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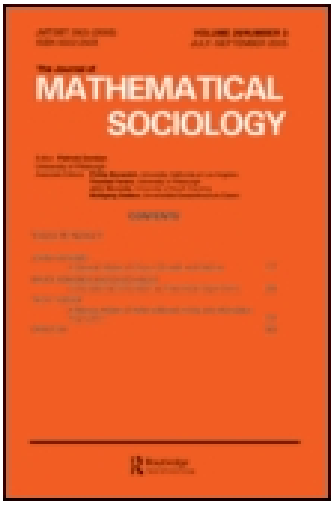

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Dissonance Minimization as a Microfoundation of Social Influence in Models of Opinion Formation

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DISSONANCE MINIMIZATION AS A MICROFOUNDATION OF SOCIAL INFLUENCE IN MODELS OF OPINION FORMATION

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Models of opinion formation are used to investigate many collective phenomena. While social influence often constitutes a basic mechanism, its implementation differs between the models. In this article, we provide a general framework of social influence based on dissonance minimization. We only premise that individuals strive to minimize dissonance resulting from different opinions compared to individuals in a given social network. Within a game theoretic context, we show that our concept of dissonance minimization resembles a coordination process when interactions are homogeneous. We further show that different models of opinion formation can be represented as best response dynamics within our framework. Thus, we offer a unifying perspective on these heterogeneous models and link them to rational choice theory.

Keywords: conventions, coordination, opinion dynamics, social influence

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1. INTRODUCTION

Models of opinion formation reflect the evolution of opinions, behaviors, beliefs or attitudes (in the following only referred to as "opinions"). These models are used to study collective phenomena, such as the formation of consensus and bipolarization (Hegselmann & Krause, 2002; Deffuant, Neau, Amblard, & Weisbuch, 2000), minority opinion spreading (Galam, 2002; Tessone, Toral, Amengual, Wio, & San Miguel, 2004) and the emergence of political parties (Ben-Naim, 2005) or extremism (Deffuant, Neau, Amblard, & Weisbuch, 2002) Real world processes of collective opinion formation include "wisdom of crowds" (Surowiecki, 2004) due to "swarm intelligence" (Bonabeau, Dorigo, & Theraulaz, 1999) or imperfect prediction markets (K.-Y. Chen, Fine, & Huberman, 2004),

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decision making in committees of experts (Visser & Swank, 2007) or participation processes in democratic societies (de Sousa Santos, 1998).[1]

In multi-agent models of opinion formation, opinions evolve according to prescribed rules of social interaction. Often the value of holding an opinion only depends on the opinions of others. The rules of social interaction and individual change differ between models as well as the set of possible opinions or the underlying interaction networks. However, there are also common characteristics. Following Flache and Macy (2008) and Flache and Mäs (2008), one can identify two basic social psychological mechanisms in many models of opinion formation: *social influence* and *homophily*. According to social influence, interaction between individuals leads to the adaptation of their respective opinions (Abelson, 1964; Brass, Galaskiewicz, Greve, & Tsai, 1998; Kerr & Tindale, 2004; Strang & Soule, 1998); that is, interaction increases the similarity of opinions. Here, interaction takes place in a social network of individuals where agents interact locally with adjacent individuals (their *neighbors*).

The implementation of social influence in an opinion formation model usually depends on the set of possible opinions. In the model of Axelrod (1997), an opinion is a discrete vector representing an agent's attitudes to different aspects, and an agent randomly adopts a component of her counterpart's opinion during interaction. In the voter model introduced by Holley and Liggett (1975), opinions are binary, and agents choose an opinion with a probability that corresponds to its frequency in the agent's neighborhood.

If opinions are continuous (e.g., Abelson, 1964; DeGroot, 1974; Friedkin & Johnson, 1990; Davis, 1996; Krause, 2000; Deffuant et al., 2000), that is, real numbers or real vectors, adaptation is usually attained by a weighted average of the opinions of interacting agents. Lehrer and Wagner (1981) provide an axiomatic characterization, which concludes that weighted arithmetic averaging is often the only appropriate method of aggregation. Also, based on experiments, the theory of information integration (Anderson, 1971) poses weighted arithmetic averaging as a general mechanism of attitude change. For a fixed network, social influence induced by weighted averaging in a connected network always leads to consensus (i.e., all agents finally exhibit identical opinions) in the models of French (1956), Harary (1959), Abelson (1964), and DeGroot (1974). This lead to the following complaint by Abelson (1964): "Since universal ultimate agreement is an ubiquitous outcome of a very broad class of models, we are naturally led to inquire what on earth one must assume in order to generate the bimodal outcome of community cleavage studies" (p. 153). The model of Friedkin and Johnsen (1990) was partly developed as a response to that complaint. It will turn out that our framework of dissonance minimization usually does not imply ultimate agreement as the only stable outcome, although we pose dissonance to rise monotonically with distance and also model an agent's aggregation of dissonance with respect to neighbors as a weighted sum.

The fact that consensus among the actors is a fixed point of the respective dynamics in many models resembles a basic property of a coordination process towards a particular type of social norm, namely a *convention* (Lewis, 1969) or *coordination norm* (Ullmann-Margalit, 1977), respectively. According to the game theoretic characterization of Ullmann-Margalit (1977), the main feature of a convention is that there is no incentive for deviation once it has been established. Hence, models of

[1]For a more detailed overview of applications of opinion formation, see Lorenz (2007b, Section 2.1).

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opinion formation can be seen as an instance of a coordination process, in particular if they lead to consensus as is the case in many models with global interaction.

Social networks as the conduits for social influence are however typically not fixed but emerge out of the interaction of individuals (Granovetter, 1985). This is where homophily comes into play. According to this concept, more similar agents tend to interact more frequently (McPherson, Smith-Lovin, & Cook, 2001; Byrne, 1971; Kandel, 1978; Rogers & Bhowmik, 1970; Lazarsfeld & Merton, 1954). Axelrod (1997) incorporated homophily in his model and showed by means of computer simulations that this may result in a combination of local convergence and global diversity of opinions: By allowing interaction between adjacent agents only if their opinions are similar enough, the interaction network constantly changes. On the one hand, homophily reinforces the effect of social influence as similarity of opinions leads to more interaction and interaction increases similarity. But, on the other hand, it inhibits the interaction between agents with sufficiently different opinions. This results in clusters of agents with identical opinions while maintaining diversity as the opinions in distinct clusters may differ. In models of continuous opinion dynamics, a similar mechanism is known as *bounded confidence* (Hegselmann & Krause, 2002; Lorenz, 2007a). Here, the interaction network is determined by the agents' opinions: Two agents are adjacent if their opinions are sufficiently similar.

Recent research in opinion formation has investigated the effect of different network structures (Weisbuch, 2004; Stauffer & Meyer-Ortmanns, 2005; Fortunato, 2004, 2005; Suchecki, Eguiluz, & San Miguel, 2005), agent heterogeneity with respect to their level of homophily (Weisbuch, Deffuant, Amblard, & Nadal, 2002; Deffuant et al., 2002; Weisbuch, Deffuant, & Amblard, 2005; Lorenz, 2010) and a coevolution of opinions and network (Centola, Gonzalez-Avella, Eguiluz, & San Miguel, 2007; Vazquez, Eguiluz, & San Miguel, 2008; Groeber et al., 2009). However, there is no canonical implementation of social influence. Often opinions evolve by repeated arithmetic averaging of a subset of opinions in a population. Hegselmann and Krause (2005) show that different ways of averaging (e.g., a geometric or a power mean) may significantly influence the dynamics within bounded confidence models. Another mechanism is the adoption of another individual's opinion with a certain probability, in particular if the set of possible opinions does not allow for averaging (e.g., if opinions are dichotomous). Depending on the implementation, social influence leads to different effects at the macro level, for example consensus or a variety of opinions.

Due to the multitude of possible mechanisms to formalize social influence, a deeper analysis of its possible first principles is required. Deriving social influence implementations from a guiding first principle would make the conditions in models more explicit and would allow for their mathematical analysis. In this article, we provide a possible microfoundation of social influence based on the concept of dissonance minimization. After introducing this framework in Section 2, our formal analysis shows that it can be interpreted as a coordination process under very strict conditions (Section 3) and, in accordance with social influence, leads to increasing similarity of opinions under general assumptions (Section 4). As we only premise that individuals strive to minimize dissonance, our framework can be interpreted as a microfoundation of social influence based on a rational choice approach (Coleman, 1990). Note that the interaction network including the weights that determine intensity of interactions are given a priori and are not subject to a rational choice. In

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Section 5, we further exemplify how different established models of opinion formation can be represented as special cases of our framework and thereby offer a unifying perspective. We conclude the article with a discussion of our results and possible extensions to our framework concerning other types of interaction.

2. FROM DISSONANCE MINIMIZATION TO SOCIAL INFLUENCE

According to the concept of social influence, an agent adjusts her opinion to the opinions of her interaction partners (the *in-group*). In a specific model of a process of opinion formation the rules for changes of agents' opinions are set ad hoc as a function of the agents' and their neighbors' opinions (e.g., building an arithmetic mean). Often the ad hoc rules are based on some implicit assumptions on what agents want to achieve by applying them (e.g., move to the "center" of relevant opinions). In the following we propose *dissonance minimization* as a general individual incentive which can be the basis of many models of opinion formation processes. The proposition of an interaction rule is thus replaced by stating an agent's distance dissonance function, which is influenced by her in-group.

The idea of dissonance minimization is inspired by the social psychological concept of *cognitive dissonance* (Festinger, 1957), but we do not aim to place our model within the framework of the theory of cognitive dissonance. According to this concept, an agent strives toward consistency (*consonance*) in her cognition which can be opinions, beliefs or knowledge. Festinger's (1957) basic hypotheses are that "[t]he existence of dissonance, being psychologically uncomfortable, will motivate the person to try to reduce the dissonance and achieve consonance" and that "[w]hen dissonance is present, in addition to trying to reduce it, the person will actively avoid situations and information which would likely increase the dissonance" (p. 3). Inconsistency in the theory of cognitive dissonance is usually between individual actions and some self-concept, as, for example, smoking while being aware of the health risk incurred. Instead, in our framework, an agent's dissonance results from the inconsistency between different opinions in her in-group.

Further on, Festinger (1957) states that "the strength of the pressures to reduce the dissonance is a function of the magnitude of the dissonance" (p. 18). Taking this analogy over to our framework of dissonance minimization, we assume that the magnitude of dissonance an agent experiences from an in-group member with another opinion increases with the distance between both opinions and with the intensity of her relations in that group. In that sense, our framework formalizes dissonance as an aversive stimulus whose minimization is a motivation for Festinger's older social comparison theory on changes of opinions.[2]

One can imagine different possible mechanisms how dissonance can be reduced. In our framework, we assume that agents can only change their opinions. Other possibilities like structural changes in their respective in-groups (by including or excluding other agents or reconsidering the importance of their relationships) are not considered. The reason is that we want to isolate the process of social influence and therefore restrict the focus to interactions within a given social context, that is, fixed exogenous in-groups. However, additional options to reduce dissonance would

[2]Dissonance reduction is a motivation which one could assume also in the balance theory (Heider, 1946) in triadic relations.

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be relevant for a microfoundation of homophily or another concept that refers to the influence of opinions on the social network of the agents in a joint framework of opinion formation.

In the following, we provide a formal framework using the above assumptions. Let us consider n agents whereas agent i's opinion is denoted by $x_i \in X$ with $i \in \{1, \ldots, n\}$. X is the *opinion space* which may be finite or infinite. A vector $(x_1, \ldots, x_n)$ of all opinions in the population is referred to as *opinion profile*. We require that the opinion space X is *metric*; that is, there exists a *distance function* d: $X \times X \to \mathbb{R}_+$ with the following standard properties:

$$\begin{aligned} &\text{(i)} \quad d(x,y) = 0 \Leftrightarrow x = y, \\ &\text{(ii)} \quad d(x,y) = d(y,x), \\ &\text{(iii)} \quad d(x,y) \leq d(x,z) + d(z,y). \end{aligned} \tag{1}$$

The first property reflects that a distance of zero between two opinions is only possible if the opinions are equal. Further, the order of the opinions does not influence the distance. The third property reflects the so-called *triangle inequality*: The distance between two opinions is at least as high as the sum of the distances of each opinion to a third opinion. Note that one can define the discrete metric d_0 with $d_0(x, y) = 0$ if $x = y$ and $d_0(x, y) = 1$ otherwise on an arbitrary opinion space. We refer to $d(x, y)$ as the *opinion distance* between opinions $x, y \in X$. In particular, we allow for multidimensional opinions, i.e., vectors from $X = X_1 \times \ldots \times X_m$, where the components correspond to the opinions on m different issues. If the opinions are real vectors ($X \subset \mathbb{R}^m$), we assume that d is the Euclidean distance if not stated otherwise, that is,

$$d(x,y) = \| x - y \|_2 = \sqrt{\sum_{1 \leq j \leq m} |x^{(j)} - x^{(j)}|^2}, \quad x = (x^{(1)}, \ldots, x^{(m)}),$$
$$y = (y^{(1)}, \ldots, y^{(m)}).$$

Further, we denote agent i's in-group by $I_i \subset I = \{1, \ldots, n\}$, and the agents in I_i are i's *neighbors*. The in-groups induce a graph W on the set of agents: For agents i and j, (i, j) is in the set of edges $E(W)$ of W if j is in i's in-group, i.e., if $j \in I_i$. Note that the microfoundation of rules with respect to how I_i is constituted is not subject of this article. It may be defined by a fixed external interaction network as in the voter model (Holley & Liggett, 1975), by the opinion profile as in bounded confidence models (e.g., Hegselmann & Krause, 2002) or may evolve over time according to a certain network formation process (e.g., Centola et al., 2007; Vazquez et al., 2008; Groeber et al., 2009).

We assume that the value of holding a certain opinion is a sole function of the opinion of others. Thus, there is no such thing as an unknown "true" opinion or a certain external value an agent draws from holding a certain opinion. Thus, the dissonance of an agent i with her opinion $x_i \in X$ only depends on the dissonance caused by the opinion difference of x_i to opinions of the agents in i's in-group. Considering agent i and her neighbor j with opinions x_i and x_j, we assume that the magnitude of dissonance is a function of the opinion distance; that is, it is captured by $f^-(d(x_i, x_j))$

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whereas we refer to f^-: $\mathbb{R}_+ \to \mathbb{R}_+$ as the *distance dissonance function*. The minus indicates that this function is intended to be minimized.

As we assume that the magnitude of total dissonance strictly increases with the opinion distance, we require that f^- is strictly increasing. Further, the magnitude may depend on the *intensity* (or the *weight*) of the in-group relation between i and j which we represent by $a_{ij} \in \mathbb{R}_+$. We assume that the higher the intensity a_{ij}, the higher the magnitude of dissonance caused by the opinion difference of i and j. More precisely, it is $a_{ij}f^-(d(x_i, x_j))$. As an agent's in-group may consist of more than one agent, the overall extent of dissonance u_i^- of i is the sum of the dissonances with each of her neighbors. We assume the *dissonance function* of agent i to be

$$u_i^-(x_1, \ldots, x_n) = \sum_{j \in I_i} a_{ij} f^-(d(x_i, x_j)), \tag{2}$$

Thus, we assume that dissonances caused by difference to opinions of others are additively separable. Note that additive separation and dissonance monotonically increasing with opinion distance are not the only possible properties of the dissonance function and they are not in every case empirically supported (see Davis, 1996). For the purpose of this article—presenting a common framework for different opinion dynamics models—these simple properties are however sufficient.

Dissonance as a weighted sum has similarities to the overt response in integration theory (Anderson, 1971) being a weighted sum of all relevant informational stimuli. This is similar to weighted sums as mechanism to form new opinions in the model of Abelson (1964), DeGroot (1974), Lehrer and Wagner (1981), and Friedkin and Johnsen (1990). In all these models opinions are weighted and summed up leading directly to a change in an agent's opinion. In contrast, in our framework distance dissonances are weighted and summed up. In the following, we analyze which actions may result from this. We will see that weighted arithmetic averaging is only one possible outcome and occurs only under very specific conditions.

3. SOCIAL INFLUENCE AS A COORDINATION GAME

As mentioned before, models of opinion formation exhibit basic properties of coordination processes in the sense of Lewis (1969) and Ullmann-Margalit (1977). In the context of our framework we can confirm this by a game theoretical analysis of the n-player game G^- with strategy space X and utility function $u_i = -u_i^-$ for each agent (player).[3] As the maximization of utility u_i is equivalent to the minimization of the dissonance u_i^- for all players, we can investigate the process of social influence by analyzing G^-. First we consider a scenario with perfectly rational agents where the respective utility functions (including the distance dissonance function and the intensity weights) are common knowledge.

[3]Of course, in these models, dissonance minimization is a rational choice of the agent, which is not the idea of the psychological concept of cognitive dissonance. The mental process of reducing dissonance is not necessarily a conscious one. Nevertheless it is interesting to study group constellations where dissonance becomes minimal, regardless of whether the adjustment mechanisms are consciously applied by the agents or not.

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In this context, we first neglect local effects by assuming that an agent's utility of a certain opinion depends on all other agents' opinions, i.e., $I_i = I \backslash \{i\}$. As f^- is strictly increasing, a state where all agents have identical opinions is a strict Nash equilibrium of G^-. We denote such a state as *consensus* and define the *consensus set* as

$$C = \{(x, \ldots, x) \in X^n | x \in X\}. \tag{3}$$

The elements of C are referred to as *consensus states* or *consensus profiles*. Further, we denote the *set of Nash equilibria* of G^- by $\Theta(G^-)$. As stated above, it holds $C \subset \Theta(G^-)$, but not every Nash equilibrium needs to be a consensus as the following example shows:

Example 1. Consider four agents with opinion space $X = \mathbb{R}$ and opinions $x_1 = x_2 = 0$, $x_3 = x_4 = 1$ whereas the dissonance caused by deviation in opinion is proportional to the opinion distance, that is, $f^-(z) = cz$ for all $z > 0$ with a constant $c > 0$. If agent 1 perceives a higher dissonance for deviating from the opinion of agent 2 compared to the deviance to the opinions of agent 3 and agent 4 due to a more intense in-group relation with agent 2 (e.g., by $a_{12} = 1$, $a_{13} = a_{14} = 0.1$), the optimal decision for agent 1 given the other agents' opinions is x_1. For analogous assumptions with respect to the other agents, x_i is optimal for agent i given the other agents' opinion for $i = 1, \ldots, 4$; that is, the nonconsensus state $(x_1, \ldots, x_4)$ is a strict Nash equilibrium of G^-.

This example demonstrates that our framework is not subject to Abelson's observation that in most weighting models a connected network enforces ultimate convergence to consensus.

We can provide a sufficient condition with respect to the distance dissonance function f^- and the intensities a_{ij} of in-group relations so that consensus states are the only Nash equilibria of G^- in case of global interaction.

Proposition 2. If $I_i = I \backslash \{i\}$ *and* $a_{ij} = a$, *then the set of Nash equilibria is equal to the set of consensus states*:

$$\Theta(G^-) = C.$$

The proof can be found in the Appendix. The proposition states that all Nash equilibria are consensus states in case of homogeneous intensities of in-group relation. This condition cannot be omitted as demonstrated by Example 1. With inhomogeneous intensities, non-consensual Nash equilbria become possible. These Nash equilibria are inefficient in the sense that the total dissonance for all agents is larger than in a consensual state. Thus, with inhomogeneous intensities, group lock-in situations are possible. That means that agents can get stuck in a situation where total dissonance is not zero, while every single agent would only increase her dissonance by deviating.

In the case of homogeneous intensities of in-group relations, we can interpret G^- as a pure coordination game in the sense that the Nash equilibria (which coincide

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with the consensus states) exhibit the same payoffs whereas the payoffs for all agents are also identical and maximal (no nonequilibrium state leads to a higher payoff for any agent).

However, there may be focal points (Schelling, 1960), that is, opinions that are in a sense more reasonable than others although the payoff in the corresponding consensus state is identical. If agent i assumes that all other agents choose their respective opinion ξ_j randomly according to a uniform distribution on the opinion space, she prefers an opinion x_i^* that minimizes the expected dissonance; that is,

$$x_i^* \in \operatorname{argmin} \left\{ x | \sum_{j \in I_i} a_{ij} E(f^-(d(x, \xi_j))) \right\}. \tag{4}$$

For 2×2 coordination games, this corresponds to the concept of risk dominance of Harsanyi and Selten (1988). If we choose $X = [0, 1]$ according to many models of continuous opinion dynamics (cf. Hegselmann & Krause, 2002) and assume again global interaction and a uniform distribution of opinions, we obtain that for $x < 0.5$,

$$\begin{aligned} E(f^-(d(x, \xi_j))) &= \int_0^1 f^-(d(x, y))\, dy \\ &= \int_0^{x+0.5} f^-(d(x, y))\, dy + \int_{x+0.5}^1 f^-(d(x, y))\, dy \\ &= \int_{0.5-x}^1 f^-(d(0.5, y))\, dy + \int_{x+0.5}^1 f^-(d(x, y))\, dy \\ &> \int_{0.5-x}^1 f^-(d(0.5, y))\, dy + \int_0^{0.5-x} f^-(d(0.5, y))\, dy \\ &= E(f^-(d(0.5, \xi_j))). \end{aligned}$$

Analogously, one can show that the same holds for $x > 0.5$. Hence, x_i^* is the unique focal point in the sense of Eq. (4). As this holds for all agents, the profile where all agents exhibit the average opinion of the opinion space is the most reasonable equilibrium in case of global interaction given the aforementioned constellation. This holds independently of the distance dissonance function f^- because of the simple structure of the opinion space. However, if we consider multidimensional (but convex)[4] opinion spaces, the average over all possible opinions is not necessarily a focal point with respect to Eq. (4).

For less structured opinion spaces focal points may not exist. In particular, there are no focal points if any two different opinions lead to identical dissonance, that is, when d is the discrete metric. Then, all opinions are equivalent with respect to expected dissonance if agents assume a uniform random distribution of opinions. Hence, no focal point exists in this case.

[4]A set $X \subset \mathbb{R}^m$ is convex if for any $x, y \in X$ and $\alpha \in [0, 1]$ it holds that $\alpha x + (1 - \alpha) y \in X$; that is, for any two points form the set, the line between them is also part of the set.

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As a conclusion, the set of Nash equilibria only coincides with the set of consensus states under very strict conditions: It requires equal intensities of relations in a fully connected social network. Further on, some opinion spaces exhibit focal points and thus make some of the equilibrium profiles more reasonable to others according to that concept.

4. MYOPIC BEST RESPONSE DYNAMICS

The game theoretic analysis of the former section is based on the assumption of perfect rationality of all agents. This assumption is not realistic with respect to many processes of opinion formation and in particular not in the context of the theory of cognitive dissonance. Therefore, we now shift the focus to agents with bounded rationality in the sense that they reduce their dissonance only with respect to the current opinion profile in their neighborhood assuming all other agents will stick to their opinion. This assumption is closer to the idea of cognitive dissonance in the sense that the opinion of an agent adjusts with respect to the stimuli represented by the current opinions of others. In game theory, this adjustment behavior is called *myopic best response strategy* (see Ellison, 1998). When agents apply a myopic best response strategy, an overall equilibrium usually requires more than one round of adjustments. Thus, with agents applying myopic best response, their opinion formation process becomes a multiperiod dynamical model. Then, an agent's own opinion may also be included in the set of opinions that are relevant for her decision-making process. This means that deviating from her own opinion may also cause dissonance for an agent.

In Section 5 we reproduce or sketch four well known classes of models of opinion formation as myopic best response dynamics within our framework. But before that, we investigate how certain assumptions for the opinion space and the distance dissonance function determine basic properties of the best response dynamics. In particular, we show that dissonance minimization implies that adjusted opinions remain in the convex hull[5] of all opinions if the opinion space is convex (Proposition 3). We show that a distance dissonance function which is strictly convex[6] ensures a unique best response for each agent (Proposition 5). Finally, we demonstrate that a quadratic distance dissonance function implies arithmetic averaging as adjustment behavior (Proposition 7).

Myopic best response leads to the following dynamic model: At time t, agent i chooses an opinion $x_i(t+1)$ that minimizes dissonance with respect to the opinion profile $x(t) = (x_1(t), \ldots, x_n(t))$ at time t. Here, "myopic" means that agents do not take into account that other agents will do the same.[7] We assume that $x(t)$ is common knowledge at every time step. The dissonance depends on the opinion distance according to the distance dissonance function f^{-}. It further increases with the agent's confidence in her own opinion which is captured by a_{ii}. Thus, in contrast to the static

[5]The convex hull of a set of points is the smallest possible convex set which contains all points.

[6]A function f: $\mathbb{R} \to \mathbb{R}$ is strictly convex when it holds for any $x, y \in \mathbb{R}$ and $\alpha \in]0, 1[$that $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$; that is, the graph of f is below the line between any two points on the graph.

[7]In the context of games with a finite number of strategies, alternative dynamics that also include the probability of random mistakes by the agents are provided by Young (1993). Ellison (1993), and Blume (1995) investigate the effect of local interactions in this context.

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setting with perfectly rational agents (Section 3), agent i is part of her in-group $I_i(t)$. Agent i's perceived dissonance caused by opinion y at time $t+1$ with opinion profile $x(t)$ at time t is therefore

$$u_i^-(y, x(t)) = \sum_{j \in I_i(t)} a_{ij}(t) f^-(d(y, x_j)). \tag{5}$$

Note that although the agent's in-group $I_i(t)$ including the weights $a_{ij}(t)$ may vary over time, we assume that this evolution of her personal network follows given external rules and is independent of her decision making: At each timestep, an agent only adjusts her own opinion and does not deliberately change her in-group.

The opinion $x_i(t+1)$ minimizes dissonance with respect to the opinion profile $x(t)$, that is,

$$x_i(t+1) \in \underset{y \in X}{\operatorname{argmax}}\ u_i^-(y, x(t)). \tag{6}$$

We also refer to $x_i(t+1)$ as a *best response* to the opinion profile $x(t)$ and the in-group $I_i(t)$. Note that the above best response dynamics exhibit a *synchronous update* of the agents' opinions. Alternatively, one could choose one agent (randomly or in sequence) per time step whose opinion is updated while the remaining agents' opinions do not change. We refer to the latter case as *asynchronous update* of the agents' opinions. Both update mechanisms appear in models in the literature, and can be implemented in our framework.

The following proposition shows that minimizing dissonance does not broaden the opinion spectrum if the opinion space is convex.[8]

Proposition 3. Assume a convex opinion space $X \subset \mathbb{R}^m$. Then for any agent i, a best response x_i^* to an opinion profile $x = (x_1, \ldots, x_n)$ is in the convex hull of x; that is,

$$x_i^* \in \left\{ y \in X | y = \sum_{j \in I_i} \lambda_j x_j, \lambda_j \geq 0, \sum_{j \in I_i} \lambda_j = 1 \right\}.$$

This proposition supports the social influence hypothesis that interaction increases similarity of the agents' opinions, in the sense that opinions do not tend to get more dissimilar. It holds under very general conditions, for example, independently of the distance dissonance functions. In particular, it implies that the maximum distance over all opinions does not increase during the best response dynamics. If opinions are one dimensional ($X \subset \mathbb{R}$), the proposition implies that the best response x_i^* to an opinion profile $(x_1, \ldots, x_n)$ is a *partial abstract mean* according to the definition of Hegselmann and Krause (2005). The proposition does not hold if the opinion space is not convex, as the following example shows.

Example 4. Consider the discrete opinion space $X = \{(0, 3), (6, 0), (6, 6), (7, 3)\}$, a population of $n = 3$ agents and assume that $f^-(z) = z^2$. Further, assume that the

[8]A set is convex if the line between any two points of the set is contained in the set.

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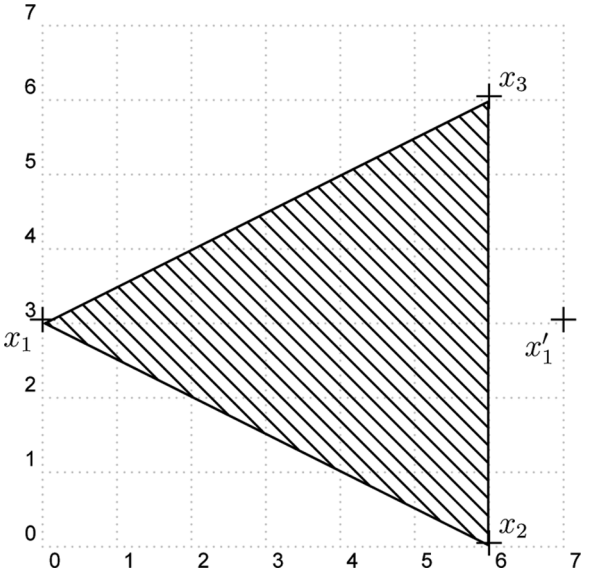

FIGURE 1 Visualization of Example 4: The dissonance minimizing opinion for agent 1 is x_1' although it is not contained in the convex hull of the opinions x_1, x_2 and x_3.

opinion profile at a time is $x = (x_1, x_2, x_3)$ with $x_1 = (0, 3)$, $x_2 = (6, 0)$, $x_3 = (6, 6)$ (see Figure 1). If the agents interact globally (i.e., $I_i = I$) and the intensity of the in-group relations is homogeneous (i.e., $a_{ij} = a$ and without a loss of generality[9] $a = 1$), we obtain that

$$\begin{aligned} u_1^-((0,3),x) &= (36+9)+(36+9) = 90, \\ u_1^-((6,0),x) &= (36+9)+36 = 81, \\ u_1^-((6,6),x) &= (36+9)+36 = 81, \\ u_1^-((7,3),x) &= 49+(1+9)+(1+9) = 69; \end{aligned}$$

that is, agent 1 perceives minimum dissonance if she chooses opinion $x_1' = (7, 3)$ which is not in the convex hull of the opinion profile x.

A real world example of a convex, multidimensional opinion space is the spectrum of political positions with dimensions like left–right and liberal–conservative, see Hermann and Leuthold (2003). The positions of a finite number of political parties, however, constitute a nonconvex opinion space (as in Example 4): With respect to an election, each voter can only choose those opinions that are reflected by a political party.

Example 4 shows that a nonconvex opinion space allows for optimal opinions that are not contained in the convex hull of the current opinion profile. Moreover, the optimal opinion may also be more extreme than all current opinions in one

[9]See the proof of Proposition 2.

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dimension: Agent 1 chooses a new opinion more to the right than all other agents' opinions.

Further, Proposition 3 guarantees the existence of a best response for each agent to any opinion profile if the distance dissonance function f^- is continuous. However, this best response is not necessarily unique. Uniqueness can be attained if we require strict convexity for f^-, as the following proposition shows.

Proposition 5. *Assume a convex opinion space $X \subset \mathbb{R}^m$. If the distance dissonance function f^- is continuous and strictly convex, there is always a unique best response for each agent to a given opinion profile $x = (x_1, \ldots, x_n)$.*

Strict convexity of f^- implies that the dissonance caused by opinion distance grows disproportionately. An increase of distance implies a proportionally larger increase in dissonance. This property is necessary: A linear function is convex but not strictly convex. Consequently, a linear function does not ensure a unique best response as the following example shows. Certainly, it can be argued that this assumption of strict convexity of f^- does not always hold. In fact, there might be real world situations where a diminishing marginal dissonance seems to be a more appropriate assumption. However, as we will point out in Section 5, we can reformulate many models of opinion formation as best response dynamics within our framework using a distance dissonance function that is strictly convex. Hence, one could argue that this assumption is also inherently in these models.

Example 6. Consider the discrete opinion space $X = [0, 1]$, a population of $n = 2$ agents and assume that $f^-(z) = z$. Further, assume that the opinion profile at a time is $x = (x_1, x_2)$ with $x_1 = 0$, $x_2 = 1$. If the agents interact globally (i.e., $I_i = I$) and the intensity of the in-group relations is homogeneous (i.e., $a_{ij} = a$), we obtain that $u_1^-(x_1', x) = |0 - x_1'| + |1 - x_1'|$. As $x_1' \in [0, 1]$ it holds $u_1^-(x_1', x) = x_1' + 1 - x_1' = 1$. Thus, the dissonance is independent of the choice x_1' and any value between 0 and 1 is a best response.

So far, we did not specify a particular functional form of the distance dissonance function f^-. In the following, we determine the location of the best response opinion for a given opinion profile if f^- is quadratic.

Proposition 7. *Let $X \subset \mathbb{R}^m$ be a convex opinion space. Further, let the distance dissonance function be quadratic; that is, $f^-(z) = z^2$. Then the best response to an opinion profile $x = (x_1, \ldots, x_n)$ for agent i with in-group I_i is*

$$x^* = \frac{\sum_j a_{ij} x_j}{\sum_j a_{ij}}.$$

Hence, the best response is the weighted average of current opinions with the weights corresponding to the relative intensities of the in-group relations. This coincides with the implementation of social influence (as a behavioral rule) in many continuous models of opinion dynamics (e.g., Friedkin & Johnsen, 1990; Hegselmann & Krause, 2002; Deffuant et al., 2000). We therefore use this proposition in the

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following section to show that the respective models can be reproduced by myopic best response dynamics within our framework for a particular parameter configuration.

Quadratically increasing distance dissonance functions are not the only reasonable assumption. As we have seen, they imply arithmetic averaging as a best response. For other distance dissonance functions, like linear ones, we know from Proposition 5 that a best response does not have to be unique. Thereby, the possibility of random choice enters models with linear dissonance functions even with rational (but myopic) choice. In models with arithmetic averaging as agent's adjustment rule, researchers make an implicit assumption that an opinion which is further away causes a disproportionally large dissonance. Our framework enables us to study this implicit assumption from the perspective of dissonance minimization.

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5. APPLICATION TO MODELS OF OPINION FORMATION

We will show that the dynamics of different models of opinion formation can be obtained as special cases from our framework of myopic best response dynamics with a particular definition of the distance dissonance function f^- and in-groups I_i. Thus, our concept of dissonance minimization provides a common microfoundation for all these models, although they differ significantly in their opinion spaces and interaction rules. With the social influence network model of Friedkin and Johnsen (1990), we obtain a classical linear averaging model (with external influence) as special case. In this model the opinion space is continuous. This also holds for bounded confidence models as proposed by Krause (2000) and Deffuant et al. (2000). In these models, the definition of in-groups that depend on the current opinion profile needs to be reflected to be obtained by our framework. Further on, we sketch how the model of cultural dissemination of Axelrod (1997) can be obtained, where the opinion space contains vectors of discrete sets. Finally, we show how a class of linear and non-linear voter models can also be obtained from the framework. In these models the opinion space only contains two opposing opinions. Thus, all these models can be mathematically microfounded in the framework of dissonance minimization. However, this microfoundation is not exclusive: An alternative mathematical microfoundation of a model would offer another consistent explanation.

5.1. General Social Influence Network Model by Friedkin and Johnsen

First, we consider the model of Friedkin and Johnsen (1990) (of which DeGroot [1974] is a special case) in its most general form. The opinion space is a multidimensional Euclidean space, but for simplicity one might think of $X = \mathbb{R}$ or $X = [0, 1]$. In the model, an agent adopts the (weighted) arithmetic mean of her opinion and the opinions of her interaction partners at each time step. The interaction network is represented by its weighted adjacency matrix $W(t) = (w_{ij}(t))$ at time t. In its most general form, the interaction network may change over time according to a particular process. The coefficient $w_{ij}(t)$ captures the intensity of the relation between agents i and j. The in-group of $I_i(t)$ manifests by all j for which $w_{ij}(t) > 0$ holds. Further on, the agent is influenced by an exogenous variable $z_i(t)$ from the opinion space such

as mass media or her initial opinion. The relative strength of the endogenous influence compared to the exogenous influence for agent i is given by $a_i(t) \in [0, 1]$. The relative strengths of all agents' endogenous influence are collected in the diagonal matrix $A(t)$. Note that in the most general form of the model the external influence as well as the individual relative strength of the endogenous influence may change over time.

In matrix notation the dynamics of the opinion profile reads,

$$x(t+1) = A(t)\, W(t)\, x(t) + (I - A(t))\, z(t),$$

where I is the identity matrix. In many versions of the model, $z(t)$ is the constant profile of initial opinions (Friedkin & Johnsen, 1990, 1999; Friedkin, 1998). This model is obtained as myopic best response dynamics with respect to the dissonance function

$$u_i(y, x(t)) = a_i \sum_j w_{ij}(t)\, d(y, x_j(t))^2 + (1 - a_i)\, d(y, z_i(t))^2, \quad y \in X,$$

with $x(t)$ denoting the opinion profile at time t. Proposition 7 guarantees that the best response profile to $x(t)$ is the weighted arithmetic mean corresponding to the weights of the quadratic distance dissonance functions: The weight of opinion $x_j(t)$ for agent i is $a_i\, w_{ij}$ whereas the external influence is weighted by $(1 - a_i)$. This is a weighting of the two components with respect to the perception of dissonance. This is mathematically analog to the weighting of behavioral response of opinion change with respect to the two components in the original model formulation.

Friedkin and Johnsen (1990) claimed that the convergence of agents' opinions to the mean of initial opinions in their model is what Festinger (1954) predicted in his social comparison theory. The above dissonance function provides a motivational microfoundation in dissonance reduction.

5.2. Bounded Confidence Models

Models of continuous opinion dynamics under bounded confidence are similar to the model of Friedkin and Johnsen (1990)[10] because agents average the opinions with respect to the agents in their current interaction network. The interaction network in bounded confidence models is determined endogenously by the opinion profile: Agents only interact if their opinions are sufficiently similar. At each time step, an agent adopts the (weighted) arithmetic mean of her opinion and the opinions of her interaction partners. The two prominent models of bounded confidence are the *Hegselmann-Krause model* (Krause 2000; Hegselmann and Krause, 2002) and the *Deffuant-Weisbuch model* (Deffuant et al., 2000; Weisbuch et al., 2002), which we refer to as the HK model and the DW model, respectively. The difference between the models is the update procedure of the agents' opinions: In the HK model, agents update their opinions simultaneously at each time step by adopting the arithmetic mean (by component) of their in-group members. Here, the in-group $I_i^{\mathrm{HK}}(t)$ of agent i at time t depends on the current opinion profile $x(t)$: It contains those agents whose

[10]External influence as in Friedkin and Johnsen (1990) is not a component of bounded confidence models.

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opinion distance to i's opinion at time t does not exceed a threshold ε; that is,

$$I_i^{\mathrm{HK}}(t) = \{j | d(x_i(t), x_j(t)) < \varepsilon_i\}.$$

Often, one assumes homogeneous bounds of confidence, meaning that all agents have identical thresholds ($\varepsilon_i = \varepsilon$). With $W^{\mathrm{HK}}(t)$ denoting the graph induced by the in-groups at time t, the new opinion profile at time step $t+1$ can be formulated as

$$x^{\mathrm{HK}}(t+1) = A(W^{\mathrm{HK}}(t))\, x(t),$$

whereas the components $A_{ij}(W)$ of the *confidence matrix* $A(W)$ for in-groups $I_1, \ldots, I_n$ and the corresponding graph W are defined by

$$A_{ij}(W) = \begin{cases} \frac{1}{|I_i|} & \text{if } j \in I_i \\ 0 & \text{otherwise} \end{cases}.$$

The update mechanism of the DW model is similar in the sense that an agent's new opinion is a weighted arithmetic mean of her in-group members' opinions. However, only two randomly chosen distinct agents can interact (if their opinion distance does not exceed the threshold) at each time step while all other agents keep their respective opinion. Formally, agent i's in-group at time t is defined as

$$I_i^{DW}(t) = \begin{cases} \{Z_1(t), Z_2(t)\} & \text{if}(i = Z_1(t) \text{ or } i = Z_2(t)) \text{ and } d(x_{Z_1(t)}, x_{Z_2(t)}) < \varepsilon_i \\ \{i\} & \text{otherwise.} \end{cases}$$

whereas $Z = (Z_1, Z_2)$ denotes a discrete time stochastic process with $Z(t)$ uniformly distributed on I^2.[11] With respect to the average of the two opinions, the weight of an agent's own opinion is usually specified by $1-\mu$ with $\mu \in [0,\ 0.5]$ while the weight of the other agent's opinion is μ. Hence, the new opinion profile at time step $t+1$ is

$$x^{\mathrm{DW}}(t+1) = A^{(\mu)}(W^{\mathrm{DW}}(t))\, x(t),$$

with $W^{\mathrm{DW}}(t)$ denoting the graph induced by the in-groups $I_i^{\mathrm{DW}}(t)$ and

$$A_{ij}^{(\mu)}(W) = \begin{cases} 1-\mu & \text{if} j = i \\ \frac{\mu}{|I_i|-1} & \text{if} j \in I_i \backslash \{i\}, \\ 0 & \text{otherwise} \end{cases}$$

denoting the bounded confidence matrix for in-groups $I_1, \ldots, I_n$ and the corresponding graph W. Note that only two of these n in-groups contain two agents, all other in-groups only trivially contain the agent itself.

[11]Sometimes, the stochastic process is defined on $I^2 \backslash \{(i, i) | i \in I\}$; that is, only distinct agent are chosen.

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According to this representation, we can reformulate both models as a myopic best response dynamics with respect to the dissonance function

$$u_i^{-\mathrm{HK}}(y, x(t)) = \sum_{j \in I_i^{\mathrm{HK}}(t)} d(y, x_j)^2, \quad y \in X^m$$

for the HK model and

$$u_i^{-\mathrm{DW}}(y, x(t)) = \sum_{j \in I_i^{\mathrm{DW}}(t)} A_{ij}^{(\mu)}(W^{\mathrm{DW}}(t))\, d(y, x_j)^2, \quad y \in X^m$$

for the DW model with $x(t)$ denoting the opinion profile at time t.[12] Proposition 7 guarantees that the best response profile to $x(t)$ is the weighted average corresponding to the weights of the quadratic distance dissonance functions.

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5.3. Cultural Dissemination

As Flache and Macy (2006a) pointed out, the famous model of Axelrod (1997) on the dissemination of culture also implements a bounded confidence mechanism although its opinion space and the distance function are discrete. In this model, each opinion is a list of F features.[13] For each feature, there are Q traits which represent alternative values of the feature. Further, Axelrod assumes a fixed network of agents where the probability of interaction for two adjacent agents is proportional to their similarity measured by the proportion of identical features. If two agents interact, one of them adopts the trait of a randomly selected feature of her interaction partner.

To analyze this implementation of social influence from the perspective of our framework, we abstain from a formal derivation of the model dynamics as it is not unique for Axelrod's model. Instead, we sketch how the main properties of its implementation of social influence can be derived from our framework.

We can define the distance between two opinions as the number of different traits of the opinions. However, we can only determine whether two traits are equal or not: no further distinction of different traits is possible. Hence, there is in general no unique best response to an opinion profile as the reduction of dissonance caused by the adoption of a (distinct) trait of the interaction partner does not depend on the feature to which the trait refers. Nevertheless, one can define a random choice of any of the possible best responses within our framework. This would resemble the random choice of the copied trait in the original model formulation. Further on, the fact that only one feature is adopted instead of all can be modelled within our framework by an appropriately large weight on the dissonance caused by deviating from the own current opinion compared to other in-group members; that is, $a_{ii} > a_{ij}$ with

[12]Note that $A_{ij}(W^{\mathrm{HK}}(t))$ is constant for all agents j in agent i's in-group within the HK model. Hence, this factor can be omitted in the utility function as strictly increasing transformations do not affect the induced preferences.

[13]Axelrod (1997), however, uses the term "culture" instead of "opinion."

$j \neq i$. Finally, the distance dissonance function f^- has to increase slowly as only one feature is adjusted in case of interaction regardless of the number of different features between the agents.

In that way Axelrod's model of cultural dissemination is also a model which can be microfounded by dissonance minimization. From this perspective, agents in Axelrod's model put a large weight on the dissonance caused by deviating from their own culture but also have distance dissonance functions which do not increase fast with the number of different traits they observe in their interaction partner. With a faster increasing distance dissonance function, copying more than one trait at a time would be the agent's best response.

5.4. Voter Models

Another approach to opinion formation can be found in *voter models*.[14] In these models the opinion space consists of only two opinions. At each time step, a randomly selected agent i chooses a particular opinion j with a probability $p = g(r_i^{(j)}(x))$ which depends on the local frequency $r_i^{(j)} \in [0,1]$ of that opinion in i's neighborhood I_i for a given opinion profile x. Here, g is called the *response function*. If g is linear, we have a *linear voter model*. The linear voter model with $g(r_i(x)) = r_i^{(j)}(x)$ was introduced independently by Clifford and Sudbury (1973) and Holley and Liggett (1975). Examples of nonlinear voter models can be found in Molofsky, Durrett, Dushoff, Griffeath, and Levin (1999), Schweitzer and Behera (2009), and Stark, Tessone, and Schweitzer (2008). In these models, the in-group structure usually does not change over time; that is, there is no influence of the opinions on the network. Such an influence could be caused by homophily.[15]

Let us assume that agent i's dissonance u_i^- caused by deviance from a given opinion profile is defined according to Eq. (5). Further, we assume homogeneous intensity of in-group relations ($a_{ij} = a$). If the agents are restricted to a deterministic choice of opinions, i's best response to a given opinion profile $x = (x_1 \ldots, x_i)$ is to choose x_i^* as

$$x_i^* = \begin{cases} 0 & \text{if } r_i^{(0)} > r_i^{(1)} \\ 1 & \text{if } r_i^{(1)} > r_i^{(0)}, \end{cases}$$

that is, the opinion with maximum frequency in her in-group. If the local frequency of both opinions is identical, the best response is not unique and we have to specify a tie breaker rule or let the agent choose randomly.

In the following, we allow for mixed strategies instead of restricting the agents to a deterministic choice. Each agent i chooses a probability distribution on the opinion space. As there are only two opinions, a distribution can be represented by the vector $(p_i^{(0)}, p_i^{(1)})$ with $p_i^{(x)}$ denoting the probability of choosing opinion $x \in X$. As

[14]Related two-state model are the social impact model (Latanè, 1981) which are analyzed within the framework of cellular automata (Lewenstein, Nowak, & Latané, 1992) and models of social pressure and polarization (Macy, Kitts, Flache, & Steve, 2003) which is analyzed within the framework of Hopfield networks. In both models social impact (respectively social pressure) adds up as the opinions of neighbors.

[15]An exception is the model of Vazquez et al. (2008).

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$p_i^{(0)} + p_i^{(1)} = 1$, we can define $p_i = p_i^{(1)}$ whereas $p_i^{(0)} = 1 - p_i$. However, only the final opinion is observed by the agents at the next time step. If the agents minimize their expected dissonance according to their respective probability distribution, the best response is still to choose the opinion with maximum frequency in an agent's in-group with probability one as

$$E_p(u_i^-(Y,x)) = (1-p)\sum_{\substack{j\in I_i \\ x_j=0}} f^-(1) + p\sum_{\substack{j\in I_i \\ x_j=1}} f^-(1),$$

with Y denoting the random choice which is distributed according to p. In particular, the best response does not depend on the distance dissonance function f^-.

Instead of referring to expected dissonance, we assume that the agents minimize the cumulative dissonance caused by the *expected deviance* to their in-group members' opinions according to the probability distribution $p = p^{(1)}$:

$$\begin{aligned}\tilde{u}_i^-(p,x) &= \sum_{j\in I_i} f^-(pd(1,x_j) + (1-p)d(0,x_j)) \\ &= \sum_{\substack{j\in I_i \\ x_j=0}} f^-(p) + \sum_{\substack{j\in I_i \\ x_j=1}} f^-(1-p).\end{aligned}$$

Without affecting the induced preferences, we can normalize the dissonance perceived by an agent by the population size n and redefine

$$\tilde{u}_i^-(p,x) = (1-r_i(x))f^-(p) + r_i(x)f^-(1-p), \tag{7}$$

with $r_i(x)$ denoting the proportion of agent i's in-group members with opinion one in the opinion profile x. In particular, the best response to a given opinion profile now crucially depends on the distance dissonance function f^-. If $f^-(z) = z^2$, the optimal distribution p_i^* for agent i with respect to an opinion profile x has to satisfy the first order condition

$$(1-r_i(x))p_i^* = r_i(x)(1-p_i^*).$$

Therefore, we obtain

$$p_i^* = r_i(x);$$

that is, the best response is to choose the probability distribution that corresponds to the local frequency of the opinions in an agent's in-group. Thus, for a quadratic distance dissonance function f^-, the myopic best response dynamics with respect to the utility function in Eq. (7) is identical to the linear voter model if at each time step one agent is randomly selected to update.

In general, the shape of f^- analogously induces a voter model where the probability for an agent to select a particular opinion depends non-linearly on the local frequency of that opinion in the agent's in-group. By Proposition 5 it follows that the optimal distribution with respect to any opinion profile is always unique if f^- is

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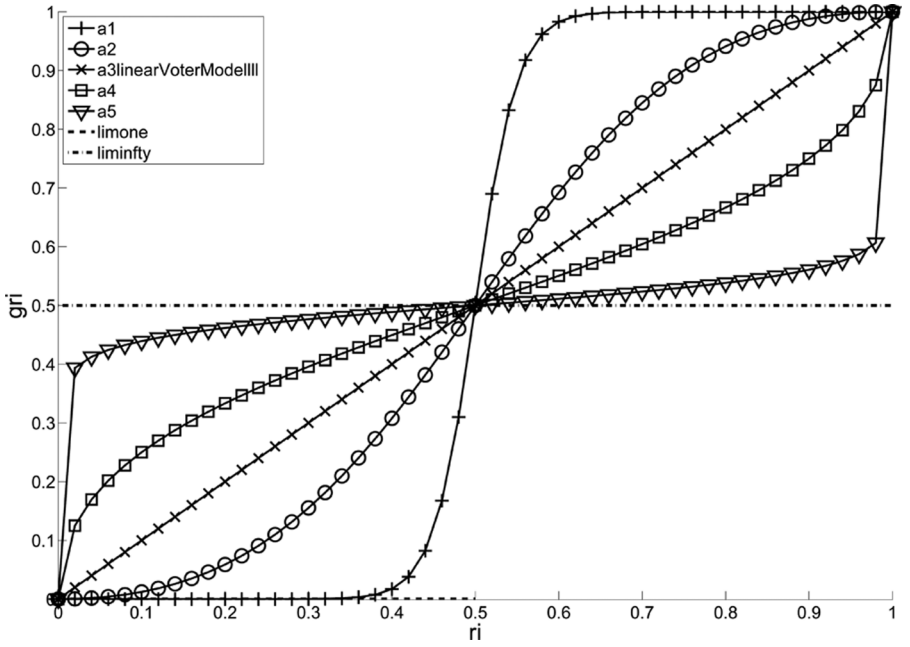

FIGURE 2 Agent i's optimal probability $g_\alpha(r_i)$ of choosing an opinion with respect to a local frequency r_i of that opinion for $f^-(z) = z^\alpha$ and various values of α. The myopic best response dynamics according to Eq. (7) correspond to (nonlinear) voter models with response function g_α.

strictly convex.[16] Further, we derive several properties of the best response to a given local frequency of the opinions in the following proposition:

Proposition 8. *Let f^- be convex and let $p_i^* = g(r_i(x))$ denote an agent's best response to given local opinion frequencies in the opinion profile x. Then the following statements hold*:

$$
\begin{aligned}
&(i) \quad g(1 - r_i(x)) = 1 - g(r_i(x));\\
&(ii) \quad g(0) = 0,\ g(0.5) = 0.5,\ g(1) = 1;\\
&(iii) \quad g \textit{ is increasing};\\
&(iv) \quad g(p) \in (0, 1) \textit{ for } p \in (0, 1).
\end{aligned}
$$

In Figure 2 we depict the decision function g induced by distance dissonance functions $f^-(z) = z^\alpha$ for various parameters $\alpha > 1$ which implies strict convexity of f^-. We observe that as α increases, a small difference in the local frequencies of the two opinions is less reflected in the respective opinions' selection probabilities. The reason is that an increase in α leads to a decreased dissonance caused by small

[16]Although the basic opinion space $X = \{0, 1\}$ is not convex, we can fulfill the requirements of Proposition 5 by extending it to the interval [0, 1]. Here, we consider only best responses to "pure" opinion profiles $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1\}$.

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opinion differences. Hence, for $\alpha \to \infty$, the agents choose both opinions with equal probability unless there is consensus in their respective in-group. For small values of α, we observe the opposite effect as then the dissonance caused by a small opinion difference is high. Thus, in the limit for $a \to 1$, the agents choose the majority opinion with probability one. Note that the linear voter model is obtained for $\alpha = 2$.

If we only account for dissonance resulting from opinion difference to in-group members, the response function is always increasing according to Proposition 8. However, also voter models with decreasing or nonmonotone response function are investigated (e.g., Schweitzer & Behera, 2009; Molofsky et al., 1999). Note that these types of non-linear voter models can be obtained within our framework if we incorporate consonant perception (see Section 6).

An alternative way to deduce the dynamics of the linear voter model from the utility functions in Eq. (5) is to extend a result from Kosfeld (2002) in the context of a 2×2 coordination game without risk-dominant equilibrium. Here, following the framework of Rosenthal (1989) and the *proportional imitation rule* introduced by Schlag (1998), it was shown that the linear voter model corresponds to a stochastic process where the probabilities of switching between the two strategies are proportional to the payoff difference.[17] Similarly, we can show that an analogous mechanism with respect to the n-player coordination game with strategy spaces $S_i = \{0, 1\}$ and payoffs according to Eq. (5) leads to the dynamics of the linear voter model. Here, let $p_i^{(j)}(x) j = 0, 1$, denote the probability that agent i chooses opinion j for a given opinion profile x. Following Rosenthal (1989), we define $u_i(y, x) = -u_i^-(y, x)$ for $y \in \{0, 1\}$ and require that

$$p_i^{(1)}(x) - p_i^{(0)}(x) = \lambda(u_i(1, x) - u_i(0, x)), \tag{8}$$

with $\lambda > 0$; that is, the difference of the probabilities for the two opinions is proportional to the respective payoff difference. With respect to the latter term we obtain

$$(u_i(1, x) - u_i(0, x)) = \sum_{\substack{j \in I_i \\ x_j = 1}} f^-(1) - \sum_{\substack{j \in I_i \\ x_j = 0}} f^-(1) = (r_i^{(1)}(x) - r_i^{(0)}) f^-(1).$$

Using Eq. (8) for $\lambda = f^-(1)^{-1}$ and $p_i^{(1)}(x) + p_i^{(0)}(x) = 1 = r_i^{(1)}(x) + r_i^{(0)}$, this leads to

$$p_i^{(1)}(x) = r_i^{(1)}(x);$$

that is, each agent's best response to a given opinion profile x is to adopt an opinion with the probability that corresponds to that opinion's relative frequency in her in-group according to x. Thus, if at each time step an agent is selected to choose the optimal opinion distribution, this process coincides with the linear voter model.

[17] If there is a risk dominant equilibrium, the process corresponds to a *biased voter model* (Schwartz, 1977).

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6. CONCLUSION

In this article, we provide a microfoundation of social influence with a general framework inspired by the social psychological concept of cognitive dissonance. Our main assumption is that deviation from the opinions of in-group members leads to dissonance which the agents want to minimize by adjusting their opinions. We showed that in this framework, social influence can be interpreted as a coordination process in a game theoretical context. While social influence is a premise in many models of opinion formation, we can derive the fact that interaction does not decrease the similarity of opinions from the assumption that the agents strive to minimize dissonance. Further, we formulated different models of opinion formation as myopic best response dynamics according to our framework of dissonance minimization and thereby provided a common basis for them. Hence, we contributed to their theoretical foundation and developed a link to rational choice theory.

With respect to opinion formation, our framework is restricted to the effect of the agent network on the opinions while the network is not deliberately changed by the agents but evolves according to given external rules. Many models additionally assume that a change in opinions feeds back on the agent network which then evolves endogenously. Our framework could be extended by allowing for a change of an agent's in-group structure in order to reduce dissonance (akin to Centola et al., 2007; Vazquez et al., 2008; or Groeber, Schweitzer, & Press, 2009). This change could either consist in a reduction of relation intensity or the elimination of an agent from the in-group.

Further, following Flache and Mäs (2008), the concepts of social influence and homophily are not sufficient to explain why opinions sometimes drift away from moderate to extreme positions as only attractive forces between opinions are considered. Therefore, some models (Jager & Amblard, 2005; Fent, Groeber, & Schweitzer, 2007; Flache & Macy, 2006b; Kitts, 2006; Salzarulo, 2006; Baldassarri & Bearman, 2007) incorporate the complementary concepts of *rejection* and *heterophobia*. According to rejection, agents change their opinions to become more dissimilar to interaction partners they do not like (Abelson, 1964; Kitts, 2006; Tsuji, 2002). Heterophobia states that agents dislike agents with sufficiently different opinions (Byrne, Clore, & Smeaton, 1986; F. Chen & Kenrick, 2002; Pilkington & Lydon, 1997; Rosenbaum, 1986). The concept of rejection can be integrated in our general framework by allowing that agents perceive *consonance* in case of deviation from the opinions of interaction partners they dislike (cf. Festinger, 1957). In this case, interaction leads to an interplay of attractive and repulsive forces. Such a mechanism has been implemented in the model of Fent et al. (2007) which leads to consensus, polarization or a broad, multimodal distribution of opinions depending on the structure of the interaction network.

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APPENDIX: PROOFS OF PROPOSITIONS 2, 3, 5, 7, AND 8

Proof of Proposition 2. We only have to show $\Theta(G^-) \subset C$. Without a loss of generality, we assume that $a = 1$ as we obtain

$$u_i^-(x_1, \ldots, x_n) = \sum_{j \in I_i} f^-(d(x_i, x_j))$$

for opinions $x_1, \ldots, x_n$ in case the in-group influence is homogeneous. Hence, $a > 0$ does not affect the order of opinion profiles induced by dissonance.

Now consider a nonconsensus opinion profile $x = (x_1, \ldots, x_n)$. First suppose that we have

$$u_{\max}^- = \max_{j \in I} u_j^-(x) > \min_{j \in I} u_j^-(x) = u_{\min}^-.$$

Now choose an agent k with $x_k = u_{min}^-$. With $x' = (x_1, \ldots, x_n)$ whereas $x_i' = x_k$ and $x_j' = x_j$ for $j \neq i$, we obtain

$$\begin{aligned} u_i^-(x') &= \sum_{j \in I \setminus \{i\}} f^-(d(x_i', x_j')) \\ &= \sum_{j \in I \setminus \{k\}} f^-(d(x_k', x_j')) \\ &\leq \sum_{j \in I \setminus \{k\}} f^-(d(x_k, x_j)) \\ &= u_k^-(x) \\ &< u_i^-(x). \end{aligned}$$

In case there is $u_{\max}^- = u_{\min}^-$, we choose agents i, k with $x_i \neq x_k$ which is possible as x is not a consensus profile. With $x' = (x_1, \ldots, x_n)$ whereas $x_i' = x_k$ and $x_j' = x_j$ for $j \neq i$, we obtain

$$\begin{aligned} u_i^-(x') &= \sum_{j \in I \setminus \{i\}} f^-(d(x_i', x_j')) \\ &= \sum_{j \in I \setminus \{k\}} f^-(d(x_k', x_j')) \\ &< \sum_{j \in I \setminus \{k\}} f^-(d(x_k, x_j)) \\ &= u_k^-(x) \\ &= u_i^-(x) \end{aligned}$$

where the inequality is strict as $x_i \neq x_k$. Hence, x is not a Nash equilibrium as agent i can reduce her perceived dissonance for the given opinions of the other agents. □

Proof of Proposition 3. Let $y \in X$ denote an opinion that is not in the convex hull H of x. As H is compact and $d(\cdot, y)$ is continuous (d is the Euclidean metric), there is an opinion $z \in H$ with (positive) minimum distance to y; that is, for all $z' \in H$ we have

$$0 < d(z, y) \leq d(z', y). \tag{A.1}$$

Now consider an agent j. The opinion distance $d(y, x_j)$ is greater than the distance $d(z, x_j)$: Let y_p denote the orthogonal projection of y on the line l defined by x_j and z; that is, we have

$$y_p = \lambda x_{j_1} + (1 - \lambda) z$$

with $\lambda \in \mathbb{R}$. If there was $\lambda > 1$, it would follow that $d(x_j, y_p) < d(z, y_p)$ and therefore (by the Pythagorean theorem) $d(x_j, y) < d(z, y)$ which is a contradiction to Eq. (1). If there was $0 < \lambda \leq 1$, the Pythagorean theorem would yield that $d(y_p, y) < d(z, y)$. As $y_p \in H$ according to the convexity of H, this is again a contradiction to Eq. (1). Hence, we obtain $\lambda \leq 0$. Then, the Pythagorean theorem yields

$$d(y, x_j)^2 = d(y, y_p)^2 + (d(x_j, z) + d(z, y_p))^2$$

and therefore $d(y, x_j) > d(z, x_j)$ as $d(y, y_p) = d(z, y_p) = 0$ would imply $z = y$ which is a contradiction to $y \notin H$.

As f^- is strictly increasing in the opinion distance, this implies $u_i^-(z, x) < u_i^-(y, x)$ so that y cannot be a best response to the opinion profile x. □

Proof of Proposition 5. As $u_i^-(\cdot, x)$ is continuous[18] on the (compact) convex hull H of x, there exists an optimal opinion $x_i^* \in H$ with

$$u_i(x_i^*, x) \geq u_i(y, x)$$

for all $y \in X$ according to Proposition 3. To guarantee the uniqueness of the optimal opinion, it is sufficient to show that $u_i^-(\cdot, x)$ is strictly convex for any opinion profile.

[18]The continuity of $u_i(\cdot, x)$ on the interior of X is already guaranteed by the convexity of the distance dissonance function.

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Therefore, let $d_{x_j}(y)$ denote the distance of opinion $y \in X$ and the fixed opinion x_j. As d_{x_j} is convex and f^- is increasing and strictly convex, we obtain

$$\begin{aligned} f^-(d(\lambda y + (1-\lambda)z, x_j)) &\leq f^-(\lambda d(y, x_j) + (1-\lambda)d(z, x_j)) \\ &< \lambda f^-(d(y, x_j)) + (1-\lambda) f^-(d(z, x_j)) \end{aligned}$$

for all $j \neq i$, y, $z \in X$ and $\lambda \in [0, 1]$. Hence, the composition $f^- \circ d$ is strictly convex. Therefore, $u_i^-(\cdot, x)$ is strictly convex as it is a positive linear combination of strictly convex functions. □

Proof of Proposition 7. Let $x = (x_1, \ldots, x_n)$ denote an opinion profile with $x_i = (x_{i1}, \ldots, x_{im})$. For an agent i and $y = (y_1, \ldots, y_m) \in X$ we define $\tilde{u}_i^-(y) = u_i^-(y, x)$. According to our assumptions on f^-, we have

$$\begin{aligned} \tilde{u}_i^-(y) &= \sum_{j \in I_i} a_{ij} f^-(d(y, x_j)) \\ &= \sum_{j \in I_i} a_{ij} f^- \left(\sqrt{\sum_{1 \leq k \leq m} (y_k - x_{jk})^2} \right) \\ &= \sum_{j \in I_i} a_{ij} \sum_{1 \leq k \leq m} (y_k - x_{jk})^2. \end{aligned}$$

To determine the utility maximizing opinion $x_i^* = (x_{i1}^*, \ldots, x_{im}^*)$, we derive the first order conditions $(\frac{\partial \tilde{u}_i}{\partial y_j}(x^*) = 0)$ and obtain

$$x_{ik}^* \sum_{j \in I_i} a_{ij} = \sum_{j \in I_i} a_{ij} x_{jk}, \quad 1 \leq k \leq m.$$

Further, the Hessian of $\tilde{u}_i$ is positive definite as

$$\frac{\partial^2 \tilde{u}_i^-}{\partial y_k \partial y_l} \equiv \begin{cases} 2 \sum_{j \in I_i} a_{ij} & \text{if } k = l \\ 0 & \text{if } k \neq l, \end{cases}$$

so that x_i^* is a local minimum of $\tilde{u}_i^-$. As f^- is strictly convex, the same holds for $\tilde{u}_i^-$ so that x_i^* is a unique global minimum of $\tilde{u}_i^-$. □

Proof of Proposition 8. (*i*) is an immediate consequence of the equivalence of both opinions and the fact that $p_i^{(1)} + p_i^{(1)} = 1$. Thus, we have in particular $g(0.5) = 0.5$. Further, as $f^-(z) > 0$ for $z > 0$ implies $\tilde{u}_i(p, x) > 0$ for $p > 0$, we obtain $g(0) = 0$. Using (*i*), this also leads to $g(1) = 1$. With respect to (*iii*), we denote $p^* = g(r)$ with $r \in [0, 1)$; that is,

$$(1-r) f^-(p) + r f^-(1-p) > (1-r) f^-(p^*) + r f^-(1-p^*)$$

for $p \neq p^*$. For $r' > r$ and $p < p^*$ this implies

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$$\begin{aligned}\tilde{u}^-(p) - \tilde{u}^-(p^*) &= (1-r)f^-(p) + rf^-(1-p) - (1-r')f^-(p^*) - r'f^-(1-p^*) \\ &> (r'-r)[f^-(p) + f^-(1-p^*) - f^-(1-p) - f^-(p^*)] \\ &> 0\end{aligned}$$

as $f^- \geq 0$ and f^- is strictly increasing. Hence, g is increasing. If there is no consensus in the in-group; that is, if $r \in (0, 1)$, the convexity of f^- implies

$$f^-(p) = f^-(p \cdot 1 + (1-p) \cdot 0) \leq pf^-(1),$$

and therefore

$$\tilde{u}^-(1, x) = (1-r)f^-(1) > pf^-(1) \geq f^-(p) > \tilde{u}^-(p, x)$$

for $p \in (0,\ 1-r)$. Thus, using (*i*), it follows that it is optimal to choose one opinion with probability one if and only if every in-group member exhibits that opinion. □

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