Heat Equation

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Demonstration

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We consider heat propagation in a medium (e.g. a solid) far away from any state transition and we define:

$$\begin{aligned} \epsilon &: \text{internal energy density} \\ c &: \text{specific heat} \\ \rho &: \text{mass density} \\ T &: \text{temperature} \end{aligned}$$

By virtue of the first thermodinamic principle ($\Delta U = \Delta Q - L = \Delta Q$):

$$d\left[\int_\Omega c\rho T \,\mathrm{d}V\right] = \left[-\int_{\partial\Omega} \vec{J}\cdot\vec{n}\,\mathrm{d}S\right]\mathrm{d}t \qquad (1)$$

Demonstration

$$d\left[\int_\Omega c\rho T\,\mathrm{d}V\right] = \left[-\int_{\partial\Omega} \vec{J}\cdot\vec{n}\,\mathrm{d}S\right]\mathrm{d}t$$

Newton-Fourier Law, heat spreads and temperature becomes uniform throughout the medium:

$$\vec{J} = -k\vec{\nabla}T \tag{2}$$

By substituting equation (2) in (1) and applying the *divergence theorem* one gets the **heat equation**:

$$\frac{\partial T}{\partial t} - \frac{k}{c\rho}\Delta T = 0 \tag{3}$$

Heat Equation

$$\frac{\partial T}{\partial t} - \frac{k}{c\rho}\Delta T = f(\vec{x}, t)$$

- First order in time
- Second order in space
- If we consider a heating system, we must add $f(\vec{x}, t)$

Numerical Solution

$$\frac{\partial T}{\partial t} - \frac{k}{c\rho}\Delta T = f(\vec{x}, t)$$

Discretize space and time: foreward difference for time, central difference for space:

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \eta \left[T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4\,T_{i,j}^{n} \right] \tag{4}$$

where n is the time index, i, j are x-index and y-index, $\eta = \frac{k\Delta t}{c\rho(\Delta x)^2}$.

General Structure

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \eta \left[T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4\,T_{i,j}^{n} \right]$$