Supplementary Information: Forces from Kink Sheets and Internal Rotors

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S1 Constitutive Calibration from Kink Sheets

Sheet solution recap. The kink-sheet profile obeys the same equations as in the main text: thickness $\lambda_* = \sqrt{\kappa/\beta_{\rm pot}}/w_*$ and tension $\sigma_* = c_{\rm sheet}\, w_* \sqrt{\kappa\beta_{\rm pot}}$ with $c_{\rm sheet} \approx 4$ from the Euler–Lagrange solution (main-text Appendix A).

Matching to Maxwell form. Treat a single 2π sheet as a slab of width λ_* with nearly constant phase gradient $|\partial_z\phi| \approx 2\pi/\lambda_*$. Identify $E_z := \partial_z\phi$; the energy per area stored in that slab is

$$\sigma_{\rm grad} \;\approx\; \frac{1}{2}\,\varepsilon_0^{\rm eff}\, E_z^2\,\lambda_* \;=\; \frac{1}{2}\,\varepsilon_0^{\rm eff}\left(\frac{2\pi}{\lambda_*}\right)^2\lambda_*\,. \tag{1}$$

Equating $\sigma_{\rm grad} = \sigma_*$ gives

$$\varepsilon_0^{\rm eff} \;=\; \frac{\sigma_*\,\lambda_*}{2\pi^2} \;=\; \frac{c_{\rm sheet}}{2\pi^2}\,\kappa, \qquad \mu_0^{\rm eff} \;=\; \frac{1}{c_s^2\,\varepsilon_0^{\rm eff}} \;=\; \frac{2\pi^2}{c_{\rm sheet}}\,\frac{1}{\kappa^2 w_*^2}\,, \tag{2}$$

using $\sigma_*\,\lambda_* = c_{\rm sheet}\kappa$ and $c_s^2 = \kappa w_*^2$. The order-one factor $c_{\rm sheet}$ encodes the detailed profile; smoother profiles shift only this prefactor.

Checks. (i) $\varepsilon_0^{\rm eff}\mu_0^{\rm eff} = 1/c_s^2$ automatically. (ii) In local conformal units, $\kappa w_*^2 = c_s^2$ is invariant, so $\varepsilon_0^{\rm eff}$ and $\mu_0^{\rm eff}$ co-vary to keep light cones fixed. (iii) The calibration is insensitive to the microscopic amplitude dip as long as $\Delta\phi = 2\pi$ across λ_*.

S2 Isotropy ⇒ SU(3) Connection

Degenerate subspace. In a coarse grain with orthonormal spatial frame $\{\boldsymbol{e}_a\}$, the three *soft* gradient eigenmodes are degenerate by isotropy. Concretely, choose an orthonormal basis $\{u_a\}_{a=1}^3$ spanning the $\ell = 1$ subspace of the grain kernel (the "gradient" sector). Any projected phase fluctuation in this subspace can be written as $\delta\phi = \sum_{a=1}^3 \Psi_a\, u_a$ with complex coefficients $\Psi_a \in \mathbb{C}$. Collect the coefficients into $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \mathbb{C}^3$. The overall scale $|\Psi|$ is set by the window/energy of the excitation, while the low-energy degree of freedom is the *orientation* of Ψ inside the degenerate subspace. This orientation is acted on by SU(3).

Connection from parallel transport. Minimizing misalignment between neighbouring grains yields, in the continuum, a Lie-algebra-valued connection. The clean route is discrete-first: define a local rotor state $\Psi_i \in \mathbb{C}^n$ (with $n = 2$ on surfaces and $n = 3$ in bulk) and an alignment energy between neighbouring grains

$$E_{ij}(U_{ij}) \;=\; \|\Psi_i - U_{ij}\Psi_j\|^2, \qquad U_{ij} \in {\rm SU}(n)\,. \tag{3}$$

For fixed (Ψ_i, Ψ_j) this is the unitary Procrustes problem: the minimizer is the unitary that maximizes $\mathrm{Re}(\Psi_i^\dagger U_{ij} \Psi_j)$. Under a local change of basis in the degenerate subspace, $\Psi_i \to V_i \Psi_i$ with $V_i \in \mathrm{SU}(n)$, the optimal link transforms as a lattice gauge field,

$$U_{ij} \;\to\; V_i\, U_{ij}\, V_j^{-1}\,. \tag{4}$$

On slowly varying configurations one writes $U_{i,i+\mu} \approx \exp\{-iaA_\mu(x)\}$, defining a continuum connection $A_\mu(x) \in \mathfrak{su}(n)$, and the plaquette holonomy gives curvature $\prod_\Box U_{ij} \approx \exp\{-ia^2 F_{\mu\nu}\}$. In a local trivialization this reduces to the familiar form

$$A_\mu \;=\; -\,i\,U^{-1}\partial_\mu U \;\in\; \mathfrak{su}(3), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{5}$$

On a 2D submanifold only two gradients remain degenerate, restricting U to SU(2); along a 1D filament only the longitudinal gradient survives, giving U(1). There is no fourth independent gradient in $d = 3$, so the ladder stops at SU(3).

Equivalent spectral view (Wilczek–Zee). The same connection arises from the spectral language natural in this framework. Let $P(x)$ be the rank-n projector onto the degenerate soft subspace of the grain kernel within the observational window, and choose a local orthonormal frame $\{|u_a(x)\rangle\}_{a=1}^n$ spanning $\mathrm{Im}\,P(x)$. Parallel transport within the eigenbundle is governed by the non-Abelian Berry (Wilczek–Zee) connection

$$(A_\mu)_{ab} \;=\; i\,\langle u_a(x)\,|\,\partial_\mu u_b(x)\rangle, \tag{6}$$

with the same gauge freedom $|u_a\rangle \to V_{ab}(x)|u_b\rangle$ and $A_\mu \to V A_\mu V^{-1} - i(\partial_\mu V)V^{-1}$. We include this as a consistency lens: it is the same emergent gauge structure expressed in eigenbundle (spectral) coordinates rather than alignment-energy (variational) coordinates.

Rotor stiffness and coupling. Expanding the gradient energy in covariant form gives the Yang–Mills term with effective coupling $g_{\mathrm{eff}}^{-2} \propto \kappa w_*^2 L_{\mathrm{rot}}^2$, where L_{rot} is the coarse-grain size. This scaling feeds into G_F^{eff} and σ_{string} quoted in the main text.

S3 Anomaly Constraints as Topological Consistency

In Quantum Field Theory, anomalies signal the breakdown of current conservation ($\partial_\mu J^\mu \neq 0$) at the loop level. In the Soliton–Noise framework, conservation laws are topological (winding numbers). A topological invariant cannot be "broken a little bit"—it is either conserved (integer) or ill-defined (field tear).

Hypothesis: Anomaly Freedom = Graph Consistency. We propose that the Infinite-Clique Graph cannot support a phase field configuration that corresponds to an anomalous particle content. An "anomaly" in the continuum limit manifests as a topological obstruction on the discrete graph (e.g., the inability to define a single-valued phase map globally).

- **Cancellation Condition:** The requirement that the global phase winding on a compact graph sums to zero is rigorous.
- **Spectrum Selection:** This topological consistency likely acts as a selection rule, allowing only those sets of defects (charges) whose anomalies cancel exactly.

The standard cancellation conditions:

$$\mathrm{U}(1)^3: \quad \sum q_i^3 = 0, \qquad \mathrm{grav}^2\mathrm{U}(1): \quad \sum q_i = 0, \tag{7}$$

$$\mathrm{SU}(2)^2\mathrm{U}(1): \quad \sum q_i\, T_2 = 0, \qquad \mathrm{SU}(3)^2\mathrm{U}(1): \quad \sum q_i\, T_3 = 0. \tag{8}$$

Scope and correction. We *do not* claim that a naive "grade-to-charge" toy assignment automatically satisfies these equalities. Rather, the role of this section is to state the structural expectation: if the continuum limit admits an effective chiral gauge description, then the underlying graph/topology must enforce the analogue of anomaly freedom as a *consistency condition* on the admissible spectrum and charge embedding. In practical terms, this becomes a selection rule on how U(1) winding (sheet charge) can be combined with SU(2)/SU(3) rotor representations across all stable defect types so that the emergent long-range currents remain exactly conserved in the low-energy window. Making this explicit for a concrete charge assignment is an open item.

S4 Magnetism from Boosted Sheets

We rigorously derive the magnetic field $\mathbf{B}$ from the Lorentz boost of a static sheet bundle, confirming the Ampère–Maxwell prefactor.

Static Configuration. Consider a bundle of kink sheets with density ρ_s oriented with normal $\hat{\mathbf{n}}$ in the rest frame K. The phase-defect current 4-vector is purely temporal (representing static charge density):

$$J^{\mu}_{(K)} = (\rho_c c_s, \mathbf{0}), \quad \text{where } \rho_c \propto \rho_s. \tag{9}$$

In this frame, the coarse-grained electric field is $\mathbf{E} = E_0 \rho_s \hat{\mathbf{n}}$ and $\mathbf{B} = 0$.

Boosted Frame. Boost to a frame K' moving with velocity $\mathbf{v}$ relative to K. The Lorentz transformation Λ^{μ}_{ν} yields the new current:

$$J'^{\mu} = \Lambda^{\mu}_{\nu} J^{\nu} = (\gamma \rho_c c_s, -\gamma \rho_c \mathbf{v}). \tag{10}$$

Here $\gamma = (1 - v^2/c_s^2)^{-1/2}$. The spatial component represents a current density $\mathbf{J'} = -\gamma \rho_c \mathbf{v}$.

Field Transformation. The phase-sector stress-energy tensor implies the fields transform as components of $F_{\mu\nu}$. Explicitly, the transverse magnetic field arising from the boost is:

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c_s^2} \right) = -\gamma \frac{\mathbf{v} \times \mathbf{E}}{c_s^2}. \tag{11}$$

Since the current $\mathbf{J'}$ effectively generates this field, we check consistency with the Ampère–Maxwell law $\nabla \times \mathbf{B} = \mu_0^{\text{eff}} \mathbf{J}$. For a sheet moving at $\mathbf{v}$, the effective current is transverse to the normal. The factor $1/c_s^2$ appears naturally from the boost, matching the constitutive requirement $\varepsilon_0^{\text{eff}} \mu_0^{\text{eff}} = 1/c_s^2$ derived in Section S1. Thus, magnetism emerges as the relativistic kinematic consequence of moving phase defects.

S5 The SU(3) Ceiling: Spectral Gap Derivations

We quantify the "Geometric Ceiling" argument by comparing the energy cost of linear gradient modes (spanning SU(3)) versus higher-order deformation modes in an elastic grain.

Mode Spectrum. Consider a grain of radius R governed by the phase stiffness action $S = \frac{\kappa w_*^2}{2} \int (\nabla \phi)^2 dV$. We expand the phase fluctuations in spherical harmonics $\phi(r, \theta, \varphi) = f(r) Y_{\ell m}(\theta, \varphi)$.

- **Dipole/Gradient Modes ($\ell = 1$):** These correspond to linear gradients $\phi \sim x, y, z$. For a linear profile, $\nabla \phi = \text{const}$. The energy density is uniform.
- **Quadrupole/Shear Modes ($\ell = 2$):** These correspond to deformations like $\phi \sim xy$. The gradient is $\nabla \phi \propto r$. The energy density scales as r^2.

Energy Gap Calculation. Solving the Laplace eigenvalue problem $-\nabla^2\phi = \lambda\phi$ on the ball with Neumann boundary conditions (free surface):

- The $\ell = 1$ eigenvalue is determined by the first root of $j_1'(kR) = 0$. $x_{1,1} \approx 2.08$. Energy $E_1 \propto (2.08/R)^2 \approx 4.33/R^2$.
- The $\ell = 2$ eigenvalue is determined by the first root of $j_2'(kR) = 0$. $x_{2,1} \approx 3.34$. Energy $E_2 \propto (3.34/R)^2 \approx 11.16/R^2$.

The relative spectral gap is large:

$$\frac{\Delta E}{E_1} = \frac{E_2 - E_1}{E_1} \approx \frac{11.16 - 4.33}{4.33} \approx 1.58. \tag{12}$$

The $\ell = 1$ subspace (3 modes) is separated from the $\ell = 2$ subspace (5 modes) by a gap of $\sim 150\%$ the ground state energy. In a thermal or noisy background, the 3-dimensional $\ell = 1$ tangent space is effectively isolated, protecting the SU(3) symmetry while suppressing higher-dimensional gauge groups (which would require mixing with $\ell = 2$). This confirms SU(3) as the energetic ceiling for internal symmetries in a 3D vacuum.