Assignment Block B

Bianca Neubert and Franziska Wehrmann

Theory

Markov random fields

Let Y denote a GMRF with mean vector μ and precision matrix Q of dimension $n \in \mathbb{N}$.

i) To show:

$$Y_i|Y_{-i} \sim \mathcal{N}\left(\mu_i - \frac{1}{q_{ii}} \sum_{j \in \mathrm{ne}(i)} q_{ij}(y_j - \mu_j), \frac{1}{q_{ii}}\right)$$

Proof: Since Y is a GMRF with expectation μ and precision matrix Q, we have for the conditional density of $Y_i|Y_{-i}$

$$\pi(y_i|y_{-i}) = \pi(y_i|y_{\mathrm{ne}(i)})$$

where $y_{\mathrm{ne}(i)}$ denotes the vector with indices in ne(i). Because the MRF is Gaussian, we have further for every $A \subset \{1, ..., n\}$

$$Y_A \sim \mathcal{N}(\mu_A, Q_A^{-1})$$

where $Y_A = (Y_i)_{i \in A}$, $\mu_A = (\mu_i)_{i \in A}$ and $Q_A = (q_{ij})_{i,j \in A}$ for the entries q_{ij} of Q.

Combining this, we get for the density of $Y_i|Y_{-i}$, since $Y_{\mathrm{ne}(i)}$ is also normally distributed and hence has strictly positive density, that

$$\begin{aligned}
\pi(y_i|y_{-i}) &= \frac{\pi(y_{\{i\}\cup \mathrm{ne}(i)})}{\pi(y_{\mathrm{ne}(i)})} \\
&\propto \exp\left\{-\frac{1}{2}(y_{\{i\}\cup \mathrm{ne}(i)} - \mu_{\{i\}\cup \mathrm{ne}(i)})^T Q_{\{i\}\cup \mathrm{ne}(i)}(y_{\{i\}\cup \mathrm{ne}(i)} - \mu_{\{i\}\cup \mathrm{ne}(i)}) \right. \\
&\quad \left. + \frac{1}{2}(y_{\mathrm{ne}(i)} - \mu_{\mathrm{ne}(i)})^T Q_{\mathrm{ne}(i)}(y_{\mathrm{ne}(i)} - \mu_{\mathrm{ne}(i)})\right\} \\
&= \exp\left\{-\frac{1}{2}\left[q_{ii}y_{ii}^2 + 2\sum_{j \in \mathrm{ne}(i)} y_i q_{ij} y_j + q_{ii}\mu_i^2 + 2\sum_{j \in \mathrm{ne}(i)} \mu_i q_{ij}\mu_j \right.\right. \\
&\quad \left.\left. -2q_{ii}\mu_i y_i - 2\sum_{j \in \mathrm{ne}(i)} \mu_i q_{ij} y_j - 2\sum_{j \in \mathrm{ne}(i)} y_i q_{ij}\mu_j\right]\right\} \\
&= \exp\left\{-\frac{q_{ii}}{2}\left[(y_i - \mu_i)^2 + \frac{2}{q_{ii}}(y_i - \mu_i)\sum_{j \in \mathrm{ne}(i)} q_{ij}(y_j - \mu_j)\right]\right\} \\
&\propto \exp\left\{-\frac{q_{ii}}{2}\left[y_i - \mu_i + \frac{1}{q_{ii}}\sum_{j \in \mathrm{ne}(i)} q_{ij}(y_j - \mu_j)\right]^2\right\}
\end{aligned}$$

which is the kernel of a normal distribution with the given expecation and variance.

iii) To show: $\mathbb{C}\text{or}[Y_i, Y_j | Y_{-ij}] = -\frac{q_{ij}}{\sqrt{q_{ii} q_{jj}}}$

Proof:

Local charcteristics

Suppose S is a finite set equipped with a symmetry relation $\sim$. For count outcomes, a specification commonly used in disease mapping is the *auto-Poisson* model where

$$
\begin{aligned}
\pi(y_i | y_{S \setminus i}) &= \exp(-\mu_i) \frac{\mu_i^{y_i}}{y_i} \\
\log(\mu_i) &= - \sum_{j \in \text{ne}(i)} y_j \;\; i \neq j
\end{aligned}
$$

for $y \in \mathbb{N}_0$, $i \in S$. Determine the canonical potential with respect to $a = 0$.

Computation

Guerry's data on social morals

This task is based on Guerry's data on moral statistics including aggregated numbers of suicide (`suicids`). Let us first import and inspect the data.

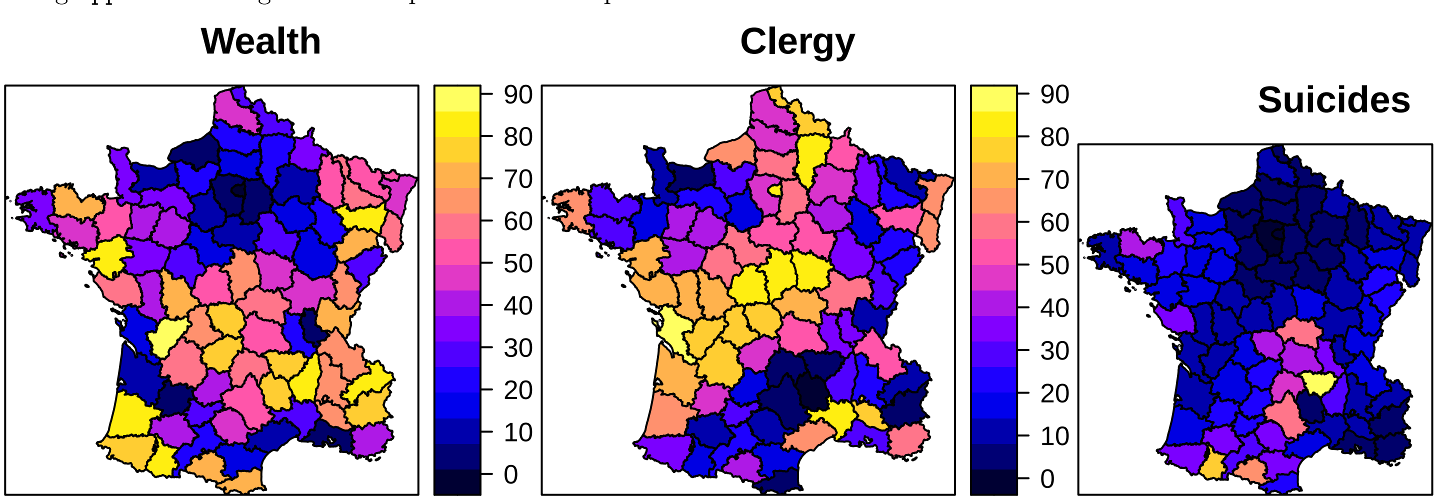
```
guerry <- readOGR("guerry/Guerry.shp")
```

```
## OGR data source with driver: ESRI Shapefile
## Source: "/Users/franziska/SpatStat/guerry/Guerry.shp", layer: "Guerry"
## with 85 features
## It has 26 fields
## Integer64 fields read as strings:  dept Crm_prs Crm_prp Litercy Donatns Infants Suicids MainCty Weal
```

We see that the variables are stored in form of strings, so we first change the class to integer of the three variables of interest in this assignment: `Suicids`, `Wealth` and `Clergy`

```
guerry$Suicids <- as.integer(guerry$Suicids)
guerry$Wealth <- as.integer(guerry$Wealth)
guerry$Clergy <- as.integer(guerry$Clergy)
```

Using `spplot` we can get a first impression on the spatial distribution of these variables.



Specify a (non-spatial) regression model of `Wealth` and `Clergy` on `Suicids`.

```r
reg <- lm(Suicids ~ Wealth + Clergy, data=guerry)
summary(reg)
```

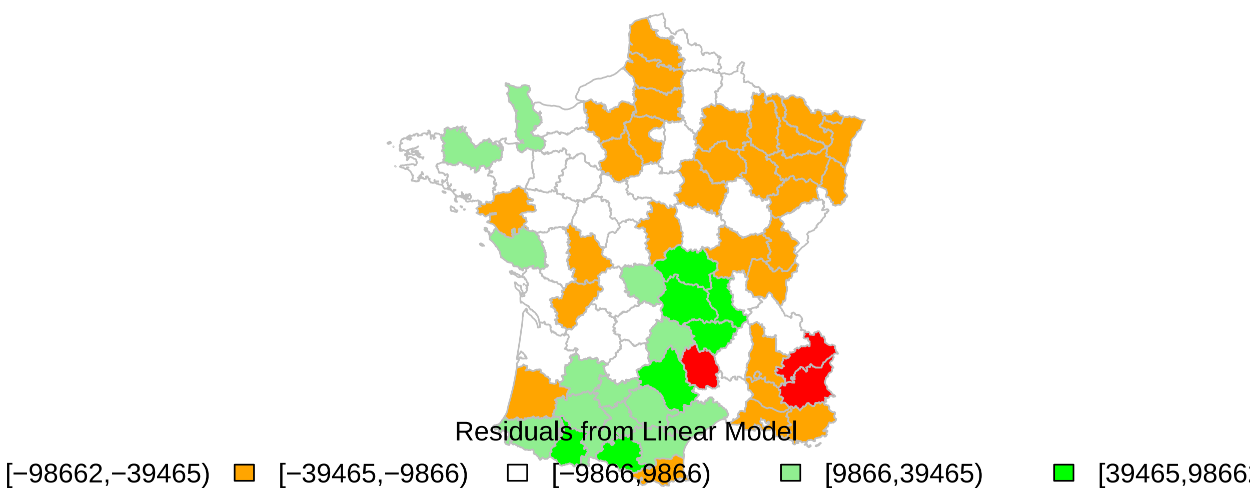
```
##
## Call:
## lm(formula = Suicids ~ Wealth + Clergy, data = guerry)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -56593 -14816  -5650   7680  98662
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  30919.3     8476.1   3.648 0.000464 ***
## Wealth         486.7      120.5   4.038 0.000121 ***
## Clergy        -355.4      122.5  -2.902 0.004760 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 27560 on 82 degrees of freedom
## Multiple R-squared:  0.2527, Adjusted R-squared:  0.2345
## F-statistic: 13.86 on 2 and 82 DF,  p-value: 6.508e-06
```

We can see, that in this the coefficients of Clergy and Wealth are not significant which at first does not indicate a relationship between these and Suicides. This model does not explain the variation in Suicides very well (e.g. Multiple R-squared = 0.011).

We can plot the residuals: First, we fix color palette once for all plots of this type. This way it is easier to compare the results later.

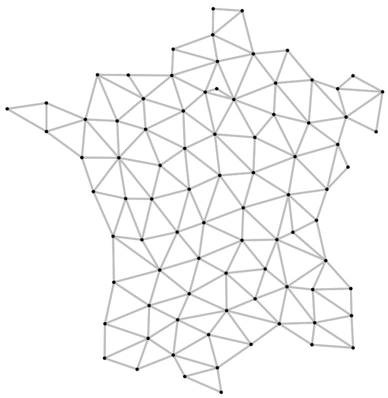
```r
m = 98661.66  # maximum residual of (lm, car, sar, sdm)
breaks = round(c(-m, -0.4*m, -0.1*m, 0.1*m, 0.4*m, m), digits=0)
res.palette <- colorRampPalette(c("red","orange","white", "lightgreen","green"), space = "rgb")
pal <- res.palette(5)
```

Now we can plot the residuals of the linear model.



To test for autocorrelation in the data, we can use Moran's I and Geary's C. For this, we need to specify a neighborhood structure in a matrix.

```r
xy <- coordinates(guerry)
W_cont_el <- poly2nb(guerry, queen=FALSE)
W_cont_el_mat <- nb2listw(W_cont_el, style="B", zero.policy=TRUE)
plot(W_cont_el_mat, coords=xy, cex=0.1, col="gray")
```



For the Moran's I ad Geary's C we get the following results

```
##
##  Moran I test under randomisation
##
## data:  guerry$Suicids
## weights: W_cont_el_mat
##
## Moran I statistic standard deviate = 6.4289, p-value = 6.428e-11
## alternative hypothesis: greater
## sample estimates:
## Moran I statistic       Expectation          Variance
##       0.404857670      -0.011904762       0.004202492
```

```
##
##  Geary C test under randomisation
##
## data:  guerry$Suicids
## weights: W_cont_el_mat
##
## Geary C statistic standard deviate = 4.2472, p-value = 1.082e-05
## alternative hypothesis: Expectation greater than statistic
## sample estimates:
## Geary C statistic       Expectation          Variance
##        0.55293883        1.00000000        0.01107975
```

The Moran's I test statistic and its expectation are both close to zero (0.074 and -0.011). However, the Moran's I test statistic is larger than the expectation which is an indication for a positive autocorrelation. If we look the given p-value we can reject the null of no autocorrelation for a 0.1-niveau, but not for a 0.05-niveau. On the other hand, Geary's C is close to 1 and the p-value with 0.197 suggests that the null of no autocorrelation can not be rejeceted for a 0.1-niveau. To sum this up, there seems to be very little global autocorrelation in the data, if there is any at all.

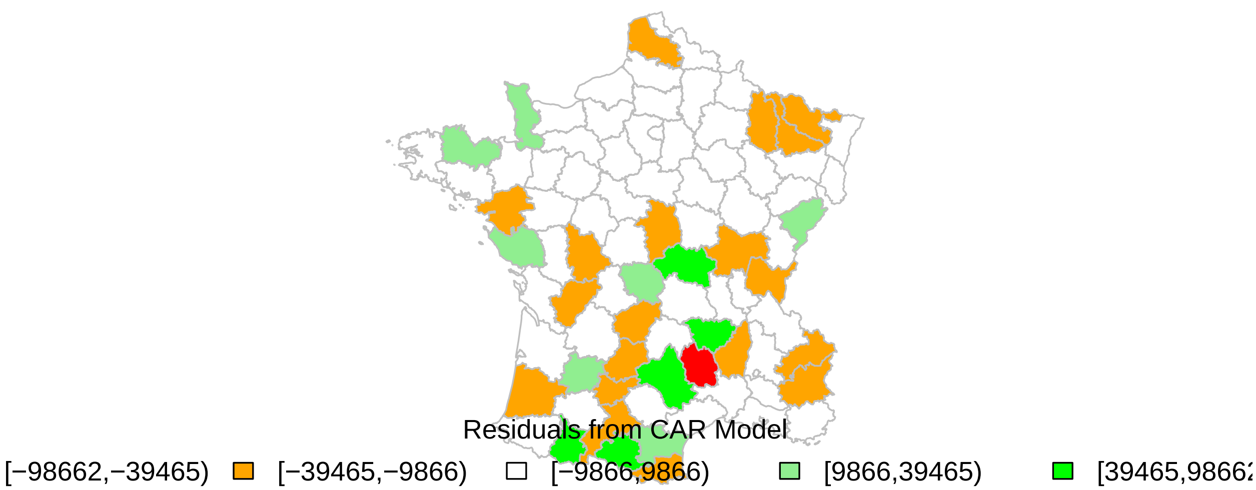
CAR

Let us compute a CAR model.

```r
car.out <- spdep::spautolm(Suicids ~ Wealth + Clergy, data=guerry,
                           listw=W_cont_el_mat, family="CAR")
mod.car <- fitted(car.out)
summary(car.out)
```

```
##
## Call: spdep::spautolm(formula = Suicids ~ Wealth + Clergy, data = guerry,
##     listw = W_cont_el_mat, family = "CAR")
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -81951.2  -8586.6  -2583.9   3722.5  94835.8
##
## Coefficients:
##             Estimate Std. Error z value  Pr(>|z|)
## (Intercept) 27409.16    9856.88  2.7807 0.0054240
## Wealth        421.99     123.21  3.4250 0.0006147
## Clergy       -275.75     128.56 -2.1449 0.0319596
##
## Lambda: 0.15913 LR test value: 15.459 p-value: 8.4312e-05
## Numerical Hessian standard error of lambda: 0.018933
##
## Log likelihood: -980.4015
## ML residual variance (sigma squared): 551060000, (sigma: 23475)
## Number of observations: 85
## Number of parameters estimated: 5
## AIC: 1970.8
```

Again, the coefficients for Wealth and Clergy are not significant. The plot of the residuals:



SAR

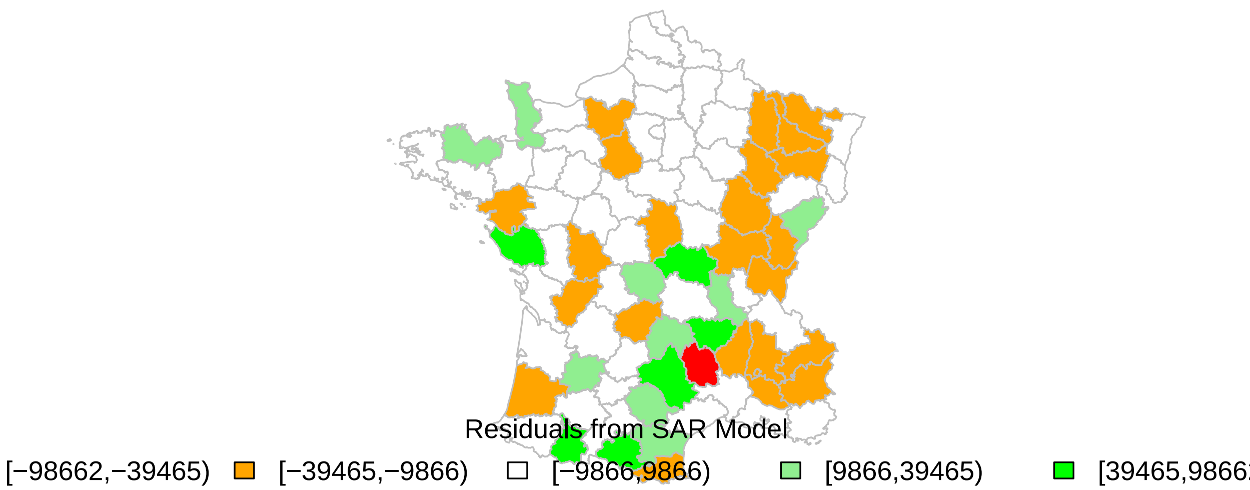
Now, the same for SAR

```r
mod.sar <- lagsarlm(Suicids ~ Wealth + Clergy, data=guerry,
                    listw=W_cont_el_mat, zero.policy=T, tol.solve=1e-12)
summary(mod.sar)
```

```
##
```

```
## Call:lagsarlm(formula = Suicids ~ Wealth + Clergy, data = guerry,
##     listw = W_cont_el_mat, zero.policy = T, tol.solve = 1e-12)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -71691.5 -10439.1  -4823.3   6113.9  94298.9
##
## Type: lag
## Coefficients: (numerical Hessian approximate standard errors)
##             Estimate Std. Error z value  Pr(>|z|)
## (Intercept) 16317.90    8188.54  1.9928 0.0462863
## Wealth        400.52     107.90  3.7121 0.0002056
## Clergy       -279.20     109.10 -2.5591 0.0104940
##
## Rho: 0.082595, LR test value: 15.313, p-value: 9.1111e-05
## Approximate (numerical Hessian) standard error: 0.019307
##     z-value: 4.2781, p-value: 1.8852e-05
## Wald statistic: 18.302, p-value: 1.8852e-05
##
## Log likelihood: -980.4748 for lag model
## ML residual variance (sigma squared): 588440000, (sigma: 24258)
## Number of observations: 85
## Number of parameters estimated: 5
## AIC: 1970.9, (AIC for lm: 1984.3)
```

Again, the coefficients for Wealth and Clergy are not significant.The plot of the residuals:



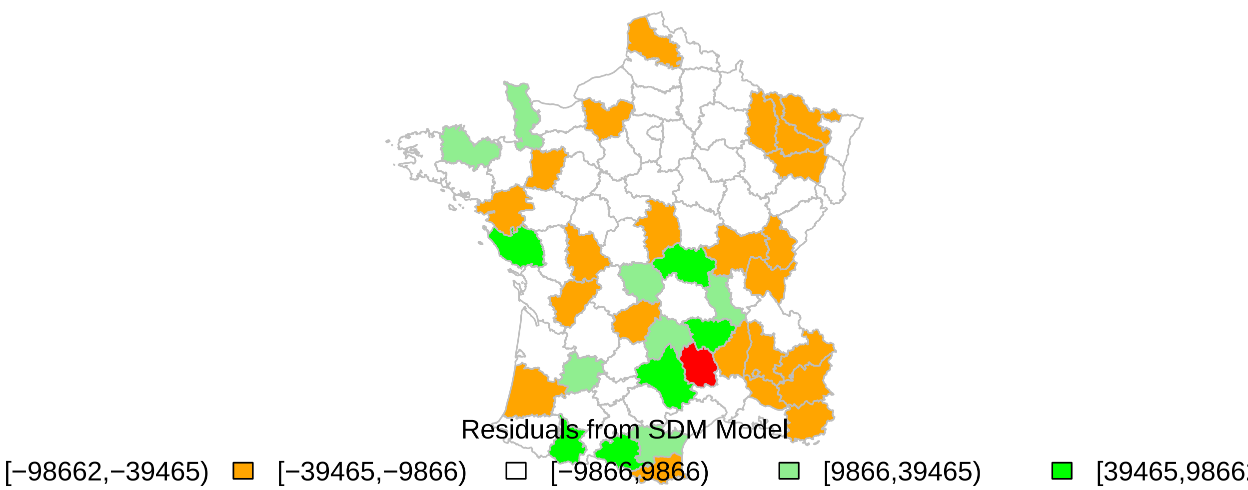
SDM

And SDM

```
mod.sdm <- lagsarlm(Suicids ~ Wealth + Clergy, data=guerry, listw=W_cont_el_mat,
                    zero.policy=T, type="mixed", tol.solve=1e-12)
summary(mod.sdm)
```

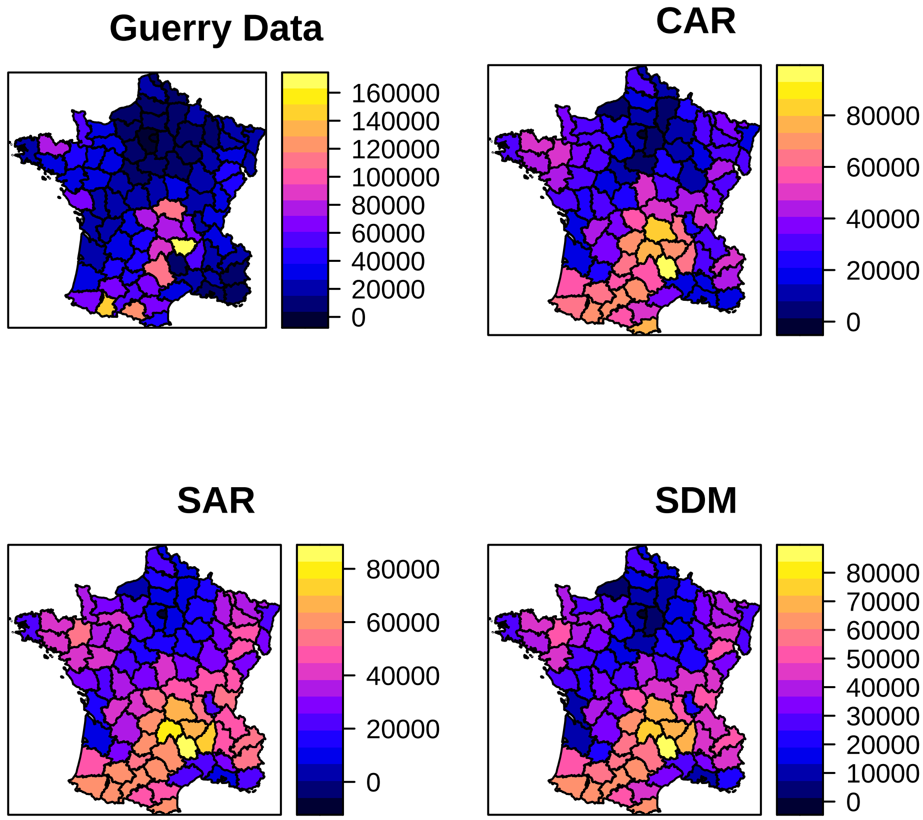
```
##
## Call:lagsarlm(formula = Suicids ~ Wealth + Clergy, data = guerry,
##     listw = W_cont_el_mat, type = "mixed", zero.policy = T, tol.solve = 1e-12)
##
## Residuals:
```

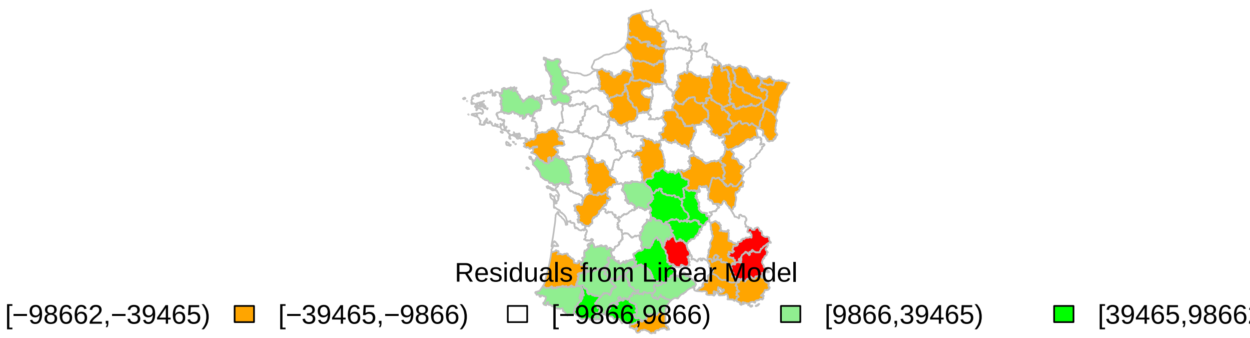
```
##      Min       1Q   Median       3Q      Max
## -72844.2 -10442.8  -5159.5   4116.9  94379.8
##
## Type: mixed
## Coefficients: (numerical Hessian approximate standard errors)
##                   Estimate Std. Error z value Pr(>|z|)
## (Intercept)     20408.816  12688.840  1.6084 0.107746
## Wealth            417.431    132.596  3.1481 0.001643
## Clergy           -211.493    140.755 -1.5026 0.132954
## lag.(Intercept)  -158.197   3430.786 -0.0461 0.963222
## lag.Wealth        -21.019     47.436 -0.4431 0.657698
## lag.Clergy        -21.646     45.160 -0.4793 0.631706
##
## Rho: 0.094743, LR test value: 12.159, p-value: 0.00048859
## Approximate (numerical Hessian) standard error: 0.023939
##     z-value: 3.9577, p-value: 7.5666e-05
## Wald statistic: 15.664, p-value: 7.5666e-05
##
## Log likelihood: -979.7269 for mixed model
## ML residual variance (sigma squared): 570060000, (sigma: 23876)
## Number of observations: 85
## Number of parameters estimated: 8
## AIC: 1975.5, (AIC for lm: 1985.6)
```

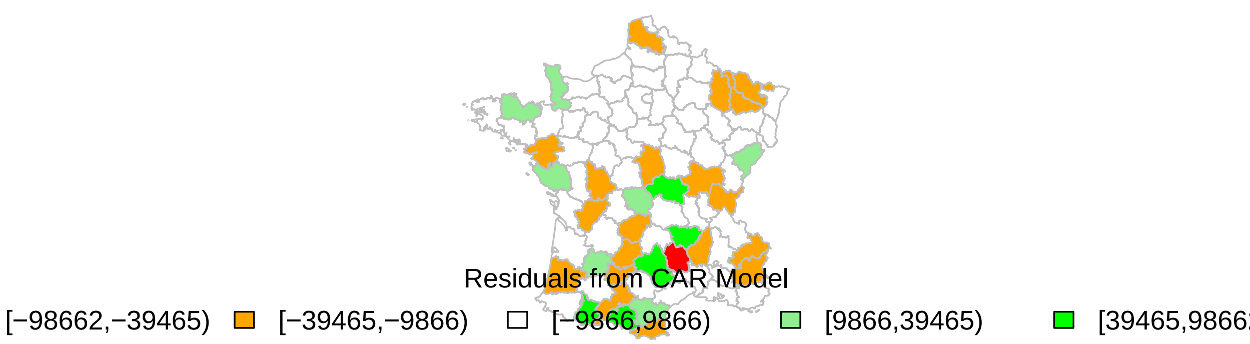


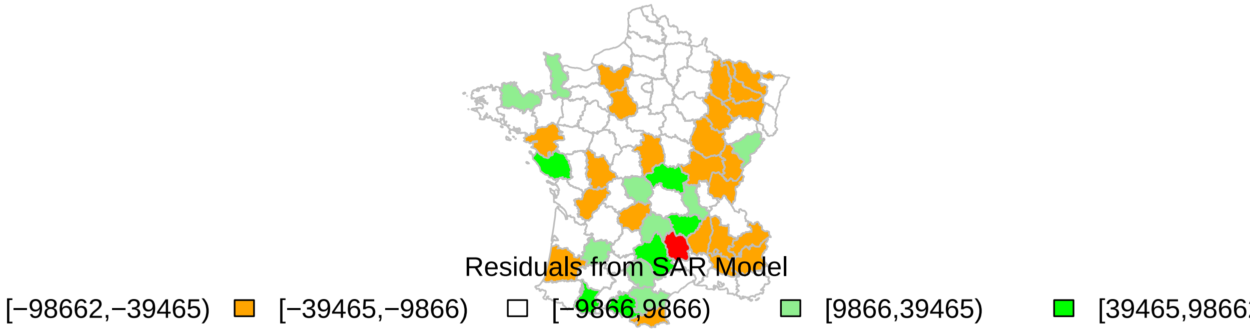
Comparison

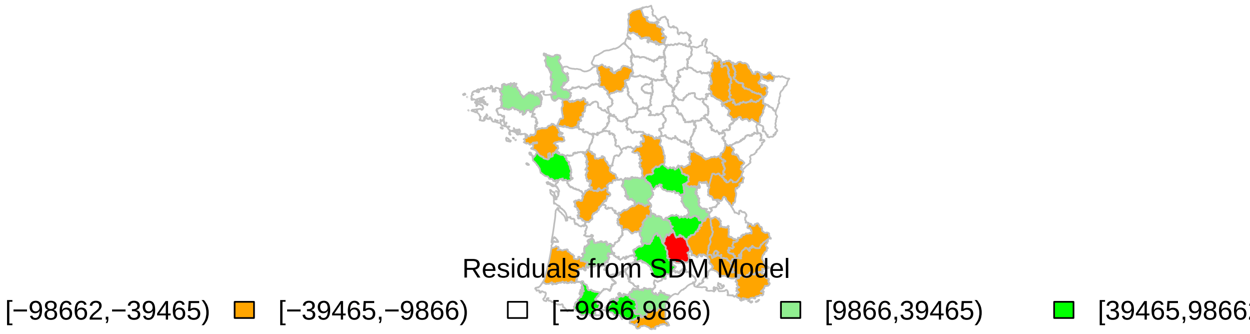
- In all regression models it is obvious that the most extreme values got mitigated, which is especially noticable since the scale of the original data goes up to 16.000 Suicides, whereas the regressed data only reaches 9.000
- SDM seems more precise than SAR (Pattern of predicted values and plot of residuals), which is not surprising since it (SDM) has the additional term of spatially lagged covariates over the SAR model.
- The Linear Regression has pretty huge residuals, especially around the extreme Suicid-Values, which is not surprising since it does not take any spatial relationships into account.
- CAR?


Guerry Data
160000
140000
120000
100000
80000
60000
40000
20000
0
CAR
80000
60000
40000
20000
0
SAR
80000
60000
40000
20000
0
SDM
80000
70000
60000
50000
40000
30000
20000
10000
0


Residuals from Linear Model
[−98662,−39465)
[−39465,−9866)
[−9866,9866)
[9866,39465)
[39465,9866


Residuals from CAR Model
[−98662,−39465)
[−39465,−9866)
[−9866,9866)
[9866,39465)
[39465,9866


Residuals from SAR Model
[−98662,−39465)
[−39465,−9866)
[−9866,9866)
[9866,39465)
[39465,9866


Residuals from SDM Model
[−98662,−39465)
[−39465,−9866)
[−9866,9866)
[9866,39465)
[39465,9866