

Supplementary Information: Particles from Rotor Textures and Link Energetics

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S1 Overlap Suppression with a Concrete Texture Basis

Consider the orthonormal Laguerre basis $\Xi_n(x) = L_n(x)$ with weight e^{-x} on $x \in [0, \infty)$. A small metric deformation $w_*^2(x) = e^{-x}(1 - \eta x)$ perturbs the overlaps to

$$I_{nm} = \int_0^\infty \Xi_n^*(x) \Xi_m(x) w_*^2(x) dx = \delta_{nm} - \eta[(2n+1)\delta_{nm} - n\delta_{n,m+1} - (n+1)\delta_{n+1,m}]. \quad (1)$$

For $\eta = 0.15$ the overlap matrix becomes

$$I \approx \begin{pmatrix} 0.85 & 0.15 & 0 \\ 0.15 & 0.55 & 0.30 \\ 0 & 0.30 & 0.25 \end{pmatrix}, \quad (2)$$

showing (i) diagonal suppression for higher textures and (ii) off-diagonals that remain small but nonzero, generating hierarchical masses and controlled mixing when inserted into $M_f \propto I$ up to sector weights.

S2 Toy Mass/Mixing Computation

Input matrices. The matrices used in the main text are

$$M_u \propto \begin{pmatrix} 1 & 0.02 & 0.02 \\ 0.02 & 0.20 & 0.02 \\ 0.02 & 0.02 & 0.04 \end{pmatrix}, \quad M_d \propto \begin{pmatrix} 0.98 & 0.08 & 0.04 \\ 0.08 & 0.98 \times 0.25 & 0.04 \\ 0.04 & 0.04 & 0.98 \times 0.25^2 \end{pmatrix}, \quad (3)$$

$$M_e \propto 0.95 M_u, \quad M_\nu \propto \begin{pmatrix} 0.70 & 0.05 & 0.025 \\ 0.05 & 0.70 \times 0.22 & 0.05 \\ 0.025 & 0.05 & 0.70 \times 0.22^2 \end{pmatrix}. \quad (4)$$

Eigenvalues. Power-iteration diagonalization gives (arbitrary units)

$$m_u \approx (1.001, 0.202, 0.0372), \quad m_d \approx (0.991, 0.243, 0.0523), \quad (5)$$

$$m_e \approx (0.951, 0.192, 0.0354), \quad m_\nu \approx (0.706, 0.166, 0.0157). \quad (6)$$

Mixing. The unitary rotations that diagonalize (M_u, M_d) and (M_e, M_ν) yield

$$|V_{CKM}| \approx \begin{pmatrix} 0.996 & 0.085 & 0.024 \\ 0.087 & 0.994 & 0.071 \\ 0.018 & 0.073 & 0.997 \end{pmatrix}, \quad |V_{PMNS}| \approx \begin{pmatrix} 0.997 & 0.074 & 0.018 \\ 0.076 & 0.973 & 0.220 \\ 0.001 & 0.221 & 0.975 \end{pmatrix}. \quad (7)$$

The pattern matches the qualitative goal: small Cabibbo-like quark mixing and large atmospheric-like leptonic mixing driven by the larger off-diagonals in M_ν .

S3 Pöschl–Teller check for three bound states

For an analytic benchmark, take $w^2(r)$ such that the radial equation reduces to the Pöschl–Teller problem

$$-\psi'' - \frac{2}{r}\psi' + \frac{\ell(\ell+1)}{r^2}\psi - \frac{\lambda(\lambda+1)}{R^2} \operatorname{sech}^2 \frac{r}{R} \psi = -k^2\psi, \quad (8)$$

with effective depth set by $\lambda(\lambda+1)/R^2$. The discrete spectrum is

$$k_{n\ell}^2 = \frac{1}{R^2} \left(\lambda - 1 - n - \ell \right)^2, \quad n = 0, 1, \dots < \lambda - \ell - 1. \quad (9)$$

We identify the particle generations with the distinct *geometric grades* corresponding to angular momentum sectors $\ell = 0$ (Line), $\ell = 1$ (Surface), and $\ell = 2$ (Volume). Radial excitations ($n > 0$) are treated as unstable resonances or heavy partners.

Choosing the depth parameter $\lambda \in (3, 4)$ (e.g., $\lambda = 3.5$) ensures that exactly three angular sectors admit bound states:

- $\ell = 0$ (Grade 1): Ground state exists ($n = 0$).
- $\ell = 1$ (Grade 2): Ground state exists ($n = 0$).
- $\ell = 2$ (Grade 3): Ground state exists ($n = 0$).
- $\ell \geq 3$: No bound states exist ($3 > 3.5 - 1$, violating condition).

This rigorously demonstrates that a smooth soliton core of finite depth naturally truncates the spectrum to exactly three geometric generations, independent of radial excitations.

S4 Mass Formula from Geometric Self-Energy

We quantify the "Moduli Hierarchy" hypothesis by calculating the self-energy of defects with different geometric grades. The mass of a defect is its self-energy $E = \int T_{00} d^3x$.

Grade 1 (Electron): Vector Field Strength. A Line defect (Grade 1) is a source of vector flux ($\nabla\phi$). Its energy density is dominated by the phase stiffness:

$$u_1 \propto E^2 \propto (\nabla\phi)^2. \quad (10)$$

Integrating over the defect volume, the mass scales with the softest modulus:

$$m_e \sim \kappa \int (\nabla\phi)^2 dV \propto \kappa. \quad (11)$$

Grade 2 (Muon): Bivector Field Strength. A Surface defect (Grade 2) involves a twist in the metric or connection, represented by a bivector field strength $B = \nabla \wedge A$. This requires bending the vacuum amplitude profile $w(x)$. The energy density includes a curvature term:

$$u_2 \propto B^2 + \text{Bend}^2 \propto (\nabla A)^2 + \alpha_{\text{grad}} (\nabla w)^2. \quad (12)$$

The mass scales with the intermediate curvature stiffness:

$$m_\mu \sim m_e + \beta_{\text{curv}} \int (\nabla w)^2 dV \propto \kappa + \beta_{\text{curv}}. \quad (13)$$

Grade 3 (Tau): Trivector Field Strength. A Volume defect (Grade 3) forces a compression of the lattice unit cell, represented by a trivector distortion V . This displaces the amplitude w from its vacuum minimum w_* over a bulk region. The energy density is dominated by the potential term:

$$u_3 \propto V^2 \propto (\delta w)^2. \quad (14)$$

The mass scales with the hardest modulus (the bulk modulus/Higgs mass):

$$m_\tau \sim B_{\text{vol}} \int V(w) dV \propto B_{\text{vol}} \gg \beta_{\text{curv}} \gg \kappa. \quad (15)$$

Resulting Hierarchy. The observed lepton mass ratios allow us to calibrate the vacuum moduli ratios:

$$\frac{\text{Curvature Stiffness}}{\text{Phase Stiffness}} \sim \frac{m_\mu}{m_e} \approx 200, \quad \frac{\text{Bulk Modulus}}{\text{Curvature Stiffness}} \sim \frac{m_\tau}{m_\mu} \approx 17. \quad (16)$$

This confirms that the vacuum is "stiff" geometrically (hard to bend) and "very stiff" volumetrically (hard to compress), explaining the generation gaps without fine-tuned Yukawas.

S5 Origin of the Chiral Double-Well from Graph Energetics

We derive the double-well potential for the chiral order parameter θ starting from the microscopic graph alignment energy. We show that the effective potential coefficient $a(\rho_N)$ changes sign as the noise field/stiffness evolves, driving a spontaneous symmetry breaking.

Microscopic Order Parameter. Let each coarse grain i contain a triad of internal gradient modes $R_i \in \text{Frame}(\mathbb{C}^3)$. We define the local chiral order parameter (triad handedness) as the Ising-like variable:

$$\chi_i = \text{sgn}(\det R_i) \in \{+1, -1\}. \quad (17)$$

This is a parity-odd scalar. A state with $\langle \chi \rangle \neq 0$ breaks mirror symmetry.

Graph Hamiltonian. The total free energy of the graph patch is a sum of alignment energies between linked grains and local entropic costs. For the reduced variable χ , this takes the form of an effective Ising model:

$$\mathcal{H}[\chi] = - \sum_{\langle ij \rangle} J(\rho_N) \chi_i \chi_j - \sum_i h_i \chi_i, \quad (18)$$

where $J(\rho_N) > 0$ is the effective ferromagnetic coupling. It arises because smooth connections (low curvature) are energetically cheaper than connections with a handedness flip (domain wall). A mismatch $\chi_i \neq \chi_j$ implies a twist in the connection across the link, incurring an energy cost proportional to the rotor stiffness. Thus, $J(\rho_N) \propto \kappa(\rho_N) w_*^2(\rho_N)$.

Mean-Field Free Energy. In the mean-field approximation, we replace the neighbor interaction with an average field $\theta = \langle \chi \rangle$. The effective free energy per grain $f(\theta)$ balances the alignment energy against the entropy of the binary choice:

$$f(\theta) \approx \underbrace{-\frac{z}{2} J(\rho_N) \theta^2}_{\text{Alignment Energy}} + \underbrace{T_{\text{eff}}(\rho_N) \int_0^\theta \tanh^{-1}(x) dx}_{\text{Configurational Entropy}}. \quad (19)$$

Here z is the coordination number and T_{eff} is the effective temperature (noise level). Expanding for small θ :

$$f(\theta) \approx \frac{1}{2} (T_{\text{eff}} - z J(\rho_N)) \theta^2 + \frac{T_{\text{eff}}}{12} \theta^4 + \mathcal{O}(\theta^6). \quad (20)$$

Spontaneous Symmetry Breaking. The quadratic coefficient is $a(\rho_N) = T_{\text{eff}}(\rho_N) - zJ(\rho_N)$.

- **High Noise (Early Universe):** ρ_N is large, so T_{eff} is high and the stiffness J is low (or washed out). $a(\rho_N) > 0$. The potential is convex with a single minimum at $\theta = 0$. The vacuum is parity-symmetric on average.
- **Low Noise (Cooling):** As the universe evolves and amplitude cores stabilize, the noise T_{eff} drops and the link stiffness J increases. At a critical density ρ_c , $a(\rho_N)$ crosses zero.
- **Broken Phase:** For $a(\rho_N) < 0$, the potential develops a double well with minima at $\theta = \pm\theta_0$. The system must choose a handedness.

Domain Formation. Random fluctuations nucleate local domains of + and -. As the barrier height grows (cooling), domain walls become energetically costly. The system undergoes a coarsening process (Ising model kinetics) where domains merge. In our observable patch, a single domain (e.g., $\theta \approx +\theta_0$) wins out, establishing the "Global Chiral Bias" formalized as a Lorentz-safe pseudoscalar condensate.