$\boxed{\text{PGAS INTEGRATING OUT } H}$

$$w_t(x_{1:t}) = \frac{p(x_{1:t}|y_{1:t})}{p(x_{1:t-1}|y_{1:t-1})\; r_t(x_t|x_{1:t-1})}$$

$$\propto \frac{p(y_{1:t}|x_{1:t})\;\cancel{p(x_t|x_{1:t-1})}\;\cancel{p(x_{1:t-1})}}{p(y_{1:t-1}|x_{1:t-1})\;\cancel{p(x_{1:t-1})}\;\cancel{p(x_t|x_{1:t-1})}}$$

$$\propto \frac{mlik(x_{1:t})}{mlik(x_{1:t-1})}$$

$$\tilde{w}^{i}_{t-1|T} = w^{i}_{t-1}\,\frac{p(x^{i}_{1:t-1}, x'_{t:T}|y_{1:T})}{p(x^{i}_{1:t-1}|y_{1:t-1})}$$

$$\propto w^{i}_{t-1}\cdot p(x'_t|x^{i}_{t-1})\,\frac{mlik(x^{i}_{1:t-1}, x'_{t:T})}{mlik(x^{i}_{1:t-1})}$$

This is practical only if we can compute mlik ratios in reasonable time.

mLik

$$\underline{h}^{(d)} = \begin{pmatrix} (h_1^1)_d \\ \vdots \\ (h_1^L)_d \\ (h_2^1)_d \\ \vdots \\ (h_2^L)_d \\ \vdots \\ (h_M^1)_d \\ (h_M^L)_d \end{pmatrix}_{ML\times 1} \qquad \underset{t\times 1}{\underline{y}_t^{(d)}} = \begin{pmatrix} y_1^{(d)} \\ \vdots \\ y_t^{(d)} \end{pmatrix}$$

$$\underset{t\times ML}{X_t^{ext}} = \left(X_t^{(1)}, \dots, X_t^{(M)} \right)$$

$$X_t^{(m)} = \underset{t\times L}{\begin{pmatrix} x_{1m} & 0 & \cdots & 0 \\ x_{2m} & x_{1m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{tm} & x_{(t-1)m} & \cdots & x_{(t-L+1)m} \end{pmatrix}}$$

$$p\left(\underline{h}^{(d)}\right) = \frac{1}{\pi^{ML} (\sigma_1^2)^M \cdots (\sigma_L^2)^M} \exp\left\{ - \left\| \Gamma_{PRIOR}^{-1/2} \underline{h}^{(d)} \right\|^2 \right\}$$

$$\Gamma_{PRIOR} = I_{M\times M} \otimes \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_L^2 \end{pmatrix}$$

$$p\left(\underline{y}_t^{(d)} \,\middle|\, \underline{h}^{(d)}, X_t^{ext}\right) = \frac{1}{\pi^t (\sigma_y^2)^t} \exp\left\{ -\frac{1}{\sigma_y^2} \left\| \underline{y}_t^{(d)} - X_t^{ext} \underline{h}^{(d)} \right\|^2 \right\}$$

$$p\left(\underline{y}_t^{(d)} \,\middle|\, X_t^{ext}\right) = \int p\left(\underline{y}_t^{(d)} \,\middle|\, \underline{h}^{(d)}, X_t^{ext}\right) p\left(\underline{h}^{(d)}\right) d\underline{h}^{(d)}$$

$$= \int \frac{1}{\pi^{ML} (\sigma_1^2)^M \cdots (\sigma_L^2)^M} \cdot \exp\left\{ - \left\| \Gamma_{PRIOR}^{-1/2} \underline{h}^{(d)} \right\|^2 \right\}$$

$$\times \frac{1}{\pi^t (\sigma_y^2)^t} \exp\left\{ -\frac{1}{\sigma_y^2} \left\| \underline{y}_t^{(d)} - X_t^{ext} \underline{h}^{(d)} \right\|^2 \right\} d\underline{h}^{(d)}$$

$$\boxed{\begin{array}{l} \underline{h}^{(d)} \to \underline{h} \\ \Gamma_{PRIOR} \to \Gamma \\ \underline{y}_t^{(d)} \to \underline{y} \\ X_t^{ext} \to X \end{array}}$$

$$= \frac{1}{\pi^{ML} \pi^t (\sigma_1^2 \cdots \sigma_L^2)^M (\sigma_y^2)^t} \int \exp\left\{ - \underline{h}^H \Gamma^{-1} \underline{h} - \frac{1}{\sigma_y^2} \underline{y}^H \underline{y} - \underline{h}^H X^H X \underline{h} \frac{1}{\sigma_y^2} + \frac{1}{\sigma_y^2} \underline{y}^H X \underline{h} + \frac{1}{\sigma_y^2} \underline{h}^H X^H \underline{y} \right\} d\underline{h}$$

$$\mu_{POST} = \frac{1}{\sigma_y^2} \Gamma_{POST} \left(X^{ext}\right)^H \underline{y}^{(d)}$$

$$\Gamma_{POST} = \left(\Gamma_{PRIOR}^{-1} + \frac{1}{\sigma_y^2} \left(X_t^{ext}\right)^H X_t^{ext} \right)^{-1}$$

$$= \frac{1}{\pi^{ML} \pi^t (\sigma_1 \cdots \sigma_L)^M (\sigma_y^2)^t} e^{-\frac{1}{\sigma_y^2} \underline{y}^H \underline{y}}$$

$$\times \int e^{-\underline{h}^H \left(\Gamma^{-1} + X^H X \frac{1}{\sigma_y^2}\right) \underline{h}} \; e^{\frac{1}{\sigma_y^2}\left(\underline{y}^H X \underline{h} + \underline{h}^H X^H \underline{y}\right)} \, d\underline{h}$$

$$= \frac{1}{\pi^{ML} \pi^t (\sigma_1 \cdots \sigma_L)^M (\sigma_y^2)^t} e^{-\frac{1}{\sigma_y^2} \underline{y}^H \underline{y}} \times e^{+\mu_{POST}^H \Gamma_{POST}^{-1} \mu_{POST}}$$

$$\int e^{-\left\| \Gamma_{POST}^{-1/2} \left(\underline{h} - \mu_{POST}\right) \right\|^2} d\underline{h}$$

$$= \frac{\left\| \det\left(\Gamma_{POST}\right) \right\|}{\pi^t (\sigma_1 \cdots \sigma_L)^M (\sigma_y^2)^t} e^{-\frac{1}{\sigma_y^2} \underline{y}^H \underline{y}} \; e^{\frac{1}{\sigma_y^4} \underline{y}^H X \Gamma_{POST} X^H \underline{y}}$$

Hence:

$$W_t(x_{1:t}) \propto \frac{\dfrac{\left|\det\left(\Gamma_{POST}(X_t)\right)\right|}{\cancel{\pi^t (\sigma_1 \cdots \sigma_L)^M (\sigma_y^2)^t}} e^{-\frac{1}{\sigma_y^2} \sum_{i=1}^{t} \|y_t^{(d)}\|^2} \; e^{+\frac{1}{\sigma_y^4} \overset{1\times t}{\underline{y}_t^H} \overset{t\times ML}{X_t} \overset{ML\times ML}{\Gamma_{POST}} \overset{ML\times t}{X_t^H} \overset{t\times 1}{\underline{y}_t}}}{\dfrac{\left|\det\left(\Gamma_{POST}(X_{t-1})\right)\right|}{\cancel{\pi^{t-1} (\sigma_1 \cdots \sigma_L)^M (\sigma_y^2)^{t-1}}} e^{-\frac{1}{\sigma_y^2} \sum_{i=1}^{t-1} \|y_t^{(d)}\|^2} \; e^{+\frac{1}{\sigma_y^4} \underset{1\times(t-1)}{\underline{y}_{t-1}^H} \underset{(t-1)\times ML}{X_{t-1}} \underset{ML\times ML}{\Gamma_{POST}} \underset{ML\times(t-1)}{X_{t-1}^H} \underset{(t-1)\times 1}{\underline{y}_{t-1}}}}$$

$$\propto \underbrace{\frac{1}{\pi \sigma_y^2} e^{-\frac{1}{\sigma_y^2} |y_t^{(d)}|^2}}_{\substack{\text{These terms don't}\\ \text{depend on the particle}\\ \text{index } i \text{ (can be removed)}}} e^{\frac{1}{\sigma_y^4}\left(\underline{y}_t^H X_t \Gamma_{POST}(X_t) X_t^H \underline{y}_t - \underline{y}_{t-1}^H X_{t-1} \Gamma_{POST}(X_{t-1}) X_{t-1}^H \underline{y}_{t-1} \right)}$$

$$\times \left| \frac{\det\left(\Gamma_{POST}(X_t^{ext})\right)}{\det\left(\Gamma_{POST}(X_{t-1}^{ext})\right)} \right|$$