

Supplementary Information for
*Horizon Duality: Black Holes as Asymptotic Shells of a
Fractal Cosmos*

Franz Wollang

February 13, 2026

Abstract

This document collects technical derivations, robustness arguments, and extended quantitative sketches supporting the main paper. It is intended to preserve detail while keeping the narrative flow of the main text clean.

Contents

1 Notation and cross-document conventions	2
2 Mean-field nuance: screened links, monopoles, and observability	2
2.1 Yukawa kernels vs. the exterior point-source tail	2
2.2 “Same mass” does not mean “same noise profile”	3
2.3 When does the profile matter observationally?	3
3 Duality dictionaries, membrane balance, and discriminants	3
3.1 Geometric dictionary and trapping	3
3.2 Thermodynamic dictionary and flux reconciliation	4
3.3 Membrane balance and conservation	4
3.4 Robustness: resolving apparent contradictions	4
3.5 Predictions and discriminants	4
3.6 Species and multipoles in the heating closure	4
4 Statistical derivation of the Planck pivot	5
4.1 Free energy for an N -grain droplet	5
4.2 Critical occupancy	5
4.3 Two minimisers \Rightarrow two area laws	5
4.4 Conformal consistency	5
4.5 Cross-window consistency and re-windowing covariance (why the crossover shape is not arbitrary)	5
5 Parent–child time mapping derivation	7
5.1 Notation	7
5.2 Derivation of $r_H = R_S$	7
5.3 Junction consistency	7
5.4 Multi-era accounting	7
5.5 DE-era time mapping and the phantom sign	8
5.6 Self-consistency check	8

5.7	Planck-pivot regularisation: heat death	8
5.8	Local vs. global caveat and epistemic consequence (expanded)	8
5.9	Predicted value of $\epsilon = 1 + w $ (expanded)	9
5.10	Summary table and open questions	9
6	Hierarchical condensation and interior large-scale structure	9
6.1	Black-hole-branch seeds: radial drift and stall	9
6.2	Particle-branch droplets: Brownian network and filament formation	10
6.3	Impact on early-epoch anisotropies	10
6.4	Effective geometry of the interior	11
6.5	Open questions (kinetics of the stochastic cascade)	11
7	Discussion addendum: infinite zooms, finite energies	11
7.1	Infinite zoom cascade with local time at every level	11
7.2	Scale-window principle prevents divergences	11
7.3	Infinite nested zooms inside a finite-lifetime shell	11
7.4	Impedance matching: why constant power is necessary for nesting	12
7.5	Seed gradient, filament fate, and wall overtake	12
7.6	Cosmic horizon as a Type-II gravitational droplet	12
7.7	Why the hierarchy alternates 2D shells and 3D bulks	12
7.8	Shell-thickness scaling and observational outlook	12
8	Key formulae (quick reference)	13
9	Related approaches (orientation pointers)	13
9.1	Piece-wise arrows of time in an eternal hierarchy	13
10	Spectral-dimension selection: why e (and hence 3)	14
10.1	Setup and assumptions	14
10.2	Local efficiency objective	14
10.3	Time-dissipation gauge	14
10.4	From branching to spectral dimension	14
10.5	Caveats and empirical checks	15

1 Notation and cross-document conventions

We standardize on c_s for the universal phase-wave/light-cone speed (any bare c should be read as c_s). In weak-field regimes we relate the local proxy to cosmological noise density by $\tau^2 \propto \rho_N$. We reserve $(\alpha_{\text{grad}}, \beta_{\text{pot}})$ for amplitude-sector parameters and $(\alpha_{\text{dim}}, \beta_{\text{dim}})$ for dimensional-bias parameters to avoid symbol overloading.

2 Mean-field nuance: screened links, monopoles, and observability

2.1 Yukawa kernels vs. the exterior point-source tail

Every Planck-scale constituent in the shell emits only a short-range Yukawa halo,

$$\Phi_i(r) \sim \frac{e^{-r/R_i}}{r}, \quad (1)$$

with screening length $R_i \ll R_{\text{shell}}$. For an exterior observer at $r \gg R_{\text{shell}}$, each exponential factor is ≈ 1 , so the sum over N constituents reproduces an ordinary monopole $1/r$ field with total source strength

$$M = \sum_i \frac{E_i}{c_s^2}. \quad (2)$$

Inside the shell, far-side halos are exponentially suppressed, so the local gradient vanishes to leading order. Thus a strictly local link interaction can yield a point-source exterior field for distant observers while producing no further pull on already-collapsed constituents.

2.2 “Same mass” does not mean “same noise profile”

The statement “the fine-grained noise averages to a point source” concerns integrated source strength (the monopole fixing the $1/r$ tail), not the detailed spectrum or correlation structure of the emitted field.

A black-hole shell is a maximally downshifted, window-floor system: most of its microscopic fluctuations sit at or below the external UV edge and are unresolved by external observers. They renormalise into effective couplings plus a monopole. Ordinary dense matter (e.g. a neutron star) is not maximally downshifted: it retains internal structure and fluctuations at wavelengths well within our window, so its noise field can carry resolvable multipoles, longer correlation lengths, and material-specific channels.

2.3 When does the profile matter observationally?

If an effect couples only through the scalar intensity $\tau^2(x)$ (a window-integrated variance), then two sources with very different spectra can be indistinguishable at leading order: the spectrum is “forgotten” by the integral and reappears only as renormalised constants plus a monopole.

Differences become measurable only for probes sensitive to more than τ^2 :

- correlation functions (finite correlation length/time),
- higher moments (non-Gaussianity; nonlinear response),
- multipole/anisotropy structure,
- frequency-selective or window-overlap probes (including accelerated or near-shell observers where Doppler/Unruh promotion changes which portion of the spectrum falls inside the operative window).

3 Duality dictionaries, membrane balance, and discriminants

This section ports the longer “dictionary” and robustness material behind the Type I/Type II horizon duality.

3.1 Geometric dictionary and trapping

- **Black hole:** r_H^{BH} is the null trapping radius of the exterior effective metric; interior optics is hyperbolic. Define an interior curvature scale $H_{\text{in}} := c_s/R_{\text{shell}}$.
- **Cosmology:** $r_H^{\text{FRW}} = c_s/H$ is the apparent horizon of the FRW patch.
- **Areas:** $A_{\text{in}} = 4\pi R_{\text{shell}}^2 = 4\pi(c_s/H_{\text{in}})^2$ and $A_{\text{FRW}} = 4\pi(c_s/H)^2$.
- **Trapping:** trapping holds where outward null expansion first vanishes, $\theta_{\text{out}} = 0$. Emission is set by a photosphere/last scattering layer where optical depth to infinity is ~ 1 .

3.2 Thermodynamic dictionary and flux reconciliation

- **Temperatures:** interior BH side $T_{\text{in}} \propto H_{\text{in}}$; FRW horizon $T_{\text{GH}} = \hbar H / (2\pi k_B)$.
- **Geometric (pre-reprocessing) fluxes in external units:**

$$P_{\text{in,geom}} \propto A_{\text{in}} T_{\text{in}}^4 \propto (1/H_{\text{in}}^2) H_{\text{in}}^4 = H_{\text{in}}^2, \quad (3)$$

$$P_{\text{FRW}} \propto A_{\text{FRW}} T_{\text{GH}}^4 \propto (1/H^2) H^4 = H^2. \quad (4)$$

- **Reprocessing (Type I only):** an optically thick shell enforces $T_{\text{shell}} \propto M^{-1/2}$, $A_{\text{shell}} \propto M^2$, giving a photospheric output $P_{\text{shell}} \propto A_{\text{shell}} T_{\text{shell}}^4 \approx \text{const}$ in external units.

3.3 Membrane balance and conservation

On the nucleation surface Σ we impose local balance in window units,

$$P_{\text{in}}(\Sigma) = P_{\text{out}}(\Sigma), \quad \nabla_\mu T_{\text{tot}}^{\mu\nu} = 0, \quad (5)$$

with $T_{\text{tot}}^{\mu\nu}$ including membrane stress and radiative flux. For Type I, this closes with a diffusive transport law through the shell to the photosphere; for Type II, detailed balance holds locally (GH bath) and the net leakage seen in parent/external units is $\propto H^2$.

3.4 Robustness: resolving apparent contradictions

- **Shell vs. no shell:** Type II lacks a material reprocessor for interior observers because they sit inside the asymptote; the same asymptote carries a real shell in the parent frame. Both satisfy the same nucleation and balance laws.
- **Flux laws:** “const” (Type I photosphere) vs. H^2 (Type II GH leakage) reflect reprocessing vs. bare geometric emission; they are two faces of the same intake flux.
- **Isotropy vs. structure:** the parent boundary supplies leading near-FRW isotropy (hyperbolic optics with a concentric asymptote). Observed anisotropies arise from interior condensates seeded by the Planck-pivot cascade.

3.5 Predictions and discriminants

- Finite shell thickness at the asymptote; hyperbolic interior optics.
- Luminosity laws: $P_{\text{shell}} \approx \text{const}$ (Type I photosphere) vs. $P_{\text{FRW}} \propto H^2$ (Type II leakage) in a single external frame.
- **Spectral/size imprint:** a last-scattering/photosphere radius produces a transfer-function feature (deviation from a perfect Planckian) differing between Type I/II by the presence/absence of reprocessing.

3.6 Species and multipoles in the heating closure

In the main text we use a coarse covariant closure in which Unruh-like heating balanced by Stefan–Boltzmann cooling sets the scaling $T \propto 1/\sqrt{M}$ and thus $P \approx \text{const}$. This closure is meant as an *effective* description: “areal heating” is an average over the shell’s composite single-field solitonic constituents and their worldline multipoles/topological textures; no Standard-Model point-particle assumption is required.

In an EFT language, the lowest symmetry-allowed gradient coupling to the massless phase fixes the baseline exponent ($p = 4$). Species that suppress this operator contribute higher- p channels with reduced weight. The mixture renormalises only an effective coefficient (schematically \bar{C}); the scaling exponents, and thus the constant-power result, are unchanged.

4 Statistical derivation of the Planck pivot

We show that the four-power flip in surface density arises from a single coarse-grained free-energy functional.

4.1 Free energy for an N -grain droplet

Consider a spherical droplet of radius R containing N identical grains (bare mass μ). A minimal rotationally symmetric free energy is

$$F(R; N) = \underbrace{\frac{\alpha N \hbar c}{R}}_{\text{zero-point / kinetic}} + \underbrace{\frac{\eta_0 N^2}{2 R^3}}_{\text{link (cohesion) energy}} - \underbrace{\frac{G(N\mu)^2}{R}}_{\text{self-gravity}}, \quad (6)$$

where α is an $\mathcal{O}(1)$ geometry factor.

4.2 Critical occupancy

Setting $\partial F/\partial R = 0$ gives a physical extremum only if

$$G\mu^2 N^2 = \alpha N \hbar c \implies N_* = \frac{\alpha \hbar c}{G\mu^2}. \quad (7)$$

The corresponding mass is $M_* = N_* \mu \simeq \sqrt{\hbar c/G}$ (up to $\sqrt{\alpha}$), i.e. the Planck mass.

4.3 Two minimisers \Rightarrow two area laws

- **Quantum regime ($N < N_*$):** kinetic + link balance gives $R \propto 1/N$, so $A \propto 1/N^2$ and $\rho_{\text{surf}} \propto N^3$.
- **Gravitational regime ($N > N_*$):** link + gravity balance gives $R \propto N$, so $A \propto N^2$ and $\rho_{\text{surf}} \propto 1/N$.

The mass-dependence of area flips by four powers at N_* . Temperature inherits this via $T \propto \sqrt{\rho_{\text{surf}}}$: particles get hotter with mass ($T \propto m^{3/2}$) while black holes get colder ($T \propto 1/\sqrt{M}$).

4.4 Conformal consistency

Because both γ (micro) and G (macro) arise from the same normalised links, they renormalise with ambient noise in lock-step: $G, \hbar \propto \rho_N$. Their ratio, and hence the pivot mass, is invariant in every conformal frame.

4.5 Cross-window consistency and re-windowing covariance (why the crossover shape is not arbitrary)

The “window” language in the main paper is doing more than bookkeeping. It functions as the EFT principle that *overlapping descriptions must agree on the same physical story* in their overlap domain. We make that implicit constraint explicit here and explain what extra assumption would be needed to promote a qualitative “smooth crossover” into a more unique functional form.

Scope note (assumption used for the pivot closure). In the present work we will *assume* the strong form of this principle *locally near the Planck pivot*, i.e. in the regime where the particle-like and shell-like windows most directly overlap and must patch. This should be read as a *closure assumption* (to be justified more microscopically by an explicit RG treatment of the coarse-grained free energy and its beta functions), not as a theorem proven here. The motivation is internal: (i) the framework treats windows as coarse-grained descriptions of one underlying graph/free-energy dynamics (no privileged window), so the overlap region should admit a fixed-form universality-class description; and (ii) without such a fixed-form assumption, the crossover shape contains an essentially arbitrary interpolation freedom that cannot be constrained by the two asymptotics alone. Away from the pivot, we allow finite-window/non-adiabatic corrections that can spoil exact covariance while preserving the weaker overlap-consistency constraints.

Cross-window consistency (implicit, already used). Let \mathcal{W}_1 and \mathcal{W}_2 be two coarse-grained descriptions (two observers, or the same observer using two different effective resolutions) whose domains of validity overlap. The framework repeatedly relies on the requirement:

In the overlap, \mathcal{W}_1 and \mathcal{W}_2 must agree on invariants and on the sign/ordering of physical effects (existence of a pivot, direction of monotonic trends, absence of catastrophic runaways), even if they parametrise them with different variables and renormalised couplings.

This is the hidden logic behind several “no-go” arguments in the main text (e.g. the impossibility of extending one asymptotic scaling law through the domain where a neighbouring window would predict the opposite trend).

Exact re-windowing covariance (a natural strengthening). The natural next tightening is to demand not just agreement in overlaps, but fixed *form* under change of window:

Under re-windowing (coarse-graining), the effective equations retain the same functional form; only parameters/couplings run.

This is an RG-style covariance principle. It is not strictly required for the Planck pivot to exist, but it is consistent with the fractal/nested ethos: no particular window is ontologically privileged, so the effective laws should live in a universality class stable under re-windowing.

What covariance does *not* fix by itself. Even exact covariance does not automatically single out one unique interpolating curve for $T(M)$; it fixes a *class* of allowed smooth crossovers. To pick a specific shape one needs an additional structural closure—for example an involutive “UV \leftrightarrow IR” mapping (a self-duality-like constraint) or an explicit rule for how independent contributions combine in the coarse-grained free energy.

A minimal closure that yields a “hyperbolic bounce”. One economical way to encode “saturation + minimisation” is to model the effective hardness/drive as two competing channels (particle-like localisation and shell-like extensification) that combine by a harmonic-mean rule. For instance, if a proxy $X(M)$ satisfies

$$X_{\text{part}}(M) \propto M^3, \quad X_{\text{BH}}(M) \propto \frac{1}{M}, \quad (8)$$

then the simplest smooth interpolant that reproduces both limits without introducing a new scale is

$$\frac{1}{X(M)} = \frac{1}{X_{\text{part}}(M)} + \frac{1}{X_{\text{BH}}(M)}. \quad (9)$$

If one takes $X \propto T^2$ (as suggested by the local scaling $T \propto \sqrt{\rho_{\text{surf}}}$), Eq. (9) yields a single smooth maximum near the equality point $X_{\text{part}} = X_{\text{BH}}$, i.e. near the pivot $M \sim M_P$. In this sense the ‘bounce’ is not an arbitrary curve-fit; it is the minimal analytic closure that (i) respects the two asymptotics, (ii) encodes gradual saturation of marginal localisation benefit, and (iii) is stable under reparametrisation across overlapping windows.

What a true self-duality would add (optional, stronger). If one further postulates an *involutive* re-windowing map exchanging the two sides, e.g. a reciprocity $M \mapsto M_P^2/M$ (or an equivalent statement about core vs. shell measures), then the pivot becomes a fixed point of a symmetry rather than merely a crossover. Such a requirement can further restrict the allowed interpolants, often selecting additive/quadrature forms as the simplest functions compatible with the involution. The present paper does not assume this stronger constraint; it is best viewed as a plausible next axiom if one wishes to make the crossover *more* unique than the generic smooth class already implied by windowing + minimisation.

5 Parent–child time mapping derivation

This appendix derives the quantitative relationship between the parent black hole’s evaporation and the child universe’s cosmological evolution.

5.1 Notation

t_p	Parent coordinate time (external ‘lab’ frame)
τ	Child cosmic time (interior FRW proper time)
$M(t_p)$	Parent BH mass; $M(t_p) = M_0 - (P/c_s^2)t_p$
$H(\tau)$	Child Hubble parameter
P	Constant evaporation power
$t_e = M_0 c_s^2 / P$	Total parent evaporation time

5.2 Derivation of $r_H = R_S$

From the Friedmann equation $H^2 = (8\pi G/3)\rho$ and the energy in a Hubble volume $E_H = (4\pi/3)r_H^3\rho c_s^2 = c_s^5/(2GH)$, setting $Mc_s^2 = E_H$ yields

$$H = \frac{c_s^3}{2GM}, \quad r_H = \frac{2GM}{c_s^2} = R_S. \quad (10)$$

5.3 Junction consistency

For a quasi-static shell separating Schwarzschild exterior from FRW interior, the Israel junction condition reads

$$\beta_+ - \beta_- = -\frac{4\pi G\sigma R}{c_s^4}, \quad \beta_{\pm} = \sqrt{1 - \frac{2GM}{c_s^2 R}} \text{ or } \sqrt{1 - \frac{H^2 R^2}{c_s^2}}. \quad (11)$$

At $R = R_S = r_H$ we have $\beta_+ = \beta_- = 0$, giving $\sigma = 0$. A tension-free shell is consistent with the trapping surface coinciding on both sides.

5.4 Multi-era accounting

During radiation and matter eras the cavity contains

$$\mathcal{N} = \left(\frac{R_{\text{cavity}}}{r_H} \right)^3 \gg 1 \quad (12)$$

causally disconnected Hubble patches, so the total mass distributes as $M c_s^2 = \mathcal{N} E_H$. As the Hubble sphere grows ($r_H \propto \tau$ in radiation, $\propto \tau^{2/3}$ in matter), $\mathcal{N} \rightarrow 1$ at DE onset. In the early phase, the two extreme time dilations (exterior ‘‘frozen star’’ and interior horizon slowdown) cancel, yielding $dt_p/d\tau|_{\text{early}} \sim \mathcal{O}(1)$.

5.5 DE-era time mapping and the phantom sign

From $H = c_s^3/(2GM)$ and $t_p = t_e(1 - H_0/H)$:

$$\frac{dt_p}{d\tau} = \frac{t_e H_0}{H^2} \dot{H} = -\frac{3}{2} t_e H_0 (1 + w_{\text{eff}}) = -\frac{3c_s^5}{4GP} (1 + w_{\text{eff}}), \quad (13)$$

using $t_e H_0 = c_s^5/(2GP)$. Forward-running clocks require $w_{\text{eff}} < -1$.

5.6 Self-consistency check

With $w_{\text{eff}} = -1 - \epsilon$ (constant $\epsilon > 0$), $\dot{H} = (3\epsilon/2)H^2$ gives $H(\tau) = H_*/[1 - (3\epsilon H_*/2)(\tau - \tau_*)]$ and

$$\Delta t_p = \frac{3c_s^5 \epsilon}{4GP} \cdot \frac{2}{3\epsilon H_*} = \frac{c_s^5}{2GPH_*} = \frac{M_* c_s^2}{P} = t_e(M_*). \checkmark \quad (14)$$

5.7 Planck-pivot regularisation: heat death

Across the Planck pivot the evaporation power flips:

Regime	M range	T	A	P
Gravitational	$M > M_P$	$\propto 1/\sqrt{M}$	$\propto M^2$	const
Quantum	$M < M_P$	$\propto M^{3/2}$	$\propto 1/M^2$	$\propto M^4$

In the quantum regime $dM/dt_p = -\alpha M^4/c_s^2$, so the remaining lifetime $\Delta t_p = (c_s^2/3\alpha)(1/M_f^3 - 1/M_P^3) \rightarrow \infty$ as $M_f \rightarrow 0$: a Planck remnant forms.

On the child side, UV exhaustion forces $\epsilon \rightarrow 0$ as $H \rightarrow H_P$. With $\epsilon(H) \approx \epsilon_0(1 - H/H_P)$, linearising near H_P yields an exponential approach to H_P in infinite child time. The Big Rip is softened into an asymptotic heat death.

5.8 Local vs. global caveat and epistemic consequence (expanded)

The Hubble radius $r_H(\tau) = c_s/H(\tau)$ is a local statement: in an approximately FRW bulk it is the same for all comoving observers at a given τ , but the *Hubble sphere* is always centred on the observer. In a finite cavity with a distinguished boundary (the shell), the phrase ‘‘the Hubble sphere fills the interior’’ is position-dependent: an observer at location x has filled the cavity only once

$$r_H(\tau) \gtrsim d_{\text{wall}}(x, \tau), \quad (15)$$

where d_{wall} is that observer’s proper distance to the wall.

Epistemic consequence. Because FRW symmetry makes $H(\tau)$ locally the same for all bulk observers, there is no local measurement revealing d_{wall} unless the wall enters the observer’s past light cone (producing anisotropy/inhomogeneity). In the late-time phantom phase ($w < -1$), r_H shrinks, making new causal contact with the wall *harder*, not easier: an observer who has not already seen wall-influenced signals will generically never see them.

5.9 Predicted value of $\epsilon = |1 + w|$ (expanded)

From the clock map one may write

$$\frac{dt_p}{d\tau} = \frac{3c_s^5 \epsilon}{4GP}, \quad (16)$$

where P is the evaporation power (in the constant-power regime). Writing $P \sim (c_s^5/G) f$ with f a dimensionless microphysical factor (greybody, numerical constants), gives

$$\frac{dt_p}{d\tau} \sim \frac{3\epsilon}{4f}. \quad (17)$$

For strong time dilation ($dt_p/d\tau \ll 1$) we require $\epsilon \ll f$. If $f \sim \mathcal{O}(1)$, then $\epsilon \ll 1$, consistent with current observational bounds on w .

5.10 Summary table and open questions

Result	Status
$r_H = R_S$ (Hubble radius equals Schwarzschild radius)	Derived
Junction consistency ($\sigma \approx 0$ at leading order)	Consistent
$w_{\text{eff}} < -1$ required for forward-running clocks	Derived
Early universe decoupled from evaporation ($\mathcal{N} \gg 1$)	Derived
Big Rip at constant ϵ , softened to heat death by Planck pivot	Derived
$ 1 + w = \epsilon \ll 1$ (small phantom departure)	Predicted

Open questions.

- A more detailed derivation of the $\mathcal{O}(1)$ early-era clock map (dynamical junction in the full index/noise description).
- First-principles calculation of P (and thus ϵ) from shell microphysics (greybody factors, species content, etc.).
- The mechanism that turns on window drift (DE) at the correct epoch when the Hubble sphere grows to encompass the cavity.

6 Hierarchical condensation and interior large-scale structure

This section records the more quantitative sketches behind the claim that stochastic fluctuations in the blackbody cavity seed a two-phase hierarchy of structures and, ultimately, a sponge-like cosmic web rather than a single central core.

6.1 Black-hole-branch seeds: radial drift and stall

Accretion rate. A minimal estimate (geometric cross-section plus gravitational focusing) gives

$$\dot{M} \simeq \pi R_{\text{mini}}^2 \rho_\gamma c_s \left(1 + \frac{v_{\text{esc}}^2}{c_s^2} \right), \quad v_{\text{esc}}^2 = \frac{2GM}{R_{\text{mini}}}. \quad (18)$$

Jeans limit in a photon gas. For a photon gas of temperature T ,

$$M_J(T) = \frac{\pi^{5/2}}{6} \frac{c_s^3}{\sqrt{G^3 \rho_\gamma}} \propto T^{-3/2}. \quad (19)$$

With $T \propto \rho_\gamma^{1/4} \propto M^{1/4}$ (local scaling), growth halts after $\mathcal{O}(10)$ quanta: a seed cannot grow to a macroscopic fraction of the parent shell mass.

Radial migration. The drift acceleration from anisotropic photon flux scales as $a_{\text{rad}} \propto A/M \propto N$, so a seed spirals toward the inner wall on a timescale

$$\tau_{\text{drift}} \sim \frac{R_{\text{shell}}}{a_{\text{rad}}} \propto \frac{R_{\text{shell}}}{N}. \quad (20)$$

Once parked against the wall it merges with neighbours, thickening the parent breathing layer into a sub-shell.

6.2 Particle-branch droplets: Brownian network and filament formation

For $N = 1$ droplets the net drift acceleration is parametrically smaller:

$$a_{\text{rad}} \propto \frac{A}{M} \approx \frac{L_{\text{min}}^2}{\mu}, \quad (21)$$

so motion is dominated by Brownian kicks with rms speed $v_{\text{rms}} \approx \sqrt{T/\mu}$. The mean free path

$$\lambda_B \sim \frac{1}{n_{\text{seed}} \pi R_{\text{mini}}^2} \quad (22)$$

can exceed seed–seed spacing, yielding a random walk through a time-dependent seed network.

Topology classification ($l = 0$ vs. $l \geq 1$). Droplets split into two populations:

1. **Scalar solitons ($l = 0$):** density fluctuations with no phase winding (collisionless; dark matter).
2. **Topological defects ($l \geq 1$):** droplets trapping a phase winding (dissipative; baryonic matter).

Combinatorial statistics favour the simpler $l = 0$ population by a factor ~ 5 , naturally producing $\Omega_{\text{DM}}/\Omega_b \approx 5$.

Morphological amplifier. Droplets clustered in a caustic source a higher local noise background τ , which triggers a super-linear downshifting response in neighbours ($\varepsilon \propto -\tau^\gamma$, $\gamma > 1$), deepening the potential well and capturing further matter. A minimal modified diffusion–drift model is

$$\partial_t n = D \nabla^2 n - \nabla \cdot (n \mathbf{v}_{\text{drift}}) + \alpha_{\text{morph}} n^2, \quad (23)$$

where D is set by Brownian motion and $n \mathbf{v}_{\text{drift}}$ by the seed potential. Numerical integration (future work) yields filamentary, sponge-like networks.

6.3 Impact on early-epoch anisotropies

Mini-shell seeds induce milli-Kelvin scale temperature dipoles (in external units) in the photon bath on angular scales $\theta \sim R_{\text{mini}}/R_{\text{shell}}$. A toy spectrum is

$$\frac{\Delta T}{T}(\ell) \approx \left(\frac{M_{\text{seed}}}{M_{\text{shell}}} \right)^{1/2} \ell_{\text{seed}}^2 e^{-\ell/\ell_{\text{seed}}}, \quad \ell_{\text{seed}} \sim \frac{R_{\text{shell}}}{R_{\text{mini}}}. \quad (24)$$

Because $M_{\text{seed}} \ll M_{\text{shell}}$, such anisotropies remain below the cavity’s intrinsic shot noise and do not spoil breathing equilibrium.

Local vs. external units. Internal observers measure every energy against a shrunken local unit (their $k_B T$ rescales by χ^{-1}), so the same dipole appears as $(\Delta T/T)_{\text{loc}} = \chi^{-1}(\Delta T/T)_{\text{ext}}$. Relative contrast therefore stays $\mathcal{O}(10^{-3})$ across frames; only the baseline temperature differs.

6.4 Effective geometry of the interior

If radiation energy drains preferentially into seeds, a simple refractive index profile $\chi(r)$ can decrease toward the centre. Ray trajectories diverge, giving an effective negative curvature. A minimal parameterisation,

$$\chi(r) = 1 + \epsilon \left(1 - \frac{r}{R_{\text{shell}}}\right), \quad \epsilon \sim 10^{-2}, \quad (25)$$

gives a curvature scale $K \approx -2\epsilon/R_{\text{shell}}^2$, consistent with a weak but everywhere negative curvature and an “optical horizon” in which timelike paths require arbitrarily long proper times to catch the wall.

6.5 Open questions (kinetics of the stochastic cascade)

A full kinetic description of the stochastic cascade remains open, including merger rates of sub-shells and the resulting statistical distribution of particle-branch vs. shell-branch species in the interior bath.

7 Discussion addendum: infinite zooms, finite energies

This section collects clarifications that support the eternal-hierarchy picture without requiring infinite energy, information, or computation in any single window.

7.1 Infinite zoom cascade with local time at every level

The clique graph admits an unbounded hierarchy of nested bubbles. Each zoom-in by a fixed factor opens a fresh UV band, and the scale-window carried by an observer shrinks in physical metres as it descends the hierarchy. Thus the newly opened UV band always carries finite energy/entropy in local units, and an endless cascade of downshifts is possible without violating global bounds.

Every level has a nonzero scale-force slope $\sigma(t)$, so time exists everywhere: “timelessness” from outside is merely the superposition of incommensurate local clocks.

7.2 Scale-window principle prevents divergences

An observer carries a window $[\Lambda_{\text{IR}}, \Lambda_{\text{UV}}]$ centred on its effective Planck cell $L_{\min}(\Lambda_{\text{obs}})$. Energy, entropy, and computational capacity inside the window are $\mathcal{O}(1)$ in local units, independent of how many zooms are performed. To access deeper UV information, one must shrink; the energetic cost of shrinking equals the potential energy of newly opened modes, preventing “free lunches.”

7.3 Infinite nested zooms inside a finite-lifetime shell

External lifetime $\tau_{\text{ext}} = M_{\text{init}}c_s^2/P_0$ is finite, but hyperbolic dilation near the wall makes internal proper time unbounded. An internal observer can execute countably many zoom steps before $t = \tau_{\text{ext}}$. Each level has its own constant-power cascade, and the geometric series of sub-energies sums to $M_{\text{init}}c_s^2$.

7.4 Impedance matching: why constant power is necessary for nesting

Standard Hawking radiation ($P \propto M^{-2}$) forbids nesting: an inner shell would radiate $P_{\text{inner}} \gg P_{\text{outer}}$, vaporising the parent from inside. Constant power ($P \propto M^0$) is the unique scaling that allows impedance matching: $P_{\text{inner}} = P_{\text{outer}} = P_0$.

7.5 Seed gradient, filament fate, and wall overtake

- Seeds ($N > N_*$) form preferentially near the wall where the local Jeans mass is lowest.
- Radiation-pressure drift moves seeds inward; Brownian droplets ($N = 1$) diffuse in a sponge-like filament network.
- As the shell contracts, rising noise variance and negative optical curvature compress and eventually absorb the filaments; external time for capture is finite while internal time can diverge logarithmically.

7.6 Cosmic horizon as a Type-II gravitational droplet

An apparent cosmic horizon in an FRW patch can be reinterpreted as the inner surface of a larger-scale shell (Type-II, outward-oriented) when zooming out one level. In the super-observer's units, the region outside our horizon contains deeper-UV modes (higher noise floor, shorter L_{\min}). From our interior vantage that region is stretched to infinite volume and redshifted to near-zero temperature, giving the impression of an empty cold outside.

7.7 Why the hierarchy alternates 2D shells and 3D bulks

A finite UV window forces resolved subgraphs toward integer geometries: with only short links the least-action arrangement is a regular lattice. On a closed interface the lattice is effectively 2D (a shell); in a filled region it is 3D (a bulk). Fractional-D tilings require holes (entropy loss) or weighted long links (energy cost), so the free energy “snaps” to the nearest integer.

The e conjecture (heuristic). If we weight bulk and shell with probabilities p and $1 - p$, the average dimension is $D_{\text{eff}} = 3p + 2(1 - p) = 2 + p$. Setting $D_{\text{eff}} = e$ gives $p = e - 2 \simeq 0.718$, corresponding to an approximate 3:1 split in log-depth (three bulk units per shell unit). This remains a motivated conjecture rather than a derived necessity.

7.8 Shell-thickness scaling and observational outlook

Rebalancing Stefan–Boltzmann flux (out) with Unruh absorption (in) gives

$$T(R) = \left(\frac{P_0}{4\pi\sigma_{\text{SB}}R^2} \right)^{1/4} \propto R^{-1/2}. \quad (26)$$

Requiring optical depth corresponding to a mean-free-path count $n_* \sim 10\text{--}30$ yields

$$\boxed{\frac{\delta R}{R} = K \left(\frac{M}{10 M_\odot} \right)^{-1/2}}, \quad K \simeq 0.02. \quad (27)$$

Log-step reinterpretation. Let $\Delta\sigma_{\text{shell}}$ be the drop in log-scale coordinate across one shell. For $\Delta\sigma_{\text{shell}} \ll 1$, $\delta R/R \simeq \Delta\sigma_{\text{shell}}$ (linearising $R \rightarrow Re^{-\Delta\sigma}$). Thus

$$\Delta\sigma_{\text{shell}}(M) = K \left(\frac{M}{10 M_\odot} \right)^{-1/2}. \quad (28)$$

If the bulk:shell bookkeeping remains 3:1, then $\Delta\sigma_{\text{bulk}} = 3\Delta\sigma_{\text{shell}}$ and $\Delta\sigma_{\text{cycle}} = 4\Delta\sigma_{\text{shell}}$.

Numerical check.

M/M_\odot	$\delta R/R$	$\Delta\sigma_{\text{shell}}$	$q \equiv R/\delta R$
10	0.020	0.020	50
30	0.012	0.012	83
4×10^6	3×10^{-4}	3×10^{-4}	3×10^3

Finite-shell corrections in binaries. For a companion approaching within $\lesssim 3\delta R$, tidal quadrupoles and overlapping photon baths can add corrections, but at LIGO-band separations these effects are far below current sensitivity. They become plausible targets for extreme mass-ratio inspirals grazing supermassive shells or for horizon-scale VLBI probing δR directly.

8 Key formulae (quick reference)

- **Droplet free energy:** $F(R; N) = \frac{\alpha N \hbar c}{R} + \frac{\eta_0}{2} \frac{N^2}{R^3} - \frac{G(N\mu)^2}{R}$.
- **Critical occupancy:** $N_* = \alpha \hbar c / (G\mu^2) \sim \mathcal{O}(1)$ in window units.
- **Dual temperature laws:** $T_{\text{sol}} \propto m^{3/2}$ (particle side) vs. $T_{\text{BH}} \propto 1/\sqrt{M}$ (shell side).
- **Evaporation power:** $P_{\text{rad}} \propto \text{const}$ (gravitational regime), crossing to $P \propto M^4$ below the Planck pivot.
- **Duality lock (late era):** $H = c_s^3/(2GM)$ and $r_H = c_s/H = R_S$.
- **Clock map (DE era):** $\frac{dt_p}{d\tau} = -(3c_s^5/(4GP))(1+w_{\text{eff}})$ requiring $w_{\text{eff}} < -1$.

9 Related approaches (orientation pointers)

- Gravastars / dark-energy stars (Mazur & Mottola): de Sitter core matched to Schwarzschild exterior via a thin relativistic shell.
- String fuzzballs (Mathur): horizon replaced by microstate “surface”.
- Planck stars (Rovelli & Vidotto): loop-quantum-gravity bounce core.
- Quantum N-portrait / graviton BEC (Dvali & Gomez): black hole as critical condensate.
- Regular black holes / modified gravity (Bardeen; Hayward; asymptotic safety; nonlocal models): de Sitter-like cores with transition layers.
- Membrane paradigm / firewalls (Thorne–Price–Macdonald; Almheiri et al.): effective dissipative or energetic surface near the horizon.

9.1 Piece-wise arrows of time in an eternal hierarchy

Each cavity has a locally defined arrow of time: a monotone scale direction ($\dot{\sigma} < 0$) plus dissipation (constant-power leakage) makes the dynamics irreversible. Each level has finite accessible information and energy in its window, but the hierarchy is past-and-future unbounded in scale.

10 Spectral-dimension selection: why e (and hence 3)

This appendix records a heuristic argument that a neutral information-per-cost criterion selects an effective coordination $k \approx e$, with integer feasibility biasing to $k = 3$ and thus spectral dimension $D_s \approx 3$.

10.1 Setup and assumptions

- **Linear per-node budget (volume-normalised links):** outbound maintenance/throughput is linear in link weights, with a minimum useful link weight (noise/quantisation floor) making each additional active neighbour consume approximately constant budget.
- **Local choice and path entropy:** with k active neighbours used roughly uniformly, the local branching entropy rate is $\hat{H} = \ln k$ (nats per step) while cost scales $\sim k$.
- **UV window and adiabaticity:** a minimum cell bounds admissible protocol speed via $\tau \gtrsim L/\omega_{1,P}$ for Fisher/thermodynamic distance L .
- **Window invariance (unit-fixing):** keep local units stable during protocols, implying a roughly constant excess-power ceiling and thus constant metric speed along optimal schedules.

10.2 Local efficiency objective

Define a unit-free efficiency (benefit per cost)

$$\eta(k) := \frac{\ln k}{k}. \quad (29)$$

Continuous optimisation gives

$$\eta'(k) = \frac{1 - \ln k}{k^2} = 0 \implies \ln k = 1 \implies k = e. \quad (30)$$

With integer feasibility, the nearest stable coordination is $k = 3$.

An equivalent MDL-style derivation: representing M alternatives with radix k costs $\propto k(\ln M / \ln k) = (\ln M)[k / \ln k]$, minimised at $\ln k = 1$.

10.3 Time-dissipation gauge

Finite-time thermodynamics gives a bound

$$\Sigma \gtrsim \frac{L^2}{\tau}, \quad (31)$$

where Σ is total dissipation. UV and window invariance fix a constant metric-speed gauge that saturates this bound on geodesic schedules. These constraints set the feasible time/dissipation scale; the local allocation $k \approx e$ follows from the neutral information-per-cost objective above.

10.4 From branching to spectral dimension

On near-regular, locally homogeneous graphs, effective coordination controls diffusivity/expansion and thus the spectral dimension via heat-kernel scaling. With $k \approx e$ and integer feasibility, $k = 3$ is preferred, yielding $D_s \approx 3$ in the vacuum fixed point.

10.5 Caveats and empirical checks

If per-link costs are nonlinear or minimum useful weights vary strongly, the optimum shifts. Suggested checks:

- optimise $(\ln k)/k$ under linear row-sum budgets and noise floors in vacuum-relaxation simulations and measure the resulting mean coordination,
- verify constant-metric-speed schedules numerically saturate $\Sigma = L^2/\tau$ between nearby equilibria,
- measure heat-kernel slopes to confirm $D_s \approx 3$ robustness under heterogeneity.