Accumulator payoff pricing using analytical distribution for barrier and gearing strike hit.

We consider that the underlyign equity follows an exponntial Brownian movement, and hence its log returns can be model is a normal distribution:

$$\frac{\Delta S}{S} = \mu\delta T + \sigma(S,t)\sqrt{t}\Delta Z_t$$

Under this stochastic process the following probabilies expression hold:

1. Probability of hitting a Down trigger $P_{hit} = P(S \leq BL) = \Phi(\frac{\log BL - \mu T}{\sigma\sqrt{T}})$

2. Probability of hitting a Up trigger $P_{hit} = P(S \geq BL) = 1.0 - \Phi(\frac{\log BL - \mu T}{\sigma\sqrt{T}})$

   In the context of modelling an accumulator payoff with KO and gearing features, both events can be mapped to a barrier trigger affecting the number of equities accumulator over the future time horizon.

In [461]:
```python
import numpy as np
import pandas as pd
from scipy.stats import norm
import  matplotlib.pyplot as plt
```

$P(S \leq BL)$

In [462]:
```python
#P(S <= BL)
def p_hit(T,sigma,S,BL, mu):
    H = BL/S
    x = np.log(H)-mu*T
    sT = sigma*np.sqrt(T)
    return norm.cdf(x/sT)
```

$P(\tau \leq T)$

In [463]:
```python
def p_hit_before_time(T,sigma,S,K, mu):
    H = K/S
    if H>1:
        H = 1.0/H
        mu=-mu
        tmp = S
        S=K
        K=tmp

    lnK = np.log(H)
    s2 = sigma**2
    sqrtT = np.sqrt(T)
    sigma05T = sigma*sqrtT
    num = np.exp(2*mu*lnK/s2)*norm.cdf((lnK+mu*T)/sigma05T)
    den = norm.cdf((lnK-mu*T)/sigma05T)+1e-12
    return (1+num/(den))*p_hit(T,sigma,S,K,mu)
```

$p(\tau = T)$

In [464]:
```python
def p_hit_in_t(T,sigma,S,K, mu):
    H = K/S
    if H>1:
        H = 1.0/H
        mu=-mu
        tmp = S
        S=K
        K=tmp

    dt = 1.0/260.0
    a = p_hit_before_time(T+dt,sigma,S,K, mu)
    b = p_hit_before_time(T-dt,sigma,S,K, mu)

    return (a-b)/(2*dt)
```

Accumulator Payoff with only Gearing Events

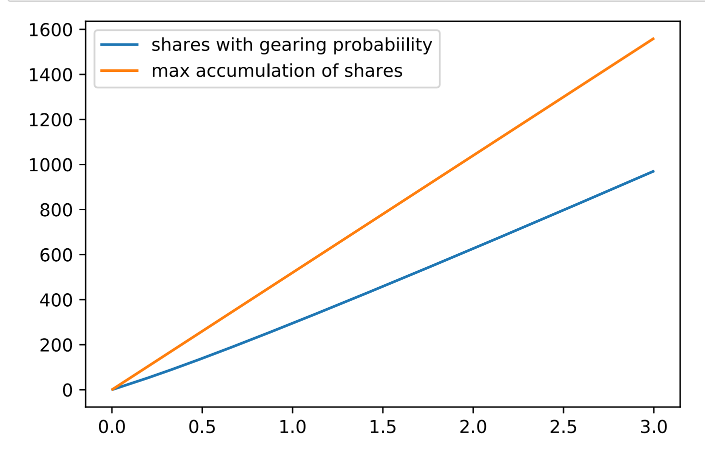
We study the case of an accumulator with a down gear strike level, H_{gear}, which increases the number of accumlated shares per day η by a factor g while the spot stays below this trigger level.

$$V(t) = \sum_{i=1}^{N} \eta \times (1 + g \times P(S \leq H_{gear}))$$

Example Accumulator gearing and without KO

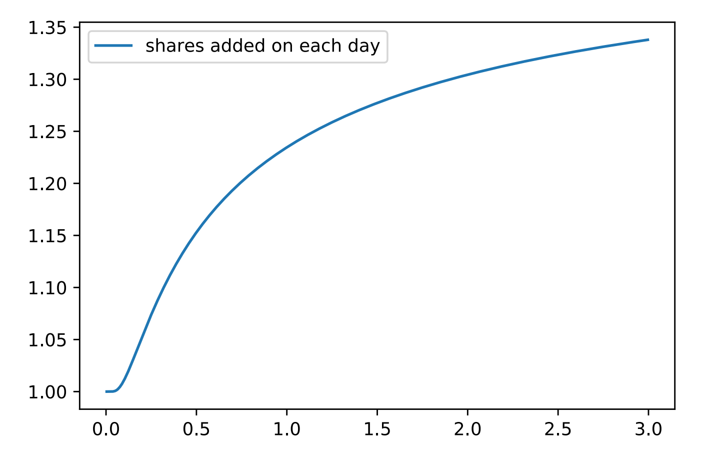
- S=100
- $H_{gear} = 98$
- $\eta = 1 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 45\%$

In [465]:
```python
sigma = 0.45/np.sqrt(260.0)
t = np.array(range(1,3*260))/260.0
dt = t[1]-t[0]
S=100.0
K=98.0
shares_per_day = 1
gearing = 1.0
shares_Accumulator_Each_day = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))  for i in t]
max_shares_Accumulator_Each_per_day = [shares_per_day*(1+gearing)  for i in t]
max_shares_Accumulator_Each_day = np.cumsum([shares_per_day*(1+gearing)  for i in t])
total_accumulated_shares = np.cumsum(shares_Accumulator_Each_day)
plt.plot(t,total_accumulated_shares,label="shares with gearing probabiility")
plt.plot(t,max_shares_Accumulator_Each_day,label="max accumulation of shares")
plt.legend()

plt.show()
```



The picture above depicts the difference in between the full share accumulation assumption (over consevative) vs the gearing with hit probability.

In [466]:
```python
plt.plot(t,shares_Accumulator_Each_day,label="shares added on each day")

plt.legend()
plt.show()
```



The curve above depicts the number of shares accumulated per day as the time passess. Since no drift is considered on the stock model, $\mu = 0$, to the 1 shares per day standard accumulation, the gearing adds one extra share with probability $P(S \leq H_{gear})$. When $t \to \infty$, the probability of hiting the gearing level tends to 50\%, meaning approximates to 1.5 shares per day as time tends to ∞.

Accumulator Payoff with Gearing and KO Events

We include the KO probability over time togethe with the feature already presented in the previous simulation.

In the current simulations the accumulation of shares is conditioned to the fact that the stock has not reached the KO barrier level BL_{KO}.

The probability of not being KO at time τ is simply defined as $P(\tau \geq T) = 1 - P(\tau \leq T)$, and the new expression of the accumulated number of shares at the maturity is:

$$V(T) = \sum_{i=1}^{N} \eta \times (1 + g \times P(S \leq H_{gear}, t_i)) \times P(\tau > t_i)$$

Example Accumulator gearing and KO

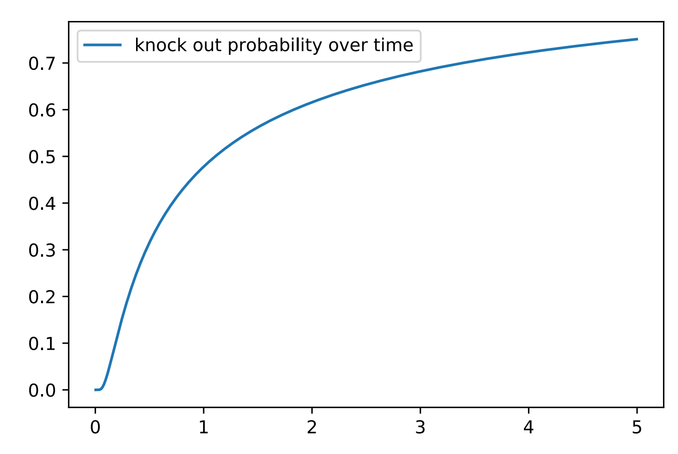
- S=100
- $H_{gear} = 98$
- $\eta = 1 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 45\%$
- $BL_{KO} = 102$

In [474]:
```python
sigma = 0.45/np.sqrt(260.0)
t = np.array(range(1,5*260))/260.0
dt = t[1]-t[0]
S=100.0
K=98.0
shares_per_day = 1
gearing = 1.0
BL_KO=102
mu=0.0
```

The figure below depicts the probability of knocking out the up barrier over time. As $t \to \infty$ the $P(S \geq BL_{KO}) \to \frac{1}{2})$

In [468]:
```python
p_ko_t = [(p_hit_before_time(i,sigma,S,BL_KO,mu))  for i in t]
plt.plot(t,p_ko_t, label="knock out probability over time")
plt.legend()
```

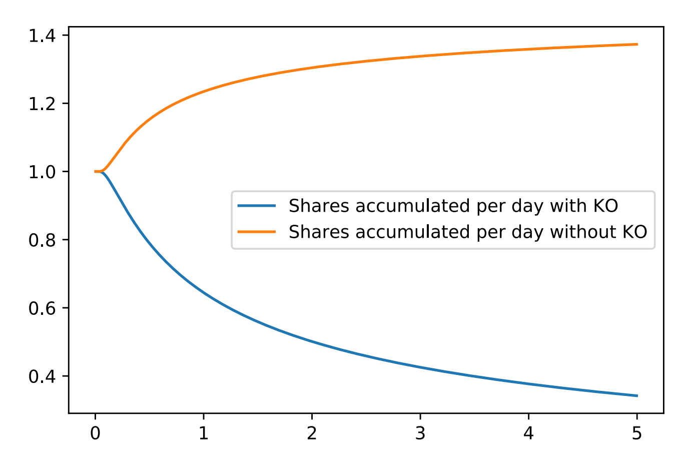
Out[468]: <matplotlib.legend.Legend at 0x7fdd6c19f110>



The number of shares accumulated each day with and without KO feature is depicted in the figure below

In [469]:
```python
shares_Accumulator_Each_day_with_KO = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))*(1.0-p_hit_before_time(i,sigm
a,S,BL_KO, mu))  for i in t]
shares_Accumulator_Each_day_without_KO = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))  for i in t]

plt.plot(t,shares_Accumulator_Each_day_with_KO,label="Shares accumulated per day with KO")
plt.plot(t,shares_Accumulator_Each_day_without_KO,label="Shares accumulated per day without KO")
plt.legend()
```

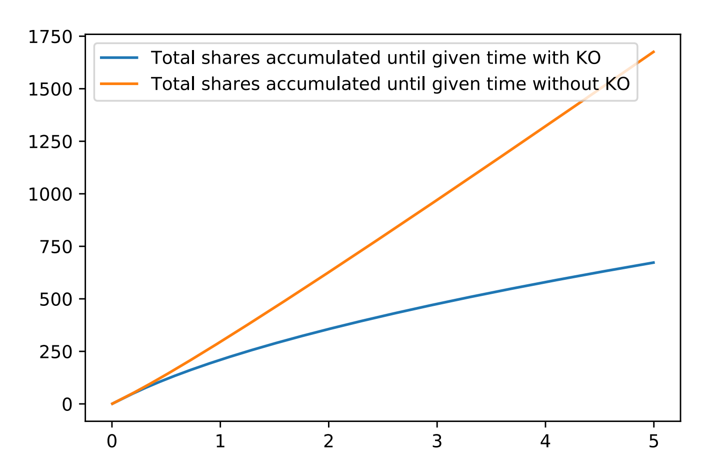
Out[469]: <matplotlib.legend.Legend at 0x7fdd6c3c9650>



The total number of share accumulated with and without KO feature is depicted in the figure below

In [470]:
```python
total_shares_Accumulator_Each_day_with_KO = np.cumsum(shares_Accumulator_Each_day_with_KO)
total_shares_Accumulator_Each_day_without_KO = np.cumsum(shares_Accumulator_Each_day_without_KO)

plt.plot(t,total_shares_Accumulator_Each_day_with_KO,label="Total shares accumulated until given time with KO")
plt.plot(t,total_shares_Accumulator_Each_day_without_KO,label="Total shares accumulated until given time without KO")
plt.legend()
```

Out[470]: <matplotlib.legend.Legend at 0x7fdd6c5f0850>



Accumulation simulation on S&P500

- S=4,509.37
- H_{gear} =S \times (1-0.05)$
- $\eta = 200 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 16\%$
- BL_{KO} =S\times (1+0.1)$
- VaR Factor = $\pm 2.35\sigma \times \sqrt{\frac{10}{260}}$

In [471]:
```python
year_sigma = 0.16
sigma = year_sigma/np.sqrt(260.0)
t = np.array(range(1,260))/260.0
dt = t[1]-t[0]
S=4509.37
K=S*(1-0.04)
shares_per_day = 5
gearing = 1.0
BL_KO=S*(1+0.1)
mu=0.0
VaR_factor = 2.32*year_sigma*np.sqrt(10/260)
Sup = S*(1+VaR_factor)
Sdown = S*(1-VaR_factor)
print("VaR="+str(VaR_factor))
print("S="+str(S))
print("Sup="+str(Sup))
print("Sdown="+str(Sdown))
```

```
VaR=0.07279830936329391
S=4509.37
Sup=4837.6445122935565
Sdown=4181.095487706443
```

Three simulation are performed to compute the up and down addons, namely:

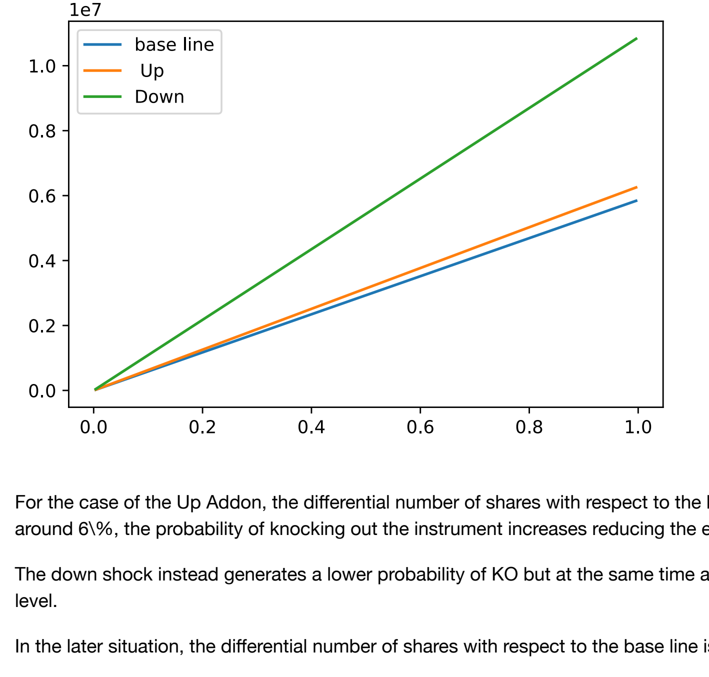
- Base Line simulation, $shares_{baseline}$, corresponding the the accumulated number of shares until maturity assuming the calculation date spot.
- Up Addon accumulated shares, $shares_{up}$, which is the accumulated number of shares for a shocked spot $S_{up} = S \times (1 + VaR)$
- Down Addon accumulated shares, $shares_{down}$, which is the accumulated number of shares for a shocked spot $S_{down} = S \times (1 - VaR)$

The up and down monetary addons are computed as follows:

- $UA = (shares_{up} - shares_{baseline}) \times S \times VaR$
- $DA = (shares_{down} - shares_{baseline}) \times S \times -VaR$

In [472]:
```python
base_addon_daily = S*np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))*(1.0-p_hit_before_time(i,sigma,S,BL_
KO, mu))  for i in t])
Up_Addon_daily = Sup*np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,Sup,K,0.0))*(1.0-p_hit_before_time(i,sigma,Sup
,BL_KO, mu))  for i in t])
Down_Addon_daily = Sdown * np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,Sdown,K,0.0))*(1.0-p_hit_before_time(i,s
igma,Sdown,BL_KO, mu))  for i in t])

plt.plot(t,base_addon_daily, label="base line")
plt.plot(t,Up_Addon_daily, label=" Up")
plt.plot(t,Down_Addon_daily, label="Down")
plt.legend()
```

Out[472]: <matplotlib.legend.Legend at 0x7fdd6c625350>



For the case of the Up Addon, the differential number of shares with respect to the base line is negative, since when the stock experiments a positive shock around 6\%, the probability of knocking out the instrument increases reducing the estimated number of shares at maturity.

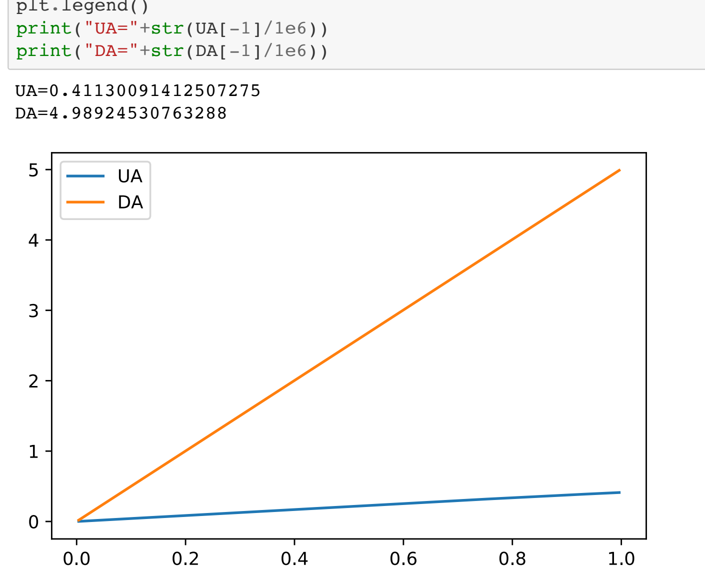
The down shock instead generates a lower probability of KO but at the same time an increased gearing probability, as the stock is closer to the gearing strike level.

In the later situation, the differential number of shares with respect to the base line is possitive and is depicted as the 'Down' curve in the figure above.

From a risk perspective, we conly consider in this Accumulator payoff the positive differential of shares with respect to the baseline, choosing then the down shock as the selected one.

In [473]:
```python
UA = Up_Addon_daily - base_addon_daily
DA = Down_Addon_daily - base_addon_daily
plt.plot(t,UA/1e6, label="UA")
plt.plot(t,DA/1e6, label="DA")
plt.legend()
print("UA="+str(UA[-1]/1e6))
print("DA="+str(DA[-1]/1e6))
```

```
UA=0.41130091412507275
DA=4.98924530763288
```



The increase losses incurred for a down shock of value $S \times (1 - VaR)$ is 5 millions.