Supplementary Information: Scalar Dark Matter (Draft)

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1 Notation and symbol map

We summarize only symbols that are used in the present dark-matter paper.

- c_s: universal signal speed (phase sector).
- $\tau(x)$: local noise proxy; $\tau^2 \propto \Phi$ (gravitational potential).
- $R_{\rm cl}, \ell$: composite source scale and screening length; Coulombic window $R_{\rm cl} \ll r \ll \ell$.
- $\sigma, \sigma_r, \beta(r)$: one-dimensional velocity dispersion, radial dispersion, and Binney anisotropy.

2 Phase Friction Mechanism

We derive the kinematic distinction between baryon (Winding) and DM (Scalar Breather) solitons.

The Friction Functional

The interaction energy between a soliton's phase field $\phi(x)$ and the vacuum noise $\tau(x)$ is given by the gradient coupling:

$$E_{\rm int} = \int \nabla\phi \cdot \nabla\tau \, d^3x. \tag{1}$$

For a moving soliton with velocity $\mathbf{v}$, the phase field is Doppler-shifted. The noise field $\tau(x)$ fluctuates stochastically with a correlation time τ_c. The resulting drag force is the time-averaged backreaction:

$$\mathbf{F}_{\rm drag} = -\langle \nabla E_{\rm int} \rangle_t. \tag{2}$$

Case 1: Baryons (Windings)

A baryon possesses a net topological winding number $n \neq 0$ (or open sheets terminating at infinity), implying a non-vanishing phase circulation $\oint \nabla\phi \cdot dl \neq 0$. Locally, $\nabla\phi$ has a sustained directionality. The coupling $\nabla\phi \cdot \nabla\tau$ contributes a net term. As the soliton moves through the fluctuating τ field, it performs work on the vacuum modes. Using the Fluctuation-Dissipation Theorem, the dissipation rate is proportional to the spectral density of the noise at the soliton's characteristic frequency $\omega \sim v/\sigma$:

$$\mathbf{F}_{\rm drag} \approx -\eta \, \mathbf{v}, \quad \eta \propto \int |\nabla\phi|^2 S_\tau(\omega) d\omega. \tag{3}$$

Since the open winding ensures a non-canceling contribution to $|\nabla\phi|^2$, $\eta > 0$. Baryons experience strong phase friction ($F \propto v$).

Case 2: Dark Matter (Scalar Solitons)

A DM scalar soliton has trivial topology at infinity ($n = 0$). It is a breather mode of the amplitude field with no net phase winding. While it has internal structure, the phase gradient $\nabla\phi$ vanishes or averages to zero over the soliton's envelope. The leading-order coupling term vanishes due to symmetry:

$$\int \nabla\phi_{\text{scalar}} \cdot \nabla\tau \, d^3x \approx 0. \tag{4}$$

Consequently, the velocity-dependent phase friction is suppressed. DM solitons experience only the universal acceleration-dependent radiative drag ($P \propto a^2$, see Gravity paper) which vanishes on geodesics. They are effectively frictionless on orbital timescales.

3 Primordial Yield Statistics

We estimate the abundance ratio $\Omega_{\text{DM}}/\Omega_{\text{B}}$ from the statistics of the primordial graph quench.

Kibble-Zurek on a Disordered Graph

The formation of topological defects ($l = 1$) is governed by the Kibble-Zurek mechanism. On a disordered graph, the critical slowing down depends on the local spectral dimension d_s and connectivity (stiffness κ).

- **Filaments (High κ):** Fast quench. The correlation length ξ is small. Defects are trapped with probability $P_{\text{defect}} \sim \xi^{-d}$.
- **Voids (Low κ):** Adiabatic quench. The correlation length diverges, $\xi \to \infty$. The field relaxes to the ground state ($l = 0$) over the entire domain.

Volume Fraction Estimate

Let $P(\kappa)$ be the probability distribution of local stiffness in the primordial graph. The fraction of volume collapsing into baryons is:

$$f_B = \int_{\kappa_c}^{\infty} P(\kappa) \, d\kappa. \tag{5}$$

The fraction remaining as scalar dark matter is:

$$f_{DM} = \int_{0}^{\kappa_c} P(\kappa) \, d\kappa. \tag{6}$$

For a scale-free graph, $P(\kappa) \sim \kappa^{-\gamma}$. We hypothesize that the critical stiffness κ_c for defect formation lies such that the mass-weighted ratio is:

$$\frac{\Omega_{DM}}{\Omega_B} \approx \frac{f_{DM}\langle M_{l=0}\rangle}{f_B\langle M_{l=1}\rangle} \approx 5. \tag{7}$$

Verifying this specific ratio is the target of the planned lattice simulations (see `simulation_plan_primordial_yie`

4 Galactic diagnostics and modeling recipes

We provide practical recipes for analyzing rotation curves in this framework.

Anisotropy and dispersion gradients

Use the general relation

$$\alpha(r) \equiv -\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} = \frac{v_c^2}{\sigma_r^2} + \frac{\mathrm{d}\ln\sigma_r^2}{\mathrm{d}\ln r} + 2\,\beta(r) \tag{8}$$

to propagate uncertainties from $\beta(r)$ and $\sigma_r(r)$ into mass profiles. When fitting Jeans models, include a linear-in-ln r term for σ_r^2 locally and a weakly informative prior on $\beta(r)$.

External-field/tidal truncation

Model truncation by imposing a taper radius r_t (from tides or external-field effects) beyond which the isothermal envelope transitions smoothly to a steeper decline; use penalized splines or an error-function taper. Predict a recovery toward the $1/r$ tail beyond r_t where the Coulombic window reopens.

4.1 Kinetic derivation of $\sigma \approx \mathrm{const}$ in local units

We sketch a Fokker – Planck derivation showing that a stationary phase bath yields an approximately radius – independent dispersion.

Setup. Consider collisionless tracers in a slowly varying potential $\Phi(r)$. Let $f(r, \mathbf{v}, t)$ obey a Fokker – Planck equation with drift from Φ and isotropic velocity – space diffusion $\mathcal{D}$:

$$\partial_t f + v_r\,\partial_r f - (\partial_r\Phi)\,\partial_{v_r} f \;=\; \partial_{v_i}\big[A_i f + \tfrac{1}{2}\partial_{v_j}(B_{ij} f)\big], \qquad B_{ij} = 2\mathcal{D}\,\delta_{ij}. \tag{9}$$

Stationarity and weak anisotropy imply $A_i \approx 0$ and $B_{ij} \approx 2\mathcal{D}\,\delta_{ij}$. A scale-invariant noise spectrum produces a diffusion coefficient $\mathcal{D}$ that is radially slowly varying.

Moment equation. The stationary second – moment equation implies

$$\rho\,\sigma_r^2\,\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} \;=\; -\,\rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;+\; 2\rho\,\mathcal{D}, \qquad \sigma_r^2 := \langle v_r^2\rangle. \tag{10}$$

Comparing with the isotropic Jeans relation shows that small $\mathcal{D}$ acts to stabilize σ_r against radial drift, yielding $\sigma_r(r) \approx \mathrm{const}$ and recovering the isothermal envelope.

5 Screening-length bands and core/truncation mapping

We map microparameters $(\alpha_{\mathrm{grad}}, \beta_{\mathrm{pot}})$ to observable halo scales via $\ell = \sqrt{\alpha_{\mathrm{grad}}/(2\beta_{\mathrm{pot}})}$.

Core mapping. When $r \lesssim \ell$, Helmholtz corrections produce a pseudo – isothermal core with $r_c \sim \mathcal{O}(\ell)$. Hence observed core radii constrain ℓ. Reported dwarf cores $r_c \sim$ kpc imply $\ell \sim$ kpc.

Environment/truncation. In strong host potentials or tides, the isothermal envelope truncates at r_t; observationally, mass discrepancy drops inside r_t and recovers outside.

6 Lensing and PPN cross-checks

In the weak-field regime we use

$$ds^2 = -\Big(1 + \tfrac{2\Phi}{c_s^2}\Big)c_s^2 dt^2 + \Big(1 - \tfrac{2\Phi}{c_s^2}\Big)d\mathbf{x}^2, \tag{11}$$

which has PPN parameters $\gamma = \beta = 1$ at leading order. Therefore light deflection matches GR expressions with $c \to c_s$. For an isothermal sphere with $\rho = \sigma^2/(2\pi G r^2)$, the projected surface density is $\Sigma(R) = \sigma^2/(2GR)$ and the reduced deflection angle is constant.

7 Simulation plan and calibration

We outline a minimal simulation program.

1. **Screening Calibration:** Calibrate ℓ by sampling $(\alpha_{\text{grad}}, \beta_{\text{pot}})$ on graphs and measuring core radii r_c.
2. **Kinetic Verification:** Verify the kinetic dispersion result by driving tracers in the stationary phase bath and confirming the flatness of $\sigma_r(r)$.
3. **Primordial Yield:** (See `simulation_plan_primordial_yield.md`) Test the hypothesis that scalar basins outnumber vector knots by 5:1.