FFBS FOR ZOUBIN'S MODEL (iICA FOR SPEAKER DIARIZ.)

Each $\underline{y}_t = \sum_{m=1}^{M} x_{tm} \underline{w}_m + \underline{n}_t$

$x_{tm} \sim \text{Laplace}(0, b) = \frac{1}{2b} e^{-\frac{|x|}{b}}$

- At each step of the FFBS, all chains are fixed except one. Define the pseudoobservations:

$$\boxed{\underline{\tilde{y}}_t = \underline{y}_t - \sum_{m' \neq m} x_t^{m'} \underline{w}^{m'}}$$

- To apply the FFBS over the m-th chain, we need:

$p(s_{tm} | s_{(t-1)m})$ ← Transition probs. given by a^m, b^m

$p(\underline{\tilde{y}}_t | s_{tm})$ ← The likelihood of $\underline{\tilde{y}}_t$ given s_{tm} (a 2×1 vector, for $s_{tm} = \{0, 1\}$).

- We focus on computing $p(\underline{\tilde{y}}_t | s_{tm})$

  - If $s_{tm} = 0$, then $\underline{\tilde{y}}_t \sim \mathcal{N}(\underline{\tilde{y}}_t | \underline{0}, \sigma_y^2 I_{D\times D})$
  - If $s_{tm} = 1$, we need to integrate out x_{tm}:

$$p(\underline{\tilde{y}}_t | s_{tm}=1) = \int \overset{\text{(GAUSS)}}{p(\underline{\tilde{y}}_t | x_{tm}, s_{tm}=1)} \, \overset{\text{(LAPLACE)}}{p(x_{tm} | s_{tm}=1)} \, dx_{tm}$$

$$= \int \frac{1}{(2\pi)^{D/2} (\sigma_y^2)^{D/2}} e^{-\frac{1}{2\sigma_y^2} (\underline{\tilde{y}}_t - x_{tm} \underline{w}^m)^\top (\underline{\tilde{y}}_t - x_{tm} \underline{w}^m)} \times$$

$$\times \frac{1}{2b} e^{-\frac{|x_{tm}|}{b}} dx_{tm}$$

$$= \frac{1}{(2\pi\sigma_y^2)^{D/2}} \cdot \frac{1}{2b} \times \left(\int_{-\infty}^{0} e^{-\frac{1}{2\sigma_y^2}\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)^T\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)} e^{\frac{x_{tm}}{b}} \, dx_{tm} \right.$$

$$\left. + \int_{0}^{+\infty} e^{-\frac{1}{2\sigma_y^2}\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)^T\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)} e^{-\frac{x_{tm}}{b}} \, dx_{tm} \right)$$

$$= \frac{1}{(2\pi\sigma_y^2)^{D/2}} \frac{1}{2b} \cdot \left(I_1 + I_2\right)$$

$$I_1 = \int_{-\infty}^{0} e^{-\frac{1}{2\sigma_y^2}\underline{\tilde{y}}_t^T\underline{\tilde{y}}_t - \frac{1}{2\sigma_y^2} x_{tm}^2 \underline{w}^{m^T}\underline{w}^m + \frac{1}{\sigma_y^2} x_{tm}\underline{w}^{m^T}\underline{\tilde{y}}_t + \frac{x_{tm}}{b}} \, dx_{tm}$$

$$= e^{-\frac{1}{2\sigma_y^2}\underline{\tilde{y}}_t^T\underline{\tilde{y}}_t} \int_{-\infty}^{0} e^{-\frac{1}{2}\frac{1}{\sigma_y^2/\underline{w}^{m^T}\underline{w}^m}\left(x_{tm} - \frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}\left(\frac{1}{\sigma_y^2}\underline{w}^{m^T}\underline{\tilde{y}}_t + \frac{1}{b}\right)\right)^2}$$

$$\times e^{+\frac{1}{2}\frac{1}{\sigma_y^2/\underline{w}^{m^T}\underline{w}^m}\left(\frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}\left(\frac{1}{\sigma_y^2}\underline{w}^{m^T}\underline{\tilde{y}}_t + \frac{1}{b}\right)\right)^2} dx_{tm}$$

$$= e^{-\frac{1}{2\sigma_y^2}\underline{\tilde{y}}_t^T\underline{y}_t} e^{\frac{1}{2}\frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}\left(\frac{1}{\sigma_y^2}\underline{w}^{m^T}\underline{\tilde{y}}_t + \frac{1}{b}\right)^2} \cdot \sqrt{2\pi}\sqrt{\frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}}$$

$$\times \int_{-\infty}^{0} N\left(x_{tm} \,\middle|\, \underbrace{\frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}\left(\frac{1}{\sigma_y^2}\underline{w}^{m^T}\underline{\tilde{y}}_t + \frac{1}{b}\right)}_{\text{mean}}, \underbrace{\frac{\sigma_y^2}{\underline{w}^{m^T}\underline{w}^m}}_{\text{variance}}\right) dx_{tm}$$

Define:

$$\boxed{\begin{aligned} \sigma_{I1}^2 &= \frac{\sigma_y^2}{\underline{w}^{m T} \underline{w}^m} \\ \mu_{I1} &= \sigma_{I1}^2 \cdot \frac{1}{\sigma_y^2}\left(\underline{w}^{m T} \underline{\tilde{y}}_t + \frac{1}{b}\right) \end{aligned}}$$

Hence:

$$I_1 = e^{-\frac{1}{2\sigma_y^2} \underline{\tilde{y}}_t^T \underline{\tilde{y}}_t} e^{\frac{1}{2\sigma_{I1}^2} \mu_{I1}^2} \cdot \text{normcdf}\left(0, \mu_{I1}, \sqrt{\sigma_{I1}^2}\right)$$

For I_2:

$$I_2 = \int_0^{+\infty} e^{-\frac{1}{2\sigma_y^2} \underline{\tilde{y}}_t^T \underline{\tilde{y}}_t - \frac{1}{2\sigma_y^2} x_{tm}^2 \underline{w}^{m T} \underline{w}^m + \frac{1}{\sigma_y^2} x_{tm} \underline{w}^{m T} \underline{\tilde{y}}_t - \frac{x_{tm}}{b}} \, dx_{tm}$$

Similarly, we get:

$$I_2 = e^{-\frac{1}{2\sigma_y^2} \underline{\tilde{y}}_t^T \underline{\tilde{y}}_t} e^{\frac{1}{2\sigma_{I2}^2} \mu_{I2}^2} \cdot \left(1 - \text{normcdf}\left(0, \mu_{I2}, \sqrt{\sigma_{I2}^2}\right)\right)$$

where

$$\boxed{\begin{aligned} \sigma_{I2}^2 = \sigma_{I1}^2 &= \frac{\sigma_y^2}{\underline{w}^{m T} \underline{w}^m} \\ \mu_{I2} &= \sigma_{I2}^2 \frac{1}{\sigma_y^2}\left(\underline{w}^{m T} \underline{\tilde{y}}_t - \frac{1}{b}\right) \end{aligned}}$$

~~Finally:~~

$$\cancel{p(\underline{\tilde{y}}_t | s_{tm}=1) = \frac{1}{(2\pi\sigma_y^2)^{D/2}} \frac{1}{2b} e^{-\frac{1}{2\sigma_y^2} \underline{\tilde{y}}_t^T \underline{\tilde{y}}_t} \times \left(e^{\frac{1}{2\sigma_{I1}^2} \mu_{I1}^2} \text{normcdf}\left(0, \mu_{I1}, \sqrt{\sigma_{I1}^2}\right) + e^{\frac{1}{2\sigma_{I2}^2} \mu_{I2}^2} \left(1 - \text{normcdf}\left(0, \mu_{I2}, \sqrt{\sigma_{I2}^2}\right)\right) \right)}$$

Finally:

$$\boxed{p(\tilde{\underline{y}}_t \mid s_{tm}=1) = \frac{1}{(2\pi\sigma_y^2)^{D/2}\cdot 2b} e^{-\frac{1}{2\sigma_y^2}\tilde{\underline{y}}_t^T \tilde{\underline{y}}_t} \times \left(e^{\frac{1}{2\sigma_{I1}^2}\mu_{I1}^2} \cdot \mathrm{normcdf}(0, \mu_{I1}, \sqrt{\sigma_{I1}^2}) + e^{\frac{1}{2\sigma_{I2}^2}\mu_{I2}^2}\left(1-\mathrm{normcdf}(0,\mu_{I2},\sqrt{\sigma_{I2}^2})\right)\right)}$$

POSTERIOR FOR x_{tm}

After sampling $\{s_{tm}\}_{t=1}^T$ for the m-th chain, we resample $\{x_{tm}\}_{t=1}^T$ from the posterior. If $s_{tm}=0$, then $x_{tm}=0$ w.p. 1. Otherwise, we need to sample x_{tm} from

$$p(x_{tm} \mid s_{tm}=1, \tilde{\underline{y}}_t) \quad \text{for } t=1,\dots,T$$

$$\begin{aligned} p(x_{tm} \mid s_{tm}=1, \tilde{\underline{y}}_t) &\propto p(\tilde{\underline{y}}_t \mid x_{tm}, s_{tm}=1)\, p(x_{tm} \mid s_{tm}=1) \\ &\propto e^{-\frac{1}{2\sigma_y^2}(\tilde{\underline{y}}_t - x_{tm}\underline{w}^m)^T(\tilde{\underline{y}}_t - x_{tm}\underline{w}^m)}\, e^{-\frac{|x_{tm}|}{b}} \\ &\propto e^{-\frac{1}{2\sigma_y^2}(\tilde{\underline{y}}_t - x_{tm}\underline{w}^m)^T(\tilde{\underline{y}}_t - x_{tm}\underline{w}^m)} \left(e^{-\frac{x_{tm}}{b}} \mathbb{I}(x_{tm}\geq 0) + e^{\frac{x_{tm}}{b}} \mathbb{I}(x_{tm}<0) \right) \end{aligned}$$

- We first sample if $x_{tm} \geq 0$ or $x_{tm} < 0$.

$$Pr\left(x_{tm} \geq 0 \mid s_{tm}=1, \underline{\tilde{y}}_t\right) \propto I_2$$

$$Pr\left(x_{tm} < 0 \mid s_{tm}=1, \underline{\tilde{y}}_t\right) \propto I_1$$

Therefore, we compute

$$I_2 \propto e^{\frac{1}{2\sigma_{I2}^2}\mu_{I2}^2}\left(1-\mathrm{normcdf}\left(0, \mu_{I2}, \sqrt{\sigma_{I2}^2}\right)\right) \triangleq P_2$$

$$I_1 \propto e^{-\frac{1}{2\sigma_{I1}^2}\mu_{I1}^2} \cdot \mathrm{normcdf}\left(0, \mu_{I1}, \sqrt{\sigma_{I1}^2}\right) \triangleq P_1 \qquad (*)$$

We sample if $x_{tm} \geq 0$ or $x_{tm} < 0$ with probabilities:

$$\begin{cases} x_{tm} \geq 0 & \text{w.p. } \dfrac{P_2}{P_1+P_2} \\ x_{tm} < 0 & \text{w.p. } \dfrac{P_1}{P_1+P_2} \end{cases}$$

where P_1, P_2 are defined in $(*)$

- Now we sample x_{tm}:

-If $x_{tm} \geq 0$, then:

$$\boxed{p\left(x_{tm} \mid s_{tm}=1, \underline{\tilde{y}}_t, x_{tm} \geq 0\right)} \propto e^{-\frac{1}{2\sigma_y^2}\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)^T\left(\underline{\tilde{y}}_t - x_{tm}\underline{w}^m\right)} e^{-\frac{x_{tm}}{b}}$$

$$\boxed{\propto N\left(x_{tm} \mid \mu_{I2}, \sigma_{I2}^2\right) \mathbb{I}\left(x_{tm} \geq 0\right)}$$

(Truncated Gaussian)

-If $x_{tm} < 0$, then:

$$\boxed{p\left(x_{tm} \mid s_{tm}=1, \underline{\tilde{y}}_t, x_{tm} < 0\right) \propto N\left(x_{tm} \mid \mu_{I1}, \sigma_{I1}^2\right) \mathbb{I}\left(x_{tm} < 0\right)}$$