Assignment Block B

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Theory

Markov random fields

Let Y denote a GMRF with mean vector μ and precision matrix Q of dimension $n \in \mathbb{N}$.

i) To show:

$$Y_i|Y_{-i} \sim \mathcal{N}\left(\mu_i - \frac{1}{q_{ii}} \sum_{j\in \mathrm{ne}(i)} q_{ij}(y_j - \mu_j), \frac{1}{q_{ii}}\right)$$

Proof: Since Y is a GMRF with expectation μ and precision matrix Q, we have for the conditional density of $Y_i|Y_{-i}$

$$\pi(y_i|y_{-i}) = \pi(y_i|y_{\mathrm{ne}(i)})$$

where $y_{\mathrm{ne}(i)}$ denotes the vector with indices in $\mathrm{ne}(i)$. Because the MRF is Gaussian, we have further for every $A \subset \{1, ..., n\}$

$$Y_A \sim \mathcal{N}(\mu_A, Q_A^{-1})$$

where $Y_A = (Y_i)_{i\in A}$, $\mu_A = (\mu_i)_{i\in A}$ and $Q_A = (q_{ij})_{i,j\in A}$ for the entries q_{ij} of Q.

Combining this, we get for the density of $Y_i|Y_{-i}$, since $Y_{\mathrm{ne}(i)}$ is also normally distributed and hence has strictly positive density, that

$$\begin{aligned}
\pi(y_i|y_{-i}) &= \frac{\pi(y_{\{i\}\cup\mathrm{ne}(i)})}{\pi(y_{\mathrm{ne}(i)})} \\
&\propto \exp\left\{-\frac{1}{2}(y_{\{i\}\cup\mathrm{ne}(i)} - \mu_{\{i\}\cup\mathrm{ne}(i)})^T Q_{\{i\}\cup\mathrm{ne}(i)}(y_{\{i\}\cup\mathrm{ne}(i)} - \mu_{\{i\}\cup\mathrm{ne}(i)}) \right. \\
&\quad \left. +\frac{1}{2}(y_{\mathrm{ne}(i)} - \mu_{\mathrm{ne}(i)})^T Q_{\mathrm{ne}(i)}(y_{\mathrm{ne}(i)} - \mu_{\mathrm{ne}(i)})\right\} \\
&= \exp\left\{-\frac{1}{2}\left[q_{ii}y_{ii}^2 + 2\sum_{j\in\mathrm{ne}(i)} y_i q_{ij} y_j + q_{ii}\mu_i^2 + 2\sum_{j\in\mathrm{ne}(i)} \mu_i q_{ij}\mu_j \right.\right. \\
&\quad \left.\left. -2q_{ii}\mu_i y_i - 2\sum_{j\in\mathrm{ne}(i)} \mu_i q_{ij} y_j - 2\sum_{j\in\mathrm{ne}(i)} y_i q_{ij}\mu_j\right]\right\} \\
&= \exp\left\{-\frac{q_{ii}}{2}\left[(y_i - \mu_i)^2 + \frac{2}{q_{ii}}(y_i - \mu_i)\sum_{j\in\mathrm{ne}(i)} q_{ij}(y_j - \mu_j)\right]\right\} \\
&\propto \exp\left\{-\frac{q_{ii}}{2}\left[y_i - \mu_i + \frac{1}{q_{ii}}\sum_{j\in\mathrm{ne}(i)} q_{ij}(y_j - \mu_j)\right]^2\right\}
\end{aligned}$$

which is the kernel of a normal distribution with the given expecation and variance.

iii) To show: $\mathbb{C}\text{or}[Y_i, Y_j|Y_{-ij}] = -\frac{q_{ij}}{\sqrt{q_{ii}q_{jj}}}$

Proof:

Local charcteristics

Suppose S is a finite set equipped with a symmetry relation $\sim$. For count outcomes, a specification commonly used in disease mapping is the *auto-Poisson* model where

$$\begin{aligned}\pi(y_i|y_{S\setminus i}) &= \exp(-\mu_i)\frac{\mu_i^{y_i}}{y_i}\\ \log(\mu_i) &= -\sum_{j\in\text{ne}(i)} y_j \;\; i \neq j\end{aligned}$$

for $y \in \mathbb{N}_0$, $i \in S$. Determine the canonical potential with respect to $a = 0$.

Computation

Guerry's data on social morals