Supplementary Information
A Sea of Noise: Relativity from a Thermodynamic Force in Scale-Space (*Draft*)

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This Supplementary Information accompanies the manuscript "A Sea of Noise: Relativity from a Thermodynamic Force in Scale-Space" and collects technical derivations and interpretive material referenced in the main text.

Contents

S1 Observation Windows and Local Units

Continuum variables in this framework are defined *relative to an observation window.* By a window we mean IR/UV cutoffs $k_{\min}(t)$ and $k_{\max}(t)$ applied to the ambient spectrum $S(k)$ of fluctuations. The windowed background measure is

$$\rho_{N,\mathrm{eff}}(t) = \int_{k_{\min}}^{k_{\max}} S(k)\, dk, \qquad \frac{d\rho_{N,\mathrm{eff}}}{dt} = S(k_{\max})\, \dot{k}_{\max} - S(k_{\min})\, \dot{k}_{\min}. \tag{1}$$

On bounded windows the induced operators are compact and have discrete spectra. This justifies: (i) the coarse grained complex field, (ii) stable soliton clocks (dominant internal mode), and (iii) the approximate coarse grain invariant $\hbar_{\mathrm{eff}} \simeq E_{\mathrm{cell}}\, \tau_{\mathrm{cell}}$ used throughout.

Operational cutoffs and their origins

We choose cutoffs that are physically motivated and close under slow drift:

$$k_{\min}(t) = \max\Big(\tfrac{2\pi}{L_{\mathrm{obs}}},\, \xi_{\mathrm{IR}}\, \tfrac{H_{\mathrm{eff}}(t)}{c_s}\Big), \qquad k_{\max}(t) = \min\Big(\tfrac{\pi}{R_{\mathrm{obs}}},\, \xi_{\mathrm{UV}}\, \tfrac{1}{\ell_{\mathrm{gap}}},\, \tfrac{2\pi}{c_s T_{\mathrm{obs}}}\Big). \tag{2}$$

Here $L_{\mathrm{obs}}, R_{\mathrm{obs}}, T_{\mathrm{obs}}$ are the spatial/temporal window scales, $\ell_{\mathrm{gap}} = \sqrt{\alpha_{\mathrm{grad}}/(2\beta_{\mathrm{pot}})}$ is the amplitude screening length, and $\xi_{\mathrm{IR}}, \xi_{\mathrm{UV}} = \mathcal{O}(1)$. The IR term H_{eff}/c_s reflects the Lorentzian phase cone and the finite causal reach of slowly evolving backgrounds; the UV terms are set by (i) the amplitude gap (no long range amplitude modes beyond $1/\ell_{\mathrm{gap}}$) and (ii) finite temporal resolution through the cone ($\omega_{\max} \sim 2\pi/T_{\mathrm{obs}} \Rightarrow k_{\max} \sim \omega_{\max}/c_s$).

Self consistent closure with H_{eff} and Λ_{eff}

Given $(k_{\min}, k_{\max})$, define the homogeneous offset captured by the window as

$$\Lambda_{\mathrm{eff}}(t) \propto \int_{k_{\min}(t)}^{k_{\max}(t)} S(k)\, dk. \tag{3}$$

In unimodular/low curvature regimes the background rate obeys $H_{\mathrm{eff}}^2 \propto \Lambda_{\mathrm{eff}}$. This yields an adiabatic fixed point map: start from $H_{\mathrm{eff}}^{(n)}$, compute $k_{\min}^{(n)}$, evaluate $\Lambda_{\mathrm{eff}}^{(n+1)}$ via the integral, and set $H_{\mathrm{eff}}^{(n+1)}$ from the proportionality. For slowly drifting windows the sequence converges and leaves *dimensionless* local predictions unchanged; only the homogeneous offset tracks the window. In this sense the apparent recurrence between the IR cutoff and Λ_{eff} is resolved by adiabatic fixed point closure.

Consequences in local units

- **Coordinate vs local speeds:** Index variations $\chi(\rho_N)$ change coordinate speeds (c_s/χ), while locally measured light speed remains c_s via co scaling of rulers and clocks.
- **Calibration of G:** The Newtonian constant is fixed operationally in the Coulombic window; sliding the window drifts dimensionful constants but not the $1/r^2$ law or dimensionless observables.
- **Homogeneous background (Λ_{eff}):** The windowed background sets a homogeneous offset that behaves as an effective cosmological term in unimodular form; bound systems are insensitive to this offset at leading order.

Acceleration induced window shifts (pointer)

Proper acceleration shifts the comoving window's effective UV edge. It is convenient to summarise this as a small Unruh like contribution with $T_U(a) = \hbar a/(2\pi c_s k_B)$ (Unruh, 1976; see also Birrell & Davies) entering the effective noise scale used for drag. Detailed consequences for non geodesic motion (the drag law and equivalence considerations) are developed in Section S2; they are not needed for the window justification itself.

S2 Drag Scaling and Justification

We quantify an acceleration-dependent drag that acts only for non-geodesic (forced) motion and vanishes on geodesics; the Lorentzian light cone already makes c_s an unattainable speed limit for massive excitations. Let $\mathbf{a}_0$ denote the apparent acceleration in the lab frame (e.g., from a background gradient), and $\hat{v}$ the unit velocity direction. The proper acceleration experienced in the comoving frame is

$$a_{\text{eff}} = \gamma_v^3 \,(\mathbf{a}_0 \cdot \hat{v}), \qquad \gamma_v = \frac{1}{\sqrt{1 - v^2/c_s^2}}. \tag{4}$$

The rate of interaction with phase-space cells is proportional to a covariant scalar built from the 4-acceleration. A minimal, frame-consistent baseline uses the proper acceleration scale in the comoving frame (with $\omega_d \propto a_s$ for a driven bound mode) and lab-frame beaming/normalisation weights to yield

$$P_{\text{rad}} \,\propto\, \gamma_v^4 \, a_s^2, \tag{5}$$

up to an overall coefficient fixed by microphysics (see Section S2.1). Balancing dissipated power with mechanical power input $P_{\text{work}} = F \cdot v$ must be carried out in a single frame (or covariantly). A nonrelativistic balance $P_{\text{work}} = mav$ mixes frames and miscounts γ_v factors. We defer the proper balance to the covariant subsection below, which yields a general relation $a(v) \propto \gamma_v^{3-p}$ when $P_{\text{rad}} \propto \gamma_v^p a^2$ in the lab frame (colinear case).

Equivalence of accelerations; isotropy vs anisotropy. In this framework, "gravitational" acceleration (from a noise/index gradient) and "mechanical" acceleration (from an external force) are equivalent at the level of the comoving-frame kinematics that enter the drag law: both produce a proper acceleration a_{eff} and hence a window shift and an effective temperature $T_{\text{eff}}(a)$. However, the induced *downshifting of soliton scale* can differ in its transient spatial pattern.

For a quasi-static noise gradient (gravitational case), $\tau(x)$ is a scalar that increases toward the source, so the equilibrium scale $\sigma^*(x) \propto 1/\tau(x)$ contracts *isotropically* to leading order. By contrast, for mechanical acceleration, the comoving-frame window shifts due to Doppler/Rindler effects preferentially along the acceleration axis. In the instantaneous comoving frame the Unruh bath is thermal and nearly isotropic for pointlike detectors, but a finite-size soliton samples mode populations and gradients differently along and transverse to $\hat{a}$, producing a small *anisotropic* excitation of internal modes.

S2.1 Covariant derivation and frame consistency

We present a covariant, frame-consistent sketch and then connect to phenomenology. Let $U^\mu = \gamma_v(1, \mathbf{v}/c_s)$ be the soliton 4-velocity and $A^\mu = dU^\mu/d\tau$ its 4-acceleration (proper time τ), with $A \cdot U = 0$ and magnitude $a \equiv \sqrt{-A^\mu A_\mu}$. The external 4-force F^μ obeys $F^\mu U_\mu = 0$ and induces $A^\mu = F^\mu/m$. Energy change is $dE/dt = F^\mu U_\mu \,\gamma_v c_s^2 = \mathbf{F} \cdot \mathbf{v}$ in the lab.

Radiation must be built from scalars formed with U^μ and A^μ. For colinear motion (acceleration parallel to velocity), the only scale is a and the lab power picks up kinematic weights from time dilation and flux: a minimal ansatz consistent with dimensional analysis and colinearity is

$$P_{\text{rad}}(v, a) \,=\, C_{\text{drag}} \,\gamma_v^p \, a^2, \tag{6}$$

with p determined by the operator content (e.g., gradient vs higher multipoles). The a^2 dependence follows from $(A{\cdot}A)$ as the lowest scalar; any odd power would violate $A^\mu \to -A^\mu$ symmetry at fixed U^μ. The coefficient $C_{\rm drag}$ carries the units required for power; in local units, microscopic scales (e.g., c_s and stiffnesses) can be absorbed into $C_{\rm drag}$.

Mechanical power in the lab is $P_{\rm work} = \mathbf{F}\cdot\mathbf{v} = (\gamma_v^3 ma)v$ for longitudinal acceleration. Steady drive imposes $P_{\rm work} = P_{\rm rad}$, hence

$$\gamma_v^3 ma\, v \ \propto\ C_{\rm drag}\, \gamma_v^p\, a^2 \quad \Rightarrow \quad a(v)\ \propto\ \gamma_v^{3-p}\, v\,. \tag{7}$$

Thus ultra relativistic suppression ($a \to 0$ as $\gamma_v \to \infty$) requires $p > 3$. Any concrete micro derivation that yields $p > 3$ is therefore compatible with the qualitative conclusion that non geodesic drag further suppresses approach to c_s.

Connecting to Kubo/EFT: linear response with a super Ohmic phase bath and worldline multipole couplings both suggest that the dominant comoving frequency scales as $\omega_d \propto a$ (first nontrivial scalar) and that lab fluxes acquire positive γ_v weights from contraction and current normalization, while time dilation suppresses rates. The net exponent p is operator dependent and must be computed covariantly; phenomenology (storage rings) then bounds C.

Fixing p under leading assumptions (baseline choice). For the present framework the leading coupling of a composite soliton to the massless phase is gradient type (first derivative). In worldline EFT, this corresponds to the lowest symmetry allowed operator linear in $\partial\phi$ (scalar dipole analogue). For colinear acceleration, boosting the comoving emission pattern to the lab introduces two powers of γ_v from beaming/solid angle compression and two from field/current normalisation relative to proper time, while a single γ_v^{-1} enters from time dilation in $P = dE/dt$. The net minimal weight is therefore γ_v^4, which we take as the *baseline* exponent for gradient couplings (cf. Unruh 1976; worldline EFT discussions by Goldberger & Rothstein, and Porto):

$$p_{\rm baseline}\ =\ 4\,. \tag{8}$$

Higher multipoles and additional tensor structure can only increase p (e.g., the electromagnetic Liénard case yields $p = 6$ for purely longitudinal acceleration). With $p = 4$, Eq. (6) gives

$$a(v)\ \propto\ \gamma_v^{-1}\, v\,, \tag{9}$$

which exhibits ultra relativistic suppression (and is a conservative lower bound on suppression strength; $p \geq 4$). We summarise complementary routes that support Eq. (6) with the derived exponent $p = 4$:

$$P_{\rm rad}(v,a)\ =\ C_{\rm drag}\, \gamma_v^4\, a^2, \qquad \omega_d\ \propto\ a\,\gamma_v^{3/2}, \tag{10}$$

where $C_{\rm drag}$ is an overall coefficient to be calibrated empirically (carrying the required units in local unit conventions), and we delineate their regime of validity.

(i) Linear response (Kubo) with Unruh bath. Let a dominant internal coordinate q couple to a bath operator B sourced by proper acceleration ($T_U = \hbar a/(2\pi c_s k_B)$; Unruh 1976; Birrell & Davies): $H_{\rm int} = -q\,B$. The dissipated power in the comoving frame is

$$P\ \simeq\ \int_0^\infty d\omega\ \omega\ {\rm Im}\,\chi_{qq}(\omega)\ S_{BB}(\omega; T_U), \tag{11}$$

with χ_{qq} the susceptibility and S_{BB} the bath spectrum obeying FDT. For the phase sector, gradient type couplings give super Ohmic spectra. Boosting to the lab multiplies the comoving result by appropriate γ_v factors from time dilation and flux; the net power matches the structure in Eq. (6) with $p = 4$ for the baseline gradient coupling.

(ii) Worldline EFT for a composite soliton. Model the soliton as a worldline with dynamical multipoles $Q_i, Q_{ij}, \ldots$ coupled to the phase field via the lowest symmetry allowed operators. The imaginary part of the dressed propagator (optical theorem) sets the radiation power $\propto (A \cdot A)$ at leading order; boosting to the lab produces Eq. (6) with $p = 4$ for the baseline gradient coupling (first-derivative operators). See, e.g., discussions of scalar radiation in worldline EFT frameworks (Goldberger & Rothstein; Porto).

(iii) Scaling in local units. In the comoving frame, the available scales are the proper acceleration a (T^{-2}) and the signal speed c_s (LT^{-1}), with the stabiliser ensuring small deformations. For a driven bound mode, the dominant frequency scales as the acceleration itself,

$$\omega_d \;\propto\; \frac{a}{c_s} \;\propto\; a, \tag{12}$$

with the $\gamma_v^{3/2}$ factor supplied by relativistic kinematics as above.

S3 Induced Gravitational Action from Microdynamics (Sketch)

This section sketches how the gravitational action arises by integrating out the phase sector defined by the microdynamics, connecting the coarse grained free energy to an induced Einstein Hilbert term plus higher curvature corrections.

Setup and assumptions

We work on near regular subgraphs where the discrete phase kernel (graph Laplacian) admits a continuum description. The coarse grained phase Lagrangian at fixed amplitude is

$$\mathcal{L}_\phi \;=\; \frac{\kappa w_*^2}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi, \qquad c_s^2 = \kappa w_*^2. \tag{13}$$

The metric $g_{\mu\nu}$ encodes the slowly varying geometry induced by vacuum relaxation of links (continuum mapping in Appendix A of the main text). Amplitude fluctuations are gapped with screening length ℓ, so at energies $E \ll \ell^{-1}$ they contribute only finite renormalisations of coefficients and can be integrated out once.

Path integral and trace log

Define the phase sector partition function on a fixed background g. The action scale is set by $\hbar_{\text{eff}}$ (the coarse grain invariant action per cycle in local units), which is the natural quantum of action for the effective low energy theory:

$$Z[g] \;=\; \int \mathcal{D}\phi\, \exp\!\left\{ -\frac{1}{\hbar_{\text{eff}}} \int d^4x\, \sqrt{|g|}\, \frac{\kappa w_*^2}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi \right\}. \tag{14}$$

Gaussian integration yields the effective action (up to an additive constant)

$$\Gamma[g] \;=\; -\hbar_{\text{eff}} \ln Z[g] \;=\; \frac{\hbar_{\text{eff}}}{2}\, \mathrm{Tr}\ln(-\Box_g)\,, \qquad \Box_g := \nabla_\mu \nabla^\mu. \tag{15}$$

Using the heat kernel representation $\mathrm{Tr}\ln(-\Box_g) = -\int_\varepsilon^\infty \frac{ds}{s}\, \mathrm{Tr}\, e^{-s(-\Box_g)}$ and the Seeley DeWitt expansion $\mathrm{Tr}\, e^{-s(-\Box_g)} = \frac{1}{(4\pi s)^2} \sum_{n\ge 0} a_n s^n$, one obtains the local curvature expansion

$$\Gamma[g] \;=\; \int d^4x\, \sqrt{|g|}\, \Big[C_0\, \Lambda_{\text{eff}}^4 \;+\; C_1\, \Lambda_{\text{eff}}^2\, R \;+\; C_2\, \ln(\Lambda_{\text{eff}}^2/\mu^2)\, \mathcal{R}^2 \;+\; \cdots \Big], \tag{16}$$

where $\mathcal{R}^2$ stands for curvature squared combinations (e.g., R^2, $R_{\mu\nu}R^{\mu\nu}$), and Λ_{eff} is an effective UV scale discussed next.

Window/cutoff and coefficient identification

In this framework the UV scale is set operationally by the observation window and the phase spectrum, $\Lambda_{\text{eff}} \sim \sqrt{\kappa}\, w_*\, k_{\max}$, with $k_{\max}$ determined by the UV edge of the sliding window (Section S1) or by the amplitude gap when more restrictive. Matching the R term to the Einstein Hilbert form gives

$$\frac{1}{16\pi G} \;=\; C_1\, \Lambda_{\text{eff}}^2. \tag{17}$$

For a standard minimally coupled real scalar one finds $C_1 = \hbar_{\text{eff}}/(96\pi^2)$ up to scheme dependent factors. Note that C_1 is scheme- and matter-content dependent (order-one factors), and the identification $1/(16\pi G) = C_1 \Lambda_{\text{eff}}^2$ is robust only up to that ambiguity; in this framework Λ_{eff} is set by the window and the phase spectrum. In the present framework the total C_1 receives additive contributions from both the massless phase and the gapped amplitude sectors. The spacetime constant piece does not source curvature in unimodular form; operationally its value is tied to the homogeneous windowed background and appears as an integration constant matched to Λ_{eff} in cosmology. This synergy makes the unimodular implementation particularly natural here. Curvature squared terms are suppressed by $\ln(\Lambda_{\text{eff}}^2/\mu^2)$ and by the small curvature regime $|R| \ll \Lambda_{\text{eff}}^2$.

Operator route (Lippmann Schwinger) and coefficient C. For completeness we give an operator derivation of the Poisson closure and identify explicitly the coefficient C in Eq. (61). Let K be the static phase operator $K = -\nabla\cdot(c_s^2 \nabla)$ with Green operator $G := K^{-1}$ (defined on the subspace orthogonal to the zero mode). A localized, static matter distribution ρ_m induces $\delta K(x) = -\nabla\cdot(\chi_c\, \rho_m(x)\, \nabla)$ (ICG SI, Matter Kernel Coupling). The perturbed resolvent obeys the Lippmann Schwinger series

$$G' \;=\; G \,-\, G\,\delta K\, G \,+\, G\,\delta K\, G\,\delta K\, G \,-\, \cdots\,, \tag{18}$$

so to first order $\delta G = -G\,\delta K\, G$. The (static) windowed fluctuation response at wavenumber k inherits this variation,

$$\delta S(x,k) \;\propto\; \int d^3x' d^3x''\; G_k(x,x')\, \delta K(x')\, G_k(x',x'')\, \Xi_k(x'')\,, \tag{19}$$

where Ξ_k encodes the local spectral estimator used for the noise proxy (cf. Sec. S8). In the Coulombic window ($k \to 0$), $G_0(r) = -1/(4\pi c_s^2)\, 1/r$ (Eq. (S S9)), and the double convolution with $\delta K = -\nabla\cdot(\chi_c\, \rho_m\, \nabla)$ reduces at long range to a single convolution with a windowed Green kernel

$$G_{\text{win}}(x-x') \;=\; A_{\text{win}}\, G_0(x-x') \;=\; -\,\frac{A_{\text{win}}}{4\pi c_s^2}\, \frac{1}{|x-x'|}\,, \qquad A_{\text{win}} > 0, \tag{20}$$

where A_{win} is a positive, dimensionless functional of the observation window and the estimator normalization. Integrating over the window gives

$$\delta\tau^2(x) \;=\; \int d^3x'\; G_{\text{win}}(x-x')\, \rho_m(x'). \tag{21}$$

Using $\nabla^2(1/r) = -4\pi\,\delta^{(3)}$ one obtains the Poisson closure

$$\nabla^2\, \delta\tau^2(x) \;=\; -\,C\, \rho_m(x)\,, \qquad C \;=\; \frac{\chi_c\, A_{\text{win}}}{c_s^2} \;>\; 0\,. \tag{22}$$

For canonical gradient-correlation estimators (SI Eq. (S 52)), $A_{\text{win}} = \mathcal{O}(1)$, so the microphysical dependence is carried by χ_c and c_s^2.

Zero modes, normalisation, and symmetries

The phase shift symmetry ($\phi \to \phi + \text{const}$) implies a zero mode on compact domains; we remove it with the standard prime determinant (or gauge condition) so that $\det'(-\Box_g)$ is finite. Local unit (conformal) rescalings move strength between $\kappa, w_*, \hbar_{\text{eff}}$ and the window edges but leave the dimensionless predictions and the induced $C_1\Lambda_{\text{eff}}^2$ combination invariant up to slow drifts captured by the window formalism.

Validity and outlook

The expansion holds for curvatures well below the window scale, $|R| \ll \Lambda_{\text{eff}}^2$, and for slowly varying backgrounds consistent with the continuum mapping. A full first principles derivation of the numerical coefficients requires the microscopic spectral density of the induced Laplacian on near regular subgraphs; this can be obtained either analytically for idealised lattices or numerically within the graph simulation programme. The sketch above shows how the microdynamics place the framework in the Einstein universality class and identifies how G and higher curvature terms encode window and microparameter dependence.

S4 Emergence of a U(1) Gauge like Structure from Phase Defects

We outline how stable phase defects in a U(1) order parameter produce an emergent Gauss law and Coulombic $1/r$ behavior in local units. The logic proceeds from microscopic topology (defects) to macroscopic field structure (sheets and flux). This chain is *predictive*: starting from continuity and $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, we derive defects and their flux law; no Gauss law behavior is assumed a priori.

Topological defects and energetic pinning

Phase winding and defects. The order parameter manifold for the phase is U(1) with fundamental group $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. Closed loops $\mathcal{C}$ in space carry an integer winding,

$$n \;=\; \frac{1}{2\pi}\oint_{\mathcal{C}} \nabla\phi\cdot d\boldsymbol{\ell} \;\in\; \mathbb{Z}\,. \tag{23}$$

Nonzero n cannot be removed by a *continuous* phase deformation unless the mapping lifts off U(1) at some point; in this framework, that requires the amplitude to vanish locally so the phase becomes undefined. Those loci where $w \to 0$ are the *amplitude cores*. For any contractible loop with $n \neq 0$, continuity thus *necessitates* at least one core inside the loop.

Energetic barrier from the amplitude gap. Around the homogeneous vacuum the amplitude mode is gapped with $m_\xi^2 = 2\beta_{\text{pot}} > 0$ (see ICG SI Sec. S2, "Fluctuation Spectrum: Gapped Amplitude and Massless Phase"), setting a screening length $\ell = \sqrt{\alpha_{\text{grad}}/m_\xi^2}$. Suppressing w to zero over a region of size $\sim \ell$ costs a finite core energy (gap plus gradients). Changing n therefore requires a localized, nonperturbative event ($w \to 0$ at the core), which energetically pins defects and renders them long lived. Equivalently, the core regularises the otherwise divergent phase gradient energy that a pure phase singularity would entail.

Phase sheets and sources

Phase jump across a sheet. Consider a codimension 1 surface Σ across which the phase jumps by 2π. The distributional gradient is

$$\nabla\phi \;=\; \nabla\phi_{\text{smooth}} \;+\; 2\pi\,\mathbf{n}\,\delta(\Sigma)\,, \tag{24}$$

with $\mathbf{n}$ the unit normal to Σ. The sheet can terminate only where the phase is undefined, i.e., on amplitude cores ($w = 0$), so sheet endpoints identify defect locations. Sheets are not postulated; they are the macroscopic encoding of allowed 2π jumps made possible by the existence of cores.

Gauss law from sheet counting

Define the coarse electric field by $\mathbf{E} = -\nabla\Phi_{\text{eff}}$, with Φ_{eff} a potential whose circulation counts net sheet crossings. For a closed surface ∂V the flux

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} \ \propto \ (\text{net sheet crossings through } \partial V) \ = \ Q_{\text{enc}} \tag{25}$$

counts the number of sheet endpoints (defects) enclosed, with orientation. For isolated point sources one obtains $|\mathbf{E}| \propto 1/r^2$ and $\Phi_{\text{eff}} \propto 1/r$ in local units. Thus defect charge (sheet endpoints) sources a Coulombic $1/r$ far field. This Gauss like behavior is therefore a *consequence* of the microscopic structure, not an input assumption.

Lorentz covariance and field representation

The phase action $\mathcal{L}_\phi = (\kappa w_*^2/2)\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi$ is Lorentz covariant with wave speed $c_s = \sqrt{\kappa} w_*$. Coarse graining the distributional $\partial_\mu\phi$ from sheet configurations and assembling into an antisymmetric tensor $F_{\mu\nu}$ with $\partial_{[\mu}F_{\nu\rho]} = 0$ yields Maxwell structure in local units; detailed constitutive relations (normalizations, couplings) require a fuller treatment beyond this sketch.

S5 Geometric Interpretations (Maps)

Geometric analogues of the "position + scale" state (x, σ) used in the scale–space force derivation. The bundle language below is offered as an interpretive map; the paper's derivations remain in the continuum field formalism.

Frame bundle (Cartan/moving frames): States are (x, e) with e an orthonormal frame in the principal $\mathrm{SO}(n)$ (or conformal $\mathrm{CO}(n) = \mathbb{R}^+ \times \mathrm{SO}(n)$) bundle. The $\mathbb{R}^+$ factor encodes local scale. Our (x, σ) corresponds to the $(x, \mathbb{R}^+)$ subbundle of the conformal frame bundle.

Weyl–Cartan (conformal gauge) geometry: Adds a local dilation gauge on top of Lorentz frames. The Weyl 1–form is the scale connection. The field $\tau(x)$ (noise/scale proxy) can be viewed as fixing a section in the $\mathbb{R}^+$ fiber; $\nabla\tau$ plays the role of a scale-connection gradient. The projection of total-space gradients to spacetime through an Ehresmann connection mirrors the force projection used here.

Parabolic Cartan geometry (conformal): A G/H geometry with $H = \mathrm{CO}(1, 3) = \mathbb{R}^+ \times \mathrm{SO}(1, 3)$. The Cartan connection splits into translational (vielbein), rotational (spin), and dilatational (scale) parts. The pair (x, σ) lives naturally in the dilation fiber; the scale–space force is the horizontal projection of a potential defined on the total space.

Jet/fibered field viewpoint: Extend the configuration bundle with an internal fiber coordinate σ; dynamics on the total space employ a connection to split horizontal (spacetime) versus vertical (scale) variations. The adiabatic reduction used in $E_{\text{eq}}(x)$ corresponds to minimizing along vertical directions and projecting the gradient onto spacetime.

S6 Thermodynamic vs Entropic Gravity (Comparison)

Gravity is emergent in both pictures, but the mechanisms differ. Here, a single bulk field with a bona fide free energy $F = E - TS$ and a noise scale $\tau(x)$ yields a conservative force $\mathbf{F} = -\nabla E_{\text{eq}}$ (with $E_{\text{eq}} \propto -\tau^2$ and $\tau^2 \propto \Phi$), and metric dynamics arise by integrating out phase fluctuations (Sakharov-like, optionally unimodular). Entropic approaches instead tie forces and field equations to horizon/screen thermodynamics (Clausius relation, Unruh temperature, area law), obtaining Einstein's equations as an equation of state and $\mathbf{F} = T\,\nabla S$ from entropy gradients. Horizons and holography are central there but optional (effective-temperature) here. Universality in this framework follows from composition-independent phase coupling (single light cone), whereas entropic routes invoke universal horizon thermodynamics.

Phenomenological distinction. This framework predicts isothermal halos with $v_{\rm flat}^2 \simeq 2\sigma^2$ (derived in Section S7) and a tiny $w \approx -1$ drift from sliding windows; entropic/elastic variants (e.g. Verlinde-type) instead relate flat curves and lensing to elastic/entanglement responses with MOND-like scalings. The isothermal prediction is a direct consequence of the thermodynamic equilibration mechanism and does not require dark matter as a separate particle species.

S7 Jeans Isothermal Equilibrium from Noise Thermodynamics

We derive the conditions under which a self-gravitating collection of solitons reaches an isothermal equilibrium with constant velocity dispersion σ, yielding flat rotation curves with $v_{\rm flat}^2 = 2\sigma^2$. The derivation proceeds through four stages: (i) separation of the noise field into equilibration and force components, (ii) FDT-based derivation of the kinetic temperature, (iii) the Jeans equation under isothermal conditions, and (iv) energy removal processes that maintain quasi-equilibrium.

Assumptions (compact).

- Quasi-static, spherically symmetric halo in the Coulombic window ($R_{\rm core} \ll r \ll \ell$).
- Solitons (amplitude excitations) act as test particles sampling the collective potential $\Phi(x)$.
- Internal soliton modes equilibrate with the phase bath on timescales $\tau_{\rm int} \ll \tau_{\rm orbit}$.
- Translational relaxation occurs over many orbits: $\tau_{\rm relax} \gg \tau_{\rm orbit}$.
- Energy removal (evaporation) at the escape boundary maintains quasi-steady state.

S7.1 Separation of Noise into Equilibration and Force Components

The noise field admits a decomposition into homogeneous and inhomogeneous parts (cf. Section S9):

$$\tau^2(x) \;=\; \tau_0^2 \;+\; \delta\tau^2(x), \qquad \delta\tau^2(x) \;=\; -\alpha\left[\Phi(x) - \Phi_{\rm ref}\right]. \tag{26}$$

These components play distinct physical roles:

- **Homogeneous background** τ_0^2: Set by the windowed phase spectrum (Section S1). In local units, this is absorbed into the definition of rulers and clocks, making it operationally constant across the halo. It determines the *equilibration temperature* for internal and kinetic degrees of freedom.

- **Inhomogeneous perturbation** $\delta\tau^2(x)$: Sourced by matter via the Poisson equation $\nabla^2(\delta\tau^2) = -C\,\rho_m$. Its gradient provides the *gravitational force*: $\mathbf{F} = -\nabla E_{\rm eq} = +\nabla(\tau^2) = \nabla(\delta\tau^2) = -\alpha\,\nabla\Phi$.

The separation is justified by the local-unit interpretation: homogeneous window shifts change τ_0^2 but not forces or dimensionless ratios. Within the halo, the ratio $|\delta\tau^2|/\tau_0^2$ is small in the weak-field regime, ensuring the perturbative treatment is valid.

S7.2 Kinetic Temperature from Fluctuation–Dissipation

The FDT (Section S8) connects the bath spectrum to an effective temperature:

$$S_{BB}(\omega; x) \;=\; \hbar\, J(\omega; x)\, \coth\!\left(\frac{\hbar\omega}{2k_B T_{\text{eff}}(x)}\right), \qquad T_{\text{eff}}(x) \;\propto\; \tau(x). \tag{27}$$

For gradient coupling (super-Ohmic, $J(\omega) \propto \omega^3$), internal soliton modes equilibrate at temperature $T_{\text{int}}(x) = T_{\text{eff}}(x)$.

Translational degrees of freedom. The center-of-mass motion of a soliton couples to the bath through acceleration-dependent radiation (Section S2). For a soliton on an orbit of period T_{orbit}, the effective kinetic temperature is set by the *orbit-averaged* bath:

$$T_{\text{kin}} \;\equiv\; \Big\langle T_{\text{eff}}\big(x(t)\big)\Big\rangle_{\text{orbit}} \;=\; \frac{1}{T_{\text{orbit}}} \oint T_{\text{eff}}\big(x(t)\big)\, dt. \tag{28}$$

In the weak-field regime where $\delta\tau^2 \ll \tau_0^2$, expanding to leading order:

$$T_{\text{kin}} \;\approx\; T_0 \;+\; \mathcal{O}\!\left(\frac{\delta\tau^2}{\tau_0^2}\right), \qquad T_0 \;\propto\; \tau_0. \tag{29}$$

Thus the kinetic temperature is approximately *constant* across the halo, with corrections suppressed by the weak-field parameter.

Velocity dispersion. For an ensemble of solitons in kinetic equilibrium at temperature T_{kin}, the equipartition theorem gives:

$$\frac{1}{2}\, m_{\text{eff}}\, \sigma^2 \;=\; \frac{1}{2}\, k_B T_{\text{kin}}, \qquad \Rightarrow \qquad \sigma^2 \;=\; \frac{k_B T_0}{m_{\text{eff}}} \;\approx\; \text{const.} \tag{30}$$

This is the **isothermal condition**: the velocity dispersion is spatially constant because it is set by the homogeneous background temperature T_0, not by the local noise $\tau(x)$.

S7.3 Jeans Equation and Isothermal Density Profile

For a spherically symmetric, self-gravitating system in hydrostatic equilibrium, the Jeans equation reads:

$$\frac{1}{\rho}\,\frac{d(\rho\sigma_r^2)}{dr} \;+\; \frac{2\beta_{\text{aniso}}(r)\,\sigma_r^2}{r} \;=\; -\frac{d\Phi}{dr}, \tag{31}$$

where σ_r is the radial velocity dispersion and $\beta_{\text{aniso}} := 1 - \sigma_\theta^2/\sigma_r^2$ is the anisotropy parameter. For an isotropic, isothermal system ($\sigma_r = \sigma_\theta = \sigma = \text{const}$, $\beta_{\text{aniso}} = 0$):

$$\sigma^2\,\frac{d\ln\rho}{dr} \;=\; -\frac{d\Phi}{dr}. \tag{32}$$

Integrating:

$$\rho(r) \;=\; \rho_0\, \exp\!\left(-\frac{\Phi(r) - \Phi_0}{\sigma^2}\right). \tag{33}$$

This is the **isothermal distribution function**: the density follows a Boltzmann distribution with "temperature" σ^2 (in units where $k_B/m_{\text{eff}} = 1$).

Singular isothermal sphere. For a self-gravitating isothermal sphere, combining Eq. (33) with the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ yields the Lane–Emden equation for the isothermal case. At large radii, the asymptotic solution is:

$$\rho(r) \;\approx\; \frac{\sigma^2}{2\pi G r^2}, \qquad \Phi(r) \;\approx\; 2\sigma^2 \,\ln(r/r_0). \tag{34}$$

The enclosed mass grows linearly: $M(<r) = 2\sigma^2 r/G$, giving the circular velocity:

$$v_c^2(r) \;=\; \frac{GM(<r)}{r} \;=\; 2\sigma^2 \;=\; \text{const.} \tag{35}$$

This is the **flat rotation curve** characteristic of isothermal halos, with $v_{\text{flat}}^2 = 2\sigma^2$.

S7.4 Energy Removal and Quasi-Equilibrium

A purely isothermal sphere has infinite mass and extent. Real halos are truncated by energy removal processes that maintain a quasi-steady state:

(i) Evaporation at the tidal boundary. Solitons with kinetic energy exceeding the local escape energy leave the system:

$$\frac{1}{2}\, m_{\text{eff}}\, v^2 \;>\; m_{\text{eff}}\, |\Phi_{\text{edge}}| \quad\Rightarrow\quad \text{escape.} \tag{36}$$

This preferentially removes high-velocity particles from the tail of the distribution, maintaining a truncated Maxwellian (King model) in the core that approaches isothermal at small radii.

(ii) Drag dissipation on non-geodesic orbits. Non-geodesic accelerations (e.g., from scattering, external perturbations) induce energy loss via the drag mechanism (Section S2):

$$P_{\text{rad}} \;=\; C_{\text{drag}}\, \gamma_v^4\, a_s^2. \tag{37}$$

This removes energy from perturbed orbits and drives the system toward the relaxed, geodesic-dominated state where drag vanishes.

(iii) Phase-channel relaxation. Internal soliton modes continuously exchange energy with the phase bath, maintaining internal equilibrium at $T_{\text{int}} \approx T_0$. This thermalization couples to translational motion through amplitude–phase backreaction, providing a slow relaxation channel.

Timescale hierarchy. For a quasi-steady isothermal halo:

$$\tau_{\text{int}} \;\ll\; \tau_{\text{orbit}} \;\ll\; \tau_{\text{relax}} \;\ll\; \tau_{\text{evap}}, \tag{38}$$

where τ_{evap} is the evaporation timescale. This hierarchy ensures that (i) internal modes are always equilibrated, (ii) orbits are approximately geodesic, (iii) the velocity distribution relaxes toward isothermal, and (iv) evaporation is slow enough for quasi-equilibrium.

S7.5 Self-Consistent Closure: The τ–Φ–ρ Triangle

The isothermal equilibrium forms a closed, self-consistent system:

$$\text{Poisson (matter sources potential):} \qquad \nabla^2\Phi \;=\; 4\pi G\,\rho, \tag{39}$$

$$\tau\text{–}\Phi \text{ link (noise tracks potential):} \qquad \tau^2 \;=\; \tau_0^2 - \alpha\,\Phi, \tag{40}$$

$$\text{FDT (temperature from noise):} \qquad T_{\text{kin}} \;\propto\; \tau_0 \;\approx\; \text{const}, \tag{41}$$

$$\text{Jeans (density from temperature):} \qquad \rho \;=\; \rho_0\, \exp(-\Phi/\sigma^2), \quad \sigma^2 \propto T_{\text{kin}}. \tag{42}$$

The closure is:

$$\nabla^2\Phi \;=\; 4\pi G\,\rho_0\,\exp\!\left(-\frac{\Phi}{\sigma^2}\right), \tag{43}$$

which is the defining equation for the isothermal sphere. The framework's contribution is to *derive* why σ^2 should be constant: it is set by the homogeneous background τ_0, not by the local potential.

S7.6 Predictions and Observational Consequences

Flat rotation curves without dark matter particles. The isothermal profile emerges from thermodynamic equilibration of amplitude solitons with the phase bath, not from a separate dark matter species. The "missing mass" is the effective gravitational response of the noise-thermodynamic mechanism.

Core–halo structure. The singular isothermal sphere ($\rho \propto r^{-2}$) develops a finite core when amplitude-gap effects or strong-field corrections become relevant at small radii. The core radius $r_c \sim \ell$ (screening length) sets the transition scale.

Velocity dispersion–circular velocity relation. The prediction $v_{\text{flat}}^2 = 2\sigma^2$ is a direct consequence of the isothermal Jeans equilibrium and can be tested observationally by comparing stellar velocity dispersions to rotation curves in the same systems.

Deviations from isothermality. Corrections arise from:

- Finite $\delta\tau^2/\tau_0^2$ (weak-field corrections to orbit-averaging);
- Anisotropy ($\beta_{\text{aniso}} \neq 0$) from incomplete relaxation;
- Truncation effects (King-model deviations at large radii);
- Non-equilibrium transients in merging or forming halos.

In the weak-field regime where $|\alpha\,\Phi/\tau_0^2| \ll 1$, we can write

$$\tau^2(x) \;=\; \tau_0^2\,[1+\epsilon(x)], \qquad |\epsilon(x)| \ll 1, \tag{44}$$

so that

$$\tau(x) \;\approx\; \tau_0\,[1+\epsilon(x)/2], \tag{45}$$

and the kinetic temperature (hence σ^2) is constant to leading order, with $O(\epsilon)$ corrections. Deviations from isothermality in $\sigma^2(r)$ or $\rho(r)$ therefore directly probe the small parameter $\epsilon(x)$.

Spectrally, τ_0^2 is set by integrating the phase spectrum over many decades in k at the RG fixed point $S(k) \propto 1/k$, so it is dominated by the full bandwidth. In contrast, $\delta\tau^2(x)$ comes only from small distortions of the long-wavelength tail induced by matter via the Poisson kernel. This makes $|\delta\tau^2|/\tau_0^2$ generically small in weak-field environments, independently of any detailed choice of α.

This is analogous to a Brownian particle in a potential $U(x)$ coupled to a homogeneous heat bath: the equilibrium density is $\rho(x) \propto e^{-U/(k_B T)}$ with fixed T determined by the bath, even though $U(x)$ varies strongly in space. Here $T \leftrightarrow \tau_0^2$ and $U \leftrightarrow \Phi$.

Galactic and cluster potentials satisfy $|\Phi|/c^2 \sim 10^{-6}$–10^{-5}, so unless τ_0^2 is tuned anomalously small relative to the microphysical scale that sets c_s^2, one expects $|\alpha\,\Phi/\tau_0^2| \ll 1$ automatically wherever the weak-field approximation is valid. These predict systematic departures from the singular isothermal model that may be constrained by high-precision rotation curve and velocity dispersion measurements. A full derivation of the $O(\epsilon)$ corrections to $\sigma^2(r)$ requires a kinetic

(Fokker–Planck) treatment or numerical simulations; here we treat the $\epsilon \to 0$ limit as the leading-order closure and regard measured deviations from isothermality as empirical probes of ϵ.

S8 Deriving the Noise Field τ from Graph Microdynamics

We outline two independent, complementary constructions that define the scalar "noise" field $\tau(x)$ from the graph microdynamics underlying the continuum limit: a spectral route based on the local fluctuation spectrum of the phase kernel, and a linear-response/Fluctuation–Dissipation (FDT) route grounded in operational dissipation measurements. Agreement between the two makes τ an operational observable rather than a mere definition.

Assumptions (compact).

- Near-regular subgraphs with slowly varying symmetric weights; dense sampling of regions of interest.
- Massless phase sector at long wavelengths; continuum generator $-\nabla\cdot(c_s^2\nabla)$ well-defined.
- Observation window in the Coulombic regime (Section S1): $k_{\min} \ll k \ll k_{\max}$, with amplitude gap ensuring phase dominance.
- Local-unit (conformal) interpretation: homogeneous window shifts change τ_0^2 only, not gradients/forces.
- Linear response holds for probe couplings; internal deformations remain small in the drag measurements.

Setup: from graph kernel to continuum

On near-regular subgraphs, the discrete phase kernel K (graph Laplacian with weights) admits a continuum limit

$$K \;\longrightarrow\; -\nabla\cdot\big(c_s^2(x)\,\nabla\,\cdot\big) \equiv -c_s^2\,\Delta + \cdots\,, \tag{46}$$

whose principal symbol fixes the light cone (Section S3). Let $\{\varphi_n(x)\}$ be localised modes of K with eigenvalues ω_n^2. Denote by $\rho(x,\omega)$ the local density of states (LDOS) and by $n(x,\omega)$ the local occupation (set by microdynamics and the observational window in Section S1).

Discrete-to-continuum details. On a weighted, undirected graph with vertices i at embedded positions x_i and symmetric weights $w_{ij} = w_{ji} \geq 0$, the phase Laplacian acts as

$$(Kf)_i \;=\; \sum_j w_{ij}\,\big(f_i - f_j\big). \tag{47}$$

On near-regular subgraphs with slowly varying weights and dense sampling, one has quadratic-form convergence

$$\sum_{i,j} w_{ij}\,\big(f_i - f_j\big)^2 \;\longrightarrow\; \int c_s^2(x)\,|\nabla f(x)|^2\,d^D x, \tag{48}$$

identifying $-\nabla\cdot(c_s^2\nabla)$ as the continuum generator (cf. Appendix A in the main text). For a mollifier $W_\ell(x)$ (width $\ell \ll$ curvature scales), define the LDOS by

$$\rho(x,\omega) \;=\; \sum_n \Big(\int W_\ell(x-x')\,|\varphi_n(x')|^2\,d^D x'\Big)\,\delta(\omega-\omega_n), \tag{49}$$

and the window selector $\Theta(k; k_{\min}, k_{\max}) = \mathbf{1}_{[k_{\min},k_{\max}]}(k)$.

Spectral route (primary)

Define the local fluctuation spectrum of phase modes

$$S(x,k) \;:=\; \rho\big(x,\omega=c_s k\big)\, n\big(x,\omega=c_s k\big), \tag{50}$$

and the windowed variance ("effective noise density")

$$\rho_{N,\mathrm{eff}}(x) \;=\; \int_{k_{\min}}^{k_{\max}} S(x,k)\, dk, \qquad (k_{\min},k_{\max}) \text{ from Section S1.} \tag{51}$$

We *define* the noise scale by

$$\tau^2(x) \;\propto\; \rho_{N,\mathrm{eff}}(x), \tag{52}$$

with the proportionality constant fixed by the local-unit convention (co-scaling; Section S1). This choice is natural because τ governs equilibration of internal soliton degrees of freedom and the radiative drag; both depend on the integrated fluctuation power accessible in the observational window.

Equivalently, in the continuum one may write a gradient-correlation estimator

$$\tau^2(x) \;\propto\; \int_{k_{\min}}^{k_{\max}} dk\, k^2\, \mathcal{P}_\phi(x,k), \qquad \mathcal{P}_\phi(x,k) := \frac{1}{V_k}\,\big\langle |\phi_k(x)|^2 \big\rangle, \tag{53}$$

so that $\tau^2 \propto \langle |\nabla\phi|^2\rangle_{\mathrm{window}}$, making explicit that τ measures phase-gradient fluctuations which control equilibration and coupling to composite solitons.

Linear-response/FDT route (independent cross-check)

Couple a probe soliton's dominant internal coordinate q to the phase sector through the lowest symmetry-allowed operator (gradient coupling), $H_{\mathrm{int}} = -\lambda\, q\, e_i\, \partial_i\phi$. The dissipated power under proper acceleration a reads (Section S2)

$$P \;=\; \int_0^\infty d\omega\; \omega\; \mathrm{Im}\,\chi_{qq}(\omega)\; S_{BB}(x;\omega), \tag{54}$$

with S_{BB} the bath spectrum set by local fluctuations and χ_{qq} the probe susceptibility. The FDT connects S_{BB} to the symmetrised correlator and hence to an *effective* local temperature/noise scale $T_{\mathrm{eff}}(x) \propto \tau(x)$ through the Unruh-like scaling in Section S2.1. Inverting the measured $P(a,v)$ scaling yields an *operational* estimate of $\tau(x)$, which can be compared pointwise with Eq. (52).

More explicitly, for the gradient coupling one finds a super-Ohmic phase-bath spectral density $J(\omega) \propto \omega^3$ (Section S2.1). The symmetrised bath spectrum satisfies

$$S_{BB}(\omega;x) \;=\; \hbar\, J(\omega;x)\, \coth\!\Big(\frac{\hbar\omega}{2k_B T_{\mathrm{eff}}(x)}\Big), \tag{55}$$

so that the frequency ω_* dominating the integrand scales as $\omega_* \propto T_{\mathrm{eff}}^{1/2}$. Measuring either the peak in the driven response or the roll-off in $P(\omega)$ allows one to infer $T_{\mathrm{eff}}(x)$ and thus $\tau(x) = \kappa_\tau\, T_{\mathrm{eff}}(x)$, where κ_τ is fixed by the local-unit convention.

Window and units; robustness

Changing $(k_{\min}, k_{\max})$ shifts $\rho_{N,\mathrm{eff}}$ by a *homogeneous* offset (Section S1). In local units, such offsets are absorbed into the background τ_0^2 and do not affect gradients or forces. The gradient field $\nabla(\tau^2)$ is invariant within the Coulombic window where the long-range tail dominates.

Corrections and regimes

Finite amplitude gap implies screened/Yukawa tails at short range, leading to a Helmholtz correction $(\nabla^2 - \lambda^2)\tau^2 \approx \cdots$ near strong sources. Nonlinearities become relevant when soliton cores overlap or the amplitude gap closes; then the full nonlocal functional should be evaluated rather than the shell approximation.

Simulation protocol (for validation). On synthetic near-regular graphs: (i) compute the LDOS and $S(x, k)$, form $\tau^2(x)$ via Eq. (52); (ii) embed a driven probe and extract $\tau(x)$ from $P(a, v)$; (iii) compare the two maps and quantify deviations outside the Coulombic window.

Background bath and vanishing limit. The windowed variance admits a homogeneous background τ_0^2 from the long-wavelength phase spectrum even in regions without nearby matter, so in practice $\tau_0^2 > 0$. Only in an idealized limit (no matter anywhere, perfectly static/flat background, zero temperature, and a degenerate/infinite window that removes all finite-band power) would one have $\tau_0^2 \to 0$. Realistic finite windows and cosmological backgrounds render τ_0^2 nonzero; localized matter adds the inhomogeneous $\delta\tau^2$ on top.

S9 Poisson Equivalence: Why $\tau^2 \propto \Phi$

We show that in the static, long-wavelength regime, the windowed fluctuation envelope $\tau^2(x)$ obeys the same Poisson problem as the Newtonian potential $\Phi(x)$, fixing their proportionality up to a boundary-dependent constant.

Assumptions (compact).

- Static, weak-field, long-wavelength limit (Coulombic window); negligible time dependence of backgrounds.
- Massless phase sector controls the long-range response; amplitude sector gapped so short-range only.
- Linear response of the fluctuation spectrum to matter density ρ_m; kernel admits a $k \to 0$ Green function.
- Common boundary conditions for Φ and τ^2 (e.g., vanishing at infinity, or neutralised periodic cell).
- Near-regularity ensures continuum Laplacian mapping; uniqueness theorem for Poisson problem applicable.

Linear response of the spectrum to matter (via δK)

Input from ICG (SI Sec. S8, *Matter–Kernel Coupling Lemma*): a static energy density $\rho_m(x)$ induces a local perturbation of the continuum phase kernel,

$$\delta K(x) \;=\; -\nabla\cdot\big(\chi_c\,\rho_m(x)\,\nabla\big) \;\propto\; \rho_m(x). \tag{56}$$

At fixed frequency, the perturbed Green function obeys $(K + \delta K)\,G' = \delta$. To leading order,

$$G' \;=\; G \;-\; G\,\delta K\,G \;+\; O(\delta K^2), \qquad G := K^{-1}. \tag{57}$$

Hence the spectral density and the local fluctuation power shift by

$$\delta S(x, k) \;\propto\; \int d^3x'\,d^3x''\;G_k(x, x')\,\delta K(x')\,G_k(x', x'') \;=\; \int d^3x'\;\mathcal{G}_k(x - x')\,\rho_m(x'), \tag{58}$$

where $\mathcal{G}_k$ is the k-resolved kernel inherited from $G\,\delta K\,G$. In the $k \to 0$ limit (Coulombic window), $\mathcal{G}_k \to \mathcal{G}_0$ with $\propto 1/|x - x'|$ spatial envelope in 3D.

Origin of the $1/r$ envelope. The massless phase sector is governed at long scales by $-\nabla\cdot(c_s^2 \nabla)$, whose static Green function in $\mathbb{R}^3$ satisfies

$$-c_s^2\,\nabla^2 G_0(x - x') \;=\; \delta^{(3)}(x - x') \quad\Rightarrow\quad G_0(r) \;=\; -\,\frac{1}{4\pi c_s^2}\,\frac{1}{r}. \tag{59}$$

Any long-wavelength, static response built from this kernel inherits the $1/r$ envelope up to multiplicative renormalisations from the window and short-distance physics.

Window integration to τ^2 and PDE

Integrating over the observational window,

$$\delta\tau^2(x) \;\propto\; \int_{k_{\min}}^{k_{\max}} dk\,\delta S(x, k) \;=\; \int d^3x'\; G(x - x')\,\rho_m(x'), \tag{60}$$

with G the windowed Green function whose long-range part is $\propto 1/|x - x'|$. Therefore $\delta\tau^2$ solves the Poisson equation

$$\nabla^2\,\delta\tau^2(x) \;=\; -\,C\,\rho_m(x), \qquad C > 0 \text{ set by microparameters and window edges.} \tag{61}$$

Impose the same boundary condition as for Φ (e.g., vanishing at infinity, or neutralisation in periodic cells). Uniqueness of solutions to Poisson's equation then gives

$$\delta\tau^2(x) \;=\; -\,\frac{C}{4\pi G}\,\Phi(x), \qquad\Rightarrow\qquad \tau^2(x) \;=\; \tau_0^2 \;-\; \alpha\,\Phi(x), \tag{62}$$

with $\alpha = C/(4\pi G)$ and τ_0^2 a homogeneous offset.

Boundary-value uniqueness (rigour). For a bounded domain Ω with Dirichlet data on $\partial\Omega$, or for $\Omega = \mathbb{R}^3$ with $f \to 0$ at infinity, the Poisson problem $\nabla^2 f = s$ has a unique weak solution. Hence if $f_1 := \delta\tau^2$ and $f_2 := -\frac{C}{4\pi G}\,\Phi$ obey the same Poisson equation with the same boundary data, then $f_1 \equiv f_2$.

Force law and calibration

From the coarse-grained free energy $E_{\text{eq}}(x) \propto -\tau^2(x)$, the effective force is $\mathbf{F} = -\nabla E_{\text{eq}} = +\nabla(\tau^2) = -\alpha\,\nabla\Phi$. Matching to Newton's law fixes α (equivalently C) once and for all. This is a single calibration, not an ad hoc mapping.

Robustness checks

- **Gauss law:** For isolated sources, $\oint \nabla(\tau^2)\cdot d\mathbf{A} \propto$ enclosed mass.
- **Superposition:** For well-separated sources in the Coulombic window, $\delta\tau^2$ is additive.
- **Boundary dependence:** τ_0^2 tracks the same reference offset as Φ under the chosen BCs.
- **Screening corrections:** In regimes with finite-range corrections, $(\nabla^2 - \lambda^2)\,\delta\tau^2 = -C\,\rho_m$ reproduces Yukawa tails; far field remains Coulombic.

Solvable lattice example (cubic near-regular)

Consider a simple cubic lattice with spacing a, nearest-neighbour weights $w_0 > 0$ (coordination $z = 6$), and the graph Laplacian $(Lf)_i = \sum_{j\sim i} w_0(f_i - f_j)$. The discrete quadratic form $\frac{1}{2}\sum_{\langle ij\rangle} w_0(f_i - f_j)^2$ converges to $\int c_s^2|\nabla f|^2 d^3x$ with principal coefficient

$$c_s^2 \;=\; \zeta\, w_0 a^2\,, \qquad \zeta \sim \mathcal{O}(1) \;\text{(geometric factor)}. \tag{63}$$

Let a localized, static excess energy density $\rho_m(x)$ modify the local bond weight to $w_0 \to w_0 + \delta w(x)$ within a bounded neighbourhood. To leading order, $\delta c_s^2(x) = (\partial c_s^2/\partial w_0)\,\delta w(x) = \zeta a^2\,\delta w(x)$. Writing $\delta w(x) = \alpha_w\,\rho_m(x)$ for a local susceptibility $\alpha_w > 0$ (near-regular, volume-normalised), one identifies

$$\delta c_s^2(x) \;=\; \chi_c\,\rho_m(x)\,, \qquad \chi_c \;=\; \zeta a^2\,\alpha_w \;>\; 0\,. \tag{64}$$

With the windowed kernel amplitude $A_{\rm win} = \mathcal{O}(1)$ (gradient-correlation estimator), the Poisson coefficient becomes

$$C \;=\; \frac{\chi_c\,A_{\rm win}}{c_s^2} \;=\; \frac{\zeta a^2\alpha_w}{\zeta w_0 a^2}\,A_{\rm win} \;=\; \frac{\alpha_w}{w_0}\,A_{\rm win} \;>\; 0\,. \tag{65}$$

Hence, on cubic near-regular lattices, C is explicitly positive and controlled by the ratio of the local bond susceptibility to the background weight, with bounded $A_{\rm win}$. This exemplifies the general discussion and provides an order-of-magnitude anchor; see ICG SI (Matter–Kernel Coupling) for the discrete-to-continuum mapping hypotheses.

Link to induced gravity coefficients. The micro-to-macro coefficient C can be cross-checked against the induced R-term coefficient C_1 in Section S3 and the window scale $\Lambda_{\rm eff}$. Consistency between the Poisson normalisation and $C_1\Lambda_{\rm eff}^2 = (16\pi G)^{-1}$ provides a nontrivial internal check.

Numerical validation plan. (i) Place a point mass (or compact source) on a near-regular graph; (ii) compute $\tau^2(x)$ via the spectral and FDT estimators of Section S8; (iii) fit the far field to Eq. (61) to extract C and verify Eq. (62); (iv) test superposition with two sources; (v) confirm Gauss-law scaling for spherical surfaces.

S10 Additional References and Pointers

Standard references for the integration of Gaussian fields on curved backgrounds and the resulting local effective action (Einstein–Hilbert plus higher curvature) include: Seeley and DeWitt coefficient expansions (e.g., DeWitt, *Dynamical Theory of Groups and Fields*; Vassilevich, "Heat kernel expansion: user's manual," Phys. Rept. 388 (2003) 279–360), QFT in curved spacetime texts (e.g., Birrell & Davies, *Quantum Fields in Curved Space*), and condensed expositions of Sakharov's induced gravity (e.g., Visser, Mod. Phys. Lett. A 17 (2002) 977; Barvinsky & Vilkovisky, Nucl. Phys. B). These sources justify the schematic form $\Gamma[g] = \int\sqrt{|g|}(C_0\Lambda^4 + C_1\Lambda^2 R + C_2\mathcal{R}^2 + \cdots)$ used in the main text.

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