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Supersymmetrical String Theories*

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ABSTRACT

A supersymmetrical light-cone-gauge string action is presented. It provides a basis for understanding the previously-studied supersymmetrical dual string theory as well as two new closed-string theories that have extended supersymmetry in ten dimensions, corresponding to N = 8 supersymmetry in four dimensions.

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In several recent papers [1-3], we have developed a new oscillator description of the supersymmetrical dual string theory [4]. It has proved to be superior to older formulations for understanding the 10-dimensional supersymmetry of the theory and for explicitly calculating scattering amplitudes. This letter describes a light-cone-gauge string action from which this oscillator formalism can be derived. The string action also leads to the discovery of two new closed-string theories with extended supersymmetry in ten dimensions (corresponding to N = 8 supersymmetry in four dimensions). Unlike any other known dual string theory, these closed-string theories turn out to be completely free of divergences in the one-loop approximation (which is all that has been checked so far).

The conventional (Veneziano) string model can be based on an action principle for the string coordinates $X^{\mu}(\sigma,\tau)$* [5]. The action can be written as a $\sigma-\tau$ reparametrization-invariant expression by using an auxiliary two-dimensional metric tensor. The local invariance allows one to choose an orthonormal gauge in which the light-cone component $X^{-}(\sigma,\tau)$ may be regarded as a dependent variable [7]. The remaining gauge freedom (plus partial use of equations of motion) allows one to set

$$X^{+}(\sigma,\tau) = x^{+} + 2\alpha' p^{+}\tau, \tag{1}$$

where x^{+} and p^{+} are constants. Once this is done, the string dynamics resides entirely in the transverse coordinates $X^{i}(\sigma,\tau)$. A description of the supersymmetrical string theories requires fermionic degrees of freedom, $S^{Aa}(\sigma,\tau)$, as well. The index $A = 1,2$ describes a spinor in the $\sigma-\tau$ space, whereas the index a

*As customary, σ is a spacelike variable that runs from 0 to π along the string, while τ is a timelike evolution parameter. μ is a vector index for the physical spacetime, 26-valued for the Veneziano model and 10-valued for the supersymmetrical models. For a collection of review articles, see Ref. [6].

is 32-valued, as appropriate for a spinor in 10-dimensional spacetime. S^{1a} and S^{2a} are Majorana-Weyl spinors in ten dimensions. For much of the analysis it is not necessary to specify whether S^1 and S^2 have the same or opposite handedness, but ultimately both cases will turn out to be of interest. Corresponding to eq. (1), the fermionic field is required to satisfy the gauge condition

$$(\gamma^+)^{ab} S^{Ab} = 0. \tag{2}$$

The postulated two-dimensional string action in the light-cone gauge is*

$$\int d\tau \int_0^\pi d\sigma \{ -\frac{1}{4\pi\alpha'} \partial_\alpha X^i \partial^\alpha X^i + \frac{i}{4\pi} \bar{S} \gamma^- \rho^\alpha \partial_\alpha S \}. \tag{3}$$

A covariant gauge-invariant action from which eqs. (1-3) can be deduced is not yet known. The equations of motion implied by (3), subject to boundary conditions to be discussed, are

$$\partial^\alpha \partial_\alpha X^i = 0 \text{ and } \rho^\alpha \partial_\alpha S = 0. \tag{4}$$

This action is invariant under the supersymmetry transformations

$$\delta X^i = \frac{1}{\sqrt{p^+}} \bar{\varepsilon} \gamma^i S \tag{5a}$$

$$\delta S = \frac{i}{\sqrt{p^+}} \gamma_- \gamma_\mu (\rho \cdot \partial X^\mu) \varepsilon, \tag{5b}$$

again assuming appropriate boundary conditions. These transformations may be contrasted with those in an earlier formulation in which ε was a spinor only in the two-dimensional sense [8]. The parameters ε^{Aa} are Majorana-Weyl in d = 10,

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*For the two-dimensional Dirac matrices, the Majorana representation $\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is convenient. We also define $\bar{S}^{Aa} = S^{+Bb} (\gamma^0)^{ba} (\rho^0)^{BA}$ and $\rho_3 = \rho^0 \rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A spacelike metric is used for both two-vectors and ten-vectors. We henceforth set $\alpha' = 1/2$.

and may either be regarded as parametrizing two supersymmetries in 10 dimensions or 16 supersymmetries in two dimensions. They may even depend on σ and τ provided that

$$\left(\frac{\partial}{\partial\tau} - \rho_3 \frac{\partial}{\partial\sigma}\right)\varepsilon = 0, \tag{6}$$

but we are primarily interested in the case of constant ε. Commuting two global supersymmetry transformations, one finds (using $\rho\cdot\partial S = 0$) that

$$[\delta_1,\delta_2]X^i = \xi^\alpha\partial_\alpha X^i + a^i \tag{7a}$$

$$[\delta_1,\delta_2]S = \xi^\alpha\partial_\alpha S \tag{7b}$$

with

$$\xi^\alpha = -\frac{2i}{p^+}\bar{\varepsilon}^{(1)}\gamma_-\rho^\alpha\varepsilon^{(2)} \tag{8}$$

$$a^i = -2i\bar{\varepsilon}^{(1)}\rho^0\gamma^i\varepsilon^{(2)}, \tag{9}$$

which may be interpreted as translations of (σ,τ) and X^i, respectively. The supersymmetry charges obtained from eqs. (3) and (5) by the Noether method are

$$Q^{Aa} = \frac{i}{\pi\sqrt{p^+}}\int_0^\pi [\gamma_\mu(\rho\cdot\partial X^\mu)\rho^0 S]^{Aa} d\sigma. \tag{10}$$

Boundary conditions needed to derive the equations of motion (4) must be considered carefully. In the case of open strings, it is necessary that S^1 and S^2 have the same handedness and

$$\frac{\partial}{\partial\sigma}X^i(0,\tau) = \frac{\partial}{\partial\sigma}X^i(\pi,\tau) = 0 \tag{11}$$

$$S^{1a}(0,\tau) = S^{2a}(0,\tau) \tag{12a}$$

$$S^{1a}(\pi,\tau) = S^{2a}(\pi,\tau). \tag{12b}$$

Possible minus signs in eq. (12) are excluded by requiring invariance under global supersymmetry transformations. In this case, one also needs $\varepsilon^{1a} = -\varepsilon^{2a}$, so that there is, in fact, just one 10-dimensional supersymmetry invariance in the open-string case. Equations (4, 11, 12) lead to the mode expansions

$$X^i = x^i + p^i\tau + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^i \cos n\sigma \, e^{-in\tau} \tag{13}$$

$$S^{1a} = \sum_{n=-\infty}^{\infty} S_n^a \, e^{-in(\tau-\sigma)} \tag{14a}$$

$$S^{2a} = \sum_{n=-\infty}^{\infty} S_n^a \, e^{-in(\tau+\sigma)}. \tag{14b}$$

Substituting these expansions in eq. (10), one finds that Q^{1a} and Q^{2a} are not separately conserved (they depend on τ) but that

$$Q^a = \frac{1}{2}(Q^{1a}+Q^{2a}) = i\sqrt{p^+}(\gamma_+ S_0)^a + \frac{i}{\sqrt{p^+}} \sum_{-\infty}^{\infty} (\gamma_i S_{-n})^a \alpha_n^i \tag{15}$$

is conserved.

Canonical quantization gives

$$[\alpha_m^i, \alpha_n^j] = m \, \delta_{m+n,0}\delta^{ij} \tag{16}$$

$$\{S_m^a, \overline{S}_n^b\} = (\gamma^+ h)^{ab} \delta_{m+n,0} \tag{17}$$

$$[\alpha_m^i, S_n^a] = 0, \tag{18}$$

where h represents a Weyl projection operator $\frac{1}{2}(1 \pm \gamma_{11})$. Equations (15-18) were discovered in Ref. [1], where it is proved that

$$\{Q^a, \overline{Q}^b\} = -2(h\gamma\cdot p)^{ab}, \tag{19}$$

provided that one imposes the mass-shell condition

$$p^- = \frac{1}{2p^+}(p^i p^i + N) \tag{20}$$

$$N = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + \frac{n}{2}\bar{S}_{-n}\gamma^- S_n). \tag{21}$$

Reference [1] gives formulas for the Lorentz generators $J^{\mu\nu}$ in terms of the α_n^i's and S_n^a's and proves that they have the correct algebra when eq. (20) is satisfied. A proof that Q^a transforms as a spinor in the 10-dimensional sense is also given there. Lorentz covariance is not at all obvious in the light-cone-gauge formalism (the action in eq. (3) only has manifest covariance in the 8-dimensional transverse space). However, functional integral methods analogous to those developed for the conventional model [9] can be used to obtain covariant S-matrix elements from the light-cone-gauge action. The results are the same as obtained by the operator methods in our previous works [1-3].

Turning attention to closed strings, we replace the boundary conditions in eqs. (11) and (12) with the requirement that X^i and S^{Aa} are periodic in σ with period π. This, together with eq. (4), leads to the closed-string mode expansions

$$X^i(\sigma,\tau) = x^i + p^i\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^i e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^i e^{-2in(\tau+\sigma)}) \tag{22}$$

$$S^{1a} = \sum_{-\infty}^{\infty} S_n^a e^{-2in(\tau-\sigma)} \tag{23a}$$

$$S^{2a} = \sum_{-\infty}^{\infty} \tilde{S}_n^a e^{-2in(\tau+\sigma)}. \tag{23b}$$

The canonical commutation relations give eqs. (16-18) for the operators with tildes as well as the ones without. Also

$$[\alpha_m^i, \tilde{\alpha}_n^j] = [\alpha_m^i, \tilde{S}_n^a] = [\tilde{\alpha}_m^i, S_n^a] = \{S_m^a, \tilde{S}_n^b\} = 0. \tag{24}$$

The handedness of S^1 and S^2 may be independently assigned. Substituting the mode expansions (22) and (23) into eq. (10) gives

$$Q^{1a} = i\sqrt{p^+}\,(\gamma_+ S_0)^a + \frac{2i}{\sqrt{p^+}} \sum_{-\infty}^{\infty} (\gamma_i S_{-n})^a \alpha_n^i \tag{25a}$$

$$Q^{2a} = i\sqrt{p^+}\,(\gamma_+ \widetilde{S}_0)^a + \frac{2i}{\sqrt{p^+}} \sum_{-\infty}^{\infty} (\gamma_i \widetilde{S}_{-n})^a \widetilde{\alpha}_n^i . \tag{25b}$$

These are separately conserved and give the algebra of extended supersymmetry in ten dimensions

$$\{Q^{Aa}, \overline{Q}^{Bb}\} = -2(h^A \gamma\cdot p)^{ab} (\rho^0)^{AB} . \tag{26}$$

The mass-shell condition in eq. (20) is replaced for closed strings by the conditions

$$L_0 = \frac{1}{8} p^2 + N = 0, \tag{27a}$$

$$\widetilde{L}_0 = \frac{1}{8} p^2 + \widetilde{N} = 0, \tag{27b}$$

where N is given in eq. (21) and $\widetilde{N}$ is the corresponding expression with $\widetilde{\alpha}$'s and $\widetilde{S}$'s. (The closed-string Regge slope is $\frac{1}{4}$ - half of the open-string slope.) Equation (27) is required both for the closure of the Lorentz algebra and for the validity of the supersymmetry algebra in eq. (26). One can define a truncation to a closed-string theory with simple supersymmetry by restricting the physical spectrum to states that are symmetrical under the interchanges $\alpha_m^i \leftrightarrow \widetilde{\alpha}_m^i$ and $S_m^a \leftrightarrow \widetilde{S}_m^a$. (This requires that S_m^a and $\widetilde{S}_m^a$ have the same handedness, $h^1 = h^2$.) In this case $Q = \frac{1}{2}(Q^1 + Q^2)$ is the only remaining supersymmetry. This truncated theory is of importance because it describes the closed strings that interact with the open strings described above, whereas untruncated closed-string models cannot interact with open strings (just as N = 4 Yang-Mills cannot couple to N = 8 supergravity).

The 256 massless physical closed-string states may be described by ket vectors $|ij\rangle$, $|ia\rangle$, $|ai\rangle$, $|ab\rangle$ where the first label refers to the S_0 space and the second to $\tilde{S}_0$ space.

$$|ai\rangle = \frac{i}{8}(\gamma_j S_0)^a |ji\rangle \tag{28a}$$

$$|ia\rangle = \frac{i}{8}(\gamma_j \tilde{S}_0)^a |ij\rangle \tag{28b}$$

$$|ab\rangle = -\frac{1}{64}(\gamma_i S_0)^a (\gamma_j \tilde{S}_0)^b |ij\rangle . \tag{28c}$$

The supersymmetry and Lorentz transformation rules for these states can be deduced from eqs. (17) and (25), as well as the Lorentz generator representations, which have the form given in Ref. [1] with identical extra terms formed from oscillators with tildes. The spin content of this multiplet is precisely that of extended supergravity in ten dimensions. There are two distinct cases. If the S and $\tilde{S}$ oscillators have opposite handedness then the two gravitinos contained in $|ai\rangle$ and $|ia\rangle$ have opposite handedness and the states $|ab\rangle$ correspond to a vector and third-rank antisymmetric tensor field. This corresponds to the spin content obtained by reduction of 11-dimensional supergravity to ten dimensions. If S and $\tilde{S}$ have the same handedness, then the two gravitinos have the same handedness and the states $|ab\rangle$ correspond to a scalar, second-rank antisymmetric tensor, and fourth-rank antisymmetric self-dual tensor. This is clearly a distinct alternative and cannot be obtained by reduction from a higher dimension. Nonetheless, the two theories are extremely close in their structure. It is amusing that even though the spin content of the two theories is different in ten dimensions, it coincides on reduction to any lower dimension.

The interacting closed-string theories can be described in the operator formalism in terms of a propagator for states satisfying eq. (27):

$$\Delta = \int_{|z|<1} d^2z \; z^{L_0-1} \, \bar{z}^{\tilde{L}_0-1} \tag{29}$$

and vertices describing the emission of massless states. The vertex formulas are given as direct products of open-string vertices [2] in the tilde and no tilde spaces. More specifically, representing the open-string vertex for emitting a massless vector with polarization vector ζ^μ as $\zeta^\mu V_\mu^B$ and the vertex for emitting a massless fermion with Dirac spinor u^a as $\bar{u}^a\, V_a^F$, the closed string vertices have the form $\zeta^{\mu\nu} V_\mu^B \tilde{V}_\nu^B$, $\bar{u}_1^{\mu a} V_\mu^B \tilde{V}_a^F$, $\bar{u}_2^{a\mu} V_a^F \tilde{V}_\mu^B$, and $\bar{\eta}^{ab}\, V_a^F \tilde{V}_b^F$. Using the vertex formulas given in Ref. [2], tree and loop amplitudes for specific processes are straightforward to calculate. One finds, for example, that one-loop graphs with three or fewer external (on mass-shell) lines are identically zero for the same reasons as in the open-string case [3], namely the vanishing of zero-mode traces.

The one-loop contribution to the N-particle connected S-matrix is given by a single Feynman graph in the case of closed strings. For the case of any four massless external states, the one-loop contribution consists of a kinematical factor involving the polarization vectors and spinors as well as the momenta of the external states times the integral

$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^5} \int \prod_{i=1}^{3} d^2\nu_i \prod_{1\le i<j\le 4} \{\exp[\frac{-\pi(\mathrm{Im}\nu_{ji})^2}{\mathrm{Im}\tau}]\,|\,\vartheta_1(\nu_{ji}\,|\,\tau)\,|\,\}^{\frac{1}{2}k_i\cdot k_j}. \tag{30}$$

In this integral ϑ_1 is a Jacobi elliptic function, $\nu_{ji} = \nu_j - \nu_i$ and $\nu_4 = \tau$. The ν_i integration regions are restricted by

$$0 \le \mathrm{Im}\nu_i \le \mathrm{Im}\tau;\; -\frac{1}{2} \le \mathrm{Re}\;\nu_i \le \frac{1}{2}. \tag{31}$$

To determine the appropriate τ integration region, one notes that eq. (30) may be written in the form $\int d^2\tau(\mathrm{Im}\tau)^{-2}F(\tau)$, where F is an automorphic function. That is, it is invariant under the modular group

$$\tau \to \frac{a+b\tau}{c+d\tau} \tag{32}$$

where a, b, c and d are integers satisfying $ad - bc = 1$. Therefore, to avoid multiple counting, one must restrict the τ integration to the "fundamental" region [10]

$$-\frac{1}{2} \le \mathrm{Re}\tau \le \frac{1}{2};\ \mathrm{Im}\tau \ge 0\ ;\ |\tau| \ge 1. \tag{33}$$

It is easy to see that the one-loop integral with these limits has the expected poles and branch cuts and is otherwise completely free from divergences. This contrasts, in particular, with the conventional Shapiro-Virasoro model in which there is a divergence associated with the corner where $\nu_1, \nu_2, \nu_3 \to \tau$ [10]. A massless scalar particle at zero momentum in the propagator connecting a tadpole to a tree graph is responsible for the divergence. This does not occur in the supersymmetrical theory, because the supersymmetry algebra makes loops with three or fewer on-shell massless states vanish identically.

In conclusion, we have found a string action that explains the oscillator description of the theory of open and closed strings with simple supersymmetry in ten dimensions. It also leads to the discovery of two new closed-string theories, both of which correspond to N = 8 supergravity in four dimensions in the limit that the radii of six compactified dimensions and the slope parameter α' are shrunk to zero. If it should turn out that N = 8 supergravity is not finite to all orders in perturbation theory, but that these closed-string theories are, then they may provide the means to make sense of the quantum theory.

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