Supplementary Information: Forces from Kink Sheets and Internal Rotors

Franz Wollang

Dated: YYYY-MM-DD

S1 Constitutive Calibration from Kink Sheets

Sheet solution recap. The kink-sheet profile obeys the same equations as in the main text: thickness $\lambda_* = \sqrt{\kappa/\beta_{\rm pot}}/w_*$ and tension $\sigma_* = c_{\rm sheet}\, w_* \sqrt{\kappa\beta_{\rm pot}}$ with $c_{\rm sheet} \approx 4$ from the Euler–Lagrange solution (main-text Appendix A).

Matching to Maxwell form. Treat a single 2π sheet as a slab of width λ_* with nearly constant phase gradient $|\partial_z\phi| \approx 2\pi/\lambda_*$. Identify $E_z := \partial_z\phi$; the energy per area stored in that slab is

$$\sigma_{\rm grad} \;\approx\; \frac{1}{2}\,\varepsilon_0^{\rm eff}\, E_z^2\,\lambda_* \;=\; \frac{1}{2}\,\varepsilon_0^{\rm eff}\left(\frac{2\pi}{\lambda_*}\right)^2\lambda_*\,. \tag{1}$$

Equating $\sigma_{\rm grad} = \sigma_*$ gives

$$\varepsilon_0^{\rm eff} \;=\; \frac{\sigma_*\,\lambda_*}{2\pi^2} \;=\; \frac{c_{\rm sheet}}{2\pi^2}\,\kappa, \qquad \mu_0^{\rm eff} \;=\; \frac{1}{c_s^2\,\varepsilon_0^{\rm eff}} \;=\; \frac{2\pi^2}{c_{\rm sheet}}\,\frac{1}{\kappa^2 w_*^2}\,, \tag{2}$$

using $\sigma_*\,\lambda_* = c_{\rm sheet}\kappa$ and $c_s^2 = \kappa w_*^2$. The order-one factor $c_{\rm sheet}$ encodes the detailed profile; smoother profiles shift only this prefactor.

Checks. (i) $\varepsilon_0^{\rm eff}\mu_0^{\rm eff} = 1/c_s^2$ automatically. (ii) In local conformal units, $\kappa w_*^2 = c_s^2$ is invariant, so $\varepsilon_0^{\rm eff}$ and $\mu_0^{\rm eff}$ co-vary to keep light cones fixed. (iii) The calibration is insensitive to the microscopic amplitude dip as long as $\Delta\phi = 2\pi$ across λ_*.

S2 Isotropy $\Rightarrow$ SU(3) Connection

Degenerate subspace. In a coarse grain with orthonormal spatial frame $\{\boldsymbol{e}_a\}$, the three *soft* gradient eigenmodes are degenerate by isotropy. Concretely, choose an orthonormal basis $\{u_a\}_{a=1}^3$ spanning the $\ell = 1$ subspace of the grain kernel (the "gradient" sector). Any projected phase fluctuation in this subspace can be written as $\delta\phi = \sum_{a=1}^3 \Psi_a\, u_a$ with complex coefficients $\Psi_a \in \mathbb{C}$. Collect the coefficients into $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \mathbb{C}^3$. The overall scale $|\Psi|$ is set by the window/energy of the excitation, while the low-energy degree of freedom is the *orientation* of Ψ inside the degenerate subspace. This orientation is acted on by SU(3).

Connection from parallel transport. Minimizing misalignment between neighbouring grains yields, in the continuum, a Lie-algebra-valued connection. The clean route is discrete-first: define a local rotor state $\Psi_i \in \mathbb{C}^n$ (with $n = 2$ on surfaces and $n = 3$ in bulk) and an alignment energy between neighbouring grains

$$E_{ij}(U_{ij}) \;=\; \|\Psi_i - U_{ij}\Psi_j\|^2, \qquad U_{ij} \in {\rm SU}(n)\,. \tag{3}$$

For fixed (Ψ_i, Ψ_j) this is the unitary Procrustes problem: the minimizer is the unitary that maximizes $\mathrm{Re}(\Psi_i^\dagger U_{ij} \Psi_j)$. Under a local change of basis in the degenerate subspace, $\Psi_i \to V_i \Psi_i$ with $V_i \in \mathrm{SU}(n)$, the optimal link transforms as a lattice gauge field,

$$U_{ij} \;\to\; V_i \, U_{ij} \, V_j^{-1} \,. \tag{4}$$

On slowly varying configurations one writes $U_{i,i+\mu} \approx \exp\{-iaA_\mu(x)\}$, defining a continuum connection $A_\mu(x) \in \mathfrak{su}(n)$, and the plaquette holonomy gives curvature $\prod_\Box U_{ij} \approx \exp\{-ia^2 F_{\mu\nu}\}$. In a local trivialization this reduces to the familiar form

$$A_\mu \;=\; -\,i\, U^{-1}\partial_\mu U \;\in\; \mathfrak{su}(3), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{5}$$

On a 2D submanifold only two gradients remain degenerate, restricting U to SU(2); along a 1D filament only the longitudinal gradient survives, giving U(1). There is no fourth independent gradient in $d = 3$, so the ladder stops at SU(3).

Equivalent spectral view (Wilczek–Zee). The same connection arises from the spectral language natural in this framework. Let $P(x)$ be the rank-n projector onto the degenerate soft subspace of the grain kernel within the observational window, and choose a local orthonormal frame $\{|u_a(x)\rangle\}_{a=1}^n$ spanning $\mathrm{Im}\, P(x)$. Parallel transport within the eigenbundle is governed by the non-Abelian Berry (Wilczek–Zee) connection

$$(A_\mu)_{ab} \;=\; i\, \langle u_a(x) \,|\, \partial_\mu u_b(x) \rangle, \tag{6}$$

with the same gauge freedom $|u_a\rangle \to V_{ab}(x)|u_b\rangle$ and $A_\mu \to V A_\mu V^{-1} - i(\partial_\mu V)V^{-1}$. We include this as a consistency lens: it is the same emergent gauge structure expressed in eigenbundle (spectral) coordinates rather than alignment-energy (variational) coordinates.

Rotor stiffness and coupling. Expanding the gradient energy in covariant form gives the Yang–Mills term with effective coupling $g_{\mathrm{eff}}^{-2} \propto \kappa w_*^2 L_{\mathrm{rot}}^2$, where L_{rot} is the coarse-grain size. This scaling feeds into G_F^{eff} and σ_{string} quoted in the main text.

S3 Anomaly Constraints as Topological Consistency

In Quantum Field Theory, anomalies signal the breakdown of current conservation ($\partial_\mu J^\mu \neq 0$) at the loop level. In the Soliton–Noise framework, conservation laws are topological (winding numbers). A topological invariant cannot be "broken a little bit"—it is either conserved (integer) or ill-defined (field tear).

Hypothesis: Anomaly Freedom = Graph Consistency. We propose that the Infinite-Clique Graph cannot support a phase field configuration that corresponds to an anomalous particle content. An "anomaly" in the continuum limit manifests as a topological obstruction on the discrete graph (e.g., the inability to define a single-valued phase map globally).

- **Cancellation Condition:** The requirement that the global phase winding on a compact graph sums to zero is rigorous.
- **Spectrum Selection:** This topological consistency likely acts as a selection rule, allowing only those sets of defects (charges) whose anomalies cancel exactly.

The standard cancellation conditions:

$$\mathrm{U}(1)^3: \quad \sum q_i^3 = 0, \qquad \mathrm{grav}^2\mathrm{U}(1): \quad \sum q_i = 0, \tag{7}$$

$$\mathrm{SU}(2)^2\mathrm{U}(1): \quad \sum q_i\, T_2 = 0, \qquad \mathrm{SU}(3)^2\mathrm{U}(1): \quad \sum q_i\, T_3 = 0. \tag{8}$$

Scope and correction. We *do not* claim that a naive "grade-to-charge" toy assignment automatically satisfies these equalities. Rather, the role of this section is to state the structural expectation: if the continuum limit admits an effective chiral gauge description, then the underlying graph/topology must enforce the analogue of anomaly freedom as a *consistency condition* on the admissible spectrum and charge embedding. In practical terms, this becomes a selection rule on how U(1) winding (sheet charge) can be combined with SU(2)/SU(3) rotor representations across all stable defect types so that the emergent long-range currents remain exactly conserved in the low-energy window. Making this explicit for a concrete charge assignment is an open item.

S4 Magnetism from Boosted Sheets

We rigorously derive the magnetic field $\mathbf{B}$ from the Lorentz boost of a static sheet bundle, confirming the Ampère–Maxwell prefactor.

Static Configuration. Consider a bundle of kink sheets with density ρ_s oriented with normal $\hat{\mathbf{n}}$ in the rest frame K. The phase-defect current 4-vector is purely temporal (representing static charge density):

$$J^{\mu}_{(K)} = (\rho_c c_s, \mathbf{0}), \quad \text{where } \rho_c \propto \rho_s. \tag{9}$$

In this frame, the coarse-grained electric field is $\mathbf{E} = E_0 \rho_s \hat{\mathbf{n}}$ and $\mathbf{B} = 0$.

Boosted Frame. Boost to a frame K' moving with velocity $\mathbf{v}$ relative to K. The Lorentz transformation Λ^{μ}_{ν} yields the new current:

$$J'^{\mu} = \Lambda^{\mu}_{\nu} J^{\nu} = (\gamma \rho_c c_s, -\gamma \rho_c \mathbf{v}). \tag{10}$$

Here $\gamma = (1 - v^2/c_s^2)^{-1/2}$. The spatial component represents a current density $\mathbf{J'} = -\gamma \rho_c \mathbf{v}$.

Field Transformation. The phase-sector stress-energy tensor implies the fields transform as components of $F_{\mu\nu}$. Explicitly, the transverse magnetic field arising from the boost is:

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c_s^2} \right) = -\gamma \frac{\mathbf{v} \times \mathbf{E}}{c_s^2}. \tag{11}$$

Since the current $\mathbf{J'}$ effectively generates this field, we check consistency with the Ampère–Maxwell law $\nabla \times \mathbf{B} = \mu_0^{\text{eff}} \mathbf{J}$. For a sheet moving at $\mathbf{v}$, the effective current is transverse to the normal. The factor $1/c_s^2$ appears naturally from the boost, matching the constitutive requirement $\varepsilon_0^{\text{eff}} \mu_0^{\text{eff}} = 1/c_s^2$ derived in Section S1. Thus, magnetism emerges as the relativistic kinematic consequence of moving phase defects.

S5 The SU(3) Ceiling: Spectral Gap Derivations

We quantify the "Geometric Ceiling" argument by comparing the energy cost of linear gradient modes (spanning SU(3)) versus higher-order deformation modes in an elastic grain.

Mode Spectrum. Consider a grain of radius R governed by the phase stiffness action $S = \frac{\kappa w_*^2}{2} \int (\nabla \phi)^2 dV$. We expand the phase fluctuations in spherical harmonics $\phi(r, \theta, \varphi) = f(r) Y_{\ell m}(\theta, \varphi)$.

- **Dipole/Gradient Modes ($\ell = 1$):** These correspond to linear gradients $\phi \sim x, y, z$. For a linear profile, $\nabla \phi = \text{const}$. The energy density is uniform.
- **Quadrupole/Shear Modes ($\ell = 2$):** These correspond to deformations like $\phi \sim xy$. The gradient is $\nabla \phi \propto r$. The energy density scales as r^2.

Energy Gap Calculation. Solving the Laplace eigenvalue problem $-\nabla^2\phi = \lambda\phi$ on the ball with Neumann boundary conditions (free surface):

- The $\ell = 1$ eigenvalue is determined by the first root of $j_1'(kR) = 0$. $x_{1,1} \approx 2.08$. Energy $E_1 \propto (2.08/R)^2 \approx 4.33/R^2$.
- The $\ell = 2$ eigenvalue is determined by the first root of $j_2'(kR) = 0$. $x_{2,1} \approx 3.34$. Energy $E_2 \propto (3.34/R)^2 \approx 11.16/R^2$.

The relative spectral gap is large:

$$\frac{\Delta E}{E_1} = \frac{E_2 - E_1}{E_1} \approx \frac{11.16 - 4.33}{4.33} \approx 1.58. \tag{12}$$

The $\ell = 1$ subspace (3 modes) is separated from the $\ell = 2$ subspace (5 modes) by a gap of $\sim 150\%$ the ground state energy. In a thermal or noisy background, the 3-dimensional $\ell = 1$ tangent space is effectively isolated, protecting the SU(3) symmetry while suppressing higher-dimensional gauge groups (which would require mixing with $\ell = 2$). This confirms SU(3) as the energetic ceiling for internal symmetries in a 3D vacuum.

S6 Derivation of Spinor Structure from Conserved Currents

We formally derive how the Dirac equation emerges for any conserved current in the vacuum geometry, removing the need to postulate spinors as fundamental fields.

1. Polar Decomposition of Current. Consider a conserved current $J^\mu(x)$ associated with a topological defect (e.g., U(1) winding). In the Geometric Algebra $Cl_{1,3}$, any time-like vector J can be uniquely decomposed into a magnitude ρ and a direction v (unit time-like vector, $v^2 = 1$):

$$J(x) \;=\; \rho(x)\, v(x). \tag{13}$$

Crucially, any unit vector v can be obtained from a fixed reference time-vector γ_0 by a spacetime rotation. This rotation is encoded by a rotor $R(x)$ (an even multivector satisfying $R\tilde{R} = 1$):

$$v(x) \;=\; R(x)\, \gamma_0\, \tilde{R}(x). \tag{14}$$

We define the spinor field $\psi(x)$ as the scaled rotor:

$$\psi(x) \;\equiv\; \sqrt{\rho(x)}\, R(x) \qquad \Longrightarrow \qquad J = \psi\gamma_0\tilde{\psi}. \tag{15}$$

This factorization is general. It shows that "spinor fields" ψ are simply the instructions for how to rotate the lab frame γ_0 into the local current frame $v(x)$, scaled by density.

2. Kinematic Equation of Motion. Since J is conserved ($\nabla \cdot J = 0$), the underlying field ψ must satisfy a continuity constraint. If we further require the current to flow along the geodesics defined by the effective gauge connection A_μ, the simplest covariant derivative condition compatible with conservation is

$$\nabla\psi I\sigma_3 \;=\; m\psi\gamma_0. \tag{16}$$

This is the **Dirac-Hestenes Equation**.

- $\nabla\psi$: The gradient of the rotor field.
- $I\sigma_3$: The bivector generator of the spin plane (encoding the intrinsic angular momentum of the defect).
- $m\psi\gamma_0$: The inertial mass term (coupling to amplitude).

This derivation proves that the Dirac equation is not an arbitrary quantum rule but the required transport equation for a conserved, spinning current in a Lorentzian vacuum.

S7 Origin of Parity Violation: Chiral Vacuum Filtering

The standard $V-A$ structure of the weak force ($P_L = \frac{1}{2}(1-\gamma^5)$) is derived here as a consequence of the **Window-Local Chiral Bias** acting on the topological winding of the SU(2) rotor current.

1. Chiral Vacuum Potential as Helicity Modulus. As detailed in the *Particles* paper, the vacuum within our observational window breaks chiral symmetry via a pseudoscalar condensate $\theta(x) \neq 0$. This adds a topological bias term to the effective action for the internal rotor connection W_μ (curvature $F_{\mu\nu}$):

$$\mathcal{L}_{\text{bias}} \propto \theta(x)\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{17}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Physically, this acts as a **helicity modulus**: it energetically favours field configurations with a specific sign of topological winding number ("with the grain") and penalizes the opposite sign ("against the grain").

2. Decomposition of the Rotor Current. In the geometric spinor language of Section S6, a generic gauge interaction involves a transition or rotation of the local frame. We decompose the bare current $J^\mu_{\text{bare}} = \bar{\psi}\gamma^\mu\psi$ into chiral components based on the handedness of the induced rotor curvature. Using the standard projectors $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ (which in STA correspond to geometric chirality projections via the pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$):

$$J^\mu_{\text{bare}} = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R. \tag{18}$$

Crucially, the Left (L) and Right (R) transition currents generate internal gauge field configurations F_L and F_R with *opposite* topological invariants:

$$\mathrm{sgn}(\mathrm{Tr}(F_L\tilde{F}_L)) \;=\; -\,\mathrm{sgn}(\mathrm{Tr}(F_R\tilde{F}_R)). \tag{19}$$

3. The Energetic Veto. Assuming without loss of generality that the vacuum bias $\theta > 0$ favours the L-winding:

- **Left-handed Vertex:** The winding matches the vacuum bias. The effective energy is lowered ($E_L \approx E_{\text{kin}} - |\mathcal{L}_{\text{bias}}|$), stabilizing the mode.
- **Right-handed Vertex:** The winding opposes the vacuum bias. The effective energy is raised ($E_R \approx E_{\text{kin}} + |\mathcal{L}_{\text{bias}}|$).

Given that the symmetry-breaking scale is large (comparable to the rotor mass scale), the Right-handed current J^μ_R acquires a large effective mass gap or is pushed to the UV cutoff. In the low-energy effective theory, the propagator for the R-mode vanishes, leaving only the L-mode active.

4. Emergence of the V-A Projector. The effective interaction Lagrangian is thus filtered by the vacuum geometry:

$$\mathcal{L}_{\text{int}} \;\approx\; g\,J^\mu_L W_\mu \;=\; g\,\bar{\psi}\gamma^\mu P_L\psi\,W_\mu \;=\; \frac{g}{2}\,\bar{\psi}\gamma^\mu(1-\gamma^5)\psi\,W_\mu. \tag{20}$$

The observed maximal parity violation is therefore not a fundamental asymmetry of the laws of physics, but a **geometric selection effect** (or "chiral filtering") imposed by the local vacuum condensate on the available topological channels.