Title

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Chapter One

Exercises Chapter 1

Exercise 1.1.1

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\,(w \Rightarrow)}{\alpha \Rightarrow \beta \to \alpha}\,(\Rightarrow\to)}{\Rightarrow \alpha \to (\beta \to \alpha)}\,(\Rightarrow\to)$$

Exercise 1.1.2

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \qquad \gamma \Rightarrow \gamma}{\beta, \beta \to \gamma \Rightarrow \gamma}\,(\to\Rightarrow)}{\beta \to \gamma, \alpha \to \beta, \alpha \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to (\beta \to \gamma), \alpha \to \beta, \alpha, \alpha \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to (\beta \to \gamma), \alpha \to \beta, \alpha \Rightarrow \gamma}\,(c \Rightarrow)}{\alpha \to (\beta \to \gamma), \alpha \to \beta \Rightarrow \alpha \to \gamma}\,(\Rightarrow\to)}{\alpha \to (\beta \to \gamma) \Rightarrow (\alpha \to \beta) \to (\alpha \to \gamma)}\,(\Rightarrow\to)$$

Exercise 1.2.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}\,(\Rightarrow \neg)}{\Rightarrow \alpha, \alpha \vee \neg\alpha}\,(\Rightarrow \vee 2)}{\Rightarrow \alpha \vee \neg\alpha, \alpha \vee \neg\alpha}\,(\Rightarrow \vee 1)}{\Rightarrow \alpha \vee \neg\alpha}\,(\Rightarrow c)$$

Exercise 1.2.2

$$
\begin{prooftree}
\AxiomC{$\alpha \Rightarrow \alpha$}
\RightLabel{$(w \Rightarrow)$}
\UnaryInfC{$\alpha,\, \beta \Rightarrow \alpha$}
\AxiomC{$\beta \Rightarrow \beta$}
\RightLabel{$(w \Rightarrow)$}
\UnaryInfC{$\alpha,\, \beta \Rightarrow \beta$}
\RightLabel{$(\Rightarrow \wedge)$}
\BinaryInfC{$\alpha,\, \beta \Rightarrow \alpha \wedge \beta$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\alpha,\, \beta,\, \neg(\alpha \wedge \beta) \Rightarrow$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\beta,\, \neg(\alpha \wedge \beta) \Rightarrow \neg\alpha$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha,\, \neg\beta$}
\RightLabel{$(\Rightarrow \vee 1)$}
\UnaryInfC{$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta,\, \neg\beta$}
\RightLabel{$(\Rightarrow \vee 2)$}
\UnaryInfC{$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta,\, \neg\alpha \vee \neg\beta$}
\RightLabel{$(\Rightarrow c)$}
\UnaryInfC{$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$}
\end{prooftree}
$$

Exercise 1.2.3

$$
\begin{prooftree}
\AxiomC{$\alpha \Rightarrow \alpha$}
\RightLabel{$(\Rightarrow w)$}
\UnaryInfC{$\alpha \Rightarrow \beta,\, \alpha$}
\RightLabel{$(\Rightarrow \rightarrow)$}
\UnaryInfC{$\Rightarrow \alpha \rightarrow \beta,\, \alpha$}
\AxiomC{$\alpha \Rightarrow \alpha$}
\RightLabel{$(\rightarrow \Rightarrow)$}
\BinaryInfC{$(\alpha \rightarrow \beta) \rightarrow \alpha \Rightarrow \alpha,\, \alpha$}
\RightLabel{$(\Rightarrow c)$}
\UnaryInfC{$(\alpha \rightarrow \beta) \rightarrow \alpha \Rightarrow \alpha$}
\end{prooftree}
$$

Exercise 1.3

Lemma 1.1. *The following propositions are equivalent:*

*(a) The sequent $\alpha_1, \alpha_2 \Rightarrow \beta_1, \beta_2$ is provable in **LK**.*

*(b) The sequent $\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1 \vee \beta_2$ is provable in **LK**.*

*(c) The sequent $\Rightarrow \alpha_1 \wedge \alpha_2 \rightarrow \beta_1 \vee \beta_2$ is provable in **LK**.*

Proof. Now we will show that (a) implies (b). Assume that $\alpha_1, \alpha_2 \Rightarrow \beta, \beta_2$ is provable in **LK**. Then there exists a proof for the sequent. Using this proof, we can construct the following proof of $\alpha_1 \wedge \alpha_2 \Rightarrow \beta \vee \beta_2$:

$$
\begin{prooftree}
\AxiomC{$\alpha_1,\, \alpha_2 \Rightarrow \beta_1,\, \beta_2$}
\RightLabel{$(\wedge 2 \Rightarrow)$}
\UnaryInfC{$\alpha_1,\, \alpha_1 \wedge \alpha_2 \Rightarrow \beta_1,\, \beta_2$}
\RightLabel{$(\wedge 1 \Rightarrow)$}
\UnaryInfC{$\alpha_1 \wedge \alpha_2,\, \alpha_1 \wedge \alpha_2 \Rightarrow \beta_1,\, \beta_2$}
\RightLabel{$(c \Rightarrow)$}
\UnaryInfC{$\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1,\, \beta_2$}
\RightLabel{$(\Rightarrow \vee 2)$}
\UnaryInfC{$\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1,\, \beta_1 \vee \beta_2$}
\RightLabel{$(\Rightarrow \vee 1)$}
\UnaryInfC{$\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1 \vee \beta_2,\, \beta_1 \vee \beta_2$}
\RightLabel{$(\Rightarrow c)$}
\UnaryInfC{$\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1 \vee \beta_2$}
\end{prooftree}
$$

If we replace the leaf containing the sequent $\alpha_1, \alpha_2 \Rightarrow \beta, \beta_2$ with the proof of this same sequent, we obtain a valid proof of $\alpha_1 \wedge \alpha_2 \Rightarrow \beta \vee \beta_2$.

Now we will show that (b) implies (c). Similarly as before. Assume that $\alpha_1 \wedge \alpha_2 \Rightarrow \beta \vee \beta_2$ is provable in **LK**. Then there exists a proof for the sequent. Using this proof, we can construct the following proof of $\alpha_1 \alpha_2 \Rightarrow \beta \vee \beta_2$:

□

Exercise 1.4

If we accept this generalized cut rule we can obtain, for example, a proof for the empty sequent $\Rightarrow$:

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}\,(\Rightarrow \neg) \qquad \dfrac{\alpha \Rightarrow \alpha}{\alpha, \neg\alpha \Rightarrow}\,(\neg \Rightarrow)}{\Rightarrow}\,(cut)$$

But this should not be possible, as the interpretation of the empty sequent $\Rightarrow$ is that "a contradiction follows from no assumptions".

Exercise 1.5

As explained in the book, it is enough to show that:

- Every initial sequent is a tautology
- For each rule of **LK**, if (each of) the upper sequent(s) is a tautology then the lower sequent is a tautology.

The initial sequents are always of the form $\alpha \Rightarrow \alpha$. But this is a tautology as the corresponding formula is $\alpha \rightarrow \alpha$ is a tautology. To see this notice that for any $a \in \{0, 1\}$ it is the case that $a \rightarrow a = 1$, and thus for any valuation h, it holds that $h(\alpha \rightarrow \alpha) = h(\alpha) \rightarrow h(\alpha) = 1$.

Now we will prove that for each rule of **LK**, if each of the upper sequents is a tautology then the lower sequent is also a tautology. More precisely, we will show that if each of the corresponding formulae of the upper sequents is a tautology then the corresponding formula of the lower sequent is a tautology.

Given a multiset of formulae Δ, we will write $\bigwedge \Delta$ instead of $\bigwedge\limits_{\alpha \in \Delta} \alpha$ and $\bigvee \Delta$ instead of $\bigvee\limits_{\alpha \in \Delta} \alpha$.

Rule $(\vee \Rightarrow)$

Assume that $\alpha \wedge \bigwedge \Gamma \to \bigvee \Pi$ and that $\beta \wedge \bigwedge \Gamma \to \bigvee \Pi$ are both tautologies. We will show that $(\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi$ is a tautology.

Let h be an arbitrary valuation. If $h(\bigvee \Pi) = 1$, then:

$$h((\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to h(\bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to 1 = 1$$

Assume now that $h(\bigvee \Pi) = 0$. Given that the upper sequents are tautologies we have that:

$$h((\alpha \wedge \bigwedge \Gamma \to \bigvee \Pi) = 1 = h((\beta \wedge \bigwedge \Gamma \to \bigvee \Pi)$$

Using the fact that $h(\bigvee \Pi) = 0$, it is easy to obtain that $h(\alpha) = 0$, $h(\beta) = 0$ and that $h(\bigwedge \Gamma) = 0$. Then $h((\alpha \vee \beta) \wedge \bigwedge \Gamma) = 0$.

And thus:

$$h((\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to h(\bigvee \Pi) = 0 \to h(\bigvee \Pi) = 1$$

Rule (cut)

Assume that $\alpha \wedge \bigwedge \Gamma \to \bigvee \Pi$ and that $\beta \wedge \bigwedge \Gamma \to \bigvee \Pi$ are both tautologies. We will show that $(\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi$ is a tautology.

Let h be an arbitrary valuation. If $h(\bigvee \Pi) = 1$, then:

$$h((\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to h(\bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to 1 = 1$$

Assume now that $h(\bigvee \Pi) = 0$. Given that the upper sequents are tautologies we have that:

$$h((\alpha \wedge \bigwedge \Gamma \to \bigvee \Pi) = 1 = h((\beta \wedge \bigwedge \Gamma \to \bigvee \Pi)$$

Using the fact that $h(\bigvee \Pi) = 0$, it is easy to obtain that $h(\alpha) = 0$, $h(\beta) = 0$ and that $h(\bigwedge \Gamma) = 0$. Then $h((\alpha \vee \beta) \wedge \bigwedge \Gamma) = 0$.

And thus:

$$h((\alpha \vee \beta) \wedge \bigwedge \Gamma \to \bigvee \Pi) = h((\alpha \vee \beta) \wedge \bigwedge \Gamma) \to h(\bigvee \Pi) = 0 \to h(\bigvee \Pi) = 1$$

The other rules are similar.

Exercise 1.6

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Exercise 1.7

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Exercise 1.8

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Exercise 1.9

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Exercise 1.10

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Exercise 1.11

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Exercise 1.12

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Exercise 1.13

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Exercise 1.14

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Exercise 1.15

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Bibliography