Accumulator payoff pricing using analytical distribution for barrier and gearing strike hit.

We consider that the underlyign equity follows an exponntial Brownian movement, and hence its log returns can be model is a normal distribution:

$$\frac{\Delta S}{S} = \mu\delta T + \sigma(S,t)\sqrt{t}\Delta Z_t$$

Under this stochastic process the following probabilies expression hold:

1. Probability of hitting a Down trigger $P_{hit} = P(S \leq BL) = \Phi(\frac{\log BL - \mu T}{\sigma\sqrt{T}})$
2. Probability of hitting a Up trigger $P_{hit} = P(S \geq BL) = 1.0 - \Phi(\frac{\log BL - \mu T}{\sigma\sqrt{T}})$

   In the context of modelling an accumulator payoff with KO and gearing features, both events can be mapped to a barrier trigger affecting the number of equities accumulator over the future time horizon.

In [288]:
```
import numpy as np
import pandas as pd
from scipy.stats import norm
import  matplotlib.pyplot as plt
```

$P(S \leq BL)$

In [289]:
```
#P(S <= BL)
def p_hit(T,sigma,S,BL, mu):
    H = BL/S
    x = np.log(H)-mu*T
    sT = sigma*np.sqrt(T)
    return norm.cdf(x/sT)
```

$P(\tau \leq T)$

In [290]:
```
def p_hit_before_time(T,sigma,S,K, mu):
    H = K/S
    if H>1:
        H = 1.0/H
        mu=-mu
        tmp = S
        S=K
        K=tmp

    lnK = np.log(H)
    s2 = sigma**2
    sqrtT = np.sqrt(T)
    sigma05T = sigma*sqrtT
    num = np.exp(2*mu*lnK/s2)*norm.cdf((lnK+mu*T)/sigma05T)
    den = norm.cdf((lnK-mu*T)/sigma05T)+1e-12
    return (1+num/(den))*p_hit(T,sigma,S,K,mu)
```

$p(\tau = T)$

In [291]:
```
def p_hit_in_t(T,sigma,S,K, mu):
    H = K/S
    if H>1:
        H = 1.0/H
        mu=-mu
        tmp = S
        S=K
        K=tmp

    dt = 1.0/260.0
    a = p_hit_before_time(T+dt,sigma,S,K, mu)
    b = p_hit_before_time(T-dt,sigma,S,K, mu)

    return (a-b)/(2*dt)
```

Accumulator Payoff with only Gearing Events

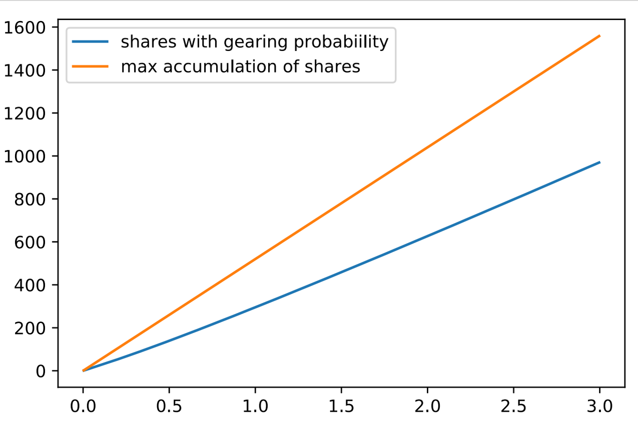
We study the case of an accumulator with a down gear strike level, H_{gear}, which increases the number of accumlated shares per day η by a factor g while the spot stays below this trigger level.

$$V(t) = \sum_{i=1}^{N} \eta \times (1 + g \times P(S \leq H_{gear}))$$

Example Accumulator gearing and without KO

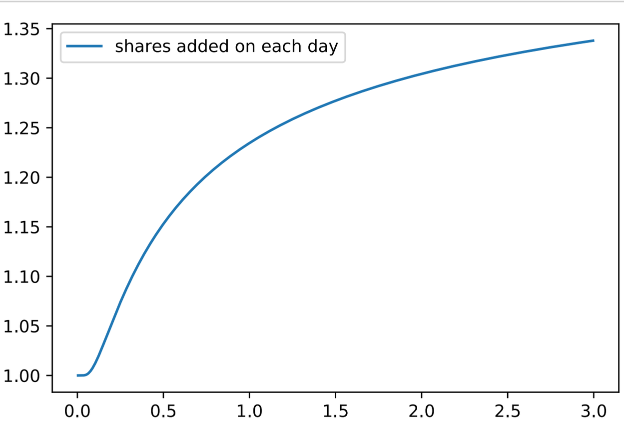
- S=100
- $H_{gear} = 98$
- $\eta = 1 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 45\%$

In [292]:
```
sigma = 0.45/np.sqrt(260.0)
t = np.array(range(1,3*260))/260.0
dt = t[1]-t[0]
S=100.0
K=98.0
shares_per_day = 1
gearing = 1.0
shares_Accumulator_Each_day = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))  for i in t]
max_shares_Accumulator_Each_per_day = [shares_per_day*(1+gearing)  for i in t]
max_shares_Accumulator_Each_day = np.cumsum([shares_per_day*(1+gearing)  for i in t])
total_accumulated_shares = np.cumsum(shares_Accumulator_Each_day)
plt.plot(t,total_accumulated_shares,label="shares with gearing probabiility")
plt.plot(t,max_shares_Accumulator_Each_day,label="max accumulation of shares")
plt.legend()

plt.show()
```



The picture above depicts the difference in between the full share accumulation assumption (over consevative) vs the gearing with hit probability.

In [293]:
```
plt.plot(t,shares_Accumulator_Each_day,label="shares added on each day")

plt.legend()
plt.show()
```



The curve above depicts the number of shares accumulated per day as the time passess. Since no drift is considered on the stock model, $\mu = 0$, to the 1 shares per day standard accumulation, the gearing adds one extra share with probability $P(S \leq H_{gear})$. When $t \to \infty$, the probability of hiting the gearing level tends to 50\%, meaning approximates to 1.5 shares per day as time tends to ∞.

Accumulator Payoff with Gearing and KO Events

We include the KO probability over time togethe with the feature already presented in the previous simulation.

In the current simulations the accumulation of shares is conditioned to the fact that the stock has not reached the KO barrier level BL_{KO}.

The probability of not being KO at time τ is simply defined as $P(\tau \geq T) = 1 - P(\tau \leq T)$, and the new expression of the accumulated number of shares at the maturity is:

$$V(T) = \sum_{i=1}^{N} \eta \times (1 + g \times P(S \leq H_{gear}, t_i)) \times P(\tau > t_i)$$

Example Accumulator gearing and KO

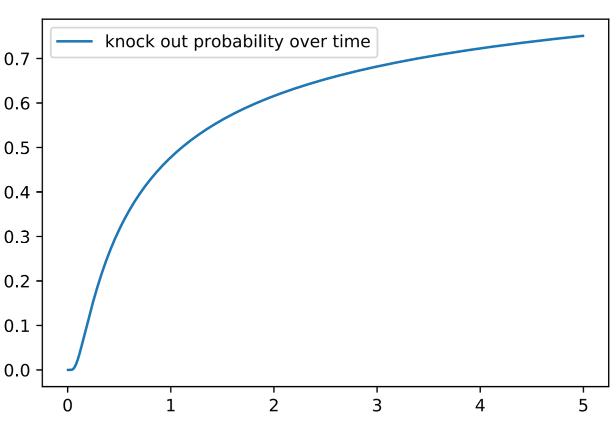
- S=100
- $H_{gear} = 98$
- $\eta = 1 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 45\%$
- $BL_{KO} = 102$

In [294]:
```
sigma = 0.45/np.sqrt(260.0)
t = np.array(range(1,5*260))/260.0
dt = t[1]-t[0]
S=100.0
K=98.0
shares_per_day = 1
gearing = 1.0
BL_KO=102
mu=0.0
```

The figure below depicts the probability of knocking out the up barrier over time. As $t \to \infty$ the $P(S \geq BL_{KO}) \to \frac{1}{2})$

In [295]:
```
p_ko_t = [(p_hit_before_time(i,sigma,S,BL_KO,mu))  for i in t]
plt.plot(t,p_ko_t, label="knock out probability over time")
plt.legend()
```

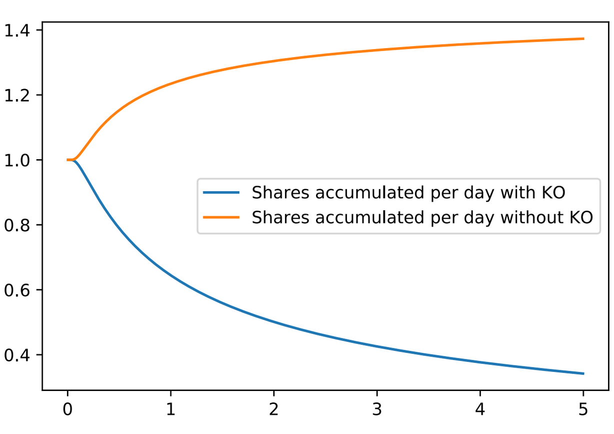
Out[295]: <matplotlib.legend.Legend at 0x7fdd65981310>



The number of shares accumulated each day with and without KO feature is depicted in the figure below

In [296]:
```
shares_Accumulator_Each_day_with_KO = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))*(1.0-p_hit_before_time(i,sigm
a,S,BL_KO, mu))  for i in t]
shares_Accumulator_Each_day_without_KO = [shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))  for i in t]

plt.plot(t,shares_Accumulator_Each_day_with_KO,label="Shares accumulated per day with KO")
plt.plot(t,shares_Accumulator_Each_day_without_KO,label="Shares accumulated per day without KO")
plt.legend()
```

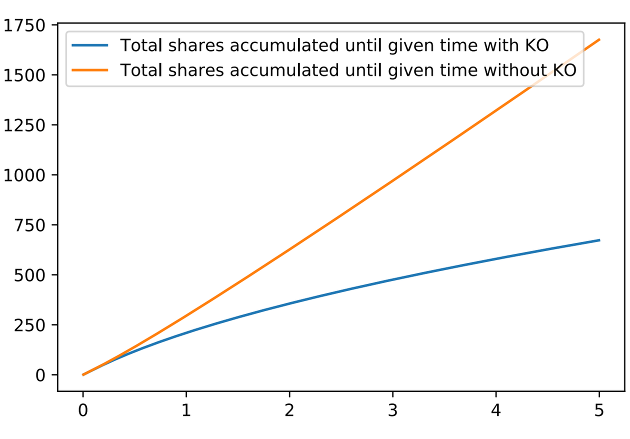
Out[296]: <matplotlib.legend.Legend at 0x7fdd65dcc710>



The total number of share accumulated with and without KO feature is depicted in the figure below

In [297]:
```
total_shares_Accumulator_Each_day_with_KO = np.cumsum(shares_Accumulator_Each_day_with_KO)
total_shares_Accumulator_Each_day_without_KO = np.cumsum(shares_Accumulator_Each_day_without_KO)

plt.plot(t,total_shares_Accumulator_Each_day_with_KO,label="Total shares accumulated until given time with KO")
plt.plot(t,total_shares_Accumulator_Each_day_without_KO,label="Total shares accumulated until given time without KO")
plt.legend()
```

Out[297]: <matplotlib.legend.Legend at 0x7fdd65ee2b90>



Accumulation simulation on S&P500

- S=4,509.37
- $H_{gear} = 4283.9$
- $\eta = 200 \times \frac{shares}{day}$
- $\mu = 0.0$
- $T = 1y$
- $\sigma = 25\%$
- $BL_{KO} = 4734.8$
- VaR Factor = 35\% $\times \sqrt{\frac{10}{260}}$

In [298]:
```
sigma = 0.25/np.sqrt(260.0)
t = np.array(range(1,260))/260.0
dt = t[1]-t[0]
S=4509.37
K=4283.9
shares_per_day = 40
gearing = 1.0
BL_KO=4734.8
mu=0.0
VaR_factor = 0.35*np.sqrt(10/260)
Sup = S*(1+VaR_factor)
Sdown = S*(1-VaR_factor)
```

Three simulation are performed to compute the up and down addons, namely:

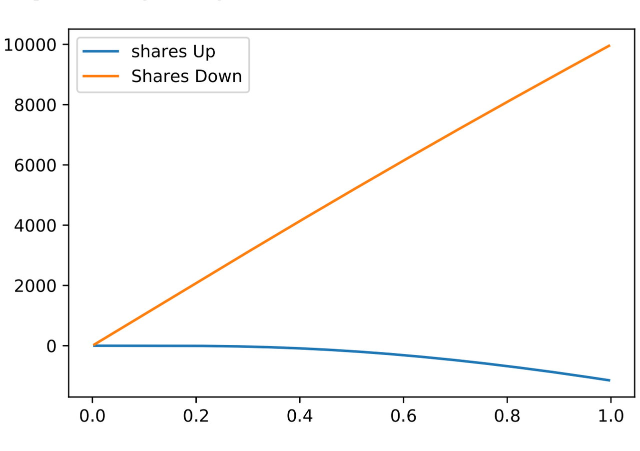
- Base Line simulation, $shares_{baseline}$, corresponding the the accumulated number of shares until maturity assuming the calculation date spot.
- Up Addon accumulated shares, $shares_{up}$, which is the accumulated number of shares for a shocked spot $S_{up} = S \times (1 + VaR)$
- Down Addon accumulated shares, $shares_{down}$, which is the accumulated number of shares for a shocked spot $S_{down} = S \times (1 - VaR)$

The up and down monetary addons are computed as follows:

- $UA = (shares_{up} - shares_{baseline}) \times S \times VaR$
- $DA = (shares_{down} - shares_{baseline}) \times S \times -VaR$

In [299]:
```
base_acc_shares = np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,S,K,0.0))*(1.0-p_hit_before_time(i,sigma,S,BL_KO,
mu))  for i in t])
Up_Addon_acc_shares = np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,Sup,K,0.0))*(1.0-p_hit_before_time(i,sigma,Su
p,BL_KO, mu))  for i in t])-base_acc_shares
Down_Addon_acc_shares = np.cumsum([shares_per_day*(1+gearing*p_hit(i,sigma,Sdown,K,0.0))*(1.0-p_hit_before_time(i,sigm
a,Sdown,BL_KO, mu))  for i in t])-base_acc_shares

plt.plot(t,Up_Addon_acc_shares, label="shares Up")
plt.plot(t,Down_Addon_acc_shares, label="Shares Down")
plt.legend()
```

Out[299]: <matplotlib.legend.Legend at 0x7fdd6682e890>



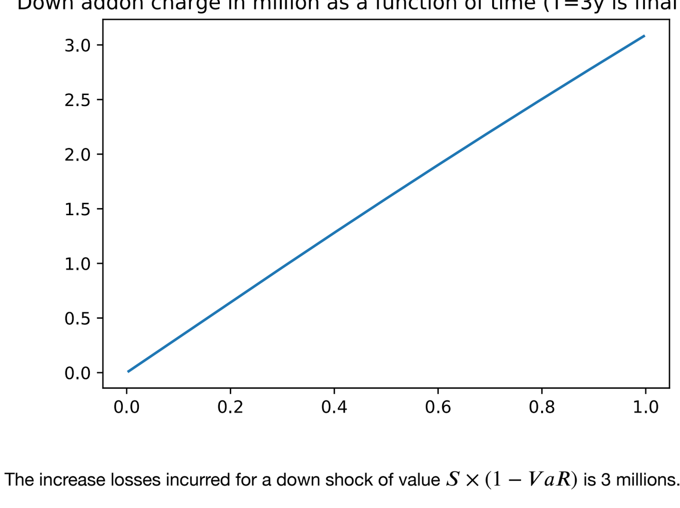
For the case of the Up Addon, the differential number of shares with respect to the base line is negative, since when the stock of shocked up, the probability of knocking out the instrument increases reducing the estimated number of share at maturity.

The down shock instead generated a lower probability of KO but at the same time an increase gearing probability, as the stock is closer to the gearing strike level. In this later situation the differential number of shares with respect to the base line is possitive and is depicted in the 'orange' curve in the figure above.

From a risk perspective, we conly consider in this Accumulator payoff the positive differential of shares with respect to the baseline, choosing then the down shock as the selected one.

In [303]:
```
Addon_down_money = Down_Addon_acc_shares*VaR_factor*S
plt.plot(t,Addon_down_money/1e6)
plt.title("Down addon charge in million as a function of time (T=3y is final addon)")
print("Addon load = " + str(Addon_down_money[-1]/1e6))
```

Addon load = 3.080210518471231



The increase losses incurred for a down shock of value $S \times (1 - VaR)$ is 3 millions.