
  - One dimension in intensity.



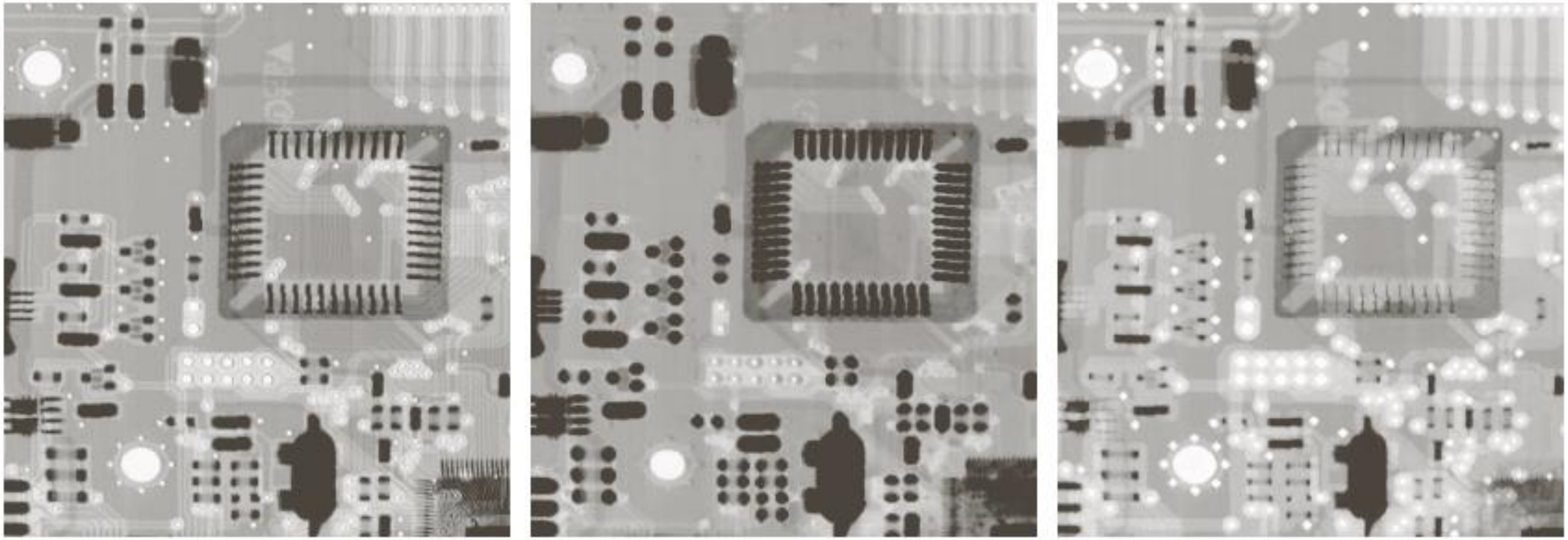
The flat one is used more often.

Gray-Scale Morphology with Flat SE

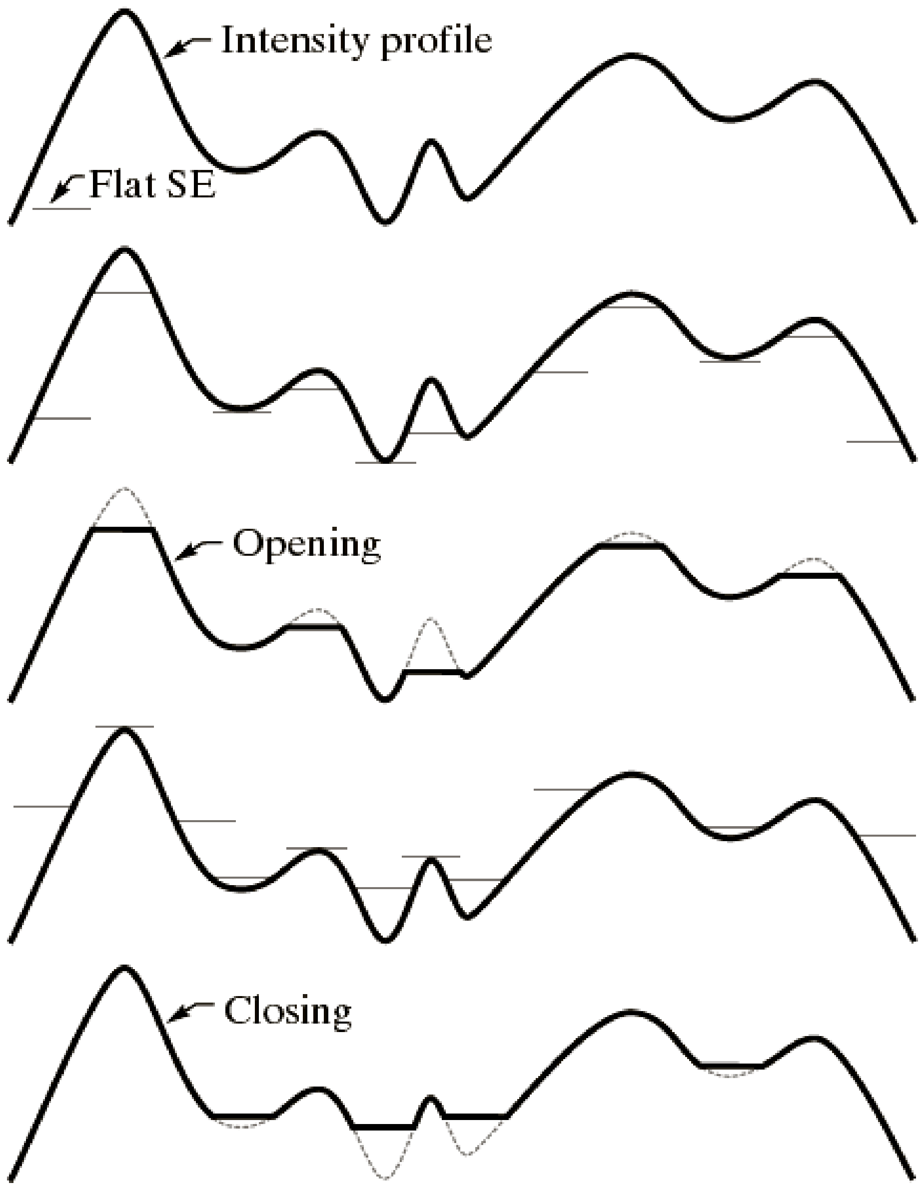
These are actually max and min filters.

- Gray-scale dilation: $[f \oplus b](x, y) = \max_{(x,y)\in b} \{ f(x-s, y-t) \}$
- Gray-scale erosion: $[f \ominus b](x, y) = \min_{(x,y)\in b} \{ f(x+s, y+t) \}$

Erosion Dilation

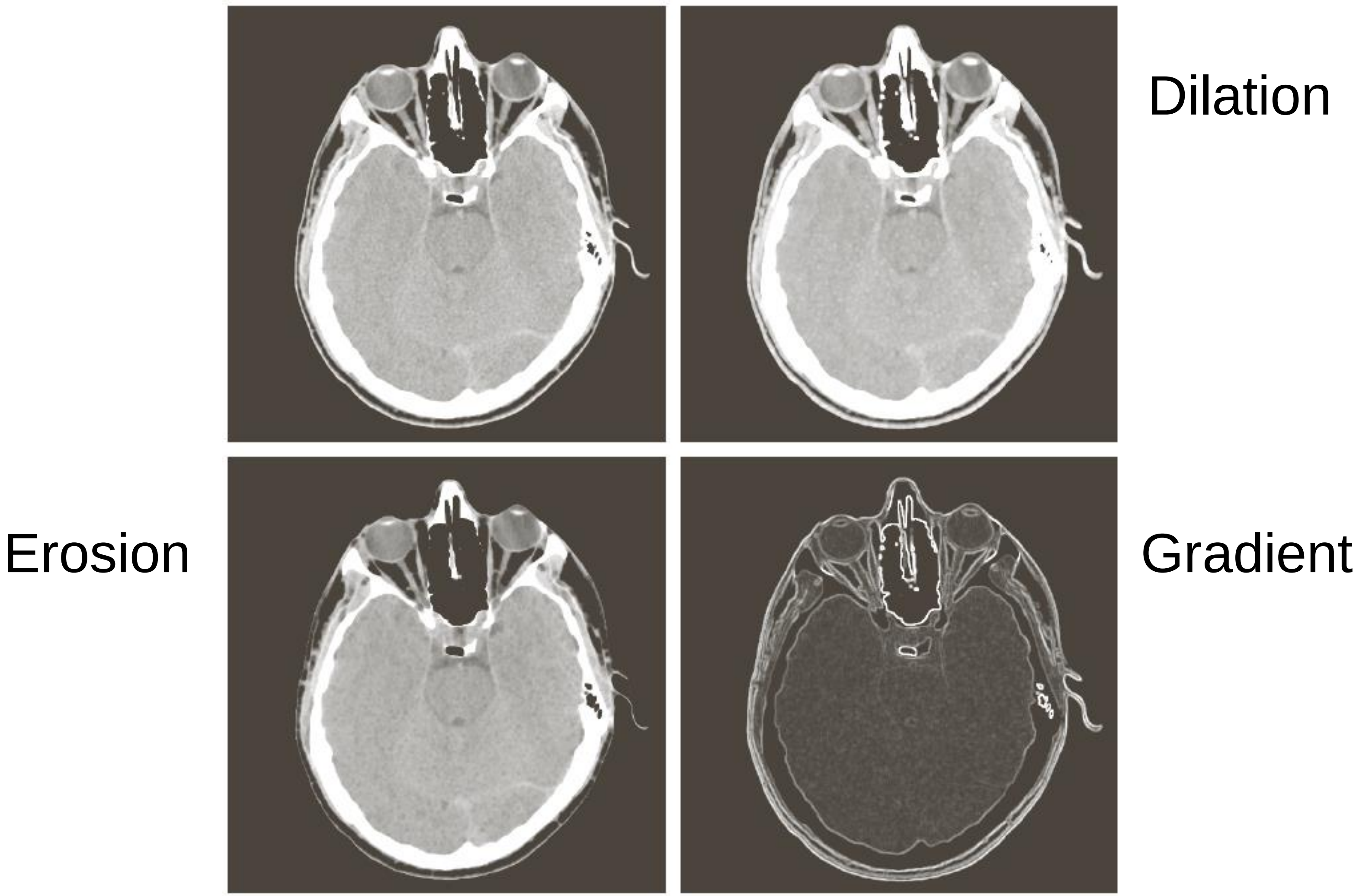


Gray-Scale Opening and Closing



Morphological Gradient

$$g = (f \oplus b) - (f \ominus b)$$



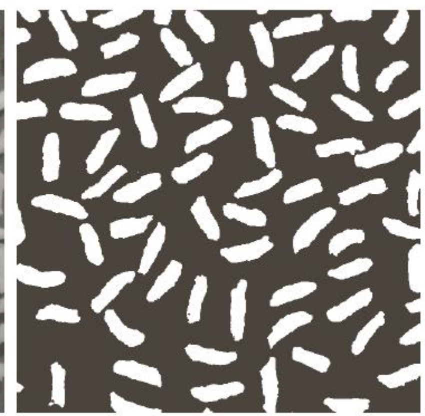
Top-Hat and Bottom-Hat

To keep only small objects (those smaller than the SE):

- Top-hat: Keep light objects: $T_{hat}(f) = f - (f \circ b)$
- Bottom-hat: Keep dark objects: $B_{hat}(f) = (f \bullet b) - f$

Thresholding

Opening



Top-hat and thresholding

Top-hat