Observing the State of a Linear System

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Summary—**In much of modern control theory designs are based on the assumption that the state vector of the system to be controlled is available for measurement. In many practical situations only a few output quantities are available. Application of theories which assume that the state vector is known is severely limited in these cases. In this paper it is shown that the state vector of a linear system can be reconstructed from observations of the system inputs and outputs.**

It is shown that the observer, which reconstructs the state vector, is itself a linear system whose complexity decreases as the number of output quantities available increases. The observer may be incorporated in the control of a system which does not have its state vector available for measurement. The observer supplies the state vector, but at the expense of adding poles to the over-all system.

I. Introduction

IN THE PAST few years there has been an increasing percentage of control system literature written from the "state variable" point of view [1]–[8]. In the case of a continuous, time-invariant linear system the state variable representation of the system is of the form:

$$\dot{y}(t) = Ay(t) + Bx(t),$$

where

$y(t)$ is an $(n \times 1)$ state vector
$x(t)$ is an $(m \times 1)$ input vector
A is an $(n \times n)$ transition matrix
B is an $(n \times m)$ distribution matrix.

This state variable representation has some conceptual advantages over the more conventional transfer function representation. The state vector $y(t)$ contains enough information to completely summarize the past behavior of the system, and the future behavior is governed by a simple first-order differential equation. The properties of the system are determined by the constant matrices A and B. Thus the study of the system can be carried out in the field of matrix theory which is not only well developed, but has many notational and conceptual advantages over other methods.

When faced with the problem of controlling a system, some scheme must be devised to choose the input vector $x(t)$ so that the system behaves in an acceptable manner. Since the state vector $y(t)$ contains all the essential information about the system, it is reasonable to base the choice of $x(t)$ solely on the values of $y(t)$ and perhaps also t. In other words, x is determined by a relation of the form $x(t) = F[y(t), t]$.

This is, in fact, the approach taken in a large portion of present day control system literature. Several new techniques have been developed to find the function F for special classes of control problems. These techniques include dynamic programming [8]–[10], Pontryagin's maximum principle [11], and methods based on Lyapunov's theory [2], [12].

In most control situations, however, the state vector is not available for direct measurement. This means that it is not possible to evaluate the function $F[y(t), t]$. In these cases either the method must be abandoned or a reasonable substitute for the state vector must be found.

In this paper it is shown how the available system inputs and outputs may be used to construct an estimate of the system state vector. The device which reconstructs the state vector is called an observer. The observer itself as a time-invariant linear system driven by the inputs and outputs of the system it observes.

Kalman [3], [13], [14] has done some work on this problem, primarily for sampled-data systems. He has treated both the nonrandom problem and the problem of estimating the state when measurements of the outputs are corrupted by noise. In this paper only the nonstatistical problem is discussed but for that case a fairly complete theory is developed.

It is shown that the time constants of an observer can be chosen arbitrarily and that the number of dynamic elements required by the observer decreases as more output measurements become available. The novel point of view taken in this paper leads to a simple conceptual understanding of the observer process.

II. Observation of a Free Dynamic System

As a first step toward the construction of an observer it is useful to consider a slightly more general problem. Instead of requiring that the observer reconstruct the state vector itself, require only that it reconstruct some constant linear transformation of the state vector. This problem is simpler than the previous problem and its solution provides a great deal of insight into the theory of observers.

Assuming it were possible to build a system which reconstructs some constant linear transformation T of the state vector y, it is clear that it would then be possible to reconstruct the state vector itself, provided that the transformation T were invertible. This is the approach taken in this paper. It is first shown that it is relatively simple to build a system which will reconstruct some linear transformation of the state vector and then it is shown how to guarantee that the transformation obtained is invertible.

The first result concerns systems which have no inputs. (Such systems are called free systems.) The situa-

Received November 2, 1963. This research was partially supported by a grant from Westinghouse Electric Corporation.
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tion which is investigated is illustrated in Fig. 1. The free system is used to drive another linear system with state vector z. In this situation it is nearly always true that z will be a constant linear transformation of the state vector of the free system.

Theorem 1 (*Observation of a Free System*): Let S_1 be a free system: $\dot{y}=Ay$, which drives S_2: $\dot{z}=Bz+Cy$. If A and B have no common eigenvalues, then there is a constant linear transformation T such that if $z(o)=Ty(o)$, then $z(t)=Ty(t)$ for all $t>0$. Or more generally,

$$z(t) = Ty(t) + e^{Bt}[z(o) - Ty(o)].$$

Proof: Notice that there is no need for A and B to be the same size; they only have to be square.

Suppose that such a transformation did exist; *i.e.*, suppose that for all t

$$z(t) = Ty(t). \tag{1}$$

The two systems are governed by

$$\begin{aligned} \dot{y} &= Ay, \\ \dot{z} &= Bz + Cy, \end{aligned} \tag{2}$$

but using the relation $z=Ty$,

$$\begin{aligned} T\dot{y} &= TAy, \\ T\dot{y} &= BTy + Cy. \end{aligned} \tag{3}$$

Now, since the left sides agree, so must the right sides. This implies that T satisfies

$$TA - BT = C. \tag{4}$$

Since A and B have no common eigenvalues, (4) will have a unique solution T, [15]. It will now be shown that T has the properties of the theorem. Using (3),

$$\dot{z} - T\dot{y} = Bz - TAy + Cy. \tag{5}$$

By using (4), this becomes

$$\dot{z} - T\dot{y} = B(z - Ty). \tag{6}$$

This is a simple first-order differential equation in the variable $z-Ty$. It has the well-known solution

$$z(t) = Ty(t) + e^{Bt}[z(o) - Ty(o)], \tag{7}$$

which proves the theorem.

The result of Theorem 1 may be easily interpreted in terms of familiar linear system theory concepts. As a simple example, consider the situation described by Fig. 2. Here both S_1 and S_2 are first order systems. It is clear from the figure that $y(t)=y(o)e^{\lambda t}$ and that $\alpha y(o)e^{\lambda t}$ is the signal which drives S_2. By elementary transform theory it may be verified that

$$z(t) = \frac{y(o)}{\lambda - \mu} e^{\lambda t} + e^{\mu t}\left[z(o) - \frac{\alpha}{\lambda - \mu} y(o)\right]. \tag{8}$$

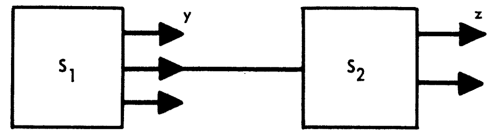

Fig. 1—A simple observer.

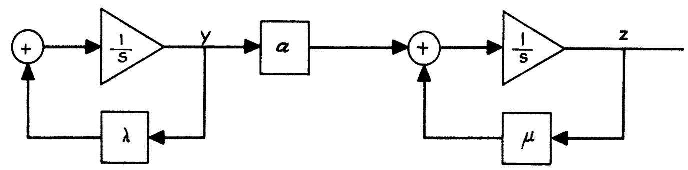

Fig. 2—Observation of a first-order system.

So, if the initial condition on $z(o)$ is chosen as

$$z(o) = \frac{\alpha}{\lambda - \mu} y(o), \tag{9}$$

then for all $t>0$,

$$z(t) = \frac{\alpha}{\lambda - \mu} y(t), \tag{10}$$

which is just a constant multiple of y. This type of reasoning may be easily extended to higher-order systems.

The results of Theorem 1 would be of little practical value if they could not be extended to include nonfree systems. Fortunately, this extension is relatively straightforward.

Assume, now, that the plant or system, S_1, that is to be observed is governed by

$$\dot{y} = Ay + Dx, \tag{11}$$

where x is an input vector. As before, an observer for this system will be driven by the state vector y. In addition, it is natural to expect that the observer must also be driven by the input vector x. Consider the system S_2 governed by

$$\dot{z} = Bz + Cy + Gx. \tag{12}$$

As before, let T satisfy $TA-BT=C$. Then, it follows that

$$\dot{z} - T\dot{y} = Bz - TAy + Cy + (G - TD)x, \tag{13}$$

or, using (4)

$$\dot{z} - T\dot{y} = B(z - Ty) + (G - TD)x. \tag{14}$$

By choosing $G=TD$ the differential equation above can be easily integrated giving

$$z(t) = Ty(t) + e^{Bt}[z(o) - Ty(o)]. \tag{15}$$

This shows that the results for free systems contained in Theorem 1 will also apply to nonfree systems provided that the input drive satisfies

$$G = TD. \tag{16}$$

This is what one might intuitively expect. The system which produces Ty is driven with an input just equal to T times the input used to drive the system which produces y.

In applications, then, an observer can be designed for a system by assuming that the system is free; then an additional input drive can be added to the observer in order to satisfy (16). For this reason it is possible to continue to develop the theory and design techniques for free systems only.

III. Observation of the Entire State Vector

It was shown in the last section that "almost" any linear system will follow a free system which is driving it. In fact, the state vectors of the two systems will be related by a constant linear transformation. The question which naturally arises now is: How does one guarantee that the transformation obtained will be invertible?

One way to insure that the transformation will be invertible is to force it to be the identity transformation. This requirement guarantees that (after the initial transient) the state vector of the observer will equal the state vector of the plant.

In the notation used here, vectors such as a are commonly column vectors, whereas row vectors are represented as transposes of column vectors, such as a'.

Assume that the plant has a single output v

$$\begin{aligned} \dot{y} &= Ay \\ v &= a'y \end{aligned} \tag{17}$$

and that the corresponding observer is driven by v as its only input

$$\dot{z} = Bz + bv \tag{18}$$

or

$$\dot{z} = Bz + ba'y. \tag{19}$$

under these conditions $z = Ty$ where T satisfies

$$TA - BT = ba'. \tag{20}$$

Forcing $T = I$ gives

$$B = A - ba' \tag{21}$$

which prescribes the observer in this case. In (21) A and a' are given as part of the plant, hence, chosing a vector b will prescribe B and the observer will be obtained.

This solution to the observer problem is illustrated in Fig. 3 and is the solution obtained by Kalman [3] using other methods. For a sampled-data system, he determined the vector b so that the transient would die out in minimum time. In the continuous case, presumably, the vector b would be chosen to make the transient die out quickly.

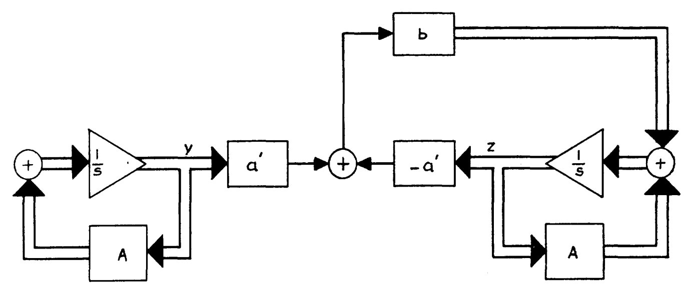

Fig. 3—Observation of the entire state vector.

IV. Reduction of Dynamic Order

The observer constructed above by requiring $T = I$ possesses a certain degree of mathematical simplicity. The state vector of the observer is equal to the state vector of the system being observed. Further examination will reveal a certain degree of redundancy in the observer system. The observer constructs the entire state vector when the output of the plant, which represents part of the state vector, is available by direct measurement.

Intuitively, it seems that this redundancy might be eliminated, thereby giving a simpler observer system. In this section it is shown that this redundancy can be eliminated by reducing the dynamic order of the observer. It is possible, however, to choose the pole locations of the observer in a fairly arbitrary manner.

The results of this section rely heavily on the concepts of controllability and observability introduced by Kalman [3] and on properties of the matrix equation $TA - BT = C$. Some new properties of this equation are developed but first a motivation for these results is given in the form of a rough sketch of the method that is used to reduce the dynamic order of the observer.

Consider the problem of building an observer for an nth-order system S_1 which has only one output. Let this system drive an $(n-1)$th-order system S_2. Then by the results of Section II each state variable of S_2 is a time-invariant, linear combination of the state variables of S_1. Thus, the $n-1$ state variables of S_2 together with the output of S_1 give n quantities each of which is a linear combination of the state variables of S_1. If these different combinations are linearly independent it is possible to find the state variables of S_1 by simple matrix (no dynamics) operations. The scheme is illustrated in Fig. 4.

Another way to describe the method is in terms of matrix inversion. The state vector of S_2 is given by $z = Ty$; but z has only $n-1$ components while y has n components. This means that T is an $(n-1) \times n$ matrix and so it cannot be inverted. However, if another component that is a linear combination of the components of y is adjoined to the z vector, one obtains an n-dimensional vector $z_1 = T_1 y$, where T_1 is now an $n \times n$ matrix which may possess an inverse. The component adjoined to the z vector in the scheme of Fig. 4 is the output of S_1.

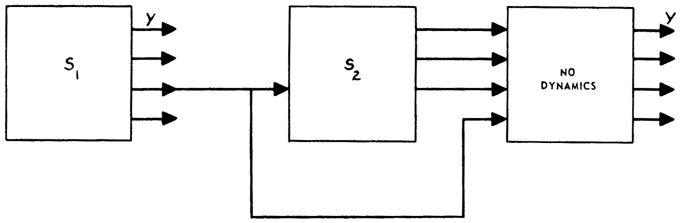

Fig. 4—Reduction of the dynamic order.

It is appropriate at this point to review the definitions of controllability and observability for linear time-invariant systems. A discussion of the physical interpretations of these definitions can be found in [3] and [16].

Definition: The nth-order system $\dot{y}=Ay+Bx$ is said to be completely controllable if the collection of vectors

$$A^k b_i \qquad k = 0, 1, 2, \cdots, n-1$$
$$i = 1, 2, \cdots, m$$

spans n dimensions. (The b_i are the m columns of the $n \times m$ matrix B.)

As a notational convenience this situation will sometimes be described by writing "(A, B) is completely controllable."

Definition: The system $\dot{y}=Ay$ with output vector $v=B'y$ is said to be completely observable if (A', B) is completely controllable. As a notational convenience, this situation will sometimes be described by writing "(A, B') is completely observable."

In the special case that A is diagonal with distinct eigenvalues and B is just a column vector there is a simple condition which is equivalent to complete controllability [16].

Lemma 1: Let A be diagonal with distinct roots. Then (A, b) is completely controllable if, and only if, each component b_i is nonzero.

The following theorem which is proved in Appendix I connects complete controllability and complete observability with the matrix equation $TA-BT=C$.

Theorem 2: Let A and B be $n \times n$ matrices with no common eigenvalues. Let a and b be vectors such that (A, a') is completely observable and (B, b) is completely controllable. Let T be the unique solution of $TA-BT=ba'$. Then T is invertible.

With this Theorem it is easy to derive a result concerning the dynamic order of an observer for a single output system.

Theorem 3: Let S_1: $\dot{y}=Ay$, $v=a'y$ be an nth-order completely observable system. Let $\mu_1, \mu_2, \cdots, \mu_n$ be a set of distinct complex constants distinct from the eigenvalues of A. An observer can be built for S_1 which is of $(n-1)$th-order and which has $n-1$ of the μ_i's as eigenvalues of its transition matrix.

Proof: As a first attempt let S_1 drive the nth-order system

$$\dot{z} = \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} v \tag{22}$$

where the μ_i are arbitrary except that $\mu_i \neq \mu_j$ for $i \neq j$ and $\mu_i \neq \lambda_k$ for all i and k. Now (under proper initial conditions) $z=Ty$ and by Theorem 2 the n rows t_i of T are independent. It is clear that there is one t_i which may be replaced by a', so that the (row) vectors $t_1, t_2, \cdots, t_{i-1}, \cdots, t_n, a'$ will be independent. (If this is not clear see Lemma 2 in Appendix II.)

By removing the ith dynamic element from the observer, an $(n-1)$th-degree system (with state vector z_1) is obtained. The state vector y may be recovered from the $n-1$ components of z_1 and the output $a'y$ since

$$\begin{bmatrix} z_1 \\ v \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_{i-1} \\ t_{i+1} \\ \cdot \\ \cdot \\ t_n \\ a' \end{bmatrix} y = Y_1 y \tag{23}$$

and the matrix on the right is invertible. This proves Theorem 3.

Note: By employing here the methods used in Appendix I in the proof of Theorem 2, it can be shown that the $n-1$ eigenvalues of the observer can, in fact, be chosen arbitrarily provided only that they are distinct from those of A.

At this point it is natural to ask whether these results can be extended to systems with more than one output. Theorem 4, which is proved in Appendix II, states that an nth-order system with m independent outputs can be observed with $n-m$ "arbitrary" dynamic elements.

Theorem 4: Let S_1 be a completely observable nth-order system with m independent outputs. Then an observer, S_2, may be built for S_1 using only $n-m$ dynamic elements. (As illustrated by the proof, the eigenvalues of the observer are essentially arbitrary.)

In order to illustrate the results obtained in this section, consider the system shown in Fig. 5. It may be expressed in matrix form as

$$\dot{y} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x. \tag{24}$$

It will be assumed that y_1 is the only measurable output.

To build an observer for this system observer eigenvalues must be chosen. According to Theorem 3, an observer can be constructed for this system using a single dynamic element. Suppose it is decided to require the observer to have -3 as its eigenvalue. The observer will have a single state variable z and will be driven by y_1 and x. The state variable z will satisfy

$$\dot{z} = -3z + [\ 1 \quad 0\ ]y + kx, \tag{25}$$

where k is determined by the input relation given by (16).[1] Then $z = Ty$, where T satisfies

$$TA + 3T = [\ 1 \quad 0\ ]. \tag{26}$$

This equation is easily solved giving $T = [1 - 1/2]$. To evaluate k (16) is used,

$$k = [\ 1 \ -1/2\ ] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1/2. \tag{27}$$

It is easy to see how to combine y_1 and z to produce y_2. The final system is shown in Fig. 6. In the figure, $\hat{y}_2$ represents the observer's estimate of y_2.

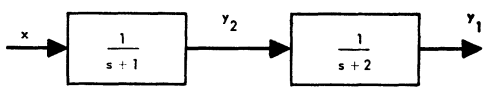

Fig. 5—A second-order plant.

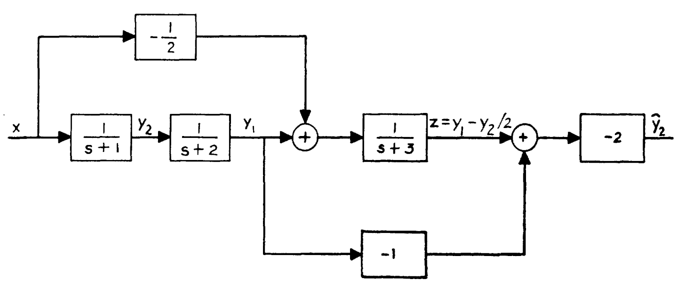

Fig. 6—Observer and plant.

V. Application to Control Problems

The primary justification for an investigation of observers is its eventual application to control system design. A control system can be regarded as performing three operations: it measures certain plant inputs and outputs; on the basis of these measurements it computes certain other input signals and it applies these input signals to the plant. The philosophy behind this paper is that the computational aspect of control should be divided into two parts. First, the state vector should be constructed; this is the job of the observer. Then, the inputs can be calculated using this value for the state vector.

A primary consideration that arises when this philosophy is used is the extent that use of the estimated state vector, rather than the actual state vector, deteriorates the performance of control. Various criteria can be used to measure this deterioration. One of the most important considerations is the effect that an observer may have on pole locations. It would be undesirable, for example, if an otherwise stable control design became unstable when an observer is used to realize it. It is shown in this section that an observer has no effect on a control system's closed-loop pole locations other than to add the poles of the observer itself.

Consider a linear plant: $\dot{y} = Ay + Dx$, which has all of its state variables measurable and all of its input components available for control. It is then possible to design a linear feedback system by putting $x = Fy$. This is a feedback system without dynamics. The closed-loop plant would be governed by $\dot{y} = (A + DF)y$, so that the eigenvalues of $A + DF$ are the closed-loop poles of the system.

Suppose the same plant is given, except that not all state variables are measurable. In this case, an observer for the plant might be used to construct an estimate, $\hat{y}$, of the plant state vector. The vector $\hat{y}$ could then be used to put $x = F\hat{y}$. The closed-loop poles of this system can be found in terms of the poles of the observer and the poles of the system above. Suppose the observer is governed by $\dot{z} = Bz + Cy + TDx$, where $TA - BT = C$. Then $\hat{y}$ is a linear combination of y and z,

$$\hat{y} = Hy + Kz, \tag{28}$$

where

$$H + KT = I. \tag{29}$$

Putting $x = F\hat{y}$, the over-all system becomes

$$\begin{aligned} \dot{y} &= Ay + DF(Hy + Kz), \\ \dot{z} &= Bz + Cy + TDF(Hy + Kz), \end{aligned} \tag{30}$$

or, in matrix form,

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + DFH & DFK \\ C + TDFH & B + TDFK \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{31}$$

Theorem 5: The eigenvalues of the over-all system (31) are the eigenvalues of $A + DF$ and the eigenvalues of B.

Proof: For an eigenvalue λ,

$$Ay + DFHy + DFKz = \lambda y, \tag{32}$$

$$Cy + TDFHy + Bz + TDFKz = \lambda z. \tag{33}$$

Multiplying (32) by T and subtracting (33),

$$(TA - C)y - Bz = \lambda(Ty - z). \tag{34}$$

Using $TA - BT$ this becomes

$$B(Ty - z) = \lambda(Ty - z). \tag{35}$$

This equation can be satisfied if λ is an eigenvalue of B or if $Ty = z$. This shows that all eigenvalues of B (including multiplicity) are eigenvalues of the over-all system (31).

[1] Here $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to D, and k corresponds to G in (16).

Now if $Ty=z$ (32) becomes

$$(A + DFH + DFKT)y = \lambda y, \tag{36}$$

but using (29) this reduces to

$$(A + DF)y = \lambda y. \tag{37}$$

This equation immediately shows that all eigenvalues of $A+DF$ (including multiplicity) are also eigenvalues of the over-all system (31). This proves the theorem.

Theorem 5 demonstrates that as far as pole location problems are concerned it is possible to design a feedback system assuming the state were available and then add an observer to construct the state. There is still the problem of what feedback coefficients to use if the state were available.

For a single input system it is possible to find feedback coefficients to place the closed-loop poles anywhere in the complex plane. This result can be obtained from a canonical form given by Kalman [17], or by a simple application of Theorem 2.

Theorem 6: Given a completely controllable, single input system: $\dot{y}=Ay+bx$, and a set of complex constants $\mu_1, \mu_2, \cdots, \mu_n$; there is a vector c such that if $x=c'y$ the resulting closed-loop system will have $\mu_1, \mu_2, \cdots, \mu_n$ as its eigenvalues.

Proof: First assume that each μ_i is distinct from the eigenvalues of A. Let B be a matrix in Jordan form which has the μ_1 as its eigenvalues and has only one Jordan block associated with each distinct eigenvalue [3]. Let c_1 be any vector such that (B, c_1') is completely observable. By Theorem 2 the equation

$$TB - AT = bc_1' \tag{38}$$

has a unique solution T which is invertible. Let $c' = c_1'T^{-1}$ then

$$A + bc' = TBT^{-1} \tag{39}$$

which says $A+bc'$ is similar to B. This establishes the result.

In case some of the μ_i are not distinct from the eigenvalues of A proceed in two steps. First, choose coefficients to make the eigenvalues distinct from those of A and from the μ_i. Then move the eigenvalues of the resulting system to the μ_i. This proves Theorem 6.

Finally, the results of Theorems 4–6 may be collected to obtain a result for systems that do not have their state vector available. Suppose one is given an nth-order system with m independent outputs. According to Theorem 4, an observer can be designed which has $n-m$ essentially arbitrary eigenvalues. If the state vector were available, constant feedback coefficients could be found to place the closed-loop eigenvalues arbitrarily by the method of Theorem 6. Then, according to Theorem 5, if the observer's estimate of the state is used in place of the actual state the resulting system will have the eigenvalues of the observer and the eigenvalues of the constant coefficient feedback system. This result is expressed in Theorem 7.

Theorem 7: Let S be an nth-order, single input, completely controllable, completely observable system with m independent outputs. Then a feedback network can be designed for S which is $(n-m)$th-order and the resulting $2n-m$ poles of the closed-loop system are essentially arbitrary.

VI. Conclusions

It has been shown that the state vector of a linear system can be reconstructed from observations of its inputs and outputs. The observer which performs the reconstruction is itself a linear system with arbitrary time constants. It has been shown that the dynamic order of an observer which observes an nth-order system with m outputs is $n-m$. Hence, when more outputs are available a simpler observer may be constructed.

Observers may be incorporated in the design of control systems. If a feedback system has been designed based on knowledge of the state, then incorporation of an observer to construct the state does not change the pole locations of the system. The observer simply adds its own poles to the system. Much work remains, however, in the area of incorporation of observers in control system design. The effects of parameter variations, use of design criteria other than pole location and consideration of systems which are "marginally" observable should be investigated.

Most of the results given can be easily extended to include sampled-data systems. The necessary proofs are in fact often simpler in the sampled case. Likewise, many of the results can be extended to include time-varying linear systems.

Appendix I

Theorem 2: Let A and B be $n\times n$ matrices with no common eigenvalues. Let a and b be vectors such that (A, a') is completely observable and (B, b) is completely controllable. Let T be the unique solution of $TA-BT = ba'$. Then T is invertible.

Proof: Without loss of generality it may be assumed that A is in Jordan Canonical Form [13], [18]. A will consist of several Jordan blocks but since (A, a') is completely observable no two blocks are associated with the same eigenvalue [3]. Furthermore, the component of the vector a which corresponds to the top of a Jordan block must be nonzero [19], [20]. Partition the matrix T into columns

$$T = [t_1 | t_2 | \cdots | t_n].$$

Then if a particular Jordan block with eigenvalue λ is located with its top row in the kth row of A and extends to row $k+q$ it is possible to express the corresponding columns of T as

$$\begin{aligned} t_k &= a_k(\lambda I - B)^{-1}b, \\ t_i &= (\lambda I - B)^{-1}(a_ib - t_{i-1}) \quad k < i \leqq k + q - 1. \end{aligned} \tag{40}$$

Hence, the vectors t_i will be linearly dependent only if for some set of α_{il} not all zero

$$\sum_i \sum_{l=1}^{q_i} \alpha_{il}(I\lambda_i - B)^{-l}b = 0.$$

This equation can be multiplied by the nonsingular matrix

$$\prod_i (I\lambda_i - B)^{q_i}$$

to obtain

$$P(B)b = 0,$$

where P is a polynomial of degree $n-1$ or less. But unless $P=0$, which implies that each α_{il} is zero, this condition contradicts the complete controllability of (B, b). Hence, the vectors t_i must be linearly independent.

Appendix II

In order to prove the general statement concerning the dynamic order of an observer the following well-known lemma [21] will be used.

Lemma 2: Let $x_1, x_2, \cdots, x_n$ be n linearly independent vectors in an n-dimensional space. Let $y_1, y_2, \cdots, y_m$ also be independent. Then there are $n-m$ x_i's such that $y_1, y_2, \cdots, y_m, x_{i_1}, x_{i_2}, \cdots, x_{i_{n-m}}$ are independent.

Theorem 4: Let S_1 be a completely observable nth-order system with m independent outputs. Then an observer, S_2, may be built for S_1 using only $n-m$ dynamic elements. (As illustrated by the proof, the eigenvalues of the observer are essentially arbitrary.)

Proof: Let the m outputs be given by $a_1'y, a_2'y, \cdots, a_m'y$. Then since S_1 is completely observable the collection of vectors

$$(A')^i a_j \qquad i = 0, 1, 2, \cdots, n-1$$
$$j = 1, 2, \cdots, m$$

spans n dimensions.

Let p be the order of the minimal polynomial of A. To each output of S_1 connect a completely controllable pth-order system with distinct eigenvalues. Consider the system driven by $a_1'y$. The p state variables of this system (under proper initial conditions) are

$$z_i = [b_i(A' - u_i)^{-1}a_1]'y \tag{41}$$

where a diagonal form for this system has been assumed. Lemma 1 guarantees that each b_i is not zero.

It will be shown that the vectors $(A'-u_iI)^{-1}a_1$, $i=1, 2, \cdots, p$, generate the same space as the vectors $(A')^k a_1$, $k=1, 2, \cdots, n$. Assume that we can find α_i's such that

$$\sum_{i=1}^{p} \alpha_i(A' - u_iI) = 0. \tag{42}$$

This can be rewritten as

$$P(A') \prod_{i=1} (A' - u_iI)^{-1} = 0 \tag{43}$$

where P is a polynomial of degree less than p. But since each $(A'-u_iI)^{-1}$ is nonsingular this implies that

$$P(A') = 0. \tag{44}$$

Since this polynomial has a degree less than the minimal polynomial, each $\alpha_i=0$ in the original combination (42). This implies that any polynomial in A' can be written as a linear combination of the $(A'-u_iI)^{-1}$. In particular, the vectors $(A'-u_iI)^{-1}a_1$, $i=1, 2, \cdots, P$, generate the same space as the vectors $(A')^k a_1$, $k=1, 2, \cdots, n$.

This same argument applies to each a_i. Hence, the output vectors from all observing systems span n dimensions. Now, from Lemma 2 n dimensions can be spanned with m output vectors $a_1, a_2, \cdots, a_m$ and $n-m$ dynamics. This proves Theorem 4.

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