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Structure of supergravity theories[1]

Antoine Van Proeyen

Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven,
Celestijnenlaan 200D B-3001 Leuven, Belgium.
email: Antoine.VanProeyen@fys.kuleuven.ac.be

Abstract

We give an elementary introduction to the structure of supergravity theories. This leads to a table with an overview of supergravity and supersymmetry theories in dimensions 4 to 11. The basic steps in constructing supergravity theories are considered: determination of the underlying algebra, the multiplets, the actions, and solutions. Finally, an overview is given of the geometries that result from the scalars of supergravity theories.

Key words: Supergravity, gauge theories, superalgebras, Kähler geometry, quaternionic geometry
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1 Introduction

Supergravity is now mostly known as an ingredient of superstring theory, the theory that tells that the elementary particles are vibrations of a fundamental string. This theory offers a magnificent framework to study all particles and all interactions. Since 1994 there is an extended framework connecting all the superstring theories and supergravities, which is called M-theory. Many unexpected connections now emerge every year from the study of superstrings and supergravities, increasing continuously our capabilities to construct realistic models, to perform calculations on gauge theories, to discuss models of cosmology and to connect many different aspects of our knowledge of physics and mathematics.

For many applications of string theory we just need to know the structure of supergravity theories. The elementary particles that emerge from any superstring theory and that are not hidden to us by being extremely massive, nicely fit in supergravity theories. In this review, we will consider supergravity in its own right, i.e. independent of its role in superstring theory. We will consider the supergravities with fields that occur in the action up to two derivatives. This corresponds only to the first order in an α' perturbation theory of superstrings.

The part of the theory that describes the spinless fields is very illustrative for the structure of such a supergravity theory. These scalar fields determine the vacua of the theory. In the context of superstring theory they are the moduli of deformations of the compactifying dimensions. The scalar fields define a geometry, the structure of which determines also a large part of the full action due to the supersymmetry relations. We will at the end of this review consider these geometries. It turns out that the richest geometric structures are those that appear when there are 8 supersymmetries.

In the next section, we will review the basic ingredients of supergravity theory at the bosonic side: the fields that are mostly n-forms with gauge transformations, and dualities between their field strengths. The mechanism of compensating fields will be shown and we will show how Poincaré gravity is obtained as a gauge theory.

The fermionic side is introduced in section 3. The possibilities for supersymmetries are determined by a table of Clifford algebras with specific properties. The classification of these lead to a map of possible supergravities and supersymmetric theories that is explained in section 4.

We can divide the discussion on supergravity theories in 4 steps. First, in section 5, we consider the algebra of the transformations that leave the action invariant. We discuss the super-Poincaré algebra, but also generalizations as central charges, the (anti-)de Sitter algebra, the conformal algebra and their supersymmetric extensions. Apart from the spacetime symmetries and the supersymmetries, other bosonic gauge symmetries play a role. In particular there is the R-symmetry, for which the supersymmetries form a non-trivial representation. At the end of this section we show how these algebras are used in constructing supergravity theories

The second step (section 6) brings in the fields as representations of the algebra. Such a set of fields is a *multiplet*, and there is a balance between bosonic and fermionic fields with an *on-shell counting*, but often also with an *off-shell counting*. We first explain this difference, and then discuss the most important multiplet in various dimensions and with various supersymmetry extensions.

In section 7 we look at the third step: the construction of an action and its properties. We explain the duality transformations in 4 dimensions, and how the kinetic terms of the scalars determine a scalar geometry. We summarize how isometries generalize to hidden symmetries of the action, often related to duality transformations, and how a subgroup of them can be gauged.

The final step (section 8) involves the properties of some solutions of the supergravity theories. The concept of BPS solutions will appear, with charges that are at the boundary of an allowed domain.

As mentioned, the scalars of supergravity theories define a geometry, and their kinetic terms in the action determine a metric. Often these are symmetric spaces. An interesting subclass of supergravities, those with 8 supersymmetries, define *special geometries*, which can be real, Kähler or quaternionic manifolds. These geometric aspects are reviewed in section 9, before concluding with some final remarks in section 10.

2 The ingredients: fields and gauge symmetry

The bosonic fields that occur in supergravity theories are mostly antisymmetric tensors (i.e. components of n-forms), with gauge symmetries. There are exceptions, but for most theories one can restrict to these types of bosonic fields. We will repeat the relevant general formulae for gauge theories. These fields determine massless particles. Masses for fields with spin 1 or higher appear by considering spontaneously broken symmetries. The construction of Poincaré gravity as a gauge theory needs already a special treatment.

Symmetries in field theories should be distinguished in different categories. The simplest ones are *rigid symmetries*. These are transformations of the fields with a parameter, say Λ, that is the same everywhere in spacetime. E.g. a rotation of the reference frame is such a transformation. The second type are the *local symmetries* or *gauge symmetries*, where this parameter can be taken differently in any point of spacetime: $\Lambda(x)$, where x denotes the spacetime point. Furthermore, one should say what is left invariant under 'symmetries'. One can consider symmetries of the action, symmetries of field equations (transforming a field equation in a linear combination of field equations), or symmetries of solutions. A symmetry of the action will also be a symmetry of the field equations but the reverse is not necessarily true. Such symmetries are not necessarily symmetries of a solution, and that is the concept of a *spontaneously broken symmetry*.

We now restrict ourselves to symmetries of the action. When we consider a rigid symmetry, and try to promote it to a local symmetry, the action S will in general not be invariant due to terms that show the x-dependence of the parameter, i.e. δS is proportional to $\partial_\mu\Lambda(x)$,

$$\delta(\Lambda)S = \int \mathrm{d}^D x\, J^\mu\, \partial_\mu\Lambda(x)\,, \tag{2.1}$$

for a J^μ ('Noether current'). $\mu = 0, 1, \ldots, D-1$ labels spacetime coordinates. To compensate these terms, one needs a field $A_\mu(x)$ (gauge field) with

$$S_{\text{new}} = S - \int \mathrm{d}^D x\, J^\mu\, A_\mu\,, \qquad \delta A_\mu(x) = \partial_\mu\Lambda(x)\,. \tag{2.2}$$

This is the first step in an iterative procedure to construct invariant actions ('Noether procedure'). The simplest example of a gauge field is the Maxwell field that appears when promoting phase transformations of complex fields that are proportional to their electric charges, to a local symmetry. The gauge field induces a force between the particles that transform under the symmetry.

Apart from the scalar fields, all bosonic fields that appear here are gauge fields. The Maxwell field above can be seen as a 1-form $A = A_\mu \mathrm{d}x^\mu$, and its transformation is $\delta A = \mathrm{d}\Lambda$. The other bosonic fields will be n-forms $A^{(n)}$ with a symmetry parameter that is an $n-1$ form: $\delta A^{(n)} = \mathrm{d}\Lambda^{(n-1)}$. Not all the components of $\Lambda^{(n-1)} = \frac{1}{(n-1)!}\Lambda_{\mu_1\ldots\mu_{n-1}}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_{n-1}}$ are independent symmetries, as all transformations where $\Lambda^{(n-1)} = \mathrm{d}\Lambda'^{(n-2)}$ are not effective. Counting the remaining gauge-invariant components of an n form gives

$$\binom{D}{n} - \binom{D}{n-1} + \binom{D}{n-2} - \ldots = \binom{D-1}{n}, \tag{2.3}$$

the number of components of an n-form in $D-1$ dimensions. This is a general feature: the independent components form a representation of $\mathrm{SO}(D-1)$. If furthermore the field equations of massless fields are used, the independent components form representations of $\mathrm{SO}(D-2)$. The former is called *off-shell counting*, while the latter is called *on-shell counting*. The number of off-shell components is also equal to the number of components of a massive field. Indeed, the massive physical fields are also characterized by $\mathrm{SO}(D-1)$ representations, as that is the little group for a massive state, while $\mathrm{SO}(D-2)$ is the little group of massless states.

The gauge invariant degrees of freedom for an n form reside in an $(n+1)$-form field strength $F^{(n+1)} = \mathrm{d}A^{(n)}$. This obviously satisfies a Bianchi identity $\mathrm{d}F^{(n+1)} = 0$, while, for the standard Lagrangian proportional to $F^{\mu_1\ldots\mu_{n+1}}F_{\mu_1\ldots\mu_{n+1}}$, the field equation gives $\mathrm{d}^*F = 0$. This shows that ${}^*F$ satisfies the same equations as a Bianchi identity for a $(D-n-1)$-form field strength. Thus, it can be considered as the field strength of a $(D-n-2)$-form. This shows the *duality* between n-forms and $(D-n-2)$-forms. E.g. in 4 dimensions this shows that a 2-form gives physically the same as a scalar, and we can thus restrict ourselves to scalars and vectors. However, the arguments above are only true for the simplest actions, i.e. with abelian gauge fields. In non-abelian field theories, antisymmetric tensors can lead to non-equivalent theories. The duality transformations on vectors is a *self-duality* between the components of the field strengths $F_{\mu\nu}$ and transforms electric in magnetic components.

The usual gauge symmetries with 1-form gauge fields are the most important ingredients of supergravity theories. For a set of gauge transformations labelled by an index A, we write the transformations as $\delta_A(\Lambda^A)$ where $\Lambda^A(x)$ denote the parameters. They form an algebra as

$$[\delta_A(\Lambda_1^A), \delta_B(\Lambda_2^B)] = \delta_C\left(\Lambda_2^B\Lambda_1^A f_{AB}{}^C\right). \tag{2.4}$$

We discuss here bosonic transformations, but the formulas are also valid when the parameters are fermionic, i.e. for supersymmetries. We need a gauge field for any symmetry: ${A_\mu}^A$, which appear in *covariant derivatives*

$$D_\mu = \partial_\mu - \delta_A(A_\mu^A)\,, \qquad \delta(\Lambda){A_\mu}^A = \partial_\mu\Lambda^A + \Lambda^C {A_\mu}^B {f_{BC}}^A\,. \tag{2.5}$$

The covariant derivatives involve a sum over all the symmetries. Replacing ordinary derivatives with these covariant derivatives eliminates the $\partial\Lambda$ terms in the variation of the action that were discussed higher. The commutator between these covariant derivatives give rise to *curvatures*, which are the *field strength* 2-forms that we mentioned[2]:

$$[D_\mu, D_\nu] = -\delta_A({R_{\mu\nu}}^A)\,, \qquad {R_{\mu\nu}}^A = 2\partial_{[\mu}{A_{\nu]}}^A + {A_\nu}^C {A_\mu}^B {f_{BC}}^A\,. \tag{2.6}$$

So far we considered only massless vectors. If we want to describe massive vectors, we will obtain them from a combination of a massless vector with a scalar using symmetry breaking. This scalar is then called a *compensating scalar*. E.g. consider a scalar with simple Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\,. \tag{2.7}$$

There is a symmetry $\delta(\Lambda)\phi = M\Lambda$, where M is a number with dimensions of mass and Λ is the parameter, so far spacetime-independent. If we want to promote this to a local symmetry, we introduce a gauge field A_μ, and define the covariant derivative $D_\mu\phi = \partial_\mu\phi - MA_\mu$ according to (2.5). Replacing the ordinary derivative by this covariant derivative, the action becomes gauge invariant, and we can add then also gauge-invariant kinetic terms for the vector, such that

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}D_\mu\phi D^\mu\phi\,. \tag{2.8}$$

As we can now change the scalar to an arbitrary value by the gauge transformation, we can put it to an arbitrary constant value. This gauge fixing reduces the action to

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}M^2 A_\mu A^\mu\,, \tag{2.9}$$

the standard action for a massive vector. The procedure of 'gauge-fixing' can also be seen as a redefinition of the gauge field A_μ to $A_\mu - \frac{1}{M}\partial_\mu\phi$. This is a general fact: gauge fixing can also be described as a field redefinition. In principle, the gauge symmetry is never broken, but acts only on the field that is not present anymore in the action after this redefinition.

If we try to mimic the gauge theory procedure for gravity, we should consider the Poincaré group consisting of translations P_a and Lorentz rotations

[2](Anti)symmetrization is always made with 'total weight' 1, which means that $\partial_{[\mu}A_{\nu]} = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$. A similar symmetrization is indicated by $(\mu\nu)$.

M_{ab}. The indices a, b also run from 0 to $D-1$ as the coordinates of the spacetime manifold, but indicate vectors or tensors in the tangent space. Each of these should get a gauge field, leading to the scheme

P_a	M_{ab}
$e_\mu{}^a$	$\omega_\mu{}^{ab}$

(2.10)

$e_\mu{}^a$ will be the vielbein in general relativity, an invertible $D \times D$ matrix, and $\omega_\mu{}^{ab}$ the spin connection. However, in Einstein's theory, the spin connection is not an independent field. Therefore we impose a covariant constraint: imposing the vanishing of the translation curvature, defined according to (2.6):

$$R_{\mu\nu}(P^a) = 2\partial_{[\mu} e^a_{\nu]} + 2\omega_{[\mu}{}^{ab} e_{\nu]b} = 0\,. \tag{2.11}$$

This is called a *conventional constraint*, because it **defines** one of the fields, here the spin connection $\omega_\mu{}^{ab}$. This is now a function of the vielbein with the expression known in Einstein's theory. The transformation of $\omega_\mu{}^{ab}$ is modified with respect to the one that would directly follow from (2.5). You can see this either from the fact that the transformation of the spin connection now follows from its new definition, or from the fact that the transformation should be adapted such that (2.11) is invariant. This deforms the algebra. The result is very nice: the translation operation is replaced by a general coordinate transformation on all fields. In this way a gauge theory of general coordinate transformations is obtained and this is Einstein's theory.

3 The guide: Clifford algebra representations

We now turn our attention to the fermions. Therefore we first of all have to know which Clifford representations we can use. This information determines also the amount of supersymmetry that can be present in any dimension.

What do we have to know? Essentially we need the answers to 3 questions:

1. How large are the smallest spinors in each dimension?
2. What are the reality conditions?
3. Which bispinors are (anti)symmetric?

The latter question is important to know what can occur in a superalgebra. Supersymmetries are transformations with a spinor parameter ϵ. E.g. a scalar field $\phi(x)$ transforms in a fermion $\psi(x)$ depending on this parameter:

$$\delta(\epsilon)\phi(x) = \bar{\epsilon}\,\psi(x)\,. \tag{3.1}$$

The algebra of supersymmetry is that a commutator of two such transformations leads to general coordinate transformations:

$$[\delta(\epsilon_2)\delta(\epsilon_1)]\,\phi(x) = \bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\phi(x) = \xi^\mu\partial_\mu\phi(x)\,. \tag{3.2}$$

For consistency, the bispinor $\xi^\mu = \bar{\epsilon}_1\gamma^\mu\epsilon_2$ should be antisymmetric in the interchange of ϵ_1 and ϵ_2.

The analysis of gamma matrices and spinors in different dimensions can be found in many places. We refer especially to [1, section 3], which has all the arguments in the conventions that are used here. Of course, that material is not original, and is rather a convenient reformulation of earlier works. Another recent approach to the theory of spinors has been presented in [2], which is convenient also for understanding the supersymmetry algebras. We now summarize the relevant results.

A priori, a spinor Ψ has $2^{\mathrm{Int}[D/2]}$ components. One can define a product of all γ-matrices $\gamma_* \equiv (-\mathrm{i})^{\mathrm{Int}[D/2]+1}\gamma_0\gamma_1\ldots\gamma_{D-1}$, which squares to $\mathbb{1}$. Note that, though in [1] the formulae occur for an arbitrary number of timelike dimensions, here we assume always just one timelike direction, i.e. the gamma matrices satisfy

$$\gamma_{(\mu}\gamma_{\nu)} = \tfrac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) = \eta_{\mu\nu} = \mathrm{diag}(-1,+1,\ldots,+1)\,. \tag{3.3}$$

For odd dimensions $\gamma_* = \pm\mathbb{1}$, but for even dimensions it is nontrivial, allowing non-trivial left and right projections

$$P_L = \tfrac{1}{2}(\mathbb{1} + \gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1} - \gamma_*)\,. \tag{3.4}$$

Weyl spinors are the reduced spinors under such a projection, thus e.g. a left-handed spinor, which satisfies $P_L\Psi = \Psi$. This concept ('chirality') thus only makes sense in even dimensions.

One may consider the possibilities for reducing the complex spinors to spinors that satisfy a reality condition $\Psi^* = B\Psi$ for a suitable matrix B. This should be consistent with the Lorentz algebra, which is only possible in some dimensions (and this depends also on the spacetime signature). The corresponding reduced spinors are called *Majorana spinors*. If this is not possible, one needs a doublet of spinors to define reality conditions. In that case the reality conditions do not really reduce the number of independent components of a spinor, but the formulation with doublets of real spinors shows more explicitly the symmetry structure. These spinors are denoted as *symplectic Majorana spinors*. The reality condition is not always compatible with the Weyl projection. If the reality condition can be imposed on Weyl spinors, the corresponding irreducible representations of the Lorentz algebra are denoted as *Majorana-Weyl spinors* and have only $2^{D/2-1}$ components.

Table 1: *Irreducible spinors, number of components and symmetry properties.*

Dim	Spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

This leads to table 1, where according to the number of dimensions it is indicated whether Majorana (M), Majorana-Weyl (MW) symplectic (S) or symplectic-Weyl (SW) spinors can be defined, and the corresponding number of components of a minimal spinor is given (the table is for Minkowski signature and has a periodicity of 8). In the final column is indicated which bispinors are antisymmetric, e.g. a 0 indicates that $\bar\epsilon_2\epsilon_1 = -\bar\epsilon_1\epsilon_2$, and a 2 indicates that $\bar\epsilon_2\gamma_{\mu\nu}\epsilon_1 = -\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2$. This entry is modulo 4, i.e. a 0 indicates also a 4 or 8 if applicable. For the even dimensions, when there are Weyl-like spinors, the symmetry makes only sense between two spinors of the same chirality, which occurs for bispinors with an odd number of gamma matrices in these dimensions $D = 2 \bmod 4$. In the other even dimensions, $D = 4 \bmod 4$, there are always two possibilities for reality conditions and we give here the one that includes the '1' as this is the most useful one for supersymmetry in view of (3.2).

Consider as an example supersymmetry in 5 dimensions. The fact that '1' does not appear in the list of antisymmetric bispinors implies that we cannot have an algebra as in (3.2). We need anyway for the reality conditions a doublet of spinor parameters ϵ^i, $i = 1, 2$ at least. The algebra can be of the form

$$[\delta(\epsilon_2), \delta(\epsilon_1)] = \bar\epsilon_1^i\gamma^\mu\epsilon_2^j\varepsilon_{ij}\partial_\mu\,, \tag{3.5}$$

where now the antisymmetric tensor ε_{ij} cares for the antisymmetry between the two parameters. We call this the $N = 2$ theory to indicate the inherent symmetry $\mathrm{USp}(2) = \mathrm{SU}(2)$ between the supercharges, though it is the simplest one that we can have in 5 dimensions.

Table 2: *Supersymmetry and supergravity theories in dimensions 4 to 11.* An entry represents the possibility to have supergravity theories in a specific dimension D with the number of supersymmetries indicated in the top row. We first repeat for every dimension the type of spinors that can be used. Every entry allows different possibilities. Theories with more than 16 supersymmetries can have different gaugings. Theories with up to 16 (real) supersymmetry generators allow 'matter' multiplets. The possibility of vector multiplets is indicated with $\heartsuit$. Tensor multiplets in $D = 6$ are indicated by $\diamondsuit$. Multiplets with only scalars and spin-$\frac{1}{2}$ fields are indicated with $\clubsuit$. At the bottom is indicated whether these theories exist only in supergravity, or also with just rigid supersymmetry.

D	susy	32	24	20	16	12	8	4
11	M	M						
10	MW	IIA | IIB			I $\heartsuit$			
9	M	$N=2$			$N=1$ $\heartsuit$			
8	M	$N=2$			$N=1$ $\heartsuit$			
7	S	$N=4$			$N=2$ $\heartsuit$			
6	SW	$(2,2)$ | $(3,1)$ | $(4,0)$	$(2,1)$ | $(3,0)$		$(1,1)$ $\heartsuit$ | $(2,0)$ $\diamondsuit$		$(1,0)$ $\heartsuit,\diamondsuit,\clubsuit$	
5	S	$N=8$	$N=6$		$N=4$ $\heartsuit$		$N=2$ $\heartsuit,\clubsuit$	
4	M	$N=8$	$N=6$	$N=5$	$N=4$ $\heartsuit$	$N=3$ $\heartsuit$	$N=2$ $\heartsuit,\clubsuit$	$N=1$ $\heartsuit,\clubsuit$
		SUGRA			SUGRA/SUSY	SUGRA	SUGRA/SUSY	

4 The map: dimensions and supersymmetries

Table 2 gives an overview on supersymmetric theories in Minkowski spacetimes and with positive definite kinetic terms. The most relevant source in this respect is the paper of Strathdee [3] that analyses the representations of supersymmetries.

Supersymmetric field theories of this type in 4 dimensions are restricted to fields with spins ≤ 2. This is the restriction to $N \leq 8$ or up to 32 supersymmetries as an elementary (real) spinor in 4 dimensions has 4 components, see table 1. The same table shows that one can not have more than 11 dimensions if the supersymmetries are restricted to 32 (at least in spacetimes of Minkowski signature) [4] . We are considering here the supersymmetries that square to general coordinate transformations. Thus, not e.g. the special supersymmetries in the superconformal algebra, which have a different role. The 11-dimensional theory [5] is the basis of 'M-theory', and is therefore indicated as M in the table.

Going down vertically in the table is obtained by Kaluza-Klein reduction. That means that one splits all the fields in representations of the lower dimensional Lorentz group. E.g. the spinors of the 11-dimensional M-theory split in one right-handed and one left-handed spinor. This is the theory of the massless sector of IIA string theory, and that is why we have indicated it as IIA [6,7,8]. The massless sector of IIB theory involves doublets of spinors of the same chirality (thus also with 32 real supersymmetries). That theory [9,10,11] is not the dimensional reduction of an 11-dimensional theory, as indicated by its place in the table.

Elementary supersymmetric theories in 10 dimensions involve only 16 supersymmetries. They appear in superstring theories with open and closed strings. Apart from the supergravity multiplet which involves the graviton, one can also have a vector multiplet. The existence of the simplest matter multiplets (representations of supersymmetry that do not involve the graviton) is indicated in the table. Vector multiplets involve a vector, an elementary spinor and possibly scalars. Tensor multiplets involve antisymmetric tensors $A_{\mu\nu}$. As explained in section 2, for dimensions 5 and 4 these are dual to vectors and scalars. Therefore, tensor multiplets are not explicitly indicated for these lower dimensions. Also, various representations of the same physical (on-shell) theory are not indicated. The non-Abelian aspects are not indicated either. E.g. the tensor multiplets in 5 dimensions are only dual to vector multiplets when the gauge group is Abelian, but we do not indicate it separately here.

One can have theories with these matter multiplets also for ungauged supersymmetry, i.e. 'rigid supersymmetry'. Rigid supersymmetry is only possible with up to 16 supersymmetries. Again, this can be understood in 4 dimensions, because for $N > 4$ one needs fields of spin-$\frac{3}{2}$. The latter are in field theory only possible when they are gauge fields of a local supersymmetry (gravitinos), which then need gravitons for the gauge fields of the translations that appear in the commutator of two supersymmetries.

In the dimensions lower than 10 we indicate the theories by a number N that indicates the symmetry group rotating different supersymmetries. The structure will be shown explicitly in the next section. For 6 dimensions, as in 10 dimensions, there are spinors of different chirality, and one has to distinguish the number of left and right-handed spinors. The simplest case is with 8 supersymmetries. They have to be all of the same chirality. The theory is then called $(1,0)$ and would be $N = 2$ in the terminology used in 5 dimensions. With 16 supersymmetries, one can have $(1,1)$ or $(2,0)$. These are the analogues of IIA and IIB, respectively, in 10 dimensions. For more supersymmetries, there is a subtlety. The $(2,1)$ and $(2,2)$ theories can be constructed using a metric tensor $g_{\mu\nu}(x)$. For the $(4,0)$, $(3,1)$ [12] or $(3,0)$ theories, this field is not present [13], but is replaced by a more complicated representation of the Lorentz group. Thus, these theories are different in the sense that they are not based on a dynamical metric tensor.

If one constructs in 4 dimensions a field theory with $N = 7$, then it automatically has an eighth local supersymmetry. That is why it is not mentioned in the table. Similarly if one constructs a rigid supersymmetric theory with $N = 3$, it automatically has a fourth supersymmetry. However, in this case, there is the possibility of having only three of the four local. Thus $N = 3$ is only meaningful in supergravity. This explains the lowest line of the table.

Finally, let me remark that the vectors of the supergravity theories can be gauge vectors for a gauge symmetry that rotates the supersymmetries. The last years, various new results have been obtained in this direction [14, 15, 16, 17, 18, 19, 20, 21]. A complete catalogue of theories is not yet known. However, we believe that all supersymmetric field theories (with a finite number of fields and field equations that are at most quadratic in derivatives) belong to one of the entries in table 2.

5 Step 1: Supersymmetry and gauge algebra

After this overview of possibilities, we will now give elementary aspects of the construction of supersymmetric theories. The first basic concept is the symmetry group.

First, we clarify the relation between transformations and generators. We have already written transformations of fields, e.g. in (3.1). This change of a field, is proportional to a parameter ϵ, and we can write[3]

$$\delta(\epsilon) = \epsilon^\alpha Q_\alpha\,, \tag{5.1}$$

i.e. the product of the parameter with an operation called the generator of the (super)symmetry. This operation is for supersymmetry also a fermionic object, such that the elementary change of a field is of the same type as the field itself. When one calculates a commutator of two transformations, one obtains an anticommutator of the generators:

$$\begin{aligned} \delta(\epsilon_1)\,\delta(\epsilon_2) &= \epsilon_1^\alpha Q_\alpha \epsilon_2^\beta Q_\beta = \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta\,, \\ [\delta(\epsilon_1),\,\delta(\epsilon_2)] &= \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta - \epsilon_1^\alpha \epsilon_2^\beta Q_\beta Q_\alpha = \epsilon_2^\beta \epsilon_1^\alpha \left(Q_\alpha Q_\beta + Q_\beta Q_\alpha\right). \end{aligned} \tag{5.2}$$

5.1 Minimal and extended superalgebras

The minimal supersymmetry algebra is the one that we saw in (3.2):

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu\,. \tag{5.3}$$

The supersymmetries commute with translations and are a spinor of Lorentz transformations:

$$[P_\mu, Q] = 0\,, \qquad [M_{\mu\nu}, Q] = -\tfrac{1}{4}\gamma_{\mu\nu} Q\,. \tag{5.4}$$

The extensions indicated in the previous section by $N > 1$ mean that there are different supersymmetries Q^i with $i = 1, \ldots, N$. The possibilities for extension of (5.3) depend on the reality properties of the spinors, which were discussed in section 3. In 4 dimensions, with Majorana spinors, and in 5 dimensions with symplectic spinors one can have

$$\begin{aligned} D = 4 \quad &: \quad \left\{Q^i_\alpha, Q_{\beta j}\right\} = \gamma^\mu_{\alpha\beta} \delta^i_j P_\mu \\ D = 5 \quad &: \quad \left\{Q^i_\alpha, Q^j_\beta\right\} = \gamma^\mu_{\alpha\beta} \Omega^{ij} P_\mu\,, \end{aligned} \tag{5.5}$$

where Ω^{ij} is an antisymmetric (symplectic) metric. The symmetries $U^i{}_j$ that rotate the supercharges by $\left[U^i{}_j, Q^k\right] = \delta^k_j Q^i$, are called *R-symmetries*. The R-symmetry group is restricted by the properties of the spinors. This gives

$$\begin{aligned} &D = 10 \;:\; \mathrm{SO}(N_L) \times \mathrm{SO}(N_R)\,, \qquad D = 9 \;:\; \mathrm{SO}(N)\,, \\ &D = 8 \text{ and } D = 4 \;:\; \mathrm{U}(N) \\ &D = 7 \text{ and } D = 5 \;:\; \mathrm{USp}(N)\,, \qquad D = 6 \;:\; \mathrm{USp}(N_L) \times \mathrm{USp}(N_R)\,. \end{aligned} \tag{5.6}$$

[3] We sometimes use spinor indices $\alpha, \ldots$ in this section. For details on the notation, see [1].

5.2 4 generalizations

Apart from these minimal possibilities, one can consider four kinds of generalizations. The first one is the possibility of central charges, as found in the classical work of Haag–Łopuszański–Sohnius [22]. The simplest example is in $N = 2$, where (5.5) can be extended to

$$\left\{ Q^i_\alpha, Q^j_\beta \right\} = \gamma^\mu_{\alpha\beta}\delta^i_j P_\mu + \varepsilon^{ij}\left[\mathcal{C}_{\alpha\beta}Z_1 + (\gamma_5)_{\alpha\beta}Z_2\right]. \tag{5.7}$$

The generators Z_1 and Z_2 commute with everything else and are thus really 'central'. They play an important role when looking for supersymmetric solutions of the theory. But the name 'central charges' has been generalized to include other generators that can appear in the anticommutator of supersymmetries. E.g. in $D = 11$ the properties of the spinors allow us to extend the anticommutator as [23]

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta}P_\mu + \gamma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu} + \gamma^{\mu_1\cdots\mu_5}_{\alpha\beta}Z_{\mu_1\cdots\mu_5}. \tag{5.8}$$

The allowed structures on the right-hand side are determined by the last entry in table 1 (remember that this is modulo 4, which thus allows the 5-index object). The 'central charges' Z are no longer Lorentz scalars, and thus do not commute with the Lorentz generators. They are therefore not 'central' in the group-theoretical meaning of the word, but play in the physical context the same role as the ones in (5.7), and therefore got the same name.

A second generalization is the extension of the Poincaré group to the (anti) de Sitter group. The spacetime with a cosmological constant is a curved space, which means that translations do not commute, but satisfy an algebra

$$[P_\mu, P_\nu] = \mp\frac{1}{2R^2}M_{\mu\nu}, \tag{5.9}$$

where R is related to the inverse of the cosmological constant. The sign in (5.9) determines the sign of the cosmological constant, and, correspondingly, whether the algebra of translations and Lorentz rotations is $\mathrm{SO}(D,1)$ or $\mathrm{SO}(D-1,2)$. The first one is the de Sitter algebra, while the second one is the anti-de Sitter algebra. Extending the first one to a superalgebra needs a non-compact R-symmetry group, which in turn needs negative kinetic terms of some of the fields, an undesirable feature. But supersymmetric extensions of anti-de Sitter algebras are well-known, see the classical work of Nahm [4] or a recent investigation in [2].

The third generalization is to (super)conformal algebras. The conformal group is the group consisting of translations P_μ, Lorentz rotations $M_{\mu\nu}$, dilatations D and special conformal transformations K_μ, which combine to

SO$(D,2)$. If one extends it with supersymmetries Q^i, the algebra requires for consistency new 'special supersymmetries' S^i (in the commutator $[Q,K]$), and the R-symmetry group mentioned above appears in the anticommutator of the Q and S supersymmetries.

The fourth generalization is to include extra 'Yang–Mills' (YM) gauge symmetries. Indeed, the spin-1 fields that appear either in the supergravity multiplet or in the extra vector multiplets, may gauge YM symmetries according to the principles expressed by (2.5). When the replacement of derivatives by covariant derivatives is performed everywhere, it is clear that one will not obtain the commutator as in (3.2), but rather

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]\,\phi=\bar\epsilon_1\gamma^\mu\epsilon_2\,D_\mu\phi=\bar\epsilon_1\gamma^\mu\epsilon_2\,\partial_\mu\phi-\bar\epsilon_1\gamma^\mu\epsilon_2\,A_\mu^I T_I\phi\,. \tag{5.10}$$

The last term shows in fact a gauge transformation with parameter $\bar\epsilon_1\gamma^\mu\epsilon_2\,A_\mu^I$. Therefore, $\gamma^\mu_{\alpha\beta}A_\mu^I$ is a field-dependent *structure function*, rather than a structure constant. This type of algebra structure appears often in supersymmetry, and is called a *soft algebra*. The (adapted) Jacobi identities then imply that more modifications to the algebra are necessary. E.g. in $N=2$, where the vector multiplets contain scalars σ^I, the algebra has an extra term

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]\,\phi=\ldots+\bar\epsilon_1\epsilon_2\sigma^I T_I\phi\,. \tag{5.11}$$

When the fields σ^I have a non-zero value for a solution of the theory, then the algebra that preserves this solution has the form of (5.7), i.e. with central charges.

This illustrates how the first generalization that we discussed above appears in solutions of the supergravity theories. Also the second generalization, super-anti-de Sitter algebras, occurs in solutions. The third and fourth generalization, on the other hand, are important in constructing supergravity theories.

5.3 Constructions

In constructing rigid supersymmetric theories, we start from a rigid Poincaré supersymmetry, i.e. the Poincaré symmetries, supersymmetry and its R symmetries. Then a YM gauge algebra can be added[4], gauged by vectors, which are parts of a 'vector multiplet'. The algebra becomes then soft and the rigid supersymmetries mix with the local YM symmetries as shown in (5.10). In

[4]In principle also gauge symmetries of antisymmetric tensors can be included, which may also have an action on the other fields. We neglect these here for simplicity, but the principles are the same as for the YM symmetries.

some cases the action can be invariant under (rigid) superconformal symmetries. Central charges are not introduced by hand, but may appear due to the mixing of supersymmetry with YM symmetries and non-zero vacuum expectation values of some fields, see (5.11).

To construct supergravity theories, there is first a straightforward way (super-Poincaré way). One gauges gravity and supersymmetry. The invariance requirements determine all the terms as well in the Lagrangian as in the transformation laws. The alternative way, called superconformal tensor calculus, is particularly useful to construct theories with matter couplings, i.e. where 'matter multiplets' are coupled to supergravity. The super-Poincaré construction leads to a lot of extra terms, which can be better understood in the context of the superconformal tensor calculus. The latter starts by gauging the full superconformal algebra. This is a generalization of the method used to gauge the Poincaré group at the end of section 2: gauge fields are associated to all the symmetries [as in (2.10)], but some of these are dependent fields using constraints on curvatures [as in (2.11)]. Then, an action is constructed that is invariant under the superconformal symmetries, but the symmetries that are superfluous are gauge-fixed. This is similar to the construction of the Lagrangian for a massive vector, where in (2.8) we constructed an action invariant under a gauge symmetry, which was then fixed to obtain the massive vector action in (2.9). For the superconformal symmetry, this means that dilatations, special conformal transformations, special supersymmetry and the R-symmetry should be gauge-fixed to obtain an action that is invariant under the super-Poincaré group. We thus need fields that will not be physical [as the scalar ϕ in (2.8)] but are *compensating* for the symmetries in the superconformal algebra that do not belong to the super-Poincaré algebra. They are part of *compensating multiplets* as they have to be in supersymmetric representations.

Also the fourth generalization has to be considered in the construction of general actions: vectors in the supergravity multiplet and vector multiplets can gauge an extra YM gauge group. In the super-Poincaré construction, one considers separately the R-symmetries and other YM symmetries. One has to define the action of R-symmetry on all the fields in the theory. In the superconformal tensor calculus, only vectors of vector multiplets can gauge an extra gauge symmetry G, commuting with supersymmetry. But one of these vector multiplets may be the compensating multiplet mentioned above. That means that some of its partners are fields that disappear by gauge conditions. The vectors are the extra vectors that appear in the 'gravity multiplet' from the super-Poincaré theory. Also, the R-symmetry is already gauged in the superconformal context as it is part of the superconformal group, but is afterwards gauge-fixed. However, the gauge fixing condition may be not invariant under some of the extra YM gauge symmetries, mixing R-symmetry and G. Schematically we have

$$\begin{array}{rcl} \text{superconformal including R} & \times & G \\ & \Downarrow & \text{gauge fixing} \\ \text{super-Poincaré} & \text{with} & G' \,, \end{array} \tag{5.12}$$

where G' is G with a mixing of superconformal R-symmetries. Therefore (part of) G' does not commute with the supersymmetries and acts as a gauged R-symmetry.

6 Step 2: Multiplets and their transformations

We did already encounter multiplets in the previous sections, especially in the overview section 4. We now give some more details, distinguishing on- and off-shell multiplets, and then giving the properties of the unique multiplets when there are 32 supersymmetries, the vector multiplets and the multiplets with spins $(\frac{1}{2}, 0)$ for lower number of supersymmetries.

We first explain the concept of *trivial symmetries*. Consider the simple action for 2 scalar fields and a gauge invariance:

$$S = \int \mathrm{d}x \left[\tfrac{1}{2}\phi^1 \Box \phi^1 + \tfrac{1}{2}\phi^2 \Box \phi^2\right], \qquad \delta_{\mathrm{triv}}\phi^1 = \epsilon \Box \phi^2 \,, \qquad \delta_{\mathrm{triv}}\phi^2 = -\epsilon \Box \phi^1 \,. \tag{6.1}$$

This can be generalized for any action, when we define transformations

$$\delta_{\mathrm{triv}}\phi^i = \epsilon\, \eta^{ij} \frac{\delta S}{\delta \phi^j} \,, \tag{6.2}$$

for any antisymmetric tensor η^{ij}. Indeed, the variation of the action is then

$$\delta_{\mathrm{triv}} S = \frac{\delta S}{\delta \phi^i}\, \epsilon\, \eta^{ij} \frac{\delta S}{\delta \phi^j} = 0 \quad \text{if} \quad \eta^{ij} = -\eta^{ji} \,. \tag{6.3}$$

The relevance of these trivial symmetries is already evident in the simplest multiplet in $D = 4$, $N = 1$ supersymmetry: the chiral multiplet. The multiplet contains a complex scalar z and a fermion χ, with (rigid) transformations

$$\delta(\epsilon) z = \bar{\epsilon} P_L \chi \,, \qquad \delta(\epsilon) P_L \chi = P_L \partial\!\!\!/ z \epsilon \tag{6.4}$$

P_L is the projection defined in (3.4), where in 4 dimensions γ_* is usually indicated as γ_5. It is sufficient to give the transformation of $P_L\chi$, as complex conjugation replaces P_L by P_R. The action

$$S = \int \mathrm{d}^4 x\, \mathcal{L} \qquad \text{with} \qquad \mathcal{L} = z^* \Box z - \tfrac{1}{2} \bar{\chi} \partial\!\!\!/ \chi \,, \tag{6.5}$$

is invariant under these transformations. When we calculate the algebra of the supersymmetries, we find

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\,z=\bar\epsilon_2\gamma^\mu\epsilon_1\partial_\mu z\,,\qquad [\delta(\epsilon_1),\delta(\epsilon_2)]\,\chi=\bar\epsilon_2\gamma^\mu\epsilon_1\left(\partial_\mu\chi-\tfrac12\gamma_\mu\not{\partial}\chi\right). \tag{6.6}$$

The first terms on the r.h.s. give the general coordinate transformations and thus represent the minimal supersymmetry algebra. The final term is proportional to $\not{\partial}\chi$, which is the field equation of the fermion. This term is of the form of (6.2). We thus find that the supersymmetry algebra (5.3) is satisfied on-shell, i.e. when the field equations are imposed.

There is a possible improvement here: we can include *auxiliary fields* h (complex). This means that we modify the transformation rules and Lagrangian to

$$\delta(\epsilon)P_L\chi=P_L\not{\partial}z\epsilon+P_Lh\epsilon\,,\quad \delta(\epsilon)h=\bar\epsilon\not{\partial}P_L\chi\,,\qquad \mathcal{L}=z^*\Box z-\tfrac12\bar\chi\not{\partial}\chi+h^*h\,. \tag{6.7}$$

The auxiliary field has no physical content as its field equation is $h=0$. However, with this modification the algebra (5.3) is realized 'off-shell', i.e. without the need of equations of motion. One additional advantage is that one can still envisage other Lagrangians invariant under the same transformation laws. Indeed, one can e.g. add to the Lagrangian a term

$$\mathcal{L}_m=m\left(zh+z^*h^*-\bar\chi\chi\right). \tag{6.8}$$

The Lagrangian is still invariant. The field equation of the auxiliary field is now $h^*=-mz$, which lead to massive scalar and spinor fields. Of course, in the formulation without auxiliary fields, one has to modify the transformation laws to allow such an extension, as the last term in (6.6) would not be a field equation any more. Therefore, when such a formalism with auxiliary fields is possible, it is certainly easier to handle, also for considering the quantum theory. The formalism with auxiliary fields can also be obtained from the concept of superspace and superfields. However, unfortunately, it is not always possible to obtain such auxiliary fields.

In general, trivial symmetries can be considered as part of the full set of transformations of the theory, and as such it makes sense to write the algebra

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\,\phi^i=\text{minimal susy algebra}+\eta^{ij}(\epsilon_1,\epsilon_2)\,\frac{\delta S}{\delta\phi^j} \tag{6.9}$$

When we can write the algebra without including the trivial symmetries, the algebra is called a *closed supersymmetry algebra.* If the trivial symmetries enter the algebra, then it is called an *open supersymmetry algebra.* In that case the algebra closes when the field equations are satisfied or when the (infinite set of) trivial symmetries are included. Thus in this case, the dynamics can be obtained already from the algebra of supersymmetry transformations, before constructing the action.

The square of the supersymmetry operation is in the minimal algebra (5.3) a general coordinate transformation. This is an invertible operation. As the supersymmetry operation transforms boson in fermion states, and vice-versa, one can conclude that the number of boson and fermion states should be the same. This is thus true when the algebra of supersymmetries gives just P. E.g. with the transformations (6.4) and algebra (6.6), we can apply this only for on-shell states, such that $\not{\partial}\chi = 0$. Then we count 2 bosonic states for the complex z, and the 4 fermionic components of χ are reduced to 2 by the field equation. So we have a $2+2$ on-shell multiplet. When the auxiliary field h is included, the algebra is also satisfied off-shell. Thus we have in this case the 4 components of χ and z and h give together also 4 components. In this case, we say that we have a $4+4$ off-shell multiplet. These two ways of counting are called *on-shell counting* and *off-shell counting*. Let me finally remark that to have the minimal algebra we have to eliminate the extra gauge terms that we illustrated in (5.10). Thus, we always have to subtract gauge degrees of freedom in all countings. E.g., a gauge vector in 4 dimensions would count off-shell for 3 degrees of freedom, and on-shell for 2 degrees of freedom [representation of SO$(D-2)$, see section 2].

We now give some general facts about the most important multiplets in 4 dimensions. The *pure supergravity* multiplet is the set of fields that represents the spacetime susy algebra and has gauge fields for the supersymmetries. The number of fields is given in table 3. These are on-shell multiplets. The fields

Table 3: *Pure supergravity multiplets in 4 dimensions according to spin s*

s	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=8$
2	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	8
1		1	3	6	10	16	28
$\frac{1}{2}$			1	4	11	26	56
0				2	10	30	70

of spin $s>0$ have 2 degrees of freedom (helicity $+s$ and $-s$). If $N \leq 4$ one can add to the theory *matter multiplets* with fields ≤ 1. Those containing a spin-1 field are called vector multiplets, and the multiplets for $N \leq 2$ with spin $\leq \frac{1}{2}$ are called hypermultiplets for $N=2$ or the already illustrated chiral multiplet for $N=1$, see table 4. An arbitrary number of these matter multiplets can

Table 4: *Matter multiplets in 4 dimensions*

s	$N=1$	$N=2$	$N=3,4$
1	1	1	1
$\frac{1}{2}$	1	2	4
0		2	6

s	$N=1$	$N=2$
$\frac{1}{2}$	1	2
0	2	4

be used for rigid supersymmetry or can be added to the gravity multiplet in local supersymmetry (supergravity). The vectors in the vector multiplets and those in the gravity multiplets can gauge an extra (possibly non-Abelian) gauge group. In rigid supersymmetry one can only have compact gauge groups if one requires positive kinetic energies, but in supergravity some non-compact gauge groups are possible without spoiling the positivity of the kinetic energies. However, the list of possible non-compact groups is restricted for any N. A number of hypermultiplets ($N = 2$) or chiral multiplets ($N = 1$) may then form a representation of these gauge groups.

When there are 32 supersymmetries, the multiplet is unique and this multiplet is only known on-shell, i.e. it is not known (and there are no-go theorems) how to add auxiliary fields to obtain off-shell closure. The basic multiplet in 11 dimensions is written in terms of just 3 fields: a graviton $g_{\mu\nu}$ (44 components, as traceless symmetric tensor of SO(9)), an antisymmetric $A_{\mu\nu\rho}$ (84 components) and a vector-spinor ψ_μ build the $128 + 128$ multiplet. Reducing this e.g. to $D = 4$ fields gives the $N = 8$ multiplet in table 3.

With 16 supersymmetries, rigid supersymmetry is possible. One can have vector multiplets, or tensor multiplets for $(2, 0)$ supersymmetry in 6 dimensions. In $D = 5$, the tensor multiplets are dual to vector multiplets at the level of zero gauge coupling constant. Gauging breaks this duality. Supergravity theories with 16 supersymmetries may contain a number of these multiplets. The model is fixed once one gives the number of matter multiplets and the gauging that is performed by the vectors.

Theories with 8 or 4 supersymmetries are not fixed by the discrete choices of number of multiplets and gauging. In these cases the model depends on some functions that can vary by infinitesimal variations. It is in these models that auxiliary fields are most useful. E.g. for the chiral multiplets that we mentioned before, a holomorphic function $W(z)$ can be introduced, which may take arbitrary values, determining a potential. Indeed, the addition (6.8) can be generalized to

$$\mathcal{L}_W = \frac{\partial W(z)}{\partial z}\, h - \frac{\partial^2 W(z)}{\partial z^2}\bar{\chi} P_L \chi + \text{h.c.}\,. \tag{6.10}$$

Similarly, the kinetic terms can be generalized depending on an arbitrary function $\mathcal{G}(z,z^*)$, which will play the role of a Kähler potential for the geometry determined by the scalars (see section 9).

7 Step 3: Actions

The next step is the determination of the action. There are some general properties of actions that we will show in this section. The kinetic terms of the scalars determine a geometry. There is also a potential for these scalars with properties that are determined by the supersymmetry. The kinetic terms of the vectors introduce the duality symmetries, which by supersymmetry imply symmetries on the full theory, leading often to hidden symmetries of the scalar geometry.

The full action of a supergravity theory is very complicated. It contains 4-fermion couplings, couplings between fermions and vectors (as dipole moments), ... We show here some general structure of the bosonic terms in 4 dimensions. The theory contains[5] the graviton, represented by the vierbein e_μ^a, a number of vectors A_μ^I with field strengths $\mathcal{F}_{\mu\nu}^I$, a number of scalars φ^u, N gravitinos ψ_μ^i, and a number of fermions λ^A. The pure bosonic terms of such an action are

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm bos} &= \tfrac12 R + \tfrac14(\mathrm{Im}\,\mathcal{N}_{IJ})\mathcal{F}_{\mu\nu}^I\mathcal{F}^{\mu\nu J} - \tfrac18(\mathrm{Re}\,\mathcal{N}_{IJ})e^{-1}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^I\mathcal{F}_{\rho\sigma}^J \\ &\quad -\tfrac12 g_{uv}(\varphi)\,D_\mu\varphi^u\,D^\mu\varphi^v - V(\varphi)\,. \end{aligned} \tag{7.1}$$

We factorized the determinant of the vierbein to the left hand side. The first term gives the pure gravity action. Then there are the kinetic terms for the spin-1 fields. They depend on two tensor functions, which we combine in a complex (symmetric) tensor $\mathcal{N}_{IJ}$. This tensor in general is a function of the scalars in the theory. The scalars have kinetic terms determined by a symmetric tensor $g_{uv}(\varphi)$. They couple 'minimally' to the vectors with a covariant derivative for the gauge symmetries as in (2.5). Finally, there is the potential $V(\varphi)$.

We will now illustrate the duality transformations in $D=4$, generalizations of the Maxwell dualities[6]. They were first discussed in [25, 26, 27, 28]. They apply only for Abelian theories and without the coupling of the vectors to other fields. Thus, e.g. we neglect the appearance of the vector in covariant derivative of the scalars in (7.1). The vectors then appear in the

[5] We neglect here the possibility of additional antisymmetric tensors, though it is not proven whether all theories with antisymmetric tensors have an equivalent description in terms of scalars.

[6] Aspects of dualities and gaugings in arbitrary dimensions are treated in [24].

action only as their field strengths. If we express the theory in terms of field strengths, we have to complement the field equations with Bianchi identities $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \mathcal{F}_{\rho\sigma} = 0$. But there is a convenient way to write the Bianchi identities and field equations:

$$\begin{aligned}
&\partial^\mu \operatorname{Im} \mathcal{F}^{+I}_{\mu\nu} = 0\,, \qquad \mathcal{F}^{\pm I}_{\mu\nu} \equiv \tfrac{1}{2}\left(\mathcal{F}^{I}_{\mu\nu} \pm \tfrac{1}{2}\mathrm{i}e\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma\, I}\right)\\
&\partial_\mu \operatorname{Im} G^{\mu\nu}_{+I} = 0\,, \qquad G^{\mu\nu}_{+I} \equiv 2\mathrm{i}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{+I}_{\mu\nu}} = \mathcal{N}_{IJ}\mathcal{F}^{+J\,\mu\nu}\,.
\end{aligned} \tag{7.2}$$

This shows that, for m vectors ($I = 1, \ldots, m$), this set of equations is invariant under $G\ell(2m, \mathbb{R})$:

$$\begin{pmatrix} \tilde{\mathcal{F}}^+ \\ \tilde{G}_+ \end{pmatrix} = \mathcal{S}\begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix}. \tag{7.3}$$

These transformations imply a new form of the tensor $\mathcal{N}$:

$$\begin{aligned}
&\tilde{G}^+ = (C + D\mathcal{N})\mathcal{F}^+ = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\tilde{\mathcal{F}}^+\\
&\Rightarrow \quad \tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,.
\end{aligned} \tag{7.4}$$

Consistency with the last of (7.2) implies that $\tilde{\mathcal{N}}$ should be symmetric. Writing out these conditions, one arrives at the conclusion that $\mathcal{S}$ should be in $\mathrm{Sp}(2m, \mathbb{R})$. Thus the vector field strengths belong to $2m$-symplectic vectors.

To understand the scalar geometry, we have to distinguish 2 manifolds. On the one hand there is the spacetime, with coordinates x^μ, $\mu = 0, 1, 2, 3$. On the other hand there is a manifold, which we indicate as $\mathcal{M}$, with dimension equal to the number of scalars in the model. The scalars φ^u give a chart in this manifold. The values of a scalar at a spacetime point x thus determine a submanifold of $\mathcal{M}$, parametrized by $\varphi^u(x)$. The metric of spacetime is $g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)$. The metric on $\mathcal{M}$ is $g_{uv}(\varphi)$. These have a different status. While the latter is part of the definition of the theory, $g_{\mu\nu}(x)$ is, together with $\varphi^u(x)$, a dynamical field. On the other hand, the induced metric on spacetime is $g_{uv}(\varphi)\,(\partial_\mu\varphi^u)(\partial_\nu\varphi^v)$ at $\varphi = \varphi(x)$, and its contraction with the (inverse) spacetime metric and its determinant $\sqrt{g}g^{\mu\nu}$ appears in the action.

The scalar manifold can have isometries, i.e. symmetries of the induced metric $\mathrm{d}s^2 = g_{uv}(\varphi)\,\mathrm{d}\varphi^u\,\mathrm{d}\varphi^v$. Usually these symmetries are extended to a symmetry of the full action (there are counterexamples, but they are rare). This group is then called the *U-duality group*. The scalars and the vectors are connected via the tensor $\mathcal{N}_{IJ}(\varphi)$. Therefore the isometries act as duality transformations in the vector sector, and as such must belong to the $\mathrm{Sp}(2m, \mathbb{R})$ group (in 4 dimensions). This gives a restriction of possible U-duality groups, which are for $D = 4$ restricted to be a subgroup of $\mathrm{Sp}(2m, \mathbb{R})$ for scalars that belong to multiplets including vectors. Actually, to count the m vectors, we have to include those of the gravity multiplet. This is natural in superconformal tensor calculus, where these vectors belong to vector multiplets of which part of the fields are compensating fields, see section 5.3. A subgroup of the isometry group, at most of dimension m, can then be gauged. This means that the vectors couple to the scalars in the covariant derivative using in (2.5) the transformations of the scalars under these isometries. There are more corrections due to this gauging in the fermionic sector. We will give more details on the geometries in section 9, but first finish the overview of the different steps of the analysis of supersymmetric theories.

8 Step 4: Solutions and their symmetry

We usually look for solutions with vanishing fermions. This is often motivated by the desire to keep at least some of the Lorentz invariance unbroken. A non-vanishing fermion is not invariant under any part of the Lorentz group. The values of the metric, vector fields and scalar fields then determine the type of solution that we are discussing. These may be Anti-de Sitter geometries, black holes, branes or pp-waves or Minkowski spaces, which all preserve some supersymmetry. When we discuss preserved supersymmetry, this means some rigid supersymmetry. There is often a confusing terminology that a solution preserves all supersymmetries. What is meant is that from all the local supersymmetries parametrized by $\epsilon^i_\alpha(x)$, indicating now as well the spinor index $\alpha = 1, \ldots, \Delta$ as the index $i = 1, \ldots, N$ for the extension, there are specific functions depending on ΔN constant parameters that are invariances of the solution.

To find preserved supersymmetries, we have to consider the transformations of the form

$$\delta(\epsilon)\,\text{boson} = \epsilon\,\text{fermion}\,, \qquad \delta(\epsilon)\,\text{fermion} = \epsilon\,\text{boson}\,. \tag{8.1}$$

For vanishing fermions, we have to consider the condition of vanishing transformations of the fermions to determine the preserved supersymmetries. A solution (a bosonic configuration) that allows non-zero parameters ϵ, is called a BPS solution. The algebra of supersymmetry implies for most of these solutions a cancellation between e.g. contributions of the energy and of the electromagnetic (or other) charges. This can be seen already from (5.10) or (5.11). For preserved supersymmetries, the right-hand side should vanish when applied to a solution. There are non-zero terms proportional to the energy determined by $\partial_0\phi$ and proportional to charges determined by $T_I\phi$. In solutions with non-zero gauge fields (e.g. charged black holes) the last term of (5.10) has to cancel the energy, while for non-zero scalars, the term in (5.11) plays this role. Thus, in any case these solutions satisfy some bounds on charges that are called Bogomol'nyi bounds.

This happens e.g. for charged black holes in $D = 4$, $N = 4$ supergravity [29]. The solutions may have electric (P) and magnetic (Q) charges. They satisfy $P^2 + Q^2 \leq M^2$, where M is the mass of the black hole. This bound is automatic for solutions of the supersymmetric theories as a consequence of the algebra, and coincides with the requirement of cosmic censorship (no naked singularities in spacetime). If there is an equality, then there are solutions for the 16 functions $\epsilon^i_\alpha(x)$ that depend on 4 constant parameters ($N = 1$ in $D = 4$). If, moreover, either P or Q is zero, then there are 8 solutions ($N = 2$).

9 Scalars and geometry

We finish these lectures by giving an overview of the geometries that appear in the scalar manifolds, as explained in section 7. The type of geometries that occur, depend on the number of supercharges. For all the theories with more than 8 supersymmetries, all the scalar manifolds are symmetric spaces. These are shown in table 5. For the theories with 4 supersymmetries ($N = 1$ in 4 dimensions, but one might also consider lower-dimensional theories), the manifold can be an arbitrary Kähler geometry, a geometry with a (closed) complex structure. The symmetric Kähler spaces are

$$\begin{array}{ccc} \dfrac{\mathrm{SU}(p,q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}\,, & \dfrac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}\,, & \dfrac{\mathrm{Sp}(2n)}{\mathrm{U}(n)}\,, \\ \dfrac{\mathrm{SO}(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}\,, & \dfrac{\mathrm{E}_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}\,, & \dfrac{\mathrm{E}_7}{\mathrm{E}_6\times\mathrm{U}(1)}\,. \end{array} \qquad (9.1)$$

Arbitrary Kähler spaces are defined by a Kähler potential $\mathcal{G}(z, z^*)$, mentioned at the end of section 6. For any Kähler manifold there is such an $N = 1$ theory.

Beautiful structures emerge for theories with 8 supercharges ($N = 2$ if in $D = 4$). These theories all belong to a class that was baptized *special geometries* [30,31], including some real [32], some Kähler geometries [33] and all the quaternionic geometries [34]. Especially, the scalars that by supersymmetry are directly related to vectors have a geometrically distinct structure, special Kähler geometry [33]. This is a subclass of the Kähler geometries discussed above, with an extra symplectic symmetry structure related to the duality transformations of the vectors shown in section 7. Scalars in hypermultiplets exhibit quaternionic structures, with many relations with special Kähler manifolds [35,36].

Specifically, the manifolds that occur in supergravity actions are

$$\begin{aligned}
D=6 &\quad:\quad \frac{\mathrm{O}(1,n)}{\mathrm{O}(n)} \times \text{quaternionic-Kähler manifold}\\
D=5 &\quad:\quad \text{very special real manifold} \times \text{quaternionic-Kähler manifold}\\
D=4 &\quad:\quad \text{special Kähler manifold} \times \text{quaternionic-Kähler manifold.}
\end{aligned} \tag{9.2}$$

A short overview of these manifolds is given in [37], especially in sections 2 and 3, where tables are given of the symmetric special geometries and homogeneous special geometries. Indeed, the work on these couplings lead to new mathematical discoveries in the field of quaternionic geometry [38, 39], especially an improvement on the classification of homogeneous quaternionic geometries [40].

These geometries determine the general couplings of supergravity to matter multiplets in $D=6$ [41,42], $D=5$ [43] and $D=4$ [44,45]. There exist also versions of these geometries for rigid supersymmetry, leading to rigid Kähler manifolds [46, 47] and hyperkähler manifolds.

Another new aspect, which has shown recently [48], is the possibility of generalization of hyperkähler to hypercomplex manifolds for rigid hypermultiplets and of quaternionic-Kähler to quaternionic manifolds for hypermultiplets in supergravity. This generalization involves theories where no invariant metric can be defined. Then the field equations do not follow from an action, but are determined by non-closure functions as in (6.9), but where the last factor is a dynamical equation E_j that can not be written as the derivative of some action $E_j \neq \frac{\delta S}{\delta \phi^j}$.

10 Final remarks

We know a lot of the general structure of supergravity theories, but still new aspects of supergravity theories are discovered every day. They lead to interesting applications in phenomenology and even cosmology these days. For those who want to study further the aspects of supergravity theories, we refer to the recent longer review of B. de Wit [49].

Table 5: *Scalar geometries in theories with more than 8 supersymmetries (and dimension ≥ 4).* The theories are ordered as in table 2. Note that the R-symmetry group, mentioned in (5.6), is always a factor in the isotropy group. For more than 16 supersymmetries, there is only a unique supergravity (up to gaugings irrelevant to the geometry), while for 16 and 12 supersymmetries there is a number n indicating the number of vector multiplets that are included.

D	32	24	20	16	12
10	$\mathrm{O}(1,1)$ | $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$				
9	$\frac{\mathrm{S}\ell(2)}{\mathrm{SO}(2)} \otimes \mathrm{O}(1,1)$			$\frac{\mathrm{O}(1,n)}{\mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
8	$\frac{\mathrm{S}\ell(3)}{\mathrm{SU}(2)} \otimes \frac{\mathrm{S}\ell(2)}{\mathrm{U}(1)}$			$\frac{\mathrm{O}(2,n)}{\mathrm{U}(1)\times\mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
7	$\frac{\mathrm{S}\ell(5)}{\mathrm{USp}(4)}$			$\frac{\mathrm{O}(3,n)}{\mathrm{USp}(2)\times\mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
6	$\frac{\mathrm{O}(5,5)}{\mathrm{USp}(4)\times\mathrm{USp}(4)}$ | $\frac{\mathrm{F}_4}{\mathrm{USp}(6)\times\mathrm{USp}(2)}$ | $\frac{\mathrm{E}_6}{\mathrm{USp}(8)}$	$\frac{\mathrm{SU}^*(4)}{\mathrm{USp}(4)}$ | $\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$		$\frac{\mathrm{O}(4,n)}{\mathrm{O}(n)\times SO(4)} \otimes \mathrm{O}(1,1)$ | $\frac{\mathrm{O}(5,n)}{\mathrm{O}(n)\times\mathrm{USp}(4)}$	
5	$\frac{\mathrm{E}_6}{\mathrm{USp}(8)}$	$\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$		$\frac{\mathrm{O}(5,n)}{\mathrm{USp}(4)\times\mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
4	$\frac{\mathrm{E}_7}{\mathrm{SU}(8)}$	$\frac{SO^*(12)}{\mathrm{U}(6)}$	$\frac{\mathrm{SU}(1,5)}{\mathrm{U}(5)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,n)}{\mathrm{SU}(4)\times\mathrm{SO}(n)}$	$\frac{\mathrm{SU}(3,n)}{\mathrm{U}(3)\times\mathrm{SU}(n)}$

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