1 Model formulation

Let θ_i denote the strength of an opinion for the ith agent of a set of N agents, noting that these N agents can be part of groups which are subsets (each with population n_g) of the total population. Consider that for a given agent, we try to express the rate of change of an opinion as

$$\dot{\theta}_i = \sum_{g=1}^{G} w_g \sum_{j=1, j\neq i}^{n_g} f_g(\theta_j) \tag{1}$$

where f_g is a group-specific function defining how an individual processes the opinion of other individuals within a given group g to form their own opinion, while w_g is the weight they attribute to the overall opinion-forming effect of a given group. Then, if f_g is a linear combination of the opinions ("θ_j"s) of the group, such as $f_g = c_j\theta_j$, we can rewrite eq. (1) above as

$$\dot{\theta}_i = \sum_{g=1}^{G} w_g \sum_{j=1, j\neq i}^{n_g} c_j\theta_j. \tag{2}$$

Now, let us define our set of agents' opinions (θ) and their rate of change ($\dot{\theta}$) as two vectors

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ ... \\ \theta_N \end{bmatrix} \qquad \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta_1} \\ \dot{\theta_2} \\ ... \\ \dot{\theta_N} \end{bmatrix} \tag{3}$$

If we assume that groups are fixed and can overlap, this means that the opinion of the kth member of the hth group can reappear in $\dot{\theta}_i$ as the opinion of the lth member of the mth group. Mathematically, we can express this by resorting to superscripts to denote group membership and subscripts to denote identity within that group, such that $\theta_h^{(k)} = \theta_m^{(l)}$. To illustrate this point, consider the expanded double sum (with these variables bolded for emphasis):

$$\begin{aligned}
\dot{\theta}_i =& \sum_{g=1}^{G} w_g \sum_{j=1, j\neq i}^{n_g} c_j^{(g)}\theta_j^{(g)} && (4)\\
=& \sum_{g=1}^{G} w_g \left[c_1^{(g)}\theta_1^{(g)} + ... + c_j^{(g)}\theta_j^{(g)} + ... + c_{n_g}^{(g)}\theta_{n_g}^{(g)} \right] && (5)\\
=& w_1 \left[c_1^{(1)}\theta_1^{(1)} + ... + c_{n_1}^{(1)}\theta_{n_1}^{(1)} \right] + ... && (6)\\
&+ w_m \left[c_1^{(m)}\theta_1^{(m)} + ... + c_l^{(m)}\boldsymbol{\theta_l^{(m)}} + ... + c_{n_m}^{(m)}\theta_{n_m}^{(m)} \right] && (7)\\
&+ w_h \left[c_1^{(h)}\theta_1^{(h)} + ... + c_k^{(h)}\boldsymbol{\theta_k^{(h)}} + ... + c_{n_1}^{(h)}\theta_{n_1}^{(h)} \right] && (8)\\
&+ ... && (9)\\
&+ w_G \left[c_1^{(G)}\theta_1^{(G)} + ... + c_{n_1}^{(G)}\theta_{n_1}^{(G)} \right] && (10)\\
& && (11)
\end{aligned}$$

From knowledge about group membership, we can substitute $\theta_h^{(k)} = \theta_m^{(l)} = \theta_n$, i.e. identify these terms as the opinion of the nth agent. If this is done for every agent, we can then write the rate of change of the opinion of the ith agent as

$$\dot{\theta}_i = \sum_{j=1, j\neq i}^{N} \alpha_j \theta_j \tag{12}$$

where here each α_j is the sum of the group weighted opinion valuation of the jth agent. For example, assuming this nth agent from earlier had no other group membership than being the kth member of the hth group and the lth member of the mth group, then $\alpha_n = w_m c_l^{(m)} + w_h c_k^{(h)}$.

This reformulation in terms of "α_j"s makes $\dot{\theta}_i$ a *linear* combination of "θ_j"s, and if we further define $\alpha_i = 0$, we can define a vector

$$\boldsymbol{\alpha}^{(i)} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_{i-1} \\ 0 \\ \alpha_{i+1} \\ \ldots \\ \alpha_N \end{bmatrix} \tag{13}$$

so that $\dot{\theta}_i = \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\theta}$, i.e. the dot-product of the ith agent's opinion-weight vector and the opinion vector. If, now, we turn to the system as a whole, we can write the "opinion dynamics"

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{\alpha}^{(1)} \\ \boldsymbol{\alpha}^{(2)} \\ \ldots \\ \boldsymbol{\alpha}^{(i)} \\ \ldots \\ \boldsymbol{\alpha}^{(N)} \end{bmatrix} \boldsymbol{\theta} = \mathbf{A}\boldsymbol{\theta} \tag{14}$$

which is a traditional linear time-invariant system characterized by the matrix $\mathbf{A}$ which encodes group membership and opinion-weighing (at the group and group-membership level). For these systems, resolved in continuous time, we find (after some extension of 1-variable ordinary differential equations) that for some initial state of population opinion $\boldsymbol{\theta}(t=0)$ the solution for some finite time t is

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0)\exp(\mathbf{A}t) \tag{15}$$

where $\exp(\mathbf{A}t)$ is the matrix exponential. Computing it usually involves various decompositions of the matrix $\mathbf{A}$. The properties of the solution (stability, etc) then depend on the properties of $\mathbf{A}$. One usual way to understand the solution $\boldsymbol{\theta}(t)$ is as a *linear* combination of eigenvectors of $\mathbf{A}$, each of which represent a principal "mode" of behavior of the system.

2 Takeaways

1. This whole linear algrebra construction assumes fixed group membership. If group membership varies, it is very likely that the expression for $\dot{\theta}_i$ would involve products of functions of θ, at which point $\dot{\boldsymbol{\theta}}$ could not be expressed as a *linear* combination of terms and so this analytic solution would break down
2. The other strong assumption is that the micro-model of opinion evaluation, here defined as "f_g" is *linear*, i.e. a proportional scaling of values of θ

3. A lot of the ”mess” of accounting for group membership in forming ”α_j”s was swept under the rug, but would need to be dealt with for even a general form of matrix-based implementation of the model
4. This does not ”implement” the saturation of an agent’s opinion, which here would means strictly bounding θ_i between $[0, 1]$ (a normalized range). Such bounding would change the overall dynamics since it bounds the potential contribution of each agent to another’s opinion.