Chapter 20

The Small-World Phenomenon

20.1 Six Degrees of Separation

In the previous chapter, we considered how social networks can serve as conduits by which ideas and innovations flow through groups of people. To develop this idea more fully, we now relate it to another basic structural issue — the fact that these groups can be connected by very short paths through the social network. When people try to use these short paths to reach others who are socially distant, they are engaging in a kind of "focused" search that is much more targeted than the broad spreading pattern exhibited by the diffusion of information or a new behavior. Understanding the relationship between targeted search and wide-ranging diffusion is important in thinking more generally about the way things flow through social networks.

As we saw in Chapter 2, the fact that social networks are so rich in short paths is known as the *small-world phenomenon*, or the "six degrees of separation," and it has long been the subject of both anecdotal and scientific fascination. To briefly recapitulate what we discussed in that earlier chapter, the first significant empirical study of the small-world phenomenon was undertaken by the social psychologist Stanley Milgram [297, 391], who asked randomly chosen "starter" individuals to each try forwarding a letter to a designated "target" person living in the town of Sharon, MA, a suburb of Boston. He provided the target's name, address, occupation, and some personal information, but stipulated that the participants could not mail the letter directly to the target; rather, each participant could only advance the letter by forwarding it to a single acquaintance that he or she knew on a first-name basis, with the goal of reaching the target as rapidly as possible. Roughly a third of the letters eventually arrived at the target, in a median of six steps, and this has since served as

D. Easley and J. Kleinberg. *Networks, Crowds, and Markets: Reasoning about a Highly Connected World.* Cambridge University Press, 2010. Draft version: June 10, 2010.

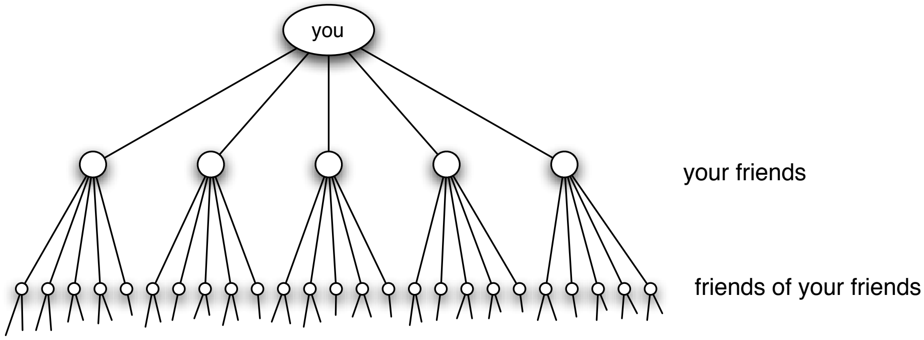
basic experimental evidence for the existence of short paths in the global friendship network, linking all (or almost all) of us together in society. This style of experiment, constructing paths through social networks to distant target people, has been repeated by a number of other groups in subsequent decades [131, 178, 257].

Milgram's experiment really demonstrated two striking facts about large social networks: first, that short paths are there in abundance; and second, that people, acting without any sort of global "map" of the network, are effective at collectively finding these short paths. It is easy to imagine a social network where the first of these is true but the second isn't — a world where the short paths are there, but where a letter forwarded from thousands of miles away might simply wander from one acquaintance to another, lost in a maze of social connections [248]. A large social-networking site where everyone was known only by 9-digit pseudonyms would be like this: if you were told, "Forward this letter to user number 482285204, using only people you know on a first-name basis," the task would clearly be hopeless. The real global friendship network contains enough clues about how people fit together in larger structures — both geographic and social — to allow the process of search to focus in on distant targets. Indeed, when Killworth and Bernard performed follow-up work on the Milgram experiment, studying the strategies that people employ for choosing how to forward a message toward a target, they found a mixture of primarily geographic and occupational features being used, with different features being favored depending on the characteristics of the target in relation to the sender [243].

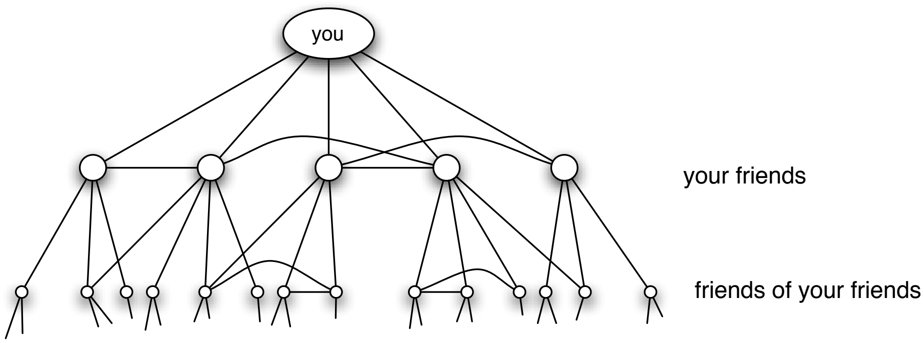
We begin by developing models for both of these principles — the existence of short paths and also the fact that they can be found. We then look at how some of these models are borne out to a surprising extent on large-scale social-network data. Finally, in Section 20.6, we look at some of the fragility of the small-world phenomenon, and the caveats that must be considered in thinking about it: particularly the fact that people are most successful at finding paths when the target is high-status and socially accessible [255]. The picture implied by these difficulties raises interesting additional points about the global structure of social networks, and suggests questions for further research.

20.2 Structure and Randomness

Let's start with models for the existence of short paths: Should we be surprised by the fact that the paths between seemingly arbitrary pairs of people are so short? Figure 20.1(a) illustrates a basic argument suggesting that short paths are at least compatible with intuition. Suppose each of us knows more than 100 other people on a first-name basis (in fact, for most people, the number is significantly larger). Then, taking into account the fact that each of your friends has at least 100 friends other than you, you could in principle be two steps away from over $100 \cdot 100 = 10,000$ people. Taking into account the 100 friends of these



(a) *Pure exponential growth produces a small world*



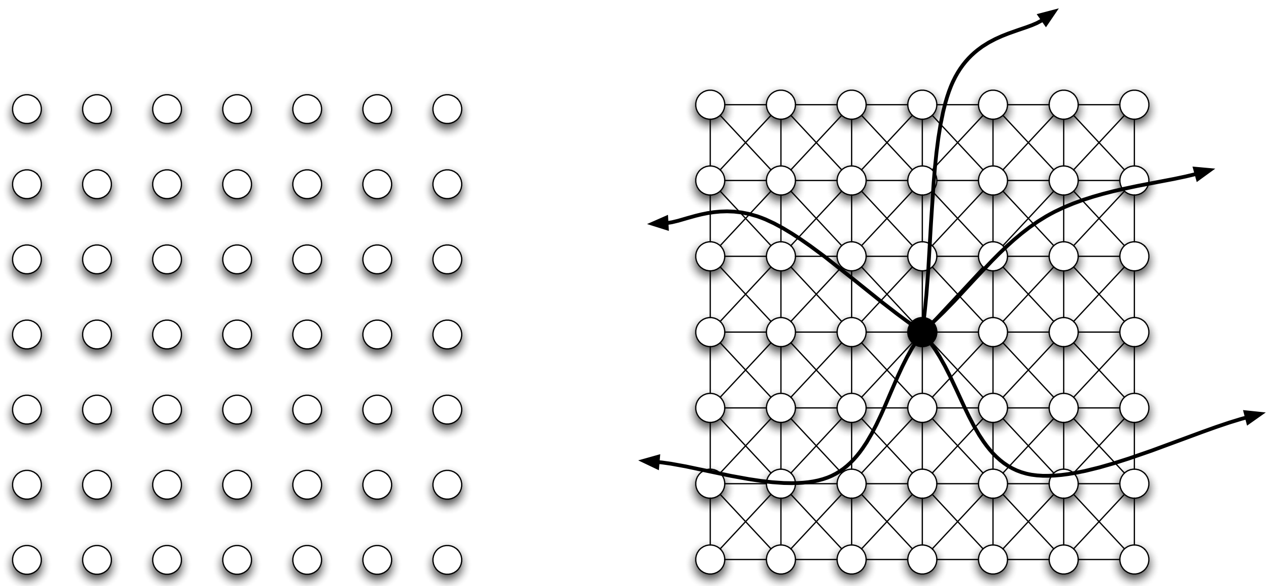
(b) *Triadic closure reduces the growth rate*

Figure 20.1: Social networks expand to reach many people in only a few steps.

people brings us to more than $100 \cdot 100 \cdot 100 = 1,000,000$ people who in principle could be three steps away. In other words, the numbers are growing by powers of 100 with each step, bringing us to 100 million after four steps, and 10 billion after five steps.

There's nothing mathematically wrong with this reasoning, but it's not clear how much it tells us about real social networks. The difficulty already manifests itself with the second step, where we conclude that there may be more than $10,000$ people within two steps of you. As we've seen, social networks abound in triangles — sets of three people who mutually know each other — and in particular, many of your 100 friends will know each other. As a result, when we think about the nodes you can reach by following edges from your friends, many of these edges go from one friend to another, not to the rest of world, as illustrated schematically in Figure 20.1(b). The number $10,000$ came from assuming that each of your 100 friends was linked to 100 *new* people; and without this, the number of friends you could reach in two steps could be much smaller.

So the effect of triadic closure in social networks works to limit the number of people you can reach by following short paths, as shown by the contrast between Figures 20.1(a)



(a) *Nodes arranged in a grid* (b) *A network built from local structure and random edges*

Figure 20.2: The Watts-Strogatz model arises from a highly clustered network (such as the grid), with a small number of random links added in.

and 20.1(b). And, indeed, at an implicit level, this is a large part of what makes the small-world phenomenon surprising to many people when they first hear it: the social network appears from the local perspective of any one individual to be highly clustered, not the kind of massively branching structure that would more obviously reach many nodes along very short paths.

The Watts-Strogatz model. Can we make up a simple model that exhibits both of the features we've been discussing: many closed triads, but also very short paths? In 1998, Duncan Watts and Steve Strogatz argued [411] that such a model follows naturally from a combination of two basic social-network ideas that we saw in Chapters 3 and 4: homophily (the principle that we connect to others who are like ourselves) and weak ties (the links to acquaintances that connect us to parts of the network that would otherwise be far away). Homophily creates many triangles, while the weak ties still produce the kind of widely branching structure that reaches many nodes in a few steps.

Watts and Strogatz made this proposal concrete in a very simple model that generates random networks with the desired properties. Paraphrasing their original formulation slightly (but keeping the main idea intact), let's suppose that everyone lives on a two-dimensional grid — we can imagine the grid as a model of geographic proximity, or potentially some more abstract kind of social proximity, but in any case a notion of similarity that guides the formation of links. Figure 20.2(a) shows the set of nodes arranged on a grid; we say that

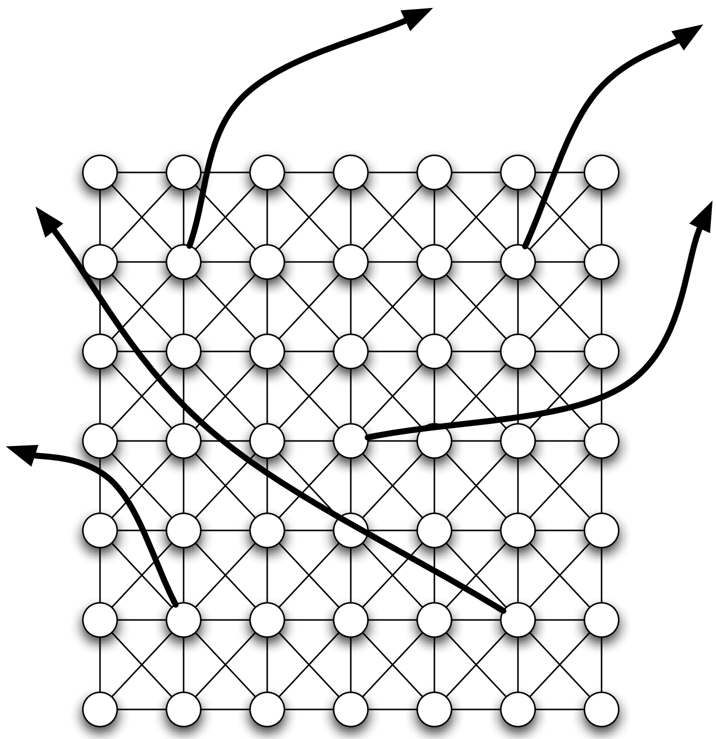


Figure 20.3: The general conclusions of the Watts-Strogatz model still follow even if only a small fraction of the nodes on the grid each have a *single* random link.

two nodes are one *grid step* apart if they are directly adjacent to each other in either the horizontal or vertical direction.

We now create a network by giving each node two kinds of links: those explainable purely by homophily, and those that constitute weak ties. Homophily is captured by having each node form a link to all other nodes that lie within a radius of up to r grid steps away, for some constant value of r: these are the links you form to people because you are similar to them. Then, for some other constant value k, each node also forms a link to k other nodes selected uniformly at random from the grid — these correspond to weak ties, connecting nodes who lie very far apart on the grid.

Figure 20.2(b) gives a schematic picture of the resulting network — a hybrid structure consisting of a small amount of randomness (the weak ties) sprinkled onto an underlying structured pattern (the homophilous links). Watts and Strogatz observe first that the network has many triangles: any two neighboring nodes (or nearby nodes) will have many common friends, where their neighborhoods of radius r overlap, and this produces many triangles. But they also find that there are — with high probability — very short paths connecting every pair of nodes in the network. Roughly, the argument is as follows. Suppose we start tracing paths outward from a starting node v, using only the k random weak ties out of each node. Since these edges link to nodes chosen uniformly at random, we are very unlikely to ever see a node twice in the first few steps outward from v. As a result, these first few steps look almost like the picture in Figure 20.1(a), when there was no triadic closure, and so a huge number of nodes are reached in a small number of steps. A mathematically precise version of this argument was carried out by Bollobás and Chung [67], who determined the typical lengths of paths that it implies.

Once we understand how this type of hybrid network leads to short paths, we in fact find that a surprisingly small amount of randomness is needed to achieve the same qualitative effect. Suppose, for example, that instead of allowing each node to have k random friends, we only allow one out of every k nodes to have a *single* random friend — keeping the proximity-based edges as before, as illustrated schematically in Figure 20.3. Loosely speaking, we can think of this model with fewer random friends as corresponding to a technologically earlier time, when most people only knew their near neighbors, and a few people knew someone far away. Even this network will have short paths between all pairs of nodes. To see why, suppose that we conceptually group $k \times k$ subsquares of the grid into "towns." Now, consider the small-world phenomenon at the level of towns. Each town contains approximately k people who each have a random friend, and so the town collectively has k links to other towns selected uniformly at random. So this is just like the previous model, except that towns are now playing the role of individual nodes — and so we can find short paths between any pair of towns. But now to find a short path between any two people, we first find a short path between the two towns they inhabit, and then use the proximity-based edges to turn this into an actual path in the network on individual people.

This, then, is the crux of the Watts-Strogatz model: introducing a tiny amount of randomness — in the form of long-range weak ties — is enough to make the world "small," with short paths between every pair of nodes.

20.3 Decentralized Search

Let's now consider the second basic aspect of the Milgram small-world experiment — the fact that people were actually able to collectively find short paths to the designated target. This novel kind of "social search" task was a necessary consequence of the way Milgram formulated the experiment for his participants. To really find the *shortest* path from a starting person to the target, one would have to instruct the starter to forward a letter to *all* of his or her friends, who in turn should have forwarded the letter to all of their friends, and so forth. This "flooding" of the network would have reached the target as rapidly as possible — it is essentially the *breadth-first search* procedure from Chapter 2 — but for obvious reasons, such an experiment was not a feasible option. As a result, Milgram was forced to embark

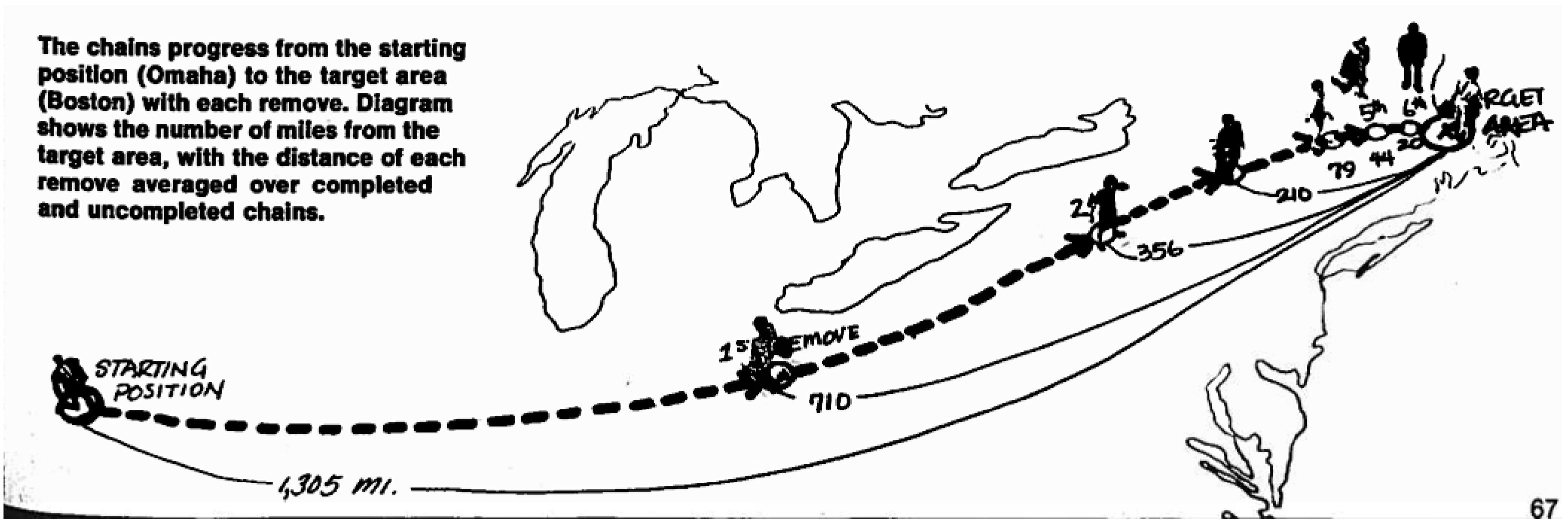
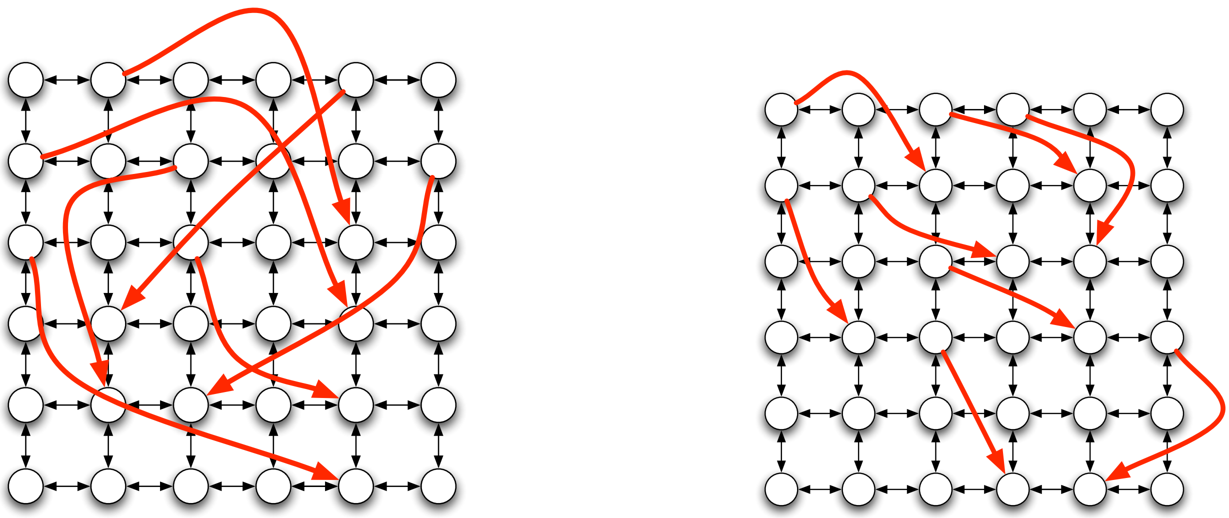


Figure 20.4: An image from Milgram's original article in *Psychology Today*, showing a "composite" of the successful paths converging on the target person. Each intermediate step is positioned at the average distance of all chains that completed that number of steps. (Image from [297].)

on the much more interesting experiment of constructing paths by "tunneling" through the network, with the letter advancing just one person at a time — a process that could well have failed to reach the target, even if a short path existed.

So the success of the experiment raises fundamental questions about the power of collective search: even if we posit that the social network contains short paths, why should it have been structured so as to make this type of *decentralized search* so effective? Clearly the network contained some type of "gradient" that helped participants guide messages toward the target. As with the Watts-Strogatz model, which sought to provide a simple framework for thinking about short paths in highly clustered networks, this type of search is also something we can try to model: can we construct a random network in which decentralized routing succeeds, and if so, what are the qualitative properties that are crucial for success?

A model for decentralized search. To begin with, it is not difficult to model the kind of decentralized search that was taking place in the Milgram experiment. Starting with the grid-based model of Watts and Strogatz, we suppose that a starting node s is given a message that it must forward to a target node t, passing it along edges of the network. Initially s only knows the location of t on the grid, but, crucially, it does not know the random edges out of any node other than itself. Each intermediate node along the path has this partial information as well, and it must choose which of its neighbors to send the message to next. These choices amount to a collective procedure for finding a path from s to t — just as the participants in the Milgram experiment collectively constructed paths to the target person.



(a) *A small clustering exponent*

(b) *A large clustering exponent*

Figure 20.5: With a small clustering exponent, the random edges tend to span long distances on the grid; as the clustering exponent increases, the random edges become shorter.

We will evaluate different search procedures according to their *delivery time* — the expected number of steps required to reach the target, over a randomly generated set of long-range contacts, and randomly chosen starting and target nodes.

Unfortunately, given this set-up, one can prove that decentralized search in the Watts-Strogatz model will necessarily require a large number of steps to reach a target — much larger than the true length of the shortest path [248]. As a mathematical model, the Watts-Strogatz network is thus effective at capturing the density of triangles and the existence of short paths, but not the ability of people, working together in the network, to actually find the paths. Essentially, the problem is that the weak ties that make the world small are "too random" in this model: since they're completely unrelated to the similarity among nodes that produces the homophily-based links, they're hard for people to use reliably.

One way to think about this is in terms of Figure 20.4, a hand-drawn image from Milgram's original article in *Psychology Today*. In order to reach a far-away target, one must use long-range weak ties in a fairly structured, methodical way, constantly reducing the distance to the target. As Milgram observed in the discussion accompanying this picture, "The geographic movement of the [letter] from Nebraska to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain" [297]. So it is not enough to have a network model in which weak ties span only the very long ranges; it is necessary to span all the intermediate ranges of scale as well. Is there a simple way to adapt the model to take this into account?

20.4 Modeling the Process of Decentralized Search

Although the Watts-Strogatz model does not provide a structure where decentralized search can be performed effectively, a mild generalization of the model in fact exhibits both properties we want: the networks contain short paths, and these short paths can be found using decentralized search [248].

Generalizing the network model. We adapt the model by introducing one extra quantity that controls the "scales" spanned by the long-range weak ties. We have nodes on a grid as before, and each node still has edges to each other node within r grid steps. But now, each of its k random edges is generated in a way that decays with distance, controlled by a *clustering exponent* q as follows. For two nodes v and w, let $d(v, w)$ denote the number of grid steps between them. (This is their distance if one had to walk along adjacent nodes on the grid.) In generating a random edge out of v, we have this edge link to w with probability proportional to $d(v, w)^{-q}$.

So we in fact have a different model for each value of q. The original grid-based model corresponds to $q = 0$, since then the links are chosen uniformly at random; and varying q is like turning a knob that controls how uniform the random links are. In particular, when q is very small, the long-range links are "too random," and can't be used effectively for decentralized search (as we saw specifically for the case $q = 0$ above); when q is large, the long-range links are "not random enough," since they simply don't provide enough of the long-distance jumps that are needed to create a small world. Pictorially, this variation in q can be seen in the difference between the two networks in Figure 20.5. Is there an optimal operating point for the network, where the distribution of long-range links is sufficiently balanced between these extremes to allow for rapid decentralized search?

In fact there is. The main result for this model is that, in the limit of large network size, decentralized search is most efficient when $q = 2$ (so that random links follow an inverse-square distribution). Figure 20.6 shows the performance of a basic decentralized search method across different values of q, for a network of several hundred million nodes. In keeping with the nature of the result — which only holds in the limit as the network size goes to infinity — decentralized search has about the same efficiency on networks of this size across all exponents q between 1.5 and 2.0. (And at this size, it's best for a value of q slightly below 2.) But the overall trend is already clear, and as the network size increases, the best performance occurs at exponents q closer and closer to 2.

A Rough Calculation Motivating the Inverse-Square Network. It is natural to wonder what's special about the exponent $q = 2$ that makes it best for decentralized search. In Section 20.7 at the end of this chapter, we describe a proof that decentralized search is efficient when $q = 2$, and sketch why search is more efficient with $q = 2$ — in the limit of

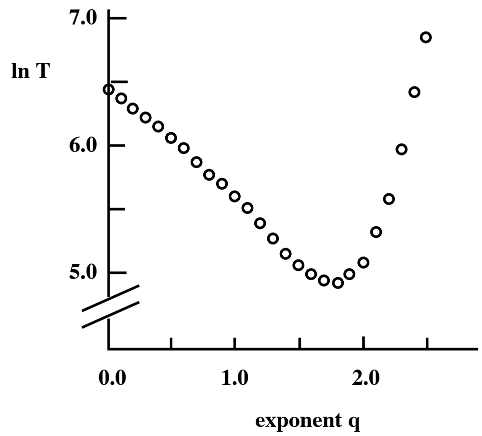


Figure 20.6: Simulation of decentralized search in the grid-based model with clustering exponent q. Each point is the average of 1000 runs on (a slight variant of) a grid with 400 million nodes. The delivery time is best in the vicinity of exponent $q = 2$, as expected; but even with this number of nodes, the delivery time is comparable over the range between 1.5 and 2 [248].

large network size — than with any other exponent. But even without the full details of the proof, there's a short calculation that suggests why the number 2 is important. We describe this now.

In the real world where the Milgram experiment was conducted, we mentally organize distances into different "scales of resolution": something can be around the world, across the country, across the state, across town, or down the block. A reasonable way to think about these scales of resolution in a network model — from the perspective of a particular node v — is to consider the groups of all nodes at increasingly large ranges of distance from v: nodes at distance 2-4, 4-8, 8-16, and so forth. The connection of this organizational scheme to decentralized search is suggested by Figure 20.4: effective decentralized search "funnels inward" through these different scales of resolution, as we see from the way the letter depicted in this figure reduces its distance to the target by approximately a factor of two with each step.

So now let's look at how the inverse-square exponent $q = 2$ interacts with these scales of resolution. We can work concretely with a single scale by taking a node v in the network, and a fixed distance d, and considering the group of nodes lying at distances between d and $2d$ from v, as shown in Figure 20.7.

Now, what is the probability that v forms a link to some node inside this group? Since area in the plane grows like the square of the radius, the total *number* of nodes in this group is proportional to d^2. On the other hand, the probability that v links to any one node in the group varies depending on exactly how far out it is, but each individual probability is proportional to d^{-2}. These two terms — the number of nodes in the group, and the

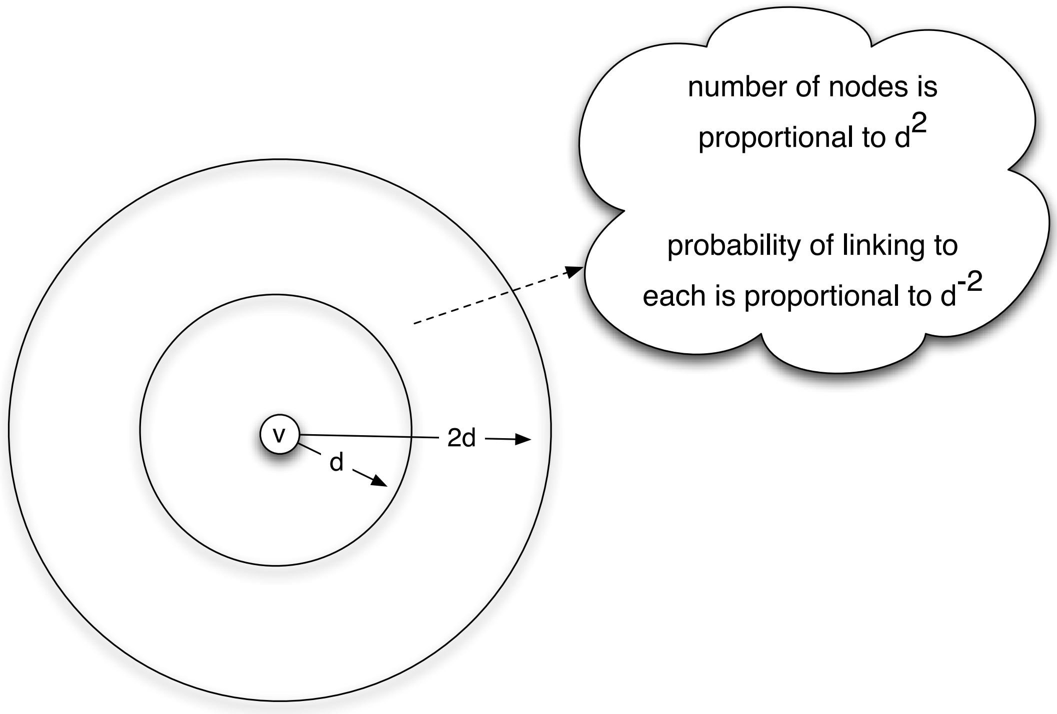


Figure 20.7: The concentric scales of resolution around a particular node.

probability of linking to any one of them — approximately cancel out, and we conclude: the probability that a random edge links into *some node* in this ring is approximately independent of the value of d.

This, then, suggests a qualitative way of thinking about the network that arises when $q = 2$: long-range weak ties are being formed in a way that's spread roughly uniformly over all different scales of resolution. This allows people fowarding the message to consistently find ways of reducing their distance to the target, no matter how near or far they are from it. In this way, it's not unlike how the U.S. Postal Service uses the address on an envelope for delivering a message: a typical postal address exactly specifies scales of resolution, including the country, state, city, street, and finally the street number. But the point is that the postal system is centrally designed and maintained at considerable cost to do precisely this job; the corresponding patterns that guide messages through the inverse-square network are arising spontaneously from a completely random pattern of links.

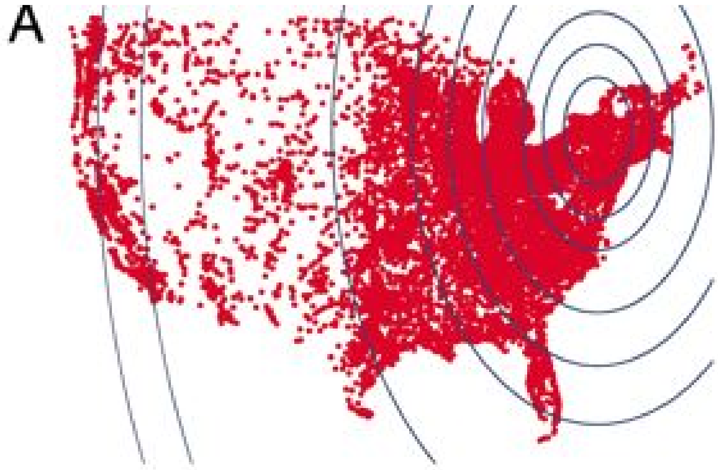


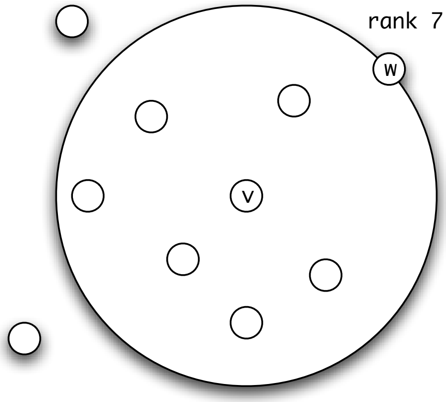
Figure 20.8: The population density of the LiveJournal network studied by Liben-Nowell et al. (Image from [277].)

20.5 Empirical Analysis and Generalized Models

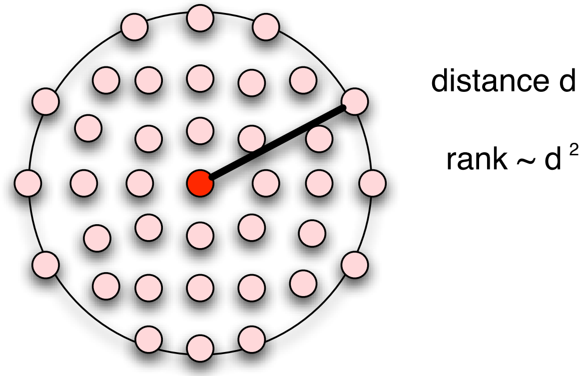
The results we've seen thus far have been for stylized models, but they raise a number of qualitative issues that one can try corroborating with data from real social networks. In this section we discuss empirical studies that analyze geographic data to look for evidence of the exponent $q = 2$, as well as more general versions of these models that incorporate non-geographic notions of social distance.

Geographic Data on Friendship. In the past few years, the rich data available on social networking sites has made it much easier to get large-scale data that provides insight into how friendship links scale with distance. Liben-Nowell et al. [277] used the blogging site LiveJournal for precisely this purpose, analyzing roughly 500,000 users who provided a U.S. ZIP code for their home address, as well as links to their friends on the system. Note that LiveJournal is serving here primarily as a very useful "model system," containing data on the geographic basis of friendship links on a scale that would be enormously difficult to obtain by more traditional survey methods. From a methodological point of view, it is an interesting and fairly unresolved issue to understand how closely the structure of friendships defined in on-line communities corresponds to the structure of friendships as we understand them in off-line settings.

A number of things have to be done in order to align the LiveJournal data with the basic grid model, and perhaps the most subtle involves the fact that the population density of the users is extremely non-uniform (as it is for the U.S. as a whole). See Figure 20.8 for a visualization of the population density in the LiveJournal data. In particular, the



(a) *w is the 7^{th} closest node to v.*



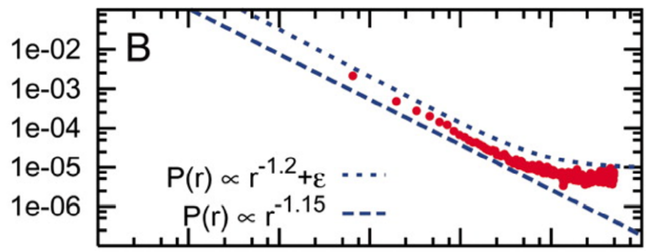
(b) *Rank-based friendship with uniform population density.*

Figure 20.9: When the population density is non-uniform, it can be useful to understand how far w is from v in terms of its *rank* rather than its physical distance. In (a), we say that w has rank 7 with respect to v because it is the 7^{th} closest node to v, counting outward in order of distance. In (b), we see that for the original case in which the nodes have a uniform population density, a node w at distance d from v will have a rank that is proportional to d^2, since all the nodes inside the circle of radius d will be closer to v than w is.

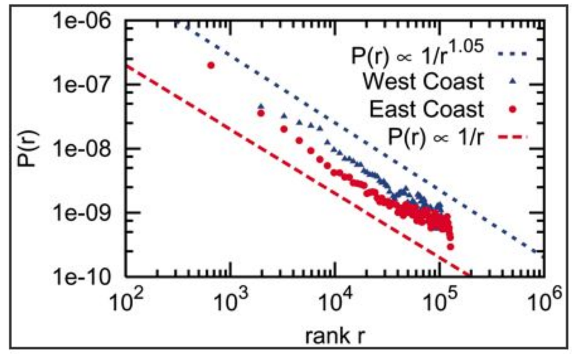
inverse-square distribution is useful for finding targets when nodes are uniformly spaced in two dimensions; what's a reasonable generalization to the case in which they can be spread very non-uniformly?

Rank-Based Friendship. One approach that works well is to determine link probabilities not by physical distance, but by *rank*. Let's suppose that as a node v looks out at all other nodes, it *ranks* them by proximity: the rank of a node w, denoted $\text{rank}(w)$, is equal to the number of other nodes that are closer to v than w is. For example, in Figure 20.9(a), node w would have rank seven, since seven others nodes (including v itself) are closer to v than w is. Now, suppose that for some exponent p, node v creates a random link as follows: it chooses a node w as the other end with probability proportional to $\text{rank}(w)^{-p}$. We will call this *rank-based friendship* with exponent p.

Which choice of exponent p would generalize the inverse-square distribution for uniformly-spaced nodes? As Figure 20.9(b) shows, if a node w in a uniformly-spaced grid is at distance d from v, then it lies on the circumference of a disc of radius d, which contains about d^2 closer nodes — so its rank is approximately d^2. Thus, linking to w with probability proportional to d^{-2} is approximately the same as linking with probability $\text{rank}(w)^{-1}$, so this suggests that exponent $p = 1$ is the right generalization of the inverse-square distribution. In fact, Liben-Nowell et al. were able to prove that for essentially any population density, if random



(a) *Rank-based friendship on LiveJournal*



(b) *Rank-based friendship: East and West coasts*

Figure 20.10: The probability of a friendship as a function of geographic rank on the blogging site LiveJournal. (Image from [277].)

links are constructed using rank-based friendship with exponent 1, the resulting network allows for efficient decentralized search with high probability. In addition to generalizing the inverse-square result for the grid, this result has a nice qualitative summary: to construct a network that is efficiently searchable, create a link to each node with probability that is inversely proportional to the number of closer nodes.

Now one can go back to LiveJournal and see how well rank-based friendship fits the distribution of actual social network links: we consider pairs of nodes where one assigns the other a rank of r, and we ask what fraction f of these pairs are actually friends, as a function of r. Does this fraction decrease approximately like r^{-1}? Since we're looking for a power-law relationship between the rank r and the fraction of edges f, we can proceed as in Chapter 18: rather than plotting f as a function of r, we can plot $\log f$ as a function of $\log r$, see if we find an approximately straight line, and then estimate the exponent p as the slope of this line.

Figure 20.10(a) shows this result for the LiveJournal data; we see that much of the body of the curve is approximately a straight line sandwiched between slopes of -1.15 and -1.2, and hence close to the optimal exponent of -1. It is also interesting to work separately with the more structurally homogeneous subsets of the data consisting of West-Coast users and East-Coast users, and when one does this the exponent becomes very close to the optimal value of -1. Figure 20.10(b) shows this result: The lower dotted line is what you should see if the points followed the distribution rank^{-1}, and the upper dotted line is what you should see if the points followed the distribution rank$^{-1.05}$. The proximity of the rank-based exponent on real networks to the optimal value of -1 has also been corroborated by subsequent research. In particular, as part of a recent large-scale study of several geographic phenomena in the Facebook social network, Backstrom et al. [33] returned to the question of rank-based friendship and again found an exponent very close to -1; in their case, the bulk of the distribution was closely approximated by rank$^{-0.95}$.

The plots in Figure 20.10, and their follow-ups, are thus the conclusion of a sequence of steps in which we start from an experiment (Milgram's), build mathematical models based on this experiment (combining local and long-range links), make a prediction based on the models (the value of the exponent controlling the long-range links), and then validate this prediction on real data (from LiveJournal and Facebook, after generalizing the model to use rank-based friendship). This is very much how one would hope for such an interplay of experiments, theories, and measurements to play out. But it is also a bit striking to see the close alignment of theory and measurement in this particular case, since the predictions come from a highly simplified model of the underlying social network, yet these predictions are approximately borne out on data arising from real social networks.

Indeed, there remains a mystery at the heart of these findings. While the fact that the distributions are so close does not necessarily imply the existence of any particular organizing mechanism [70], it is still natural to ask why real social networks have arranged themselves in a pattern of friendships across distance that is close to optimal for forwarding messages to far-away targets. Furthermore, whatever the users of LiveJournal and Facebook are doing, they are not explicitly trying to run versions of the Milgram experiment — if there are dynamic forces or selective pressures driving the network toward this shape, they must be more implicit, and it remains a fascinating open problem to determine whether such forces exist and how they might operate. One intriguing approach to this question has been suggested by Oskar Sandberg, who analyzes a model in which a network constantly re-wires itself as people perform decentralized searches in it. He argues that over time, the network essentially begins to "adapt" to the pattern of searches; eventually the searches become more efficient, and the arrangement of the long-range links begins to approach a structure that can be approximated by rank-based friendship with the optimal exponent [361].

Social Foci and Social Distance. When we first discussed the Watts-Strogatz model in Section 20.2, we noted that the grid of nodes was intended to serve as a stylized notion of similarity among individuals. Clearly it is most easily identified with geographic proximity, but subsequent models have explored other types of similarity and the ways in which they can produce small-world effects in networks [250, 410].

The notion of *social foci* from Chapter 4 provides a flexible and general way to produce models of networks exhibiting both an abundance of short paths and efficient decentralized search. Recall that a social focus is any type of community, occupational pursuit, neighborhood, shared interest, or activity that serves to organize social life around it [161]. Foci are a way of summarizing the many possible reasons that two people can know each other or become friends: because they live on the same block, work at the same company, frequent the same cafe, or attend the same kinds of concerts. Now, two people may have many possi-

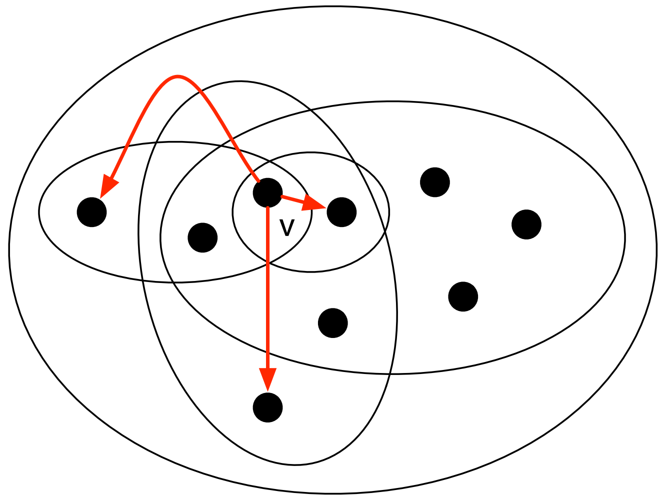


Figure 20.11: When nodes belong to multiple foci, we can define the social distance between two nodes to be the smallest focus that contains both of them. In the figure, the foci are represented by ovals; the node labeled v belongs to five foci of sizes $2, 3, 5, 7$, and 9 (with the largest focus containing all the nodes shown).

ble foci in common, but all else being equal, it is likely that the shared foci with only a few members are the strongest generators of new social ties. For example, two people may both work for the same thousand-person company and live in the same million-person city, but it is the fact that they both belong to the same twenty-person literacy tutoring organization that makes it most probable they know each other. Thus, a natural way to define the *social distance* between two people is to declare it to be *the size of the smallest focus that includes both of them.*

In the previous sections, we've used models that build links in a social network from an underlying notion of geographic distance. Let's consider how this might work with this more general notion of social distance. Suppose we have a collection of nodes, and a collection of foci they belong to — each focus is simply a set containing some of the nodes. We let $\text{dist}(v, w)$ denote the social distance between nodes v and w as defined in terms of shared foci: $\text{dist}(v, w)$ is the size of the smallest focus that contains both v and w. Now, following the style of earlier models, let's construct a link between each pair of nodes v and w with probability proportional to $\text{dist}(v, w)^{-p}$. For example, in Figure 20.11, the node labeled v construct links to three other nodes at social distances 2, 3, and 5. One can now show, subject to some technical assumptions on the structure of the foci, that when links are generated this way with exponent $p = 1$, the resulting network supports efficient decentralized search with high probability [250].

There are aspects of this result that are similar to what we've just seen for rank-based

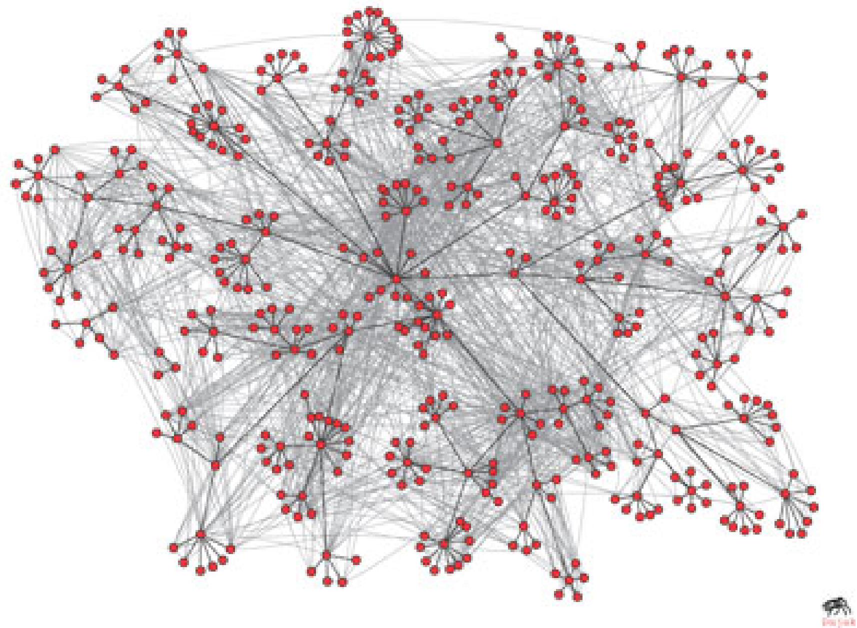


Figure 20.12: The pattern of e-mail communication among 436 employees of Hewlett Packard Research Lab is superimposed on the official organizational hierarchy, showing how network links span different social foci [6]. (Image from http://www-personal.umich.edu/ ladamic/img/hplabsemailhierarchy.jpg)

friendship. First, as with rank-based friendship, there is a simple description of the underlying principle: when nodes link to each other with probability inversely proportional to their social distance, the resulting network is efficiently searchable. And second, the exponent $p = 1$ is again the natural generalization of the inverse-square law for the simple grid model. To see why, suppose we take a grid of nodes and define a set of foci as follows: for each location v on the grid, and each possible radius r around that location, there is a focus consisting of all nodes who are within distance r of v. (Essentially, these are foci consisting of everyone who live together in neighborhoods and locales of various sizes.) Then for two nodes who are a distance d apart, their smallest shared focus has a number of nodes proportional to d^2, so this is their social distance. Thus, linking with probability proportional to d^{-2} is essentially the same as linking with probability inversely proportional to their social distance.

Recent studies of who-talks-to-whom data has fit this model to social network structures. In particular, Adamic and Adar analyzed a social network on the employees of Hewlett Packard Research Lab that we discussed briefly in Chapter 1 connecting two people if they exchanged e-mail at least six times over a three-month period [6]. (See Figure 20.12.) They then defined a focus for each of the groups within the organizational structure (i.e. a group of employees all reporting a common manager). They found that the probability of a link between two employees at social distance d within the organization scaled proportionally to $d^{-3/4}$. In other words, the exponent on the probability for this network is close to, but smaller than, the best exponent for making decentralized search within the network efficient.

These increasingly general models thus provide us with a way to look at social-network data and speak quantitatively about the ways in which the links span different levels of distance. This is important not just for understanding the small-world properties of these networks, but also more generally for the ways in which homophily and weak ties combine to produce the kinds of structures we find in real networks.

Search as an Instance of Decentralized Problem-Solving. While the Milgram experiment was designed to test the hypothesis that people are connected by short paths in the global social network, our discussion here shows that it also served as an empirical study of people's ability to collectively solve a problem — in this case, searching for a path to a far-away individual — using only very local information, and by communicating only with their neighbors in the social network. In addition to the kinds of search methods discussed here, based on aiming as closely to the target as possible in each step, researchers have also studied the effectiveness of path-finding strategies in which people send messages to friends who have a particularly large number of edges (on the premise that they will be "better connected" in general) [6, 7], as well as strategies that explicitly trade off the proximity of a person against their number of edges [370].

The notion that social networks can be effective at this type of decentralized problem-solving is an intriguing and general premise that applies more broadly than just to the problem of path-finding that Milgram considered. There are many possible problems that people interacting in a network could try solving, and it is natural to suppose that their effectiveness will depend both on the difficulty of the problem being solved and on the network that connects them. There is a long history of experimental interest in collective problem-solving [47], and indeed one way to view the bargaining experiments described in Chapter 12 is as an investigation of the ability of a group of people to collectively find a mutually compatible set of exchanges when their interaction is constrained by a network. Recent experiments have explored this issue for a range of basic problems, across multiple kinds of network structures [236, 237], and there is also a growing line of work in the design of systems that can exploit the power of collective human problem-solving by very large on-line populations [402, 403].

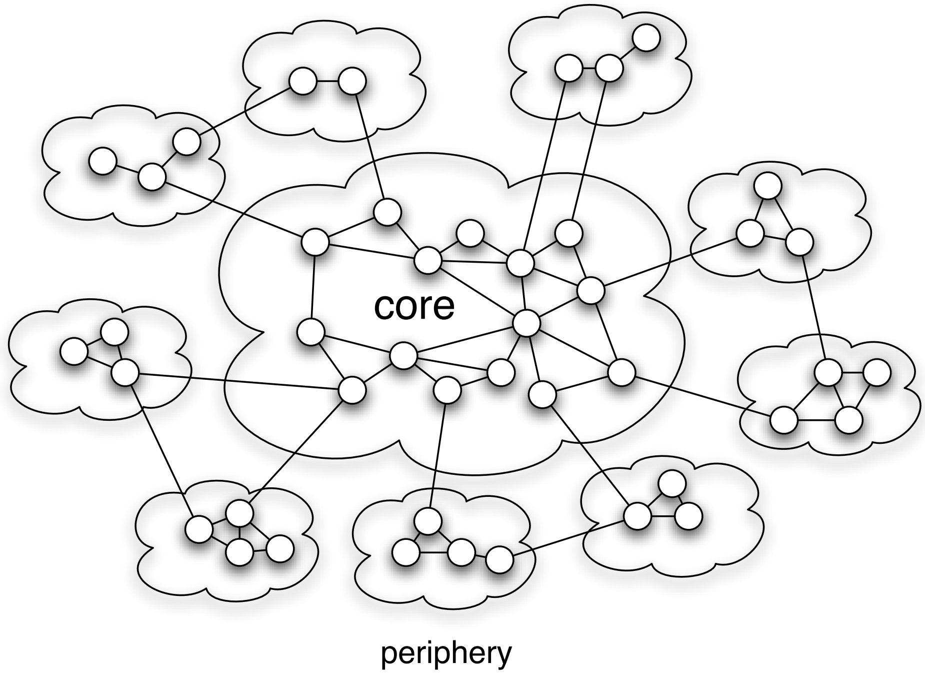


Figure 20.13: The core-periphery structure of social networks.

20.6 Core-Periphery Structures and Difficulties in Decentralized Search

In the four decades since the Milgram experiment, the research community has come to appreciate both the robustness and the delicacy of the "six degrees" principle. As we noted in Chapter 2, many studies of large-scale social network data have confirmed the pervasiveness of very short paths in almost every setting. On the other hand, the ability of people to find these paths from within the network is a subtle phenomenon: it is striking that it should happen at all, and the conditions that facilitate it are not fully understood.

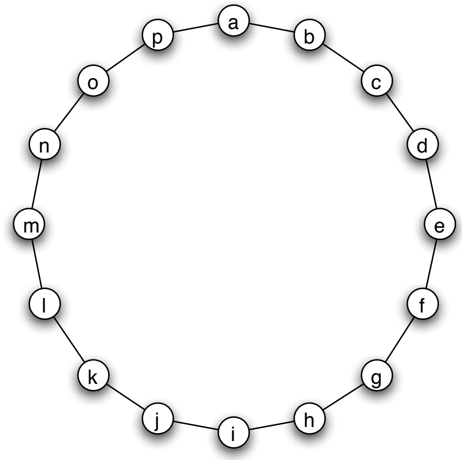
As Judith Kleinfeld has noted in her recent critique of the Milgram experiment [255], the success rate at finding targets in recreations of the experiment has often been much lower than it was in the original work. Much of the difficulty can be explained by lack of participation: many people, asked to forward a letter as part of the experiment, will simply throw it away. This is consistent with lack of participation in any type of survey or activity carried out by mail; assuming this process is more or less random, it has a predictable effect on the results, and one can correct for it [131, 416].

But there are also more fundamental difficulties at work, pointing to questions about large social networks that may help inform a richer understanding of network structure. In particular, Milgram-style search in a network is most successful when the target person is affluent and socially high-status. For example, in the largest small-world experiment to date [131], 18 different targets were used, drawn from a wide range of backgrounds. Completion rates to all targets were small, due to lack of participation in the e-mail based forwarding of messages, but they were highest for targets who were college professors and journalists, and particularly small for low-status targets.

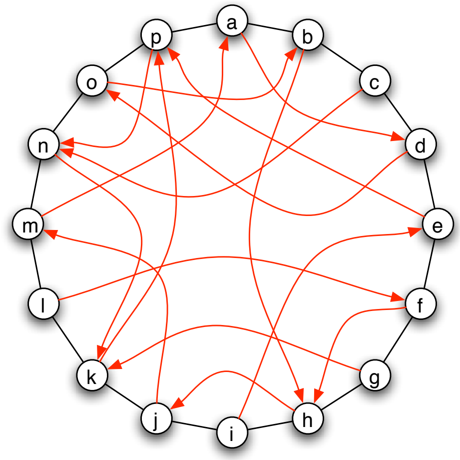
Core-Periphery Structures. This wide variation in the success rates of search to different targets does not simply arise from variations in individual attributes of the respective people — it is based on the fact that social networks are structured to make high-status individuals much easier to find than low-status ones. Homophily suggests that high-status people will mainly know other high-status people, and low-status people will mainly know other low-status people, but this does not imply that the two groups occupy symmetric or interchangeable positions in the social network. Rather, large social networks tend to be organized in what is called a *core-periphery structure* [72], in which the high-status people are linked in a densely-connected *core*, while the low-status people are atomized around the *periphery* of the network. Figure 20.13 gives a schematic picture of such a structure. High-status people have the resources to travel widely; to meet each other through shared foci around clubs, interests, and educational and occupational pursuits; and more generally to establish links in the network that span geographic and social boundaries. Low-status people tend to form links that are much more clustered and local. As a result, the shortest paths connecting two low-status people who are geographically or socially far apart will tend to go into the core and then come back out again.

All this has clear implications for people's ability to find paths to targets in the network. In particular, it indicates some of the deep structural reasons why it is harder for Milgram-style decentralized search to find low-status targets than high-status targets. As you move toward a high-status target, the link structure tends to become richer, based on connections with an increasing array of underlying social reasons. In trying to find a low-status target, on the other hand, the link structure becomes structurally more impoverished as you move toward the periphery.

These considerations suggest an opportunity for richer models that take status effects more directly into account. The models we have seen capture the process by which people can find each other when they are all embedded in an underlying social structure, and motivated to continue a path toward a specific destination. But as the social structure begins to fray around the periphery, an understanding of how we find our way through it has the potential to shed light not just on the networks themselves, but on the way that network structure is intertwined with status and the varied positions that different groups occupy in society as a whole.



(a) *A set of nodes arranged in a ring.*



(b) *A ring augmented with random long-range links.*

Figure 20.14: The analysis of decentralized search is a bit cleaner in one dimension than in two, although it is conceptually easy to adapt the arguments to two dimensions. As a result, we focus most of the discussion on a one-dimensional ring augmented with random long-range links.

20.7 Advanced Material: Analysis of Decentralized Search

In Section 20.4, we gave some basic intuition for why an inverse-square distribution of links with distance makes effective decentralized search possible. Even given this way of thinking about it, however, it still requires further work to really see why search succeeds with this distribution. In this section, we describe the complete analysis of the process [249].

To make the calculations a bit simpler, we vary the model in one respect: we place the nodes in one dimension rather than two. In fact, the argument is essentially the same no matter how many dimensions the nodes are in, but one dimension makes things the cleanest (even if not the best match for the actual geographic structure of a real population). It turns out, as we will argue more generally later in this section, that the best exponent for search is equal to the dimension, so in our one-dimensional analysis we will be using an exponent of $q = 1$ rather than $q = 2$. At the end, we will discuss the minor ways in which the argument needs to be adapted in two or higher dimensions.

We should also mention, recalling the discussion earlier in the chapter, that there is a second fundamental part of this analysis as well — showing that this choice of q is in fact the best for decentralized search in the limit of increasing network size. At the end, we sketch why this is true, but the full details are beyond what we will cover here.

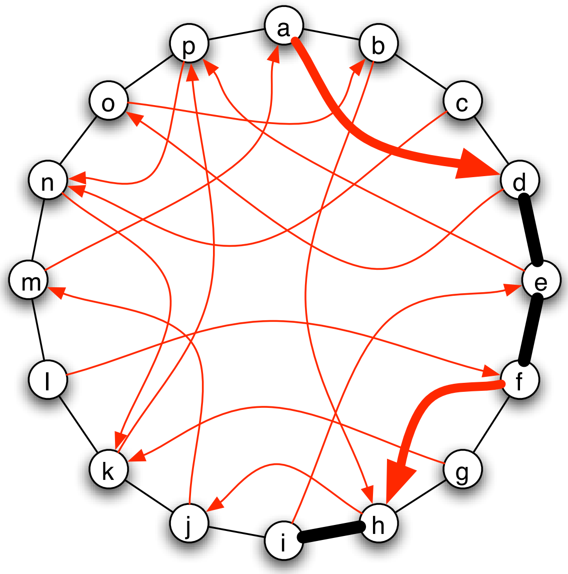


Figure 20.15: In myopic search, the current message-holder chooses the contact that lies closest to the target (as measured on the ring), and it forwards the message to this contact.

A. The Optimal Exponent in One Dimension

Here, then, is the model we will be looking at. A set of n nodes are arranged on a one-dimensional ring as shown in Figure 20.14(a), with each node connected by directed edges to the two others immediately adjacent to it. Each node v also has a single directed edge to some other node on the ring; the probability that v links to any particular node w is proportional to $d(v, w)^{-1}$, where $d(v, w)$ is their distance apart on the ring. We will call the nodes to which v has an edge its *contacts*: the two nodes adjacent to it on the ring are its *local contacts*, and the other one is its *long-range contact*. The overall structure is thus a ring that is augmented with random edges, as shown in Figure 20.14(b). Again, this is essentially just a one-dimensional version of the grid with random edges that we saw in Figure 20.5.[1]

Myopic Search. Let's choose a random start node s and a random target node t on this augmented ring network. The goal, as in the Milgram experiment, is to forward a message from the start to the target, with each intermediate node on the way only knowing the locations of its own neighbors, and the location of t, but nothing else about the full network.

The forwarding strategy that we analyze, which works well on the ring when $q = 1$, is a

[1]We could also analyze a model in which nodes have more outgoing edges, but this only makes the search problem easier; our result here will show that even when each node has only two local contacts and a single long-range contact, search can still be very efficient.

simple technique that we call *myopic search*: when a node v is holding the message, it passes it to the contact that lies as close to t on the ring as possible. Myopic search can clearly be performed even by nodes that know nothing about the network other than the locations of their friends and the location of t, and it is a reasonable approximation to the strategies used by most people in Milgram-style experiments [243].

For example, Figure 20.15 shows the myopic path that would be constructed if we chose a as the start node and i as the target node in the network from Figure 20.14(b).

1. Node a first sends the message to node d, since among a's contacts p, b, and d, node d lies closest to i on the ring.

2. Then d passes the message to its local contact e, and e likewise passes the message to its local contact f, since the long-range contacts of both d and e lead away from i on the ring, not closer to it.

3. Node f has a long-range contact h that proves useful, so it passes it to h. Node h actually has the target as a local contact, so it hands it directly to i, completing the path in five steps.

Notice that this myopic path is not the shortest path from a to i. If a had known that its friend b in fact had h as a contact, it could have handed the message to b, thereby taking the first step in the three-step a-b-h-i path. It is precisely this lack of knowledge about the full network structure that prevents myopic search from finding the true shortest path in general.

Despite this, however, we will see next that in expectation, myopic search finds paths that are surprisingly short.

Analyzing Myopic Search: The Basic Plan. We now have a completely well-defined probabilistic question to analyze, as follows. We generate a random network by adding long-range edges to a ring as above. We then choose a random start node s and random target node t in this network. The number of steps required by myopic search is now a random variable X, and we are interested in showing that $E[X]$, the expected value of X, is relatively small.

Our plan for putting a bound on the expected value of X follows the idea contained in Milgram's picture from Figure 20.4: we track how long it takes for the message to reduce its distance by factors of two as it closes in on the target. Specifically, as the message moves from s to t, we'll say that it's in *phase* j of the search if its distance from the target is between 2^j and 2^{j+1}. See Figure 20.16 for an illustration of this. Notice that the number of different phases is at most $\log_2 n$ — that is, the number of doublings needed to go from 1 to n. (In what follows, we will drop the base of the logarithm and simply write $\log n$ to denote $\log_2 n$.)

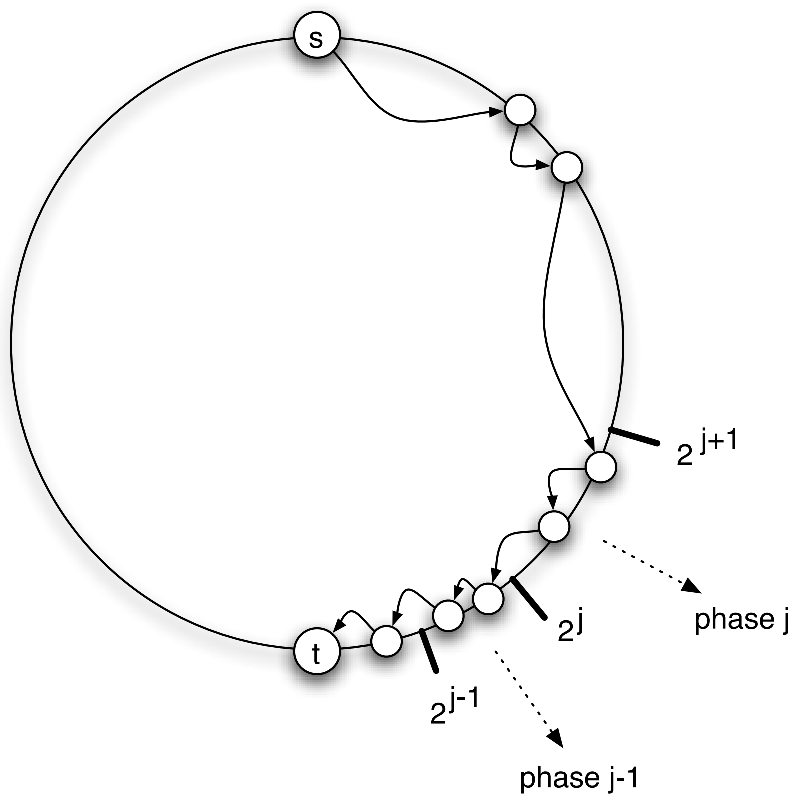


Figure 20.16: We analyze the progress of myopic search in *phases*. Phase j consists of the portion of the search in which the message's distance from the target is between 2^j and 2^{j+1}.

We can write X, the number of steps taken by the full search, as

$$X = X_1 + X_2 + \cdots + X_{\log n};$$

that is, the total time taken by the search is simply the sum of the times taken in each phase. Linearity of expectation says that the expectation of a sum of random variables is equal to the sum of their individual expectations, and so we have

$$E\left[X\right] = E\left[X_1 + X_2 + \cdots + X_{\log n}\right] = E\left[X_1\right] + E\left[X_2\right] + \cdots + E\left[X_{\log n}\right].$$

We will now show — and this is the crux of the argument — that the expected value of each X_j is at most proportional to $\log n$. In this way, $E\left[X\right]$ will be a sum of $\log n$ terms, each at most proportional to $\log n$, and so we will have shown that $E\left[X\right]$ is at most proportional to $(\log n)^2$.

This will achieve our overall goal of showing that myopic search is very efficient with the given distribution of links: the full network has n nodes, but myopic search constructs a path that is *exponentially* smaller: proportional to the square of $\log n$.

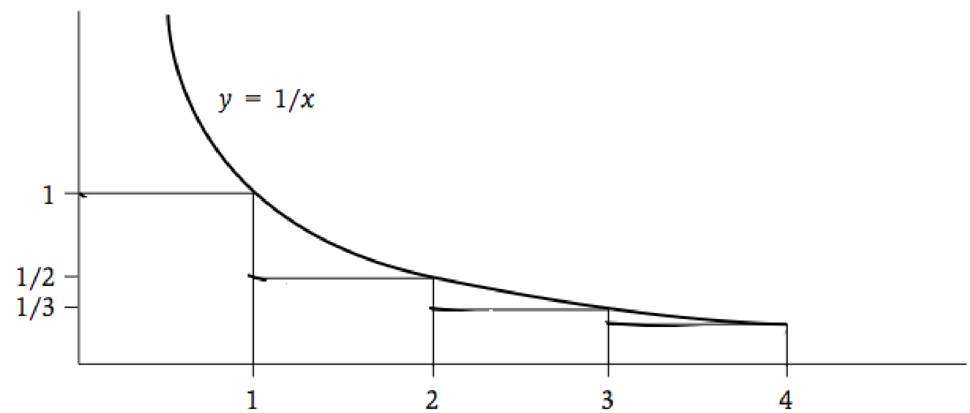


Figure 20.17: Determining the normalizing constant for the probability of links involves evaluating the sum of the first $n/2$ reciprocals. An upper bound on the value of this sum can be determined from the area under the curve $y = 1/x$.

Intermediate Step: The Normalizing Constant In implementing this high-level strategy, the first thing we need to work out is in fact something very basic: we've been saying all along that v forms its long-range link to w with probability *proportional to* $d(v,w)^{-1}$, but what is the constant of proportionality? As in any case when we know a set of probabilities up to a missing constant of proportionality $1/Z$, the value of Z is here simply the sum of $d(v,u)^{-1}$ over all nodes $u \neq v$ on the ring. Dividing everything down by this normalizing constant Z, the probability of v linking to w is then equal to $\frac{1}{Z}d(v,w)^{-1}$.

To work out the value of Z, we note that there are two nodes at distance 1 from v, two at distance 2, and more generally two at each distance d up to $n/2$. Assuming n is even, there is also a single node at distance $n/2$ from v — the node diametrically opposite it on the ring. Therefore, we have

$$Z \leq 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n/2}\right). \tag{20.1}$$

The quantity inside parentheses on the right is a common expression in probabilistic calculations: the sum of the first k reciprocals, for some k, in this case $n/2$. To put an upper bound on its size, we can compare it to the area under the curve $y = 1/x$, as shown in Figure 20.17. As that figure indicates, a sequence of rectangles of unit widths and heights $1/2, 1/3, 1/4, \ldots, 1/k$ fits under the curve $y = 1/x$ as x ranges from 1 to k. Combined with a single rectangle of height and width 1, we see that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} \leq 1 + \int_1^k \frac{1}{x}dx = 1 + \ln k.$$

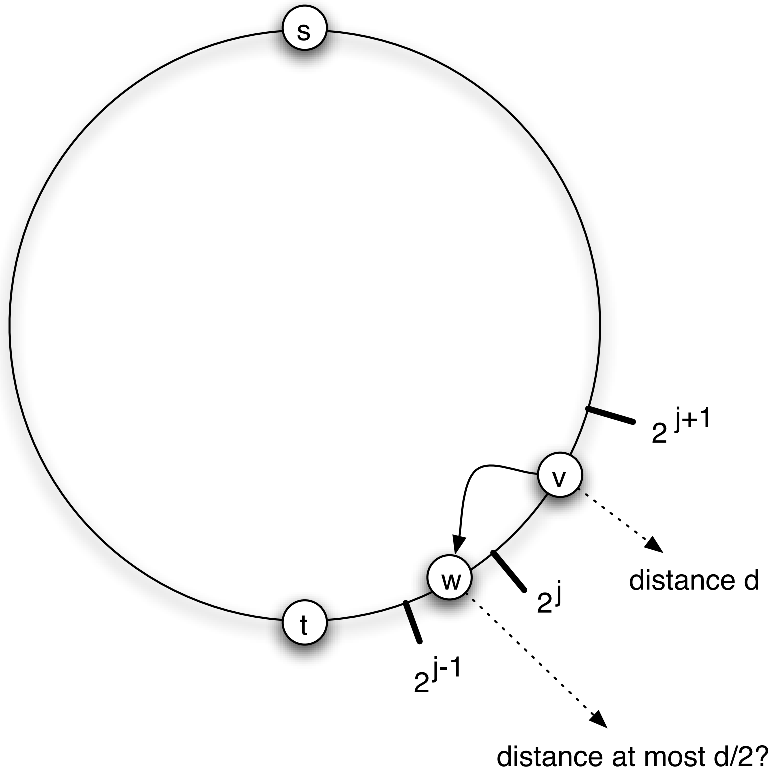


Figure 20.18: At any given point in time, the search is in some phase j, with the message residing at a node v at distance d from the target. The phase will come to an end if v's long-range contact lies at distance $\leq d/2$ from the target t, and so arguing that the probability of this event is large provides a way to show that the phase will not last too long.

Plugging in $k = n/2$ to the expression on the right-hand side of inequality (20.1) above, we get

$$Z \leq 2(1 + \ln(n/2)) = 2 + 2\ln(n/2).$$

For simplicity, we'll use a slightly weaker bound on Z, which follows simply from the observation that $\ln x \leq \log_2 x$:

$$Z \leq 2 + 2\log_2(n/2) = 2 + 2(\log_2 n) - 2(\log_2 2) = 2\log_2 n.$$

Thus, we now have an expression for the actual probability that v links to w (including its constant of proportionality): the probability v links to w is

$$\frac{1}{Z} d(v,w)^{-1} \geq \frac{1}{2\log n} d(v,w)^{-1}.$$

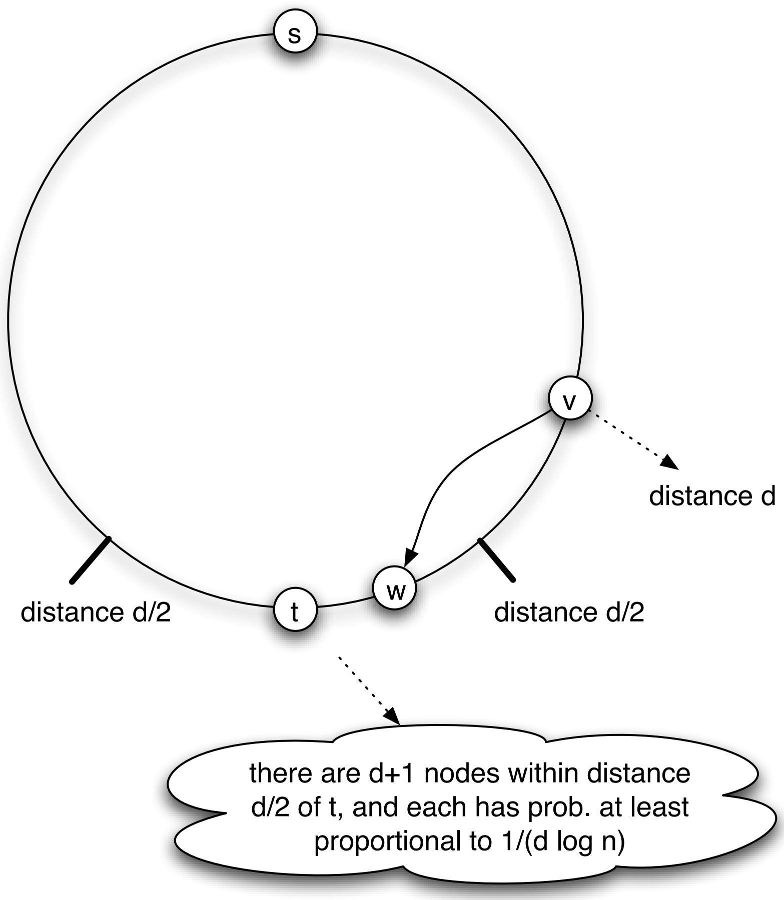


Figure 20.19: Showing that, with reasonable probability, v's long-range contact lies within half the distance to the target.

Analyzing the Time Spent in One Phase of Myopic Search. Finally, we come to the last and central step of the analysis: showing that the time spent by the search in any one phase is not very large. Let's choose a particular phase j of the search, when the message is at a node v whose distance to the target t is some number d between 2^j and 2^{j+1}. (See Figure 20.18 for an illustration of all this notation in context.) The phase will come to an end once the distance to the target decreases below 2^j, and we want to show that this will happen relatively quickly.

One way for the phase to come to an end immediately would be for v's long-range contact w to be at distance $\leq \frac{d}{2}$ from t. In this case, v would necessarily be the last node to belong to phase j. So let's show that this immediate halving of the distance in fact happens with reasonably large probability.

The argument is pictured in Figure 20.19. Let I be the set of nodes at distance $\leq \frac{d}{2}$ from

t; this is where we hope v's long-range contact will lie. There are $d+1$ nodes in I: this includes node t itself, and $d/2$ nodes consecutively on each side of it. Each node w in I has distance at most $3d/2$ from v: the farthest one is on the "far side" of t from v, at distance $d + d/2$. Therefore, each node w in I has probability at least

$$\frac{1}{2\log n} d(v,w)^{-1} \geq \frac{1}{2\log n} \cdot \frac{1}{3d/2} = \frac{1}{3d\log n}$$

of being the long-range contact of v. Since there are more than d nodes in I, the probability that *one of them* is the long-range contact of v is at least

$$d \cdot \frac{1}{3d\log n} = \frac{1}{3\log n}.$$

If one of these nodes is the long-range contact of v, then phase j ends immediately in this step. Therefore, in each step that it proceeds, phase j has a probability of at least $1/(3\log n)$ of coming to an end, independently of what has happened so far. To run for at least i steps, phase j has to fail to come to an end $i-1$ times in a row, and so the probability that phase j runs for at least i steps is at most

$$\left(1 - \frac{1}{3\log n}\right)^{i-1}.$$

Now we conclude by just using the formula for the expected value of a random variable:

$$E[X_j] = 1 \cdot \Pr[X_j = 1] + 2 \cdot \Pr[X_j = 2] + 3 \cdot \Pr[X_j = 3] + \cdots \tag{20.2}$$

There is a useful alternate way to write this: notice that in the expression

$$\Pr[X_j \geq 1] + \Pr[X_j \geq 2] + \Pr[X_j \geq 3] + \cdots \tag{20.3}$$

the quantity $\Pr[X_j = 1]$ is accounted for once (in the first term only), the quantity $\Pr[X_j = 2]$ is accounted for twice (in the first two terms only), and so forth. Therefore the expressions in (20.2) and (20.3) are the same thing, and so we have

$$E[X_j] = \Pr[X_j \geq 1] + \Pr[X_j \geq 2] + \Pr[X_j \geq 3] + \cdots \tag{20.4}$$

Now, we've just argued above that

$$\Pr[X_j \geq i] \leq \left(1 - \frac{1}{3\log n}\right)^{i-1},$$

and so

$$E[X_j] \leq 1 + \left(1 - \frac{1}{3\log n}\right) + \left(1 - \frac{1}{3\log n}\right)^2 + \left(1 - \frac{1}{3\log n}\right)^3 + \cdots$$

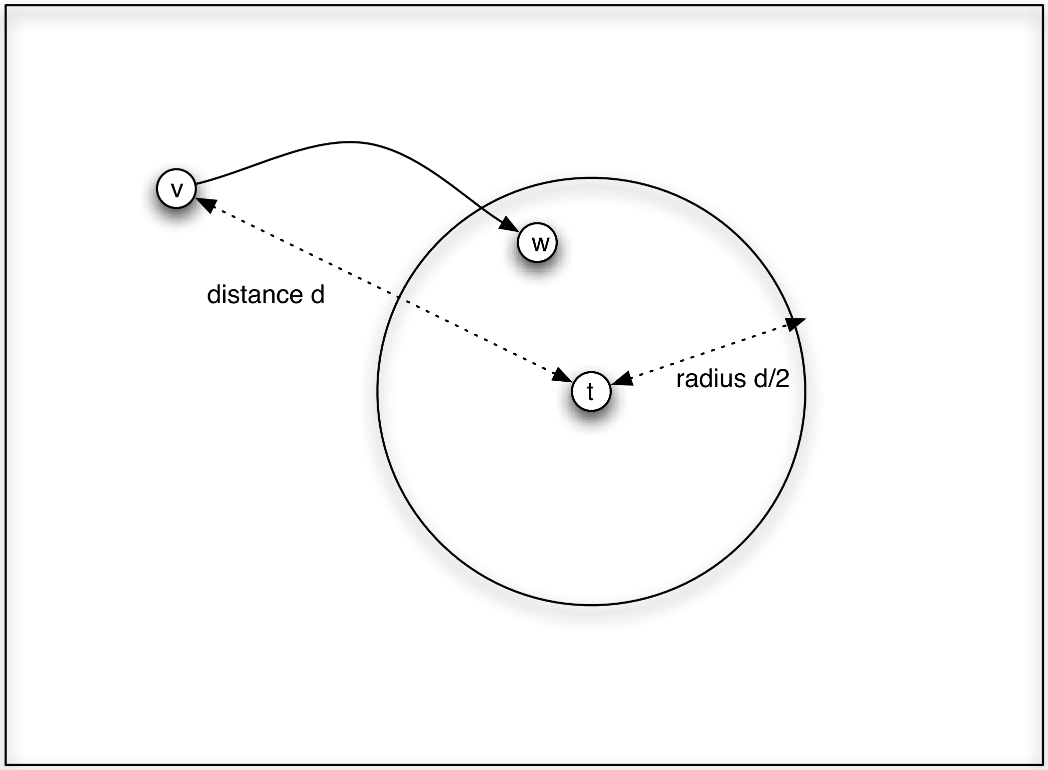


Figure 20.20: The analysis for the one-dimensional ring can be carried over almost directly to the two-dimensional grid. In two dimensions, with the message at a current distance d from the target t, we again look at the set of nodes within distance $d/2$ of t, and argue that the probability of entering this set in a single step is reasonably large.

The right-hand side is a geometric sum with multiplier $\left(1 - \dfrac{1}{3\log n}\right)$, and so it converges to

$$\frac{1}{1-\left(1-\frac{1}{3\log n}\right)} = 3\log n.$$

Thus we have

$$E\left[X_j\right] \leq 3\log n.$$

And now we're done. $E[X]$ is a sum of the $\log n$ terms $E[X_1] + E[X_2] + \cdots + E[X_{\log n}]$, and we've just argued that each of them is at most $3\log n$. Therefore, $E[X] \leq 3(\log n)^2$, a quantity proportional to $(\log n)^2$ as we wanted to show.

B. Higher Dimensions and Other Exponents

Using the analysis we've just completed, we now discuss two further issues. First, we sketch how it can be used to analyze networks built by adding long-range contacts to nodes arranged in two dimensions. Then we show how, in the limit of increasing network size, search is more efficient when q is equal to the underlying dimension than when it is equal to any other value.

The Analysis in Two Dimensions. It's not hard to adapt our analysis for the one-dimensional ring directly to the case of the two-dimensional grid. Essentially, we only used the fact that we were in one dimension in two distinct places in the analysis. First, we used it when we determined the normalizing constant Z. Second, and in the end most crucially, we used it to argue that there were at least d nodes within distance $d/2$ of the target t. This factor of d canceled the d^{-1} in the link probability, allowing us to conclude that the probability of halving the distance to the target in any given step was at least proportional to $1/(\log n)$, *regardless of the value of* d.

At a qualitative level, this last point is the heart of the analysis: with link probability d^{-1} on the ring, the probability of linking to any one node exactly offsets the number of nodes close to t, and so myopic search makes progress at every possible distance away from the target.

When we go to two dimensions, the number of nodes within distance $d/2$ of the target will be proportional to d^2. This suggests that to get the same nice cancellation property, we should have v link to each node w with probability proportional to $d(v,w)^{-2}$, and this exponent -2 is what we will use.

With the above ideas in mind, and with this change in the exponent to -2, the analysis for two dimensions is almost exactly the same as what we just saw for the one-dimensional ring. First, while we won't go through the calculations here, the normalizing constant Z is still proportional to $\log n$ when the probability of v linking to w is proportional to $d(v,w)^{-2}$. We then consider $\log n$ different phases as before; and as depicted in Figure 20.20, we consider the probability that at any given moment, the current message-holder v has a long-range contact w that halves the distance to the target, ending the phase immediately. Now we use the calculation foreshadowed in the previous paragraph: the number of nodes within distance $d/2$ of the target is proportional to d^2, and the probability that v links to each is proportional to $1/(d^2 \log n)$. Therefore, the probability that the message halves its distance to the target in this step is at least $d^2/(d^2 \log n) = 1/(\log n)$, and the rest of the analysis then finishes as before.

A similarly direct adaptation of the analysis shows that decentralized search is efficient for networks built by adding long-range contacts to grids in $D > 2$ dimensions, when the exponent q is equal to D.

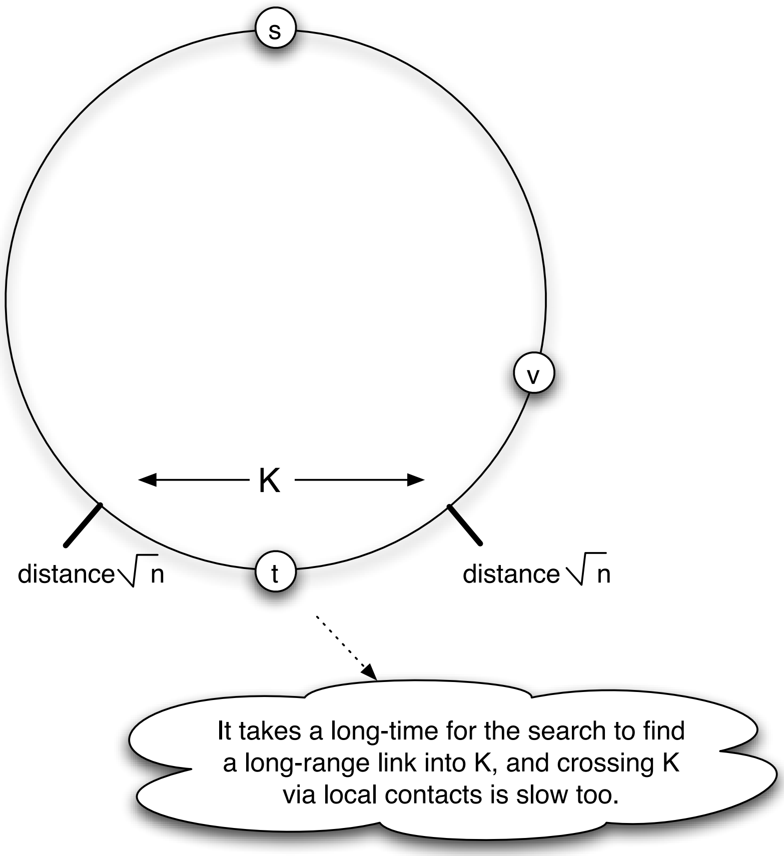


Figure 20.21: To show that decentralized search strategies require large amounts of time with exponent $q = 0$, we argue that it is difficult for the search to cross the set of $\sqrt{n}$ nodes closest to the target. Similar arguments hold for other exponents $q < 1$.

Why Search is Less Efficient with Other Exponents. Finally, let's sketch why decentralized search is less efficient when the exponent is anything else. For concreteness, we'll focus on why search doesn't work well when $q = 0$ — the original Watts-Strogatz model when long-range links are chosen uniformly at random. Also, we'll talk again about the one-dimensional ring rather than the two-dimensional grid, since things are a bit cleaner in one dimension, although again the analysis in two dimensions is essentially the same.

The key idea, as with the "good" exponent $q = 1$, is to consider the set of all nodes within some distance of the target t. But whereas in the case of $q = 1$ we wanted to argue that it is easy to enter smaller and smaller sets centered around t, here we want to identify a set of nodes centered at t that is somehow "impenetrable" — a set that is very hard for the search to enter.

In fact, it is not difficult to do this. The basic idea is depicted in Figure 20.21; we sketch how the argument works, but without going into all the details. (The details can be found in [249].) Let K be the set of all nodes within distance less than $\sqrt{n}$ of the target t. Now, with high probability, the starting point of the search lies outside K. Because long-range contacts are created uniformly at random (since $q = 0$), the probability that any one node has a long-range contact inside K is equal to the size of K divided by n: so it is less than $2\sqrt{n}/n = 2/\sqrt{n}$. Therefore, any decentralized search strategy will need at least $\sqrt{n}/2$ steps in expectation to find a node with a long-range contact in K. On the other hand, as long as it doesn't find a long-range link leading into K, it can't reach the target in less than $\sqrt{n}$ steps, since it would take this long to "walk" step-by-step through K using only the connections among local contacts. From this, one can show that the expected time for any decentralized search strategy to reach t must be at least proportional to $\sqrt{n}$.

There are similar arguments for every other exponent $q \neq 1$. When q is strictly between 0 and 1, a version of the argument above works, with a set K centered at t whose width depends on the value of q. And when $q > 1$, decentralized search is inefficient for a different reason: since even the long-range links are relatively short, it takes a long time for decentralized search to find links that span sufficiently long distances. This makes it hard to quickly traverse the distance from the starting node to the target.

Overall, one can show that for every exponent $q \neq 1$, there is a constant $c > 0$ (depending on q), so that it takes at least proportional to n^c steps in expectation for any decentralized search strategy to reach the target in a network generated with exponent q. So in the limit, as n becomes large, decentralized search with exponent $q = 1$ requires time that grows like a polynomial in $\log n$, while decentralized search at any other exponent requires a time that grows like a polynomial in n — exponentially worse.[2] The exponent $q = 1$ on the ring — or $q = 2$ in the plane — is optimally balanced between producing networks that are "too random" for search, and those that are not random enough.

20.8 Exercises

1. In the basic "six degrees of separation" question, one asks whether most pairs of people in the world are connected by a path of at most six edges in the social network, where an edge joins any two people who know each other on a first-name basis.

   Now let's consider a variation on this question. Suppose that we consider the full population of the world, and suppose that from each person in the world we create a directed edge only to their ten closest friends (but not to anyone else they know on a first-name basis). In the resulting "closest-friend" version of the social network, is it possible that for each pair of people in the world, there is a path of at most six edges

[2] Of course, it can take very large values of n for this distinction to become truly pronounced; recall Figure 20.6, which showed the results of simulations on networks with 400 million nodes.

connecting this pair of people? Explain.

2. In the basic "six degrees of separation" question, one asks whether most pairs of people in the world are connected by a path of at most six edges in the social network, where an edge joins any two people who know each other on a first-name basis.

   Now let's consider a variation on this question. For each person in the world, we ask them to rank the 30 people they know best, in descending order of how well they know them. (Let's suppose for purposes of this question that each person is able to think of 30 people to list.) We then construct two different social networks:

   (a) The "close-friend" network: from each person we create a directed edge only to their ten closest friends on the list.

   (b) The "distant-friend" network: from each person we create a directed edge only to the ten people listed in positions 21 through 30 on their list.

   Let's think about how the small-world phenomenon might differ in these two networks. In particular, let C be the average number of people that a person can reach in six steps in the close-friend network, and let D be the average number of people that a person can reach in six steps in the distant-friend network (taking the average over all people in the world).

   When researchers have done empirical studies to compare these two types of networks (the exact details often differ from one study to another), they tend to find that one of C or D is consistently larger than the other. Which of the two quantities, C or D, do you expect to be larger? Give a brief explanation for your answer.

3. Suppose you're working with a group of researchers studying social communication networks, with a particular focus on the distances between people in such networks, and the broader implications for the small-world phenomenon.

   The research group is currently negotiating an agreement with a large mobile phone carrier to get a snapshot of their "who-calls-whom" graph. Specifically, under a strict confidentiality agreement, the carrier is offering to provide a graph in which there is a node representing each of the carrier's customers, and each edge represents a pair of people who called each other over a fixed one-year period. (The edges will be annotated with the number of calls and the time at which each one happened. No personal identification will be provided with the nodes.)

   Recently, the carrier has proposed that instead of providing all the data, they'll only provide edges corresponding to pairs of people who called each other at least once a week on average over the course of the year. (That is, all nodes will be present, but there will only be edges for pairs of people who talked at least 52 times.) The carrier understands that this is not the full network, but they would prefer to release less data and they argue that this will be a good approximation to the full network.

Your research group objects, but the carrier is not inclined to change its position unless your group can identify specific research findings that are likely to be misleading if they are drawn from this reduced dataset. The leader of your research group asks you to prepare a brief response to the carrier, identifying some concrete ways in which misleading conclusions might be reached from the reduced dataset.

What would you say in your response?