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COVARIANT FIELD EQUATIONS OF CHIRAL $N = 2$ $D = 10$ SUPERGRAVITY*

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Even though chiral $N = 2$ $D = 10$ supergravity does not have a manifestly Lorentz-invariant action principle, it does have covariant field equations. The latter are obtained by first deriving the local supersymmetry transformations of the fields and then deducing the set of field equations that transform into one another and are required for closure of the algebra. One of the equations is the self-duality of a supercovariant fifth-rank field strength tensor. An SU(1,1) symmetry plays a crucial role in the analysis.

1. Introduction

Dimensional reduction of $D = 11$ supergravity [1] gives a theory in ten dimensions with two supersymmetries, one transforming as a left-handed Weyl spinor and the other as a right-handed one. This theory has an overall left-right symmetry as a consequence of the nonexistence of Weyl spinors in 11 dimensions. There exists a second $N = 2$ $D = 10$ supergravity theory, in which both supersymmetries have the same handedness, that is not derivable from a higher dimension. This theory was originally discovered in a study of superstring theories [2], where it was found that there are two "type II" theories, one of which has the chiral and the other of which has the nonchiral $N = 2$ $D = 10$ supergravity theory as its massless sector. In discussing either of these field theories it is important to regard it as an approximation to the corresponding type II superstring theory, since it has been proved at one loop that the string theory is finite whereas the ten-dimensional field theory has a nonrenormalizable quadratic divergence in physical S-matrix elements [3]. (The $D = 11$ theory is worse, of course.) Thus in discussing $D = 10$ field theories, they must be regarded as classical theories. If one wishes to study quantum corrections, it is necessary to deal with the string theories.

The chiral theory is especially attractive because the chiral structure may be helpful in obtaining the required left-right asymmetry when interpreted in four dimensions (via spontaneous compactification). Another attractive feature, of a more

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technical character, is the fact that the entire theory can be described in terms of an unconstrained scalar light-cone superfield $\Phi(x, \vartheta)$ [4]. The Grassmann coordinate ϑ^a has eight real components and transforms as a spinor of the transverse SO(8) rotation group. Therefore the field describes 128 Bose and 128 Fermi degrees of freedom. The theory has a conserved charge that can be interpreted as generating a rotation of the two supersymmetries in the super-Poincaré algebra. The charges of the components of Φ are measured by the operator $U = 2 - \frac{1}{2}\vartheta\, \partial/\partial\vartheta$. The component fields are complex, but the coefficient of ϑ^n is related to the complex conjugate of the coefficient of ϑ^{8-n} by the rule that Φ^* is equal to the Grassmann Fourier transform of Φ.

The spectrum of the theory is easily read off from the superfield. Expressed in terms of covariant fields, it contains a complex scalar A with $U = 2$, a complex Weyl spinor λ with $U = \frac{3}{2}$, a complex antisymmetric second-rank tensor $A_{\mu\nu}$ with $U = 1$, a complex Weyl gravitino ψ_μ with $U = \frac{1}{2}$, a real gravitino ("zehnbein") e_μ^r with $U = 0$, and a real fourth-rank antisymmetric tensor $A_{\mu\nu\rho\lambda}$ with $U = 0$. This last field has a field strength $F_{\mu\nu\rho\lambda\sigma}$ that is self-dual in the free theory. This rule must be suitably generalized in the interacting theory. Self-duality of a field strength is consistent with the Minkowski metric whenever the spacetime dimension is 2 mod 4. (In four dimensions one considers self-dual field strengths in euclidean space.)

The explicit construction of the chiral supergravity theory has been impeded by the fact that the field equation $F = \tilde{F}$ cannot be obtained from a manifestly covariant action principle [5]. In other words, there does not exist a set of auxiliary fields such that the Lorentz algebra closes offshell. For this reason, the construction of the theory has been investigated directly in terms of the light-cone superfield Φ, and completed to first order in κ, the gravitational coupling constant [4]. The techniques that have been developed for this purpose are cumbersome and difficult to extend to higher orders. Even if one could, however, lack of experience with the light-cone gauge would make the equations difficult to interpret. After all, a primary motivation is to look for solutions of the classical equations of motion describing spontaneous compactification of six dimensions. Techniques for finding such solutions are known in the case of covariant field equations, but their extension to light-cone-gauge field equations would require a major study Even though the theory does not possess a manifestly Lorentz-invariant action principle, it is expected to possess covariant field equations since this dichotomy already exists for the free theory. Therefore we need a technique for deducing field equations that does not require discussing lagrangians. The key to doing this is to identify the symmetries of the theory and to work out the symmetry transformation formulas of the fields In addition to the obvious local symmetries, the theory possesses a global SU(1,1) invariance. Once this fact is recognized it is not very difficult, drawing on previous experience, to derive the supersymmetry transformation formulas of each of the fields. In sect. 2 this is done in a coset formalism involving an auxiliary scalar field and a compensating local U(1) symmetry. In sect 3 the results are recast in the unitary gauge in which the auxiliary scalar field and the local U(1) symmetry are eliminated. In sect. 4 the field equations are deduced by requiring that they transform into one another under supersymmetry variations The results are given to all orders in Bose fields, but only to leading order in Fermi fields.

2. Formulation with a global SU(1,1) and a local U(1)

Chiral $N=2$ $D=10$ supergravity contains a complex scalar field A with charge $U=2$. The particle spectrum differs from that of the nonchiral theory, which has only one real scalar, although the spectra of the two theories does coincide after a trivial truncation to nine or fewer dimensions. In all lower dimensions it is known that the theory has global noncompact symmetries and that the scalar fields can be associated with the coset space formed by the noncompact symmetry group divided by its maximal compact subgroup [6]. Thus in $D=3$ one has $E_{8,8}/SO(16)$, in $D=4$ $E_{7,7}/SU(8)$ [7], in $D=5$ $E_{6,6}/USp(8)$ [8], and so forth. In each case the denominator group is the same one as occurs as a linear symmetry of the free-theory spectrum. Since the chiral $N=2$ $D=10$ theory has a linearly realized U(1) and a complex scalar, this suggests that it may be possible to formulate the interacting theory with a noncompact global SU(1,1) symmetry, identifying the complex scalar with the coset space SU(1,1)/U(1). This is, in fact, exactly what happens in the $N=4$ $D=4$ supergravity theory, which was the first theory of this type to be discussed [9].

There are two ways to proceed, depending on whether one wishes to add an auxiliary field and realize the SU(1,1) symmetry linearly or to use physical fields only and implement it nonlinearly. This section takes the first approach, but the next section shows how to eliminate the auxiliary field and recast the theory in the unitary gauge in which the SU(1,1) symmetry is implemented nonlinearly. This section is a somewhat expanded version of a letter recently coauthored with West [10].

In the coset approach the scalar fields are written as an SU(1,1) group matrix:

$$\begin{pmatrix} V^1_- & V^1_+ \\ V^2_- & V^2_+ \end{pmatrix} = \exp\left\{\kappa\begin{pmatrix} i\varphi & A \\ A^* & -i\varphi \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \cosh\rho + i\kappa\varphi\dfrac{\sinh\rho}{\rho} & \kappa A\dfrac{\sinh\rho}{\rho} \\ \kappa A^*\dfrac{\sinh\rho}{\rho} & \cosh\rho - i\kappa\varphi\dfrac{\sinh\rho}{\rho} \end{pmatrix}, \tag{2.1}$$

where

$$\rho^2 = \kappa^2(A^*A - \varphi^2), \tag{2.2}$$

and φ is an auxiliary real scalar field. In addition to the global SU(1, 1), the theory is formulated with a local U(1) symmetry to compensate for the extra field. Then the matrix $V^\alpha_\pm$ is regarded as having a superscript that labels a doublet representation of SU(1, 1) ($\alpha = 1, 2$) and a subscript that labels a pair of local U(1) representations $U = \pm 1$. Therefore under an infinitesimal SU(1, 1) transformation with constant parameters (α is complex and γ is real)

$$m^\alpha{}_\beta = \begin{pmatrix} i\gamma & \alpha \\ \alpha^* & -i\gamma \end{pmatrix}, \tag{2.3}$$

the variation of $V^\alpha_\pm$ is given by

$$\delta V^\alpha_\pm = m^\alpha{}_\beta V^\beta_\pm . \tag{2.4}$$

Under local U(1) transformations with infinitesimal parameter $\Sigma(x)$

$$\delta V^\alpha_\pm = \pm i \Sigma V^\alpha_\pm . \tag{2.5}$$

In the particular case of SU(1, 1), it is unnecessary to explicitly introduce the inverse matrix because

$$\varepsilon_{\alpha\beta} V^\alpha_- V^\beta_+ = \det V = 1 . \tag{2.6}$$

We also note that

$$V^\alpha_- V^\beta_+ - V^\alpha_+ V^\beta_- = \varepsilon^{\alpha\beta} . \tag{2.7}$$

The SU(1, 1) invariant combination

$$Q_\mu = -i \varepsilon_{\alpha\beta} V^\alpha_- \partial_\mu V^\beta_+ \tag{2.8}$$

acts as a U(1) gauge field, since it follows from eqs. (2.5)–(2.8) that

$$\delta Q_\mu = \partial_\mu \Sigma . \tag{2.9}$$

The SU(1, 1) invariant combination

$$P_\mu = -\varepsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_+ , \tag{2.10}$$

on the other hand, transforms covariantly as a $U = 2$ object

$$\delta P_\mu = 2 i \Sigma P_\mu . \tag{2.11}$$

These equations are simplified analogs of ones that have been introduced in the $E_{7,7} \times SU(8)$ formulation of $N = 8$ $D = 4$ supergravity [7, 11] The complex antisymmetric tensor $A_{\mu\nu}$ is described as an SU(1, 1) doublet $A^{\alpha}_{\mu\nu}$. In the SU(1, 1) basis that we are using $A^{1}_{\mu\nu} = A^{2*}_{\mu\nu}$ If one were to use an SL(2, R) basis instead, then $\alpha = 1, 2$ would label real and imaginary parts In the real basis the local U(1) group would be replaced by SO(2), which would necessitate describing the $U \neq 0$ fields as symmetric traceless tensors, resulting in a considerable proliferation of indices Fortunately, in the complex basis this unpleasantness is avoided. The fermions transform under the local U(1) group and are inert under the global SU(1, 1), as is always the case in coset formulations. Thus the only remaining nonzero SU(1, 1) $\times$ U(1) transformation formulas are

$$\delta A^{\alpha}_{\mu\nu} = m^{\alpha}{}_{\beta} A^{\beta}_{\mu\nu} , \tag{2.12}$$

$$\delta \psi_{\mu} = \tfrac{1}{2} i \Sigma \psi_{\mu} , \tag{2.13}$$

$$\delta \lambda = \tfrac{3}{2} i \Sigma \lambda \tag{2 14}$$

The gauge fields $A^{\alpha}_{\mu\nu}$ and $A_{\mu\nu\rho\lambda}$ require local gauge invariances with parameters Λ^{α}_{μ} and $\Lambda_{\mu\nu\rho}$. One of the transformation rules is exactly what one would naively expect, namely*

$$\delta(\Lambda\quad) A^{\alpha}_{\mu\nu} = 0 , \tag{2.15}$$

$$\delta(\Lambda\quad) A_{\mu\nu\rho\lambda} = 4 \partial_{[\mu} \Lambda_{\nu\rho\lambda]} . \tag{2.16}$$

The second gauge invariance turns out to act on both gauge fields:

$$\delta(\Lambda\ ) A^{\alpha}_{\mu\nu} = 2 \partial_{[\mu} \Lambda^{\alpha}_{\nu]} , \tag{2.17}$$

$$\delta(\Lambda\ ) A_{\mu\nu\rho\lambda} = -\tfrac{1}{4} i \kappa \varepsilon_{\alpha\beta} \Lambda^{\alpha}_{[\mu} F^{\beta}_{\nu\rho\lambda]} . \tag{2 18}$$

The necessity of the last term is deduced from consistency requirements of the supersymmetry algebra to be described shortly. The specific coefficient is essentially an arbitrary normalization choice, which has been arranged to agree with other choices to be made. The factor of i is required by reality since $\Lambda^{1} = \Lambda^{2*}$ and $F^{1} = F^{2*}$. The following field strengths are invariant under both gauge transformations

$$F^{\alpha}_{\mu\nu\rho} = 3 \partial_{[\mu} A^{\alpha}_{\nu\rho]} , \tag{2.19}$$

$$F_{\mu\nu\rho\lambda\sigma} = 5 \partial_{[\mu} A_{\nu\rho\lambda\sigma]} + \tfrac{5}{8} i \kappa \varepsilon_{\alpha\beta} A^{\alpha}_{[\mu\nu} F^{\beta}_{\rho\lambda\sigma]} . \tag{2 20}$$

* Square brackets denote antisymmetrization with unit weight Thus in this case, using the fact that $\Lambda_{\mu\nu\rho}$ is assumed to be antisymmetric itself, $4 \partial_{[\mu} \Lambda_{\nu\rho\lambda]} = \partial_{\mu} \Lambda_{\nu\rho\lambda} - \partial_{\nu} \Lambda_{\rho\lambda\mu} + \partial_{\rho} \Lambda_{\lambda\mu\nu} - \partial_{\lambda} \Lambda_{\mu\nu\rho}$

It is also convenient to define

$$G_{\mu\nu\rho} = -\epsilon_{\alpha\beta} V^{\alpha} F^{\beta}_{\mu\nu\rho}, \tag{2.21}$$

which is SU(1, 1) invariant and is U(1) covariant with charge $U = 1$.

The metric and Dirac matrix conventions are described in the appendix The complex Weyl spinors ψ_μ and λ have opposite handedness

$$\gamma_{11}\psi_\mu = -\psi_\mu, \tag{2.22}$$

$$\gamma_{11}\lambda = \lambda, \tag{2.23}$$

a fact that follows from the SO(8) representations implied by the light-cone superfield. In defining local supersymmetry transformations, we take

$$\delta\psi_\mu = \frac{1}{\kappa} D_\mu \varepsilon + \cdots, \tag{2.24}$$

from which it follows that the infinitesimal parameter ε is also a complex (hence $N = 2$) Weyl spinor satisfying $\gamma_{11}\varepsilon = -\varepsilon$. The derivative D_μ is covariant with respect to local Lorentz transformations and local U(1) transformations. Thus

$$D_\mu \varepsilon = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu^{rs}\gamma_{rs} - \tfrac{1}{2} i Q_\mu\right)\varepsilon. \tag{2.25}$$

It is convenient for the spin connection ω to include ψ_μ dependence in the "supercovariant" form

$$\omega_{\mu\nu\rho} = \Omega_{\nu\rho\mu} + \Omega_{\nu\mu\rho} + \Omega_{\mu\rho\nu}, \tag{2.26}$$

$$\Omega_{\mu\nu\rho} = e^r_\rho \partial_{[\mu} e_{\nu]r} + \kappa^2 \operatorname{Im}\left(\bar{\psi}_\mu \gamma_\rho \psi_\nu\right) \tag{2.27}$$

The supersymmetry variation of the zehnbein is given by the standard formula

$$\delta e^r_\mu = -2\kappa \operatorname{Im}\left(\bar{\varepsilon}\gamma^r\psi_\mu\right). \tag{2.28}$$

It follows that the $\partial\varepsilon$ terms cancel in the variation of eq. (2.27), which is the meaning of supercovariance.

Let us now consider the supersymmetry of the other Bose fields. For δV^α possible terms with the correct SU(1, 1) and U(1) properties are $V^\alpha \bar{\varepsilon}^*\lambda$, $V^\alpha \bar{\varepsilon}\gamma\cdot\psi$, and $V^\alpha_+ \bar{\varepsilon}^*\gamma\cdot\psi^*$. The latter terms are excluded, however, because a second variation would give $\partial\varepsilon$ terms, which are not permissible since $V^\alpha_\pm$ is not a gauge field or a spatial tensor. Thus the only possibility (making an arbitrary normalization choice) is

$$\delta V^\alpha_+ = \kappa V^\alpha_- \bar{\varepsilon}^*\lambda, \qquad \delta V^\alpha_- = \kappa V^\alpha_+ \bar{\varepsilon}\lambda^*. \tag{2.29}$$

In the case of $A_{\mu\nu}^{\alpha}$ and $A_{\mu\nu\rho\lambda}$, gauge fields can occur in the variation because they themselves are gauge fields. This must be arranged so that the commutator of two supersymmetry transformations gives rise to a consistent set of gauge transformations. Altogether the algebra must have the form

$$\left[\delta(\varepsilon_1), \delta(\varepsilon_2)\right] = \delta(\xi) + \delta(l) + \delta(\varepsilon) + \delta(\Lambda\quad) + \delta(\Lambda\quad\ ) + \delta(\Sigma). \tag{2.30}$$

In other words, the commutator of two local supersymmetries can give a combination of all the local symmetry transformations of the theory SU(1.1) transformations do not occur, since they are only global. The general coordinate transformation is completely determined by eqs. (2.24) and (2.28). The parameter is

$$\xi^{\mu} = 2\,\mathrm{Im}(\bar{\varepsilon}_1\gamma^{\mu}\varepsilon_2), \tag{2.31}$$

as in all supergravity theories The other local transformations that can account for $\partial\varepsilon$ terms when $[\delta(\varepsilon_1), \delta(\varepsilon_2)]$ acts on $A_{\mu\nu}^{\alpha}$ or $A_{\mu\nu\rho\lambda}$ are the Λ and Λ gauge transformations. From these considerations, as well as SU(1,1)×U(1) covariance, one deduces that (up to arbitrariness in field normalizations)

$$\delta A_{\mu\nu}^{\alpha} = V^{\alpha}\bar{\varepsilon}^{*}\gamma_{\mu\nu}\lambda^{*} + V_{-}^{\alpha}\bar{\varepsilon}\gamma_{\mu\nu}\lambda + 4iV^{\alpha}\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}^{*} + 4iV^{\alpha}\bar{\varepsilon}^{*}\gamma_{[\mu}\psi_{\nu]}, \tag{2.32}$$

$$\delta A_{\mu\nu\rho\lambda} = 2\,\mathrm{Re}\left(\bar{\varepsilon}\gamma_{[\mu\nu\rho}\psi_{\lambda]}\right) - \tfrac{3}{8}i\kappa\varepsilon_{\alpha\beta}A_{[\mu\nu}^{\alpha}\delta A_{\rho\lambda]}^{\beta}. \tag{2.33}$$

Then the commutator in eq. (2 30) contains local gauge transformations with parameters

$$\Lambda_{\mu}^{\alpha} = A_{\mu\rho}^{\alpha}\xi^{\rho} + \frac{2i}{\kappa}\left(V_{-}^{\alpha}\bar{\varepsilon}_1\gamma_{\mu}\varepsilon_2^{*} + V^{\alpha}\bar{\varepsilon}_1^{*}\gamma_{\mu}\varepsilon_2\right), \tag{2.34}$$

$$\Lambda_{\mu\nu\rho} = A_{\mu\nu\rho\lambda}\xi^{\lambda} - \frac{1}{2\kappa}\mathrm{Re}\left(\bar{\varepsilon}_1\gamma_{\mu\nu\rho}\varepsilon_2\right) + \tfrac{3}{8}\varepsilon_{\alpha\beta}A_{[\mu\nu}^{\alpha}\left(V_{-}^{\beta}\bar{\varepsilon}_1\gamma_{\rho]}\varepsilon_2^{*} + V^{\beta}\bar{\varepsilon}_1^{*}\gamma_{\rho]}\varepsilon_2\right). \tag{2.35}$$

It is this analysis that gives the need for eq (2.18), in particular. A nontrivial check of these results is provided by considering the commutator of a local supersymmetry and a Λ_{μ} gauge transformation Using eqs. (2.18) and (2.33) one finds that the result corresponds to a $\Lambda_{\mu\nu\rho}$ gauge transformation with parameter

$$\Lambda_{\mu\nu\rho} = -\tfrac{3}{16}i\kappa\varepsilon_{\alpha\beta}\Lambda_{[\mu}^{\alpha}\delta(\varepsilon)A_{\nu\rho]}^{\beta}, \tag{2.36}$$

where $\delta(\varepsilon)A$ is given by eq. (2.32).

Knowing the Bose field supersymmetry transformation formulas, one can define supercovariant field strengths. From eqs. (2.10) and (2.29) we see that

$$\delta P_{\mu} = \kappa\partial_{\mu}\bar{\varepsilon}^{*}\lambda + \cdots. \tag{2.37}$$

Therefore

$$\hat{P}_\mu = P_\mu - \kappa^2 \bar{\psi}^*_\mu \lambda \tag{2.38}$$

is supercovariant Similarly, from eqs. (2.19), (2.21), and (2.32)

$$\delta G_{\mu\nu\rho} = 3\partial_{[\mu} \bar{\varepsilon} \gamma_{\nu\rho]} \lambda - 12 i \partial_{[\mu} \bar{\varepsilon}^* \gamma_\nu \psi_{\rho]} + \cdots \tag{2.39}$$

Therefore

$$\hat{G}_{\mu\nu\rho} = G_{\mu\nu\rho} - 3\kappa \bar{\psi}_{[\mu} \gamma_{\nu\rho]} \lambda - 6 i \kappa \bar{\psi}^*_{[\mu} \gamma_\nu \psi_{\rho]} \tag{2.40}$$

From eqs. (2.20), (2.32), and (2.33)

$$\delta F_{\mu\nu\rho\lambda\sigma} = 5\partial_{[\mu} \bar{\varepsilon} \gamma_{\nu\rho\lambda} \psi_{\sigma]} + 5 \bar{\psi}_{[\mu} \gamma_{\nu\rho\lambda} \partial_{\sigma]} \varepsilon + \cdots \tag{2.41}$$

and therefore

$$\hat{F}_{\mu\nu\rho\lambda\sigma} = F_{\mu\nu\rho\lambda\sigma} - 5\kappa \bar{\psi}_{[\mu} \gamma_{\nu\rho\lambda} \psi_{\sigma]} - \tfrac{1}{16} \kappa \bar{\lambda} \gamma_{\mu\nu\rho\lambda\sigma} \lambda \tag{2.42}$$

is supercovariant. The last term in eq. (2.42) is not determined by these considerations, but it is included in the definition for later convenience.

Now let us turn to the Fermi field supersymmetry transformation formulas. The result for λ is

$$\delta\lambda = \frac{i}{\kappa} \gamma^\mu \varepsilon^* \hat{P}_\mu - \tfrac{1}{24} i \gamma^{\mu\nu\rho} \varepsilon \hat{G}_{\mu\nu\rho} \tag{2.43}$$

These two terms are the only ones compatible with SU(1,1) invariance, U(1) conservation, and supercovariance. The coefficients are easily determined by requiring closure of the algebra on the Bose fields. The ψ_μ result, also uniquely determined by requiring closure of the algebra for the Bose fields, is

$$\begin{aligned}
\delta\psi_\mu = {} & \frac{1}{\kappa} D_\mu \varepsilon + \tfrac{1}{480} i \gamma^{\rho_1 \cdots \rho_5} \gamma_\mu \varepsilon \hat{F}_{\rho_1 \cdots \rho_5} + \tfrac{1}{96} \left(\gamma_\mu{}^{\nu\rho\lambda} \hat{G}_{\nu\rho\lambda} - 9 \gamma^{\rho\lambda} \hat{G}_{\mu\rho\lambda} \right) \varepsilon^* \\
& - \tfrac{7}{16} \kappa \left(\gamma_\rho \lambda \bar{\psi}_\mu \gamma^\rho \varepsilon^* - \tfrac{1}{1680} \gamma_{\rho_1 \cdots \rho_5} \lambda \bar{\psi}_\mu \gamma^{\rho_1 \cdots \rho_5} \varepsilon^* \right) \\
& + \tfrac{1}{32} i \kappa \Big[\left(\tfrac{9}{4} \gamma_\mu \gamma^\rho + 3 \gamma^\rho \gamma_\mu \right) \varepsilon \bar{\lambda} \gamma_\rho \lambda \\
& \quad - \left(\tfrac{1}{24} \gamma_\mu \gamma^{\rho_1 \rho_2 \rho_3} + \tfrac{1}{6} \gamma^{\rho_1 \rho_2 \rho_3} \gamma_\mu \right) \varepsilon \bar{\lambda} \gamma_{\rho_1 \rho_2 \rho_3} \lambda + \tfrac{1}{960} \gamma_\mu \gamma^{\rho_1 \cdots \rho_5} \varepsilon \bar{\lambda} \gamma_{\rho_1 \cdots \rho_5} \lambda \Big] .
\end{aligned} \tag{2.44}$$

The types of terms that occur in this expression are the only ones allowed by local

U(1) conservation and SU(1,1) invariance. Supercovariance rules out explicit ψ^2 terms, but not $\psi_\mu\lambda$ terms because the commutator can involve a local supersymmetry with parameter proportional to λ. In fact one finds that the local supersymmetry transformation in eq. (2.30) is

$$\varepsilon = -\kappa\psi_\rho\xi^\rho - \tfrac{7}{16}\kappa\left(\gamma_\rho\lambda\bar\varepsilon_1\gamma^\rho\varepsilon_2^* - \tfrac{1}{1680}\gamma_{\rho_1\ldots\rho_5}\lambda\bar\varepsilon_1\gamma^{\rho_1\ldots\rho_5}\varepsilon_2^*\right). \tag{2.45}$$

The local Lorentz transformation parameter occurring in eq. (2.30) is determined by considering the commutator transformation of the zehnbein. The result is

$$\begin{aligned} l^{rs} = {}& \omega_\rho{}^{rs}\xi^\rho - \tfrac{1}{3}\kappa\hat F^{rs\mu\nu\rho}\,\mathrm{Re}(\bar\varepsilon_1\gamma_{\mu\nu\rho}\varepsilon_2) \\ & + \tfrac{3}{4}\kappa\,\mathrm{Im}\left(\hat G^{rs\rho}\bar\varepsilon_1\gamma_\rho\varepsilon_2^* + \tfrac{1}{18}\hat G_{\mu\nu\rho}\bar\varepsilon_1\gamma^{rs\mu\nu\rho}\varepsilon_2^*\right) \\ & + \mathrm{Re}\Big(3\bar\varepsilon_1\gamma^{rs\rho}\varepsilon_2\bar\lambda\gamma_\rho\lambda - \tfrac{1}{4}\bar\varepsilon_1\gamma^{rs\rho_1\rho_2\rho_3}\varepsilon_1\bar\lambda\gamma_{\rho_1\rho_2\rho_3}\lambda \\ & \qquad - \tfrac{5}{2}\bar\varepsilon_1\gamma_\rho\varepsilon_2\bar\lambda\gamma^{rs\rho}\lambda - \tfrac{1}{960}\bar\varepsilon_1\left\{\gamma^{rs},\gamma^{\rho_1\ldots\rho_5}\right\}\varepsilon_2\bar\lambda\gamma_{\rho_1\ldots\rho_5}\lambda\Big) \end{aligned} \tag{2.46}$$

The local U(1) transformation parameter is determined by the commutator transformation of V_-^α. It is

$$\Sigma = -Q_\rho\xi^\rho + 2\kappa^2\,\mathrm{Im}(\bar\varepsilon_1\lambda^*\bar\varepsilon_2^*\lambda)\,. \tag{2.47}$$

The most challenging part of the calculation is the determination of the λ^2 terms in eq. (2.44). If we denote them collectively by $\Delta\psi_\mu$, then matching λ^2 terms in $[\delta_1,\delta_2]A^\alpha_{\mu\nu}$ gives the condition

$$\begin{aligned} &\left(\bar\lambda\gamma_{\mu\nu}\varepsilon_2\bar\varepsilon_1^*\lambda + \frac{4i}{\kappa}\bar\varepsilon_1^*\gamma_{[\mu}\Delta_2\psi_{\nu]}\right) - (1\leftrightarrow 2) \\ &\qquad = -\tfrac{7}{16}\left(\bar\lambda\gamma_\rho\gamma_{\mu\nu}\lambda\bar\varepsilon_1^*\gamma^\rho\varepsilon_2 - \tfrac{1}{1680}\bar\lambda\gamma_{\rho_1\ldots\rho_5}\gamma_{\mu\nu}\lambda\bar\varepsilon_1^*\gamma^{\rho_1\ldots\rho_5}\varepsilon_2\right). \end{aligned} \tag{2.48}$$

Matching λ^2 terms in $[\delta_1,\delta_2]A_{\mu\nu\rho\lambda}$ gives the condition

$$\left[2\,\mathrm{Re}(\bar\varepsilon_2\gamma_{[\mu\nu\rho}\Delta_1\psi_{\lambda]}) - \tfrac{3}{4}\kappa\,\mathrm{Im}\left(\bar\varepsilon_1^*\gamma_{[\mu\nu}\lambda^*\bar\varepsilon_2\gamma_{\rho\lambda]}\lambda\right)\right] - (1\leftrightarrow 2) = \tfrac{1}{16}\kappa\bar\lambda\gamma_{\mu\nu\rho\lambda\sigma}\lambda\xi^\sigma\,. \tag{2.49}$$

Performing Fierz transformations and matching different types of γ matrix terms in eq. (2.48) gives four conditions for the five coefficients in $\Delta\psi_\mu$, determining all but the $\gamma^\rho\gamma_\mu\bar{\lambda}\gamma^\rho\lambda$ term. The term on the right-hand side of eq. (2.49) comes directly from the last term in eq. (2.42). It arises because the factor $\gamma^{\rho_1 \ldots \rho_5}\gamma_\mu\varepsilon$ projects out the self-dual part of $\hat{F}_{\rho_1 \ldots \rho_5}$ in the second term on the right-hand side of eq. (2.44) as a consequence of eq. (A.9) and $\gamma_{11}\varepsilon = -\varepsilon$. It is therefore convenient to define $\hat{F}$ to be self-dual, which becomes one of the field equations. Eq. (2.49) requires for its consistency that the right-hand side have the indicated size and magnitude. Therefore it effectively determines the $A_{\mu\nu\rho\lambda}$ field equation required for consistent closure of the algebra. (This is the only instance I am aware of where a field equation is needed to close the supersymmetry algebra on a Bose field.) In addition, eq. (2.49) provides four more conditions on the coefficients in $\Delta\psi_\mu$. One determines the coefficient that was not determined by eq (2.48), and the other three serve as consistent checks.

3. Reformulation in the unitary gauge

The local U(1) symmetry of the preceding section is, in a certain sense, fake inasmuch as it exactly compensates for the unphysical auxiliary scalar field that was introduced. It is therefore interesting to use the U(1) gauge freedom to eliminate the auxiliary scalar so that only physical fields and physical symmetries remain. For the present, "unitary gauge" refers only to the elimination of this one scalar, which still leaves manifestly covariant equations. (In future work it may be desirable to try to eliminate all off-shell and gauge degrees of freedom so that all local symmetries are eliminated and only fields associated with physical, propagating modes are retained. This requires choosing a noncovariant gauge.)

Referring to eqs. (2.1) and (2.2), we see that in the $\varphi = 0$ gauge

$$V^1 = V^2 = \cosh\rho\,. \tag{3.1}$$

A general SU(1,1) transformation takes V_-^α satisfying this condition into one that does not unless it is accompanied by a local U(1) transformation chosen to maintain the gauge choice. In other words

$$V_\pm^\alpha \to (V_\pm^\alpha)' = U^\alpha{}_\beta V_\pm^\beta e^{\pm i\Sigma}\,, \tag{3.2}$$

where U is an arbitrary SU(1,1) matrix and Σ is chosen so that V' satisfies eq. (3.1). For infinitesimal transformations

$$U = \begin{pmatrix} 1+i\gamma & \alpha \\ \alpha^* & 1-i\gamma \end{pmatrix} \tag{3.3}$$

with α and γ infinitesimal, so that

$$\delta V_-^1 = i\gamma\cosh\rho + \kappa\alpha A^*\frac{\sinh\rho}{\rho} - i\Sigma\cosh\rho\,. \tag{3.4}$$

We can make eq. (3.4) real by setting

$$\Sigma = \gamma + \kappa \operatorname{Im}(\alpha A^*) \frac{\tanh \rho}{\rho} . \tag{3.5}$$

It then follows that

$$\delta A = \left[2i\gamma + \frac{\kappa}{\rho^2} \operatorname{Re}(\alpha A^*) + 2i \frac{\kappa}{\rho} \operatorname{Im}(\alpha A^*) \coth 2\rho \right] A . \tag{3.6}$$

This can be recast in a more elegant form by making the standard change of variables

$$B = \kappa \frac{\tanh \rho}{\rho} A . \tag{3.7}$$

Then under infinitesimal SU(1, 1) transformations

$$\delta B = \alpha + 2i\gamma B - \alpha^* B^2 . \tag{3.8}$$

This exponentiates to give linear fractional transformations for the finite SU(1, 1) group action

$$B \to \frac{uB + v}{v^* B + u^*} , \tag{3.9}$$

where

$$uu^* - vv^* = 1 . \tag{3.10}$$

The B field has its values restricted to the interior of the unit disk.

The supersymmetry transformations in the unitary gauge can be analyzed in a similar fashion. The transformations of sect. 2 must also be accompanied by a local U(1) transformation to stay in the unitary gauge. Specifically

$$\delta V^1_- = \kappa^2 A \frac{\sinh \rho}{\rho} \bar{\varepsilon} \lambda^* - i \Sigma_\varepsilon \cosh \rho \tag{3.11}$$

is real for

$$\Sigma_\varepsilon = \kappa \operatorname{Im}(B \bar{\varepsilon} \lambda^*) . \tag{3.12}$$

It then follows that

$$\delta B = \kappa f^{-2} \bar{\varepsilon}^* \lambda , \tag{3.13}$$

where

$$f = \cosh \rho = (1 - B^* B)^{-1/2} \tag{3.14}$$

The transformation formulas of the Fermi fields are correspondingly modified to

$$\delta\lambda = \delta_0\lambda + \tfrac{3}{2}i\Sigma_\epsilon\lambda, \tag{3.15a}$$

$$\delta\psi_\mu = \delta_0\psi_\mu + \tfrac{1}{2}i\Sigma_\epsilon\psi_\mu, \tag{3.15b}$$

where δ_0 refers to the transformation rules in eqs. (2.43) and (2.44). The Fermi fields also have induced SU(1,1) transformations as a consequence of eq. (3.5). The infinitesimal rules are

$$\delta\lambda = \tfrac{3}{2}i\left[\gamma + \mathrm{Im}(\alpha B^*)\right]\lambda, \tag{3.16a}$$

$$\delta\psi_\mu = \tfrac{1}{2}i\left[\gamma + \mathrm{Im}(\alpha B^*)\right]\psi_\mu. \tag{3.16b}$$

In describing the second-rank potential it is convenient to make the identifications

$$A^1_{\mu\nu} \equiv A_{\mu\nu}, \qquad A^2_{\mu\nu} \equiv A^*_{\mu\nu}. \tag{3.17}$$

Then the SU(1,1) transformation rule takes the form

$$\delta A_{\mu\nu} = i\gamma A_{\mu\nu} + \alpha A^*_{\mu\nu}, \tag{3.18}$$

while the supersymmetry transformation rule becomes

$$\delta A_{\mu\nu} = f\left(\bar{\epsilon}\gamma_{\mu\nu}\lambda + 4i\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} + B\bar{\epsilon}^*\gamma_{\mu\nu}\lambda + 4iB\bar{\epsilon}\gamma_{[\mu}\psi^*_{\nu]}\right). \tag{3.19}$$

Eqs. (2.8), (2.10), (2.20), and (2.21) can be rewritten in the form

$$G_{\mu\nu\rho} = f\left(F_{\mu\nu\rho} - BF^*_{\mu\nu\rho}\right), \tag{3.20}$$

$$P_\mu = f^2\partial_\mu B, \tag{3.21}$$

$$Q_\mu = f^2\,\mathrm{Im}\left(B\partial_\mu B^*\right), \tag{3.22}$$

$$F_{\mu\nu\rho\lambda\sigma} = 5\partial_{[\mu}A_{\nu\rho\lambda\sigma]} - \tfrac{5}{4}\kappa\,\mathrm{Im}\left(A_{[\mu\nu}F^*_{\rho\lambda\sigma]}\right). \tag{3.23}$$

These have the SU(1,1) transformation rules:

$$\delta F_{\mu\nu\rho\lambda\sigma} = 0, \tag{3.24}$$

$$\delta G_{\mu\nu\rho} = i\left[\gamma + \mathrm{Im}(\alpha B^*)\right]G_{\mu\nu\rho}, \tag{3.25}$$

$$\delta P_\mu = 2i\left[\gamma + \mathrm{Im}(\alpha B^*)\right]P_\mu, \tag{3.26}$$

$$\delta Q_\mu = \mathrm{Im}\left(\alpha\partial_\mu B^*\right) = \partial_\mu\left[\gamma + \mathrm{Im}(\alpha B^*)\right]. \tag{3.27}$$

Except for the U(1) gauge field, Q_μ, these are SU(1,1) covariant.

The commutator of a supersymmetry and a gauge transformation is still given by eq. (2.36) The rule for the commutator of two supersymmetries is modified, however. First of all, there is no local U(1) transformation that can arise, so we only have

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \delta(\xi) + \delta(l) + \delta(\varepsilon) + \delta(\Lambda\ \ ) + \delta(\Lambda\ \ \ ). \tag{3 28}$$

The expressions for ξ and l are given by eqs. (2.31) and (2.46) without change. The supersymmetry parameter becomes

$$\varepsilon = \varepsilon_0 + \left[\tfrac{1}{2} i\kappa\varepsilon_1 \operatorname{Im}\left(B\bar{\varepsilon}_2\lambda^*\right) - (1 \leftrightarrow 2)\right], \tag{3.29}$$

where ε_0 refers to the previous expression in eq. (2.45). Λ_μ and $\Lambda_{\mu\nu\rho}$ are not changed, but they may be rewritten in the form

$$\Lambda_\mu = A_{\mu\rho}\xi^\rho - \frac{2i}{\kappa} f\left(\bar{\varepsilon}_2^*\gamma_\mu\varepsilon_1 + B\bar{\varepsilon}_2\gamma_\mu\varepsilon_1^*\right), \tag{3 30}$$

$$\Lambda_{\mu\nu\rho} = A_{\mu\nu\rho\lambda}\xi^\lambda + \frac{1}{2\kappa}\operatorname{Re}(\bar{\varepsilon}_2\gamma_{\mu\nu\rho}\varepsilon_1) - \tfrac{3}{4} f \operatorname{Re}\left[A_{[\mu\nu}\left(\bar{\varepsilon}_2\gamma_{\rho]}\varepsilon_1^* + B^*\bar{\varepsilon}_2^*\gamma_{\rho]}\varepsilon_1\right)\right] \tag{3.31}$$

The commutator of an SU(1,1) transformation and a supersymmetry gives another supersymmetry.

$$[\delta(\alpha, \gamma), \delta(\varepsilon)] = \delta(\varepsilon'), \tag{3 32}$$

$$\varepsilon' = -\tfrac{1}{2} i\left[\gamma + \operatorname{Im}(\alpha B^*)\right]\varepsilon. \tag{3.33}$$

Several identities are needed in the following section.

$$D_{[\mu}P_{\rho]} = 0, \tag{3 34}$$

$$D_{[\mu}G_{\nu\rho\lambda]} = -P_{[\mu}G^*_{\nu\rho\lambda]}, \tag{3.35}$$

$$\partial_{[\rho}Q_{\mu]} = -iP_{[\rho}P^*_{\mu]}, \tag{3.36}$$

$$\partial_{[\nu}F_{\mu_1\ \ \mu_5]} = \tfrac{2}{12} i\kappa G_{[\nu\mu_1\mu_2}G^*_{\mu_3\mu_4\mu_5]} \tag{3.37}$$

4. Equations of motion

Because this theory does not possess auxiliary fields to close the supersymmetry algebra off shell, the algebra given by eq. (2.30) is only satisfied using equations of motion This fact can be turned to advantage as a means of deducing equations of motion. For example, we showed in sect. 2 that the algebra only closes on $A_{\mu\nu\rho\lambda}$ if it satisfies the self-duality field equation

$$e\hat{F}_{\mu_1 \dots \mu_5} = g_{\mu_1\rho_1} \cdots g_{\mu_5\rho_5} \varepsilon^{\rho_1 \dots \rho_5 \lambda_1 \dots \lambda_5} \hat{F}_{\lambda_1 \dots \lambda_5}, \tag{4.1}$$

where e is the determinant of e^r_μ, $g_{\mu\rho}$ is the metric tensor

$$g_{\mu\rho} = \eta_{rs} e^r_\mu e^s_\rho, \tag{4.2}$$

and $\hat{F}$ is given by eqs. (2.20) and (2.42).

The supersymmetry algebra closes on the fields B, e^r_μ, and $A_{\mu\nu}$ without any field equations, but closure on the Fermi fields ψ_μ and λ requires field equations. To see this, consider $[\delta_1, \delta_2]\lambda$, which contains the Fermi-field derivative terms

$$[\delta_1, \delta_2]\lambda = \left(-\tfrac{1}{2}\gamma^{\mu\nu\rho}\varepsilon_2\bar{\varepsilon}_1^*\gamma_\mu D_\nu\psi_\rho + i\gamma^\mu\varepsilon_2^*\bar{\varepsilon}_1^* D_\mu\lambda - \tfrac{1}{8}i\gamma^{\mu\nu\rho}\varepsilon_2\bar{\varepsilon}_1\gamma_{\nu\rho}D_\mu\lambda \right) - (1 \leftrightarrow 2) + \cdots . \tag{4.3}$$

The only derivative term that should occur in the commutator is the transport term $\xi^\mu\partial_\mu\lambda$. All remaining Fermi derivative terms must therefore combine into forms appropriate for elimination by equations of motion. The gravitino term is suitable as it stands, because its field equation has the form

$$\gamma_{[\mu} D_\nu \psi_{\rho]} = \cdots . \tag{4.4}$$

For the λ field terms, we make a Fierz rearrangement to show that

$$\begin{aligned} \left(i\gamma^\mu\varepsilon_2^*\bar{\varepsilon}_1^* D_\mu\lambda - \tfrac{1}{8}i\gamma^{\mu\nu\rho}\varepsilon_2\bar{\varepsilon}_1\gamma_{\nu\rho}D_\mu\lambda \right) - (1 \leftrightarrow 2) = \xi^\mu D_\mu\lambda &- \tfrac{3}{8}i\left[\varepsilon_2\gamma^\rho\varepsilon_1 - (1 \leftrightarrow 2)\right]\gamma_\rho\gamma^\mu D_\mu\lambda \\ &- \tfrac{1}{96}i\left[\bar{\varepsilon}_2\gamma^{\rho_1\rho_2\rho_3}\varepsilon_1 - (1 \leftrightarrow 2)\right] \\ &\times \gamma_{\rho_1\rho_2\rho_3}\gamma^\mu D_\mu\lambda \end{aligned} \tag{4.5}$$

Thus the derivatives combine to give the transport term and the field-equation form $\gamma^\mu D_\mu\lambda$, so that an important consistency requirement is satisfied. It is quite tedious to collect all the terms and read off the field equations for ψ_μ and λ, but there is no doubt that this could be done. We have found another method of proceeding to be somewhat easier. The point is that the structure of the fermion field equations is obvious from symmetry and dimensionality considerations up to a few unknown constant coefficients. When these expressions are varied by a supersymmetry transformation, they must yield field equations again. The calculations require many "miracles" for their success, as there are many different types of structures that have to combine in special ways. Consistency, in fact, determines all the unknown coefficients in the original Fermi field equations as well as the resulting bosonic equations. These are calculations that would need to be carried out in any case to derive the bosonic field equations. Thus we are able to shortcut the unpleasant closure calculations for the Fermi fields.

Let us start with the field equation for the spinor field λ. It must have the form

$$\gamma^{\mu} D_{\mu} \lambda = i\kappa a_1 \gamma^{\rho_1 \ldots \rho_5} \lambda F_{\rho_1 \ldots \rho_5} + O(\psi^3). \tag{4.6}$$

There are no other charge-$\frac{3}{2}$ terms of correct dimensionality linear in λ that can occur (ψ_{μ} fields can only occur in supercovariant combinations.) $O(\psi^3)$ means terms cubic in Fermi fields, which we do not attempt to determine. Varying this expression under supersymmetry, we can completely determine all the terms that are independent of Fermi fields. The calculation splits up into two pieces: terms involving ε and terms involving ε^*. The ε terms give

$$\begin{aligned} &-\tfrac{1}{24} i D_{\mu} G_{\nu\rho\lambda} \gamma^{\mu} \gamma^{\nu\rho\lambda} - \tfrac{1}{96} i \gamma^{\mu} \gamma^{\rho} \left(\gamma_{\mu}{}^{\lambda\sigma\eta} - 9\delta_{\mu}^{\lambda} \gamma^{\sigma\eta} \right) G^{*}_{\lambda\sigma\eta} P_{\rho} \\ &- \frac{\kappa}{24 \times 480} \gamma^{\mu} \gamma^{\nu\rho\sigma} \gamma^{\lambda_1 \ldots \lambda_5} \gamma_{\mu} G_{\nu\rho\sigma} F_{\lambda_1 \ldots \lambda_5} = \tfrac{1}{24} a_1 \kappa \gamma^{\lambda_1 \ldots \lambda_5} \gamma^{\nu\rho\sigma} G_{\nu\rho\sigma} F_{\lambda_1 \ldots \lambda_5} + O(\psi^2). \end{aligned} \tag{4.7}$$

This can only be solved for the choice

$$a_1 = \tfrac{1}{240}. \tag{4.8}$$

Then one finds using eq. (3.35) and doing some Dirac algebra that

$$D^{\rho} G_{\mu\nu\rho} = P^{\rho} G^{*}_{\mu\nu\rho} - \tfrac{2}{3} i\kappa F_{\mu\nu\rho\lambda\sigma} G^{\rho\lambda\sigma} + O(\psi^2). \tag{4.9}$$

The ε^* terms give the relation

$$\begin{aligned} &\frac{i}{\kappa} \gamma^{\mu} \gamma^{\rho} D_{\mu} P_{\rho} - \tfrac{1}{480} \gamma^{\mu} \gamma^{\rho} \gamma^{\lambda_1 \ldots \lambda_5} \gamma_{\mu} P_{\rho} F_{\lambda_1 \ldots \lambda_5} + \frac{i\kappa}{24 \times 96} \gamma^{\mu} \gamma^{\nu\rho\lambda} \left(\gamma_{\mu}{}^{\nu'\rho'\lambda'} - 9\delta_{\mu}^{\nu'} \gamma^{\rho'\lambda'} \right) G_{\nu\rho\lambda} G_{\nu'\rho'\lambda'} \\ &\quad = -a_1 \gamma^{\lambda_1 \ldots \lambda_5} \gamma^{\rho} P_{\rho} F_{\lambda_1 \ldots \lambda_5} + O(\psi^2). \end{aligned} \tag{4.10}$$

Again, this can only be solved for the choice of a_1 given in eq. (4.8). Using eq. (3.34) and doing some Dirac algebra gives

$$D^{\mu} P_{\mu} = \tfrac{1}{24} \kappa^2 G_{\mu\nu\rho} G^{\mu\nu\rho} + O(\psi^2) \tag{4.11}$$

This exhausts the information to be obtained by varying eq. (4.6), aside from terms of higher-order in Fermi fields.

The most general possible form of the gravitino field equation is

$$\gamma^{\mu\nu\rho}D_\nu\psi_\rho = i\left(a_2\gamma^\rho\gamma^\mu + a_3\gamma^\mu\gamma^\rho\right)\lambda^* P_\rho + i\kappa\left(a_4\gamma^{\nu\rho\lambda}\gamma^\mu + a_5\gamma^\mu\gamma^{\nu\rho\lambda}\right)\lambda G^*_{\nu\rho\lambda} + \mathrm{O}(\psi^3). \tag{4.12}$$

Applying a supersymmetry transformation to this equation and keeping only terms independent of Fermi fields gives for the coefficient of ε^*

$$\begin{aligned}
&\tfrac{1}{96}\gamma^{\mu\nu\rho}D_\nu\left(\gamma_\rho{}^{\zeta\eta\sigma}G_{\zeta\eta\sigma} - 9\gamma^{\eta\sigma}G_{\rho\eta\sigma}\right)\\
&\qquad + \frac{i\kappa}{96\times 480}\gamma^{\mu\nu\rho}\left(\gamma_\rho{}^{\zeta\eta\sigma}G_{\zeta\eta\sigma} - 9\gamma^{\eta\sigma}G_{\rho\eta\sigma}\right)\gamma^{\lambda_1\cdots\lambda_5}\gamma_\nu F_{\lambda_1\cdots\lambda_5}\\
&\qquad - \frac{i\kappa}{96\times 480}\gamma^{\mu\nu\rho}\gamma^{\lambda_1\cdots\lambda_5}\gamma_\rho\left(\gamma_\nu{}^{\zeta\eta\sigma}G_{\zeta\eta\sigma} - 9\gamma^{\eta\sigma}G_{\nu\eta\sigma}\right)F_{\lambda_1\cdots\lambda_5}\\
&= -\left(a_4\gamma^{\rho\lambda\sigma}\gamma^\mu + a_5\gamma^\mu\gamma^{\rho\lambda\sigma}\right)\gamma^\nu P_\nu G^*_{\rho\lambda\sigma}\\
&\quad - \tfrac{1}{24}\left(a_2\gamma^\lambda\gamma^\mu + a_3\gamma^\mu\gamma^\lambda\right)\gamma^{\rho\eta\sigma}P_\lambda G^*_{\rho\eta\sigma} + \mathrm{O}(\psi^2).
\end{aligned} \tag{4.13}$$

After a certain amount of Dirac algebra one discovers that this can only be solved for

$$a_2 = -\tfrac{1}{2}, \qquad a_3 = 0, \tag{4.14a}$$

$$a_4 = -\tfrac{1}{48}, \qquad a_5 = 0, \tag{4.14b}$$

and that then eq. (4.9) is precisely reproduced.

The coefficient of ε in the variation of eq. (4.12) to lowest order in Fermi fields gives (using eq. (4.14))

$$\begin{aligned}
&\frac{1}{\kappa}\gamma^{\mu\nu\rho}D_\nu D_\rho\varepsilon + \tfrac{1}{480}i\gamma^{\mu\nu\rho}\gamma^{\lambda_1\cdots\lambda_5}\gamma_\rho\varepsilon\partial_\nu F_{\lambda_1\cdots\lambda_5}\\
&\quad - \tfrac{1}{96}\kappa\gamma^{\mu\nu\rho}\left(\gamma_\rho{}^{\zeta\eta\sigma} - 9\delta_\rho^\zeta\gamma^{\eta\sigma}\right)\left(\gamma_\nu{}^{\zeta'\eta'\sigma'} - 9\delta_\nu^{\zeta'}\gamma^{\eta'\sigma'}\right)\varepsilon G_{\zeta\eta\sigma}G^*_{\zeta'\eta'\sigma'}\\
&\quad + \frac{\kappa}{(480)^2}\gamma^{\mu\nu\rho}\gamma^{\lambda_1\cdots\lambda_5}\gamma_\rho\gamma^{\eta_1\cdots\eta_5}\gamma_\nu\varepsilon F_{\lambda_1\cdots\lambda_5}F_{\eta_1\cdots\eta_5}\\
&\quad + \frac{\kappa}{24\times 48}\gamma^{\rho\lambda\sigma}\gamma^\mu\gamma^{\rho'\lambda'\sigma'}\varepsilon G_{\rho\lambda\sigma}G^*_{\rho'\lambda'\sigma'} - \frac{1}{2\kappa}\gamma^\lambda\gamma^\mu\gamma^\rho\varepsilon P^*_\rho P_\lambda = \mathrm{O}(\psi^2).
\end{aligned} \tag{4.15}$$

To solve this we note that

$$D_{[\nu}D_{\rho]}\varepsilon = \tfrac{1}{8}R'^{rs}_{\nu\rho}\gamma_{rs}\varepsilon - \tfrac{1}{2}P_{[\nu}P^*_{\rho]}\varepsilon. \tag{4.16}$$

so that the total PP^* contribution in eq (4.15) becomes

$$-\frac{1}{2\kappa}\left[\gamma^{\rho}\left(P^{\mu}P^{*}_{\rho}+P^{\mu *}P_{\rho}\right)-\gamma^{\mu}P^{*}_{\rho}P^{\rho}\right]\varepsilon . \tag{4.17}$$

It also contains

$$\frac{1}{8\kappa}\gamma^{\mu\nu\rho}\gamma_{rs}R^{rs}_{\nu\rho}\varepsilon=\frac{1}{2\kappa}\gamma_{\rho}\varepsilon\left(R^{\mu\rho}-\tfrac{1}{2}Rg^{\mu\rho}\right)+\mathrm{O}(\psi^{2}) . \tag{4.18}$$

After some very painful Dirac algebra, the entire expression boils down to one in which each term contains just a single γ matrix. At this point all that remains is the equation

$$\begin{aligned} R_{\mu\rho}-\tfrac{1}{2}g_{\mu\rho}R={}&P_{\mu}P^{*}_{\rho}+P^{*}_{\mu}P_{\rho}-g_{\mu\rho}P^{\lambda}P^{*}_{\lambda}+\tfrac{1}{6}\kappa^{2}F_{\lambda_1\ \ \lambda_4\mu}F^{\lambda_1\ \ \lambda_4}{}_{\rho}\\ &+\tfrac{1}{8}\kappa^{2}\left(G_{\mu}{}^{\lambda\sigma}G^{*}_{\rho\lambda\sigma}+G^{*\lambda\sigma}_{\mu}G_{\rho\lambda\sigma}\right)-\frac{\kappa^{2}}{24}g_{\mu\rho}G^{\lambda\sigma\eta}G^{*}_{\lambda\sigma\eta}+\mathrm{O}(\psi^{2}) \end{aligned} \tag{4.19}$$

We have now obtained all the field equations to leading order in Fermi fields. As one final check we note that applying a supersymmetry transformation to eq (4.1) partially determines eq. (4.12). It gives a_2 and a_4 correctly, but does not determine a_3 and a_5. Two of the six field equations are known to all orders in Fermi fields: eq. (4.1) is complete as it stands, and all the possible ψ^3 terms in the gravitino field equation are easily determined by supercovariantization. Superspace methods may be the most convenient ones for completing the other equations and demonstrating closure of the algebra on the Fermi fields [12].

5. Discussion

We have succeeded in finding the complete supersymmetry transformations and the field equations to leading order in Fermi fields for $N=2$ $D=10$ chiral supergravity. In studying spontaneous compactification one ordinarily sets the Fermi fields equal to zero. Doing so leaves the system

$$D^{\mu}P_{\mu}=\tfrac{1}{24}\kappa^{2}G_{\mu\nu\rho}G^{\mu\nu\rho}, \tag{5.1}$$

$$D^{\rho}G_{\mu\nu\rho}=P^{\rho}G^{*}_{\mu\nu\rho}-\tfrac{2}{3}i\kappa F_{\mu\nu\rho\lambda\sigma}G^{\rho\lambda\sigma}, \tag{5.2}$$

$$\begin{aligned} R_{\mu\rho}={}&P_{\mu}P^{*}_{\rho}+P^{*}_{\mu}P_{\rho}+\tfrac{1}{6}\kappa^{2}F_{\lambda_1\ \ \lambda_4\mu}F^{\lambda_1\ \ \lambda_4}{}_{\rho}\\ &+\tfrac{1}{8}\kappa^{2}\left(G_{\mu}{}^{\lambda\sigma}G^{*}_{\rho\lambda\sigma}+G^{*\lambda\sigma}_{\mu}G_{\rho\lambda\sigma}-\tfrac{1}{6}g_{\mu\rho}G^{\lambda\sigma\eta}G^{*}_{\lambda\sigma\eta}\right), \end{aligned} \tag{5.3}$$

$$F_{\mu\nu\rho\lambda\sigma}=\tilde{F}_{\mu\nu\rho\lambda\sigma}. \tag{5.4}$$

where

$$F_{\mu\nu\rho\lambda\sigma} = 5\partial_{[\mu}A_{\nu\rho\lambda\sigma]} - \tfrac{5}{4}\kappa\,\mathrm{Im}\left(A_{[\mu\nu}F^{*}_{\rho\lambda\sigma]}\right), \tag{5.5}$$

$$P_{\mu} = f^{2}\partial_{\mu}B, \tag{5.6}$$

$$G_{\mu\nu\rho} = f\left(F_{\mu\nu\rho} - BF^{*}_{\mu\nu\rho}\right), \tag{5.7}$$

$$F_{\mu\nu\rho} = 3\partial_{[\mu}A_{\nu\rho]}, \tag{5.8}$$

$$f = (1 - B^{*}B)^{\;1/2}. \tag{5.9}$$

We also note that $D^{\mu}P_{\mu}$ contains $-2iQ^{\mu}P_{\mu}$ and $D^{\rho}G_{\mu\nu\rho}$ contains $-iQ^{\rho}G_{\mu\nu\rho}$ where

$$Q_{\mu} = f^{2}\,\mathrm{Im}\left(B\partial_{\mu}B^{*}\right). \tag{5.10}$$

These equations can now be studied for solutions giving spontaneous compactification. There is one that is easy to discover (found in discussions with N. Warner). It simply involves taking

$$F_{\mu\nu\rho\lambda\sigma} = c\varepsilon_{\mu\nu\rho\lambda\sigma} \tag{5.11}$$

for five of the dimensions. Then self-duality, eq. (5.4), requires an ε symbol in the other five dimensions as well. In this way one obtains a solution involving a product of a five-dimensional Einstein space with positive Λ and a five-dimensional Einstein space with negative Λ. This is analogous to the 4 + 7 decomposition that has been obtained for $D = 11$ supergravity [13]. It appears that to obtain other, more interesting, solutions requires doing something nontrivial with the other fields as well, if that is possible. Otherwise, it may be necessary to extend the analysis to the full type II superstring theory.

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Appendix

$D = 10$ DIRAC ALGEBRA

There exist ten anticommuting 32×32 matrices satisfying

$$\{\gamma^{r}, \gamma^{s}\} = 2\eta^{rs}. \tag{A.1}$$

where η^{rs} is the Minkowski metric diag$(+ - \cdots -)$. We define

$$\gamma^{r_1 r_2 \ldots r_N} = \gamma^{[r_1}\gamma^{r_2}\ldots\gamma^{r_N]} \tag{A.2}$$

and

$$\gamma_{11} = \gamma^0\gamma^1 \ldots \gamma^9 \tag{A.3}$$

In a Majorana representation γ^0 is antisymmetric and imaginary, γ^1 to γ^9 are symmetric and imaginary, and γ_{11} is symmetric and real. Using

$$\gamma^0\gamma_\mu^\dagger\gamma^0 = \gamma_\mu, \tag{A.4}$$

$$\gamma^0\gamma_\mu\gamma^0 = -\gamma_\mu^{\mathrm{T}}, \tag{A.5}$$

it follows that for arbitrary spinors (not necessarily Weyl or Majorana) χ_1 and χ_2

$$\begin{aligned}(\bar{\chi}_1\gamma^{r_1 \ldots r_N}\chi_2)^\dagger &= \bar{\chi}_2\gamma^{r_N \ldots r_1}\chi_1 \\ &= (-1)^N\bar{\chi}_1^*\gamma^{r_1 \ldots r_N}\chi_2^*.\end{aligned} \tag{A.6}$$

We also note the "Majorana flip" rule used in the last step

$$\bar{\chi}_1\gamma^{r_1 \ldots r_N}\chi_2 = (-1)^{N(N-1)/2}\bar{\chi}_2^*\gamma^{r_1 \ldots r_N}\chi_1^*. \tag{A.7}$$

A Majorana spinor is one which satisfies $\chi = \chi^*$ in the Majorana representation. An arbitrary spinor χ can be decomposed in terms of Majorana spinors χ_1 and χ_2 as follows

$$\chi = \chi_1 + i\chi_2. \tag{A.8}$$

The fact that γ_{11} is real in the Majorana representation demonstrates the fact that the Majorana and Weyl conditions are compatible

When Latin indices are used, the γ matrices are purely numerical (independent of the coordinates) When they are converted to Greek indices with the "zehnbein" e_μ^r, or its inverse e_r^μ, they become field-dependent functions. The tenth-rank antisymmetric ε symbol is defined, with Greek indices, to have $\varepsilon^{01 \ldots 9} = 1$ It is a tensor density A useful relation that follows from these definitions is

$$(\det e)\gamma^{\mu_1 \ldots \mu_N} = -(-1)^{N(N-1)/2}\frac{1}{(10-N)!}\varepsilon^{\mu_1 \ldots \mu_{10}}\gamma_{\mu_{N+1} \ldots \mu_{10}}\gamma_{11} \tag{A.9}$$

As an example of the use of this equation, we note that the Weyl spinor λ ($\gamma_{11}\lambda = \lambda$)

satisfies the antiduality relation

$$(\det e)\,\varepsilon^{\mu_1\ldots\mu_{10}}\bar{\lambda}\gamma_{\mu_6\ldots\mu_{10}}\lambda = -120\bar{\lambda}\gamma^{\mu_1\ldots\mu_5}\lambda\,. \tag{A.10}$$

This expression would vanish by eq. (A.7) if λ were also Majorana, but in our applications it is not.

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