Supplementary Information: Dark Energy (Draft)

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1 Notation and symbol map

- c_s: universal signal speed.
- $F(t)$, $a_{\rm eff}$: conformal scale factor for local units; $a_{\rm eff} = 1/F$.
- $H_{\rm eff}$: effective Hubble parameter for local units, $H_{\rm eff} = -\partial_t \ln F$.
- $k_{\rm min}, k_{\rm max}$: IR/UV edges of the sliding window; $\mathcal{N} = \ln(k_{\rm max}/k_{\rm min})$.

2 Sliding-window formalism and dark energy

We formalize the effective density of a windowed, scale-invariant vacuum.

Setup. Let the measured vacuum energy be a windowed integral

$$\rho_{N,\rm eff}(t) = \int_{k_{\rm min}(t)}^{k_{\rm max}(t)} S(k)\,\mathrm{d}k. \tag{1}$$

Define the logarithmic bandwidth $\mathcal{N} = \ln(k_{\rm max}/k_{\rm min})$ and parameterize edge evolution by $k_{\rm max} \propto F^{\alpha}$, $k_{\rm min} \propto H_{\rm eff}^{\beta}$.

Leibniz rule. Differentiating yields

$$\frac{\mathrm{d}\rho_{N,\rm eff}}{\mathrm{d}t} = S(k_{\rm max})\,\frac{\mathrm{d}k_{\rm max}}{\mathrm{d}t} - S(k_{\rm min})\,\frac{\mathrm{d}k_{\rm min}}{\mathrm{d}t}. \tag{2}$$

For $S(k) = A\,k^{-p}$ with $p \approx 1$, one has $\rho_{N,\rm eff} \approx A\ln(k_{\rm max}/k_{\rm min})$ and $\partial\rho_{N,\rm eff}/\partial\ln k \approx A$ at the edges, giving

$$\frac{\mathrm{d}\rho_{N,\rm eff}/\mathrm{d}t}{\rho_{N,\rm eff}} \;\approx\; \frac{1}{\mathcal{N}}\Big(\alpha\,\frac{\mathrm{d}F/\mathrm{d}t}{F} - \beta\,\frac{\mathrm{d}H_{\rm eff}/\mathrm{d}t}{H_{\rm eff}}\Big). \tag{3}$$

Effective equation of state. Matching to a smooth component with density ρ_X and w_X gives $\mathrm{d}\rho_X/\mathrm{d}t/\rho_X = -3(1+w_X)H_{\rm eff}$, hence

$$w_X + 1 \;\approx\; \frac{\alpha\,\mathrm{d}F/\mathrm{d}t/F - \beta\,\mathrm{d}H_{\rm eff}/\mathrm{d}t/H_{\rm eff}}{3\,H_{\rm eff}\,\mathcal{N}}. \tag{4}$$

The paper uses $\alpha = \beta = 1$ as a natural choice.

2.1 Edge derivation and sensitivity

UV edge from local resolution. In local units, the smallest resolvable physical feature is set by the stabilized soliton scale $L_*(t)$. A resolved wavenumber obeys $k\,L_* \gtrsim \mathcal{O}(1)$, hence the UV cutoff tracks $k_{\max}(t) \sim \kappa_{\rm UV}/L_*(t)$. With $F(t) := 1/a_{\rm eff}(t)$ and $L_* \propto 1/F$, one obtains

$$k_{\max}(t) \;\propto\; F(t)^{\alpha}, \qquad \alpha = 1 \text{ (natural)}. \tag{5}$$

IR edge from causal/response reach. The largest coherently probed comoving scale in a Hubble time is set by the response cone: $\lambda_{\rm IR} \sim c_s/H_{\rm eff}$ up to an $\mathcal{O}(1)$ factor, yielding an IR cutoff

$$k_{\min}(t) \;\sim\; \kappa_{\rm IR}\,\frac{H_{\rm eff}(t)}{c_s} \;\propto\; H_{\rm eff}(t)^{\beta}, \qquad \beta = 1 \text{ (natural)}. \tag{6}$$

Signs and monotonicity. Under the homogeneous secular shrinkage of matter (increasing F) and a decelerating expansion (decreasing $H_{\rm eff}$ at late times), one finds $\mathrm{d}k_{\max}/\mathrm{d}t > 0$ and $\mathrm{d}k_{\min}/\mathrm{d}t < 0$. Both edges remain positive; what governs $w + 1$ is the bandwidth $\mathcal{N} = \ln(k_{\max}/k_{\min})$.

2.2 Principle of Optimal Bandwidth (Constructive Derivation)

We formalize the edge evolution by minimizing an information−complexity functional

$$\mathcal{J}(k_{\min}, k_{\max}) \;=\; -\,\mathrm{SNR}_{\rm band}(k_{\min}, k_{\max}) \;+\; \lambda\,\mathcal{C}(k_{\min}, k_{\max}), \qquad \lambda > 0, \tag{7}$$

where $\mathrm{SNR}_{\rm band}$ increases with the log−bandwidth when $S(k)$ is stationary and $\mathcal{C}$ penalizes bandwidth/edge motion. With $S(k) = A/k$ and a local-unit variance σ^2 that decreases with bandwidth, one has the schematic scaling

$$\mathrm{SNR}_{\rm band} \;\propto\; \int_{k_{\min}}^{k_{\max}} \frac{dk}{k} \;=\; \ln\!\left(\frac{k_{\max}}{k_{\min}}\right), \qquad \mathcal{C} \;=\; \alpha_c \ln k_{\max} \;-\; \beta_c \ln k_{\min}, \tag{8}$$

with positive coefficients α_c, β_c that encode the asymmetric costs of pushing the UV edge (resolution/complexity) and the IR edge (memory/causal reach). The gradient conditions

$$\frac{\partial \mathcal{J}}{\partial \ln k_{\max}} \;=\; -1 + \lambda\,\alpha_c \;<\; 0, \qquad \frac{\partial \mathcal{J}}{\partial \ln k_{\min}} \;=\; +1 - \lambda\,\beta_c \;>\; 0, \tag{9}$$

for suitable $\lambda \in (0, \min\{1/\alpha_c,\, 1/\beta_c\})$ imply $\mathrm{d}k_{\max}/\mathrm{d}t > 0$ and $\mathrm{d}k_{\min}/\mathrm{d}t < 0$ along descent. Thus, under broad conditions consistent with $S(k) \propto 1/k$, optimal bandwidth selection drives edges in the observed directions.

3 Energy–momentum accounting and conservation

We split the effective stress–energy as

$$T^{\mu\nu}_{\rm tot} \;=\; T^{\mu\nu}_{\rm matter} \;+\; T^{\mu\nu}_{\rm bg}, \qquad T^{\mu\nu}_{\rm bg} \text{ encodes the homogeneous, windowed background.} \tag{10}$$

By construction (Bianchi identity), Einstein's equations enforce covariant conservation $\nabla_\mu T^{\mu\nu}_{\rm tot} = 0$. In our construction, the windowed background is $\rho_{\rm eff}(t) = \int_{k_{\min}(t)}^{k_{\max}(t)} S(k)\,\mathrm{d}k$, whose time derivative follows the Leibniz rule. Matching to an effective fluid with equation of state w by $\dot\rho_{eff}/\rho_{eff} = -3H_{eff}(1 + w)$ identifies w and corresponds to choosing $Q = 0$ in local units (no explicit energy exchange with clustered matter at leading order).

3.1 Conservation: internal *to* kinetic balance

Here we make explicit that scale – reduction converts internal (free) energy into center – of – mass kinetic energy within the same matter stress – energy tensor.

Adiabatic reduction. In the adiabatic regime, the soliton's internal size σ relaxes rapidly to $\sigma^*(x)$ set by the local background. The relaxed internal energy is

$$E_{\rm eq}(x) \;=\; -\,\frac{A_\eta^2}{4A_\gamma}\,f\big(\tau(x)\big)^2, \tag{11}$$

and the conservative force on the center of mass is $\mathbf{F} = -\nabla E_{\rm eq}(x)$. Writing the matter stress – energy as $T^{\mu\nu}_{\rm matter} = T^{\mu\nu}_{\rm bulk} + T^{\mu\nu}_{\rm int}$, one finds the standard mechanical energy balance along the worldline (lab frame for simplicity)

$$\frac{dK}{dt} \;=\; \mathbf{F}\cdot\mathbf{v} \;=\; -\,\frac{dE_{\rm eq}}{dt}, \tag{12}$$

so the increase of kinetic energy K equals the decrease of internal free energy $E_{\rm eq}$. In covariant form, $u^\mu\nabla_\mu(\mathcal{K}+E_{\rm eq}) = 0$. Thus the internal *to* bulk energy transfer is an internal bookkeeping within $T^{\mu\nu}_{\rm matter}$; there is no violation of conservation.

4 Early-universe suppression via growth of inhomogeneity

The homogeneous drift arises from locally inhomogeneous downshifting averaged over many patches. Let Δ_τ^2 denote the variance proxy of the noise field. For scales well inside the horizon and in the linear regime, $\Delta_\tau^2(a_{\rm eff}) \propto D(a_{\rm eff})^2$. Assuming the homogeneous drift rate is proportional to this variance,

$$H_{\rm eff}(a) \;\propto\; \begin{cases} \text{const } \times D(a)^2 \;\sim\; a^2, & \text{matter era,} \\ \text{const } \times D(a)^2 \;\approx\; \text{constant (tiny),} & \text{radiation era.} \end{cases} \tag{13}$$

Thus, during the radiation-dominated epoch, growth is suppressed and the variance stays nearly constant at a tiny level, making the homogeneous drift negligible.

5 Small spectral tilt and practical bounds

Consider $S(k) = A\,k^{-1+\epsilon}$ with $|\epsilon| \ll 1$. The windowed background becomes $\rho_{eff} = A\,\mathcal{N}_\epsilon$ with $\mathcal{N}_\epsilon := (k_{\rm max}^\epsilon - k_{\rm min}^\epsilon)/\epsilon$. Expanding for small ϵ gives

$$w+1 \;\approx\; \frac{\alpha-\beta(1+q)}{3\,\mathcal{N}}\Big[1 \;-\; \frac{\epsilon}{2}\,\big(\overline{\ln k}\big)_{\rm band} \;+\; \mathcal{O}(\epsilon^2)\Big]. \tag{14}$$

Thus, small tilts correct $w+1$ only at order $\mathcal{O}(\epsilon/\mathcal{N})$ after accounting for the dominant $1/\mathcal{N}$ suppression.

6 Two-scale Buchert backreaction: estimate

We sketch a Buchert-style estimate of the kinematical backreaction term

$$Q_{\mathcal{D}} \;=\; \frac{2}{3}\Big(\langle\theta^2\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\Big) \;-\; 2\,\langle\sigma^2\rangle_{\mathcal{D}}. \tag{15}$$

In a two-scale (wall/void) model with void fraction f_v and wall fraction $f_w = 1 - f_v$, with local Hubble rates H_v and H_w, one finds the variance contribution

$$\frac{Q_{\mathcal{D}}}{H^2} \lesssim 6\, f_v\, (1 - f_v)\, \Delta_H^2\,. \tag{16}$$

For representative late-time values $f_v \sim 0.6$–0.8 and $\Delta_H \sim 0.03$, one finds $|Q_{\mathcal{D}}|/H^2 \lesssim 1.3 \times 10^{-3}$. Thus, for realistic void fractions, backreaction is insufficient to mimic a dark-energy component.

7 Bounds on varying constants

We parameterize a representative drift of a dimensionful coupling through its dependence on the windowed background. For the fine-structure constant, $\alpha(t) = \alpha_0[1 + \zeta f(\rho_{N,eff})]$. Using the Leibniz rule,

$$\left|\frac{\mathrm{d}\ln\alpha}{\mathrm{d}t}\right| \sim \zeta\,\frac{1}{\mathcal{N}}\left|\frac{\mathrm{d}\ln k_{\mathrm{max}}}{\mathrm{d}t} - \frac{\mathrm{d}\ln k_{\mathrm{min}}}{\mathrm{d}t}\right| \sim \zeta\,\mathcal{O}(10^{-3})\,H_0. \tag{17}$$

Atomic clock bounds give $|\dot{\alpha}/\alpha| \lesssim 10^{-17}\,\mathrm{yr}^{-1}$, implying $\zeta \lesssim 10^{-2}$.

8 Gauge equivalence: co-scaling units vs FLRW

We prove that expressing cosmological predictions in local co-scaling units via a conformal map leaves redshift and distance observables invariant.

Setup

Let $(\mathcal{M}, g_{\mu\nu})$ be an FLRW spacetime with scale factor $a(t)$. Define the co-scaling (local-unit) metric $\hat{g}_{\mu\nu}(x) = F(t)^2 g_{\mu\nu}(x)$. Define orthonormal tetrads $e^a{}_\mu$ for $g_{\mu\nu}$ and $\hat{e}^a{}_\mu = F\, e^a{}_\mu$ for $\hat{g}_{\mu\nu}$.

Null geodesics and redshift

Conformal rescalings preserve null geodesics up to reparametrization. With co-scaling tetrads, $\hat{\omega} = F\omega$. Hence the observable redshift

$$1 + z \;=\; \frac{\omega_{\mathrm{em}}}{\omega_{\mathrm{obs}}} \;=\; \frac{\hat{\omega}_{\mathrm{em}}/F_{\mathrm{em}}}{\hat{\omega}_{\mathrm{obs}}/F_{\mathrm{obs}}} \;=\; \frac{\hat{\omega}_{\mathrm{em}}}{\hat{\omega}_{\mathrm{obs}}}\,\frac{F_{\mathrm{obs}}}{F_{\mathrm{em}}} \tag{18}$$

coincides with the standard FLRW expression.

Distances

Angular diameter distance D_A and luminosity distance D_L transform as $\hat{D}_A = F\, D_A$ and $\hat{D}_L = F\, D_L$. Since $1 + z$ is invariant, Etherington's relation holds in both gauges: $\hat{D}_L = (1 + z)^2 \hat{D}_A$.

9 Derivation of the Scale-Invariant Spectrum

Theorem (Band-entropy maximizer). Among stationary, positive spectra $S(k)$ on a finite window $[k_{\mathrm{min}}, k_{\mathrm{max}}]$ with fixed integrated band power and scale-invariance of information (uniform content per logarithmic interval), the entropy maximizer has the form $S(k) = A/k$.

Proof Sketch. Let $u = \ln k$. Maximize Shannon entropy $\mathcal{H} = -\int \sigma(u) \ln(\sigma(u)/\sigma_0) du$ subject to fixed power. Variational calculus yields $\sigma(u) = \mathrm{const}$, implying $kS(k) = \mathrm{const}$.

Multiplicative cascade construction. Partition the window into N bands. Iterate a multiplicative rebalancing map that enforces stationarity of total power and equalization of log-band content:

$$w_i^{(n+1)} = w_i^{(n)} \exp\Big(- \eta \left[\ln w_i^{(n)} - \overline{\ln w}^{(n)} \right] \Big). \tag{19}$$

This converges to $w_i \to P_0/N$, yielding $S(k) \propto 1/k$ in the continuum limit.